SURVEY ON L^2-INVARIANTS AND 3-MANIFOLDS

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Abstract. In this paper give a survey about L^2-invariants focusing on 3-manifolds.

0. Introduction

The theory of L^2-invariants was triggered by Atiyah in his paper on the L^2-index theorem [4]. He followed the general principle to consider a classical invariant of a compact manifold and to define its analog for the universal covering taking the action of the fundamental group into account. Its application to (co)homology, Betti numbers and Reidemeister torsion led to the notions of L^2-(cohomology), L^2-Betti numbers and L^2-torsion. Since then L^2-invariants were developed much further. They already had and will continue to have striking, surprizing, and deep impact on questions and problems of fields, for some of which one would not expect any relation, e.g., to algebra, differential geometry, global analysis, group theory, topology, transformation groups, and von Neumann algebras.

The theory of 3-manifolds has also a quite impressive history culminating in the work of Waldhausen and in particular of Thurston and much later in the proof of the Geometrization Conjecture by Perelman and of the Virtual Fibration Conjecture by Agol. It is amazing how many beautiful, easy to comprehend, and deep theorems, which often represent the best result one can hope for, have been proved for 3-manifolds.

The motivating question of this survey paper is: What happens if these two prominent and successful areas meet one another? The answer will be: Something very interesting.

In Section 1 we give a brief overview over basics about 3-manifolds, which of course can be skipped by someone who has already some background. We will explain the prime decomposition, Kneser's Conjecture, the Jaco-Shalen splitting, Thurston's Geometrization Conjecture, and the Virtual Fibration Conjecture. They give a deep insight into the structure of 3-manifolds. All these conjectures have meanwhile been proved. We also explain the Thurston norm and polytope, which have been connected to L^2-invariants in the recent years.

In Section 2 we briefly explain the definition of L^2-Betti numbers and of L^2-torsion including the necessary input from the theory of von Neumann algebras. For the rest of the article the concrete constructions are not relevant and can be skipped, but one has to understand the basic properties of the L^2-invariants, see Subsection 2.8. For the large variety of applications of L^2-invariants, we will refine ourselves to 3-manifolds. A discussion of all the other plentiful applications to various different fields goes beyond the scope of this survey article.

In Section 3 we discuss the main open conjectures and problems about L^2-invariants: the Atiyah Conjecture, the Singer Conjecture, the Determinant Conjecture, various conjectures about approximation and homological growth, and the

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relation between simplicial volume and L^2-invariants for aspherical closed manifolds. These conjectures make sense and are interesting in all dimensions.

In general L^2-invariants are hard to compute. We explain in Section 4 that one can compute the L^2-Betti numbers and the L^2-torsion for 3-manifolds explicitly exploring all the known results about 3-manifolds mentioned above.

In Section 5 we discuss the status of all the conjectures mentioned in Section 3 for 3-manifolds. Roughly speaking, they are essentially all known except the conjecture about homological growth, which is wide open also in dimension 3.

The remaining sections are dealing with rather new developments concerning the twisting of L^2-invariants with (not necessarily unitary or unimodular) finite-dimensional representations. The basics of this construction are presented in Section 6 and Section 7.

All these twisted invariants make sense in all dimensions and have a great potential, but concrete and interesting results are known so far only in dimension 3. Again this due to the fact that the structure of 3-manifolds is rather special and well understood nowadays. This will be carried out in Sections 8 and 9, where the Turston norm and polytope are linked to the degree of the L^2-torsion function, universal L^2-torsion, and the L^2-polytope.

In the final Section 10 we relate the conjecture of homological growth to the question, whether the L^2-torsion of an aspherical closed 3-manifold depends only on the profinite completion of the fundamental group.

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The paper is organized as follows:

Contents

1. Brief survey on 3-manifolds

In this section we give a brief survey about 3-manifolds.

For the remainder of this paper 3-manifold is to be understood to be connected compact and orientable, and we allow a non-empty boundary. The assumption orientable will make the formulation of some results easier and is no real constraint for L^2-invariants, since these are multiplicative under finite coverings and therefore one may pass to the orientation covering in the non-orientable case.

1.1. The prime decomposition and Kneser's Conjecture.

A 3-manifold M is *prime* if for any decomposition of M as a connected sum $M_1 \# M_2$, M_1 or M_2 is homeomorphic to S^3. It is *irreducible* if every embedded 2-sphere bounds an embedded 3-disk. Any prime 3-manifold is either irreducible or is homeomorphic to $S^1 \times S^2$ [40, Lemma 3.13].

