# The Farrell-Jones Conjecture for the Hecke algebras of reductive p-adic groups

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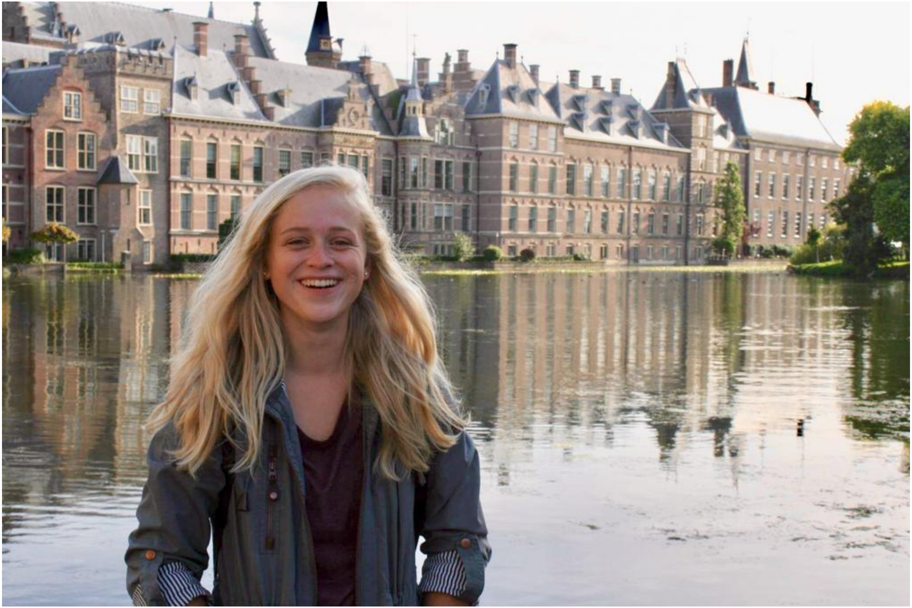
Edinburgh / Bonn, June 2021

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# Main result

- This is a joint work with Arthur Bartels.
- We formulate a version of the Farrell-Jones Conjecture for the algebraic $K$-theory of the Hecke algebra of a totally disconnected locally compact second countable Hausdorff group.
- We can prove it for any closed subgroup of a reductive $p$-adic group.
- This is interesting for the theory of smooth representations of reductive $p$-adic groups.

# Outlook

- Brief introduction to totally disconnected groups.
- Introduction to the Farrell-Jones Conjecture for discrete groups.
- Formulation of the Farrell-Jones Conjecture for the algebraic $K$-theory of the Hecke algebra of a td-group.
- Explanations why it is much harder to deal with totally disconnected groups than with discrete groups.
- Motivation for trying to extend the Farrell-Jones Conjecture from discrete groups to reductive $p$-adic groups.
- We will not say much about the actual proof which is very technical and involved.
- Our work is relies on Ranicki's work because we have to extend the notion of an assembly map.

# Dictionary

| | |
|---|---|
| td-groups | discrete groups |
| Smooth $G$-representations over $R$ | $G$-representations over $R$ |
| Hecke algebra $\mathcal{H}(G;R)$ | group ring $RG$ |
| $\exists$ approximate unit | $\exists$ unit |
| {smooth $G$-representations} = {n.d. $\mathcal{H}(G;R)$-modules} | $\{G$-representations$\}$ = $\{RG$-modules$\}$ |

| | |
|---|---|
| reductive $p$-adic groups | CAT(0)-groups |
| Examples: $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{Q}_p)$ | Examples: Fundamental groups of closed manifolds with non-sectional curvature |
| Cocompact proper smooth action on the associated Bruhat-Tits building | Cocompact proper action on a CAT(0)-space |
| Family $\mathcal{COM}$ of compact open subgroups | Family $\mathcal{FIN}$ of finite subgroups |