Every 3-manifold M has a prime decomposition, i.e., one can write M as a connected sum

$$M \quad = \quad M_1 \# M_2 \# \ldots \# M_r,$$

where each M_j is prime, and this prime decomposition is unique up to renumbering and orientation preserving homeomorphism [40, Theorems 3.15 and 3.21].

Let M be a 3-manifold with incompressible boundary whose fundamental group admits a splitting $\alpha\colon \pi_1(M) \to \Gamma_1 * \Gamma_2$. Kneser's Conjecture, whose proof can be found in [40, Chapter 7], says that there are manifolds M_1 and M_2 with Γ_1 and Γ_2 as fundamental groups and a homeomorphism $M \to M_1 \# M_2$ inducing α on the fundamental groups. Here *incompressible boundary* means that no boundary component is S^2 and the inclusion of each boundary component into M induces an injection on the fundamental groups.

1.2. **The Jaco-Shalen-Johannson splitting.** We use the definition of *Seifert manifold* given in [89], which we recommend as a reference on Seifert manifolds. The work of Casson and Gabai shows that an irreducible 3-manifold with infinite fundamental group π is Seifert if and only if π contains a normal infinite cyclic subgroup, see [37, Corollary 2 on page 395]. This together with the argument appearing in [89, page 436] implies the following statement: If a 3-manifold M has infinite fundamental group and empty or incompressible boundary, then it is Seifert if and only if it admits a finite covering $\overline{M}$, which is the total space of a S^1-principal bundle over a compact orientable surface.

Johannson [47] and Jaco and Shalen [44] have shown for an irreducible 3-manifold M with incompressible boundary the following result. There is a finite family of disjoint, pairwise-nonisotopic incompressible tori in M, which are not isotopic to boundary components and which split M into pieces that are Seifert manifolds or are *geometrically atoroidal*, meaning that they admit no embedded incompressible torus (except possibly parallel to the boundary). A minimal family of such tori is unique up to isotopy, and we will say that it gives a *toral splitting* of M.

A *graph manifold* is an irreducible 3-manifold for which all its pieces in the Jaco-Shalen-Johannson splitting are Seifert fibered spaces.

1.3. **Thurston's Geometrization Conjecture.** Recall that a manifold (possible with boundary) is called *hyperbolic* if its interior admits a complete Riemannian metric whose sectional curvature is constant -1.

Thurston's Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental groups states that the geometrically atoroidal pieces in the Jaco-Shalen-Johannson splitting carry a hyperbolic structure.

Roughly speaking, a *geometry* on a 3-manifold M is a complete locally homogeneous Riemannian metric on its interior. The precise definition is given for instance in [3, page 17]. The universal cover of the interior has a complete homogeneous Riemannian metric, meaning that the isometry group acts transitively [91]. Thurston has shown that there are precisely eight simply connected 3-dimensional geometries having compact quotients, namely S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Sol and $\mathbb{H}^3$. If a closed 3-manifold admits a geometric structure modelled on one of these eight geometries, then the geometry involved is unique.

Let M be a closed Seifert manifold. Then it has a geometry. In terms of the Euler class e of the Seifert bundle and the Euler characteristic χ of the base orbifold, the geometric structure is determined as follows [89, Theorem 5.3]

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2(\mathbb{R})}$

The geometry is S^3 if and only if $\pi_1(M)$ is finite. Moreover, M is finitely covered by the total space $\overline{M}$ of an S^1-principal bundle $p\colon \overline{M} \to F$ over an orientable closed surface F. We have $e = 0$ if and only if $e(p) = 0$, and the Euler characteristic χ of the base orbifold of M is negative, zero or positive if and only if $\chi(\overline{M}/S^1)$ has the same property, see [89, page 426, 427 and 436].

For completeness we mention that Thurston's Geometrization Conjecture implies for a closed 3-manifold with finite fundamental group that its universal covering is homeomorphic to S^3, the fundamental group of M is a subgroup of $SO(4)$ and the action of it on the universal covering is conjugated by a homeomorphism to the restriction of the obvious $SO(4)$-action on S^3. This implies, in particular, the *Poincaré Conjecture* that any homotopy 3-sphere is homeomorphic to S^3.

Many results about 3-manifolds have as hypothesis that Thurston's Geometrization Conjecture holds. Meanwhile Thurston's Geometrization Conjecture is known to be true, a proof is given in [56, 77] following the spectacular ideas of Perelman.

1.4. **The Virtual Fibration Conjecture.** Given a 3-manifold and a non-trivial element $\phi \in H^1(N;\mathbb{Q}) = \operatorname{Hom}(\pi_1(N),\mathbb{Q})$, we say that ϕ *is fibered* if there exists a locally trivial fiber bundle $p\colon N \to S^1$ with a compact surface as fiber and an element $r \in \mathbb{Q}$ such that the induced map $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $r \cdot \phi$. We say $\phi \in H^1(N;\mathbb{R})$ *is quasi-fibered* if ϕ is the limit in $H^1(N;\mathbb{R})$ of fibered classes in $H^1(N;\mathbb{Q}) \subseteq H^1(N;\mathbb{R})$. A group is *residually finite rationally solvable* (RFRS for short), if there is a filtration of π by subgroups $\pi = \pi_0 \supseteq \pi_1 \supseteq \pi_2 \supseteq \cdots$ such that

(1) $\bigcap_i \pi_i = \{1\}$;
(2) for any i the group π_i is a normal, finite-index subgroup of π;
(3) for any i the map $\pi_i \to \pi_i/\pi_{i+1}$ factors through $\pi_i \to H_1(\pi_i;\mathbb{Z})/\text{torsion}$.

If (P) is a property of groups, for instance being (RFRS), then a group is called virtually (P) if it contains a subgroup of finite index which has property (P). The following is a straightforward consequence of the Virtual Fibering Theorem of Agol [1, Theorem 5.1], see also [31, Corollary 5.2] and [53].

Theorem 1.1. *Let N be a prime 3-manifold. Suppose that $\pi_1(N)$ is virtually* RFRS*. Then there exists a finite regular cover $p\colon \widehat{N} \to N$ such that for every class $\phi \in H^1(N;\mathbb{R})$ the class $p^*\phi \in H^1(\widehat{N};\mathbb{R})$ is quasi-fibered.*

The following theorem was proved by Agol [2] and Wise [102, 103] in the hyperbolic case. It was proved by Liu [62] and Przytycki-Wise [84] for graph manifolds with boundary and it was proved by Przytycki-Wise [83] for manifolds with a non-trivial Jaco-Shalen-Johannson decomposition and at least one hyperbolic piece in the JSJ decomposition.

Theorem 1.2. *If N is a prime* 3*-manifold that is not a closed graph manifold, then $\pi_1(N)$ is virtually* RFRS*.*

This implies that any hyperbolic 3-manifold M has a finite covering $p\colon \overline{M} \to M$ such that $\overline{M}$ fibers over S^1 in the sense that there exists a locally trivial fiber bundle $\overline{M} \to S^1$ with a compact 2-manifold as fiber.

1.5. **Topological rigidity.** The fundamental group plays a dominant role in the theory of 3-manifolds. Besides Kneser's Conjecture this is illustrated by the following discussion of topological rigidity.

By the Sphere Theorem [40, Theorem 4.3], an irreducible 3-manifold is *aspherical*, i.e., all its higher homotopy groups vanish, if and only if it is a 3-disk or has infinite fundamental group. If M and N are two aspherical closed 3-manifolds, then they are homeomorphic if and only if their fundamental groups are isomorphic. Actually, every isomorphism between their fundamental groups is induced by a homeomorphism. More generally, every 3-manifold N with torsionfree fundamental group group is topologically rigid in the sense that any homotopy equivalence of closed 3-manifolds with N as target is homotopic to a homeomorphism. This follows from results of Waldhausen, see Hempel [40, Lemma 10.1 and Corollary 13.7] and Turaev [97], as explained for instance [58, Section 5].

1.6. **On the fundamental groups of 3-manifolds.** The fundamental group of a closed manifold is finitely presented. Fix a natural number $d \geq 4$. Then a group G is finitely presented if and only if it occurs as fundamental group of a closed orientable d-dimensional manifold. This is not true in dimension 3. A detailed exposition about the problem, which finitely presented groups occur as fundamental groups of closed 3-manifolds, can be found in [3]. For us it will be important that the fundamental group of any 3-manifold is residually finite, This follows from [41] and the proof of the Geometrization Conjecture. More information about fundamental groups of 3-manifolds can be found for instance in [3].

1.7. **The Thurston norm and the dual Thurston