| | |
|---|---|
| A sequence of subgroups $K \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ of a compact td-group does in general *not* stabilize after finitely many steps. | A sequence of subgroups $F \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ of a finite group $F$ stabilizes after finitely many steps. |
| The space $G/H \times G/K$ with the diagonal $G$-action is in general *not* $G$-homotopy equivalent to a $G$-$CW$-complex. | The space $G/H \times G/K$ with the diagonal $G$-action is a zero-dimensional $G$-$CW$-complex. |
| The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are *not* $G$-homotopy equivalent in general. Fortunately $E_{\mathcal{COM}}(G)$ and $J_{\mathcal{COM}}(G)$ are $G$-homotopy equivalent. | The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are $G$-homotopy equivalent. |

| | |
|---|---|
| The Hecke algebra is only functorial under open group homomorphisms. In particular it is not functorial under inclusions of subgroups $H \subseteq G$, unless $H$ is open in $G$. | The group ring $RG$ is functorial under any group homomorphism. |
| $\exists$ unit $\Leftrightarrow$ $G$ discrete | $\exists$ unit |
| Reductive $p$-adic groups contain interesting closed but not open subgroups such as the Borel subgroup | not applicable |

# Prominent Conjectures about group rings

**Conjecture (Idempotent Conjecture)**

*The Idempotent Conjecture says that for a torsionfree group $G$ and an integral domain $R$ the elements $0$ and $1$ are the only idempotents in $RG$.*

**Conjecture (Unit Conjecture)**

*The Unit Conjecture says that for a torsionfree group $G$ and an integral domain $R$ every unit in $RG$ is trivial, i.e., of the form $r \cdot g$ for $r \in R^{\times}$ and $g \in G$.*

# A stable version of the unit conjecture

**Conjecture (Stable Unit Conjecture)**

*The Stable Unit Conjecture says that for a torsionfree group $G$ and an integral domain $R$ we can find for every unit $u$ in $RG$ a trivial unit $v$ in $RG$ such that one can pass from the invertible $(1,1)$ matrix $(u)$ to the invertible $(1,1)$ matrix $(v)$ by elementary row and column operation and taking the block sum $A \mapsto A \oplus (1)$ or the inverse operation.*

- Relevance of the reduced projective class group:

  Let $P$ be a finitely generated projective $R$-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for some $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- Note that we are not asking whether $P$ itself is finitely generated free.

**Conjecture (Vanishing of reduced projective class group for torsionfree $G$)**

*If $G$ is torsionfree, then $\widetilde{K}_0(\mathbb{Z}G)$ and $\widetilde{K}_0(FG)$ for a field $F$ of characteristic zero vanish.*

- The last conjecture implies the Idempotent Conjecture.

- There is the Whitehead group $\mathrm{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup given by the trivial units $\pm g$.
- The following conjecture plays a prominent role for the classification of closed manifolds because of the $s$-Cobordism Theorem.

**Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree $G$)**

*If $G$ is torsionfree, then*

$$\mathrm{Wh}(G) = \{0\}.$$

# Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- For some time $G$ is a discrete group.
- There are $K$-groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If $G_0$ and $G_1$ are torsionfree and $R$ is regular, one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \widetilde{K}_n(R[G_0 * G_1]) &\cong \widetilde{K}_n(RG_0) \oplus \widetilde{K}_n(RG_1). \end{aligned}$$

- If $\mathcal{H}$ is any (generalized) homology theory, then

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}); \\ \widetilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \widetilde{\mathcal{H}}_n(BG_0) \oplus \widetilde{\mathcal{H}}_n(BG_1). \end{aligned}$$

- Question: Can we find $\mathcal{H}_*$ with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that $G$ is torsionfree and $R$ is regular.
- Of course such $\mathcal{H}_*$ has to satisfy $\mathcal{H}_n(\mathrm{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-;\mathbf{K}_R)$.

**Conjecture ($K$-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)**

*The $K$-theoretic Farrell-Jones Conjecture with coefficients in the regular ring $R$ for the torsionfree group $G$ predicts that the assembly map*

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

*is bijective for every $n \in \mathbb{Z}$.*

- There is also an $L$-theory version.

- The conjectures above about the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$, $\widetilde{K}_0(FG)$ and $\mathrm{Wh}(G)$ for torsionfree $G$ are consequences of the Farrell-Jones Conjecture above.

- This follows from the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_{\mathbb{Z}})$. It is a first quadrant spectral sequence with $E^2$-term

$$
\begin{array}{ccc}
K_2(\mathbb{Z}) \cong \mathbb{Z}/2 & H_1(BG; K_2(\mathbb{Z})) & H_2(BG, K_2(\mathbb{Z})) \\
 & & \\
K_1(\mathbb{Z}) \cong \{\pm 1\} & H_1(BG; K_1(\mathbb{Z})) & H_2(BG, K_1(\mathbb{Z})) \\
 & & \\
K_0(\mathbb{Z}) \cong \mathbb{Z} & H_1(BG; K_0(\mathbb{Z})) & H_2(BG, K_0(\mathbb{Z}))
\end{array}
$$

- The Farrell-Jones Conjecture implies the Stable Unit Conjecture.
- Gardam has recently shown for a group $G$ which contains $\mathbb{Z}^3$ as subgroup of finite index that the Unit Conjecture is false.
- For this group the Farrell-Jones Conjecture and hence the Stable Unit Conjecture are true.

## Definition (Topologically rigid)

A closed topological manifold $N$ is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold $M$ as source is homotopic to a homeomorphism.

## Conjecture (Borel Conjecture)

*The Borel Conjecture for $G$ predicts that an aspherical closed manifold with fundamental group $G$ is topologically rigid.*

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.
- The Farrell-Jones Conjecture implies the Borel Conjecture in dimensions $\geq 5$.

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Novikov Conjecture.
- Bass Conjecture.
- Moody's Induction Conjecture.
- Serre's Conjecture.
- Classification of manifolds.
- Poincaré duality groups.
- $\kappa$-classes for aspherical manifolds.
- Hyperbolic groups with a sphere as boundary.
- Rational calculation of the homotopy groups of the space of automorphism of aspherical closed manifolds in a certain range.
- Stable Cannon Conjecture.

# The general version the Farrell-Jones Conjecture

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups $G$ and all rings $R$.

**Conjecture ($K$-theoretic Farrell-Jones-Conjecture)**

*The $K$-theoretic Farrell-Jones Conjecture with coefficients in $R$ for the group $G$ predicts that the assembly map*

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG).$$

*is bijective for every $n \in \mathbb{Z}$.*

- $H_*^G(-;\mathbf{K}_R)$ is a $G$-homology theory satisfying

$$H_n^G(G/H;\mathbf{K}_R) \cong K_n(RH).$$

- The Farrell-Jones Conjecture is equivalent to the homotopy theoretic version that we have a weak homotopy equivalence

$$\operatorname{hocolim}_{G/V \in \mathrm{Or}_{\mathcal{VC}\mathrm{yc}}(G)} \mathbf{K}_R(G/V) \xrightarrow{\simeq} \mathbf{K}_R(G/G).$$

- There is also an $L$-theory version.

- One can also allow twisted group rings and orientation characters.

- In the sequel the Full Farrell-Jones Conjecture refers to the most general version for both $K$-theory and $L$-theory, namely, with coefficients in additive $G$-categories (with involution) and finite wreath products.

# Status of the Full Farrell-Jones Conjecture

**Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)**

*Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:*

- *Hyperbolic groups;*
- $\mathrm{CAT}(0)$*-groups;*
- *Solvable groups;*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) $d$-dimensional manifolds (possibly with boundary) for $d \leq 3$;*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field $F$;*
- *All $S$-arithmetic groups;*
- *mapping class groups.*

## Theorem (continued)

*Moreover, $\mathcal{FJ}$ has the following inheritance properties:*

- *If $G_1$ and $G_2$ belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of $G$ with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Carlsson, Davis, Farrell, Ferry, Goodwillie, Hsiang, Jones, Kasprowski, Linnell, Madsen, Pedersen, Quinn, Ranicki, Reich, Rognes, Varisco, Weinberger, Winges, Yu, Wu.

- The Farrell-Jones Conjecture is open for:
  - $\mathrm{Out}(F_n)$;
  - amenable groups;
  - Thompson's groups;

- It would be great to find a counterexample.

# Towards the Farrell-Jones Conjecture for reductive *p*-adic groups

**Conjecture (Farrell-Jones Conjecture for fields of characteristic zero)**

*Let F be a field of characteristics zero. Then the assembly map*

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_F) \xrightarrow{\cong} K_n(FG)$$

*is an isomorphism for* $n \in \mathbb{Z}$.

It implies:

**Conjecture (Moody's Induction Conjecture)**

*For every field F of characteristic zero we get an isomorphism*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

**Theorem (Bernstein, Dat)**

*Let G be a reductive p-adic group. Then the canoncial map*

$$\operatorname*{colim}_{G/K \in \mathrm{Or}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;\mathbb{C})) \xrightarrow{\cong} K_0(\mathcal{H}(G;\mathbb{C}))$$

*is rationally an isomorphism.*

- Often a smooth $G$-representation regarded as a module over $\mathcal{H}(G;\mathbb{C})$ has a finite projective resolution and hence defines an element in $K_0(\mathcal{H}(G;\mathbb{C}))$.
- There is the conjecture, often attributed to Bernstein, that every irreducible super-cuspidal representation of a reductive $p$-adic group is (compactly) induced from some compact open subgroup.

**Theorem (Bartels-Lück)**

*Let $G$ be a closed subgroup of a reductive $p$-adic group. Let $R$ be a regular ring satisfying $\mathbb{Q} \subseteq R$. Then the canoncial map*

$$\operatorname*{colim}_{G/K \in \mathrm{Or}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;R)) \xrightarrow{\cong} K_0(\mathcal{H}(G;R))$$

*is an isomorphism.*

## Conjecture (Farrell-Jones Conjecture for td-groups)

*For any td-group G and any regular ring R satisfying $\mathbb{Q} \subseteq R$ and $n \in \mathbb{Z}$ the assembly map*

$$H_n^G(E_{\mathcal{COM}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$$

*is an isomorphism.*

- $H_*^G(-; \mathbf{K}_R)$ is a smooth $G$-homology theory satisfying $H_n^G(G/H; \mathbf{K}_R) \cong K_n(\mathcal{H}(H; R))$ for every open subgroup $H \subseteq G$.

## Theorem (Bartels-Lück)

*The Farrell-Jones Conjecture holds for G if G is a closed subgroup of a reductive p-adic group.*

# Strategy of the proof in the discrete case

- The assembly map can be thought of an approximation of the algebraic $K$- or $L$-theory by a homology theory.
- The basic feature between the left and right side of the assembly map is that on the left side one has excision which is not present on the right side.
- In general excision is available if one can make representing cycles small.
- A best illustration for this is the proof of excision for simplicial or singular homology based on subdivision whose effect is to make the support of cycles arbitrary small.

- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class.

- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.

- The idea is to use the geodesic flow on the sphere tangent bundle of the universal covering to gain the necessary control and to use transfer methods which allow to work on the sphere tangent bundle.

# Challenges in the td-case

- It is not enough to consider open subgroups. In particular the reduction from covirtually cyclic subgroups to compact open subgroups is very difficult.
- Extension of the functor $G/H \mapsto \mathbf{K}(\mathcal{H}(H))$ to subgroups $H \subseteq G$ which are not open.
- The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are *not* $G$-homotopy equivalent in general.
- One needs to understand the geometry of the Bruhat-Tits building and its compactification and understand the associated geodesic flow space.

# Thanks

- Thank you for your attention!
- Many thanks for the organizers of this conference:
  Clark Barwick, Diarmuid Crowley, Oscar Randal-Williams, Ulrike Tillmann, and Gillian Kerr!
- Andrew, thank you for all your work and help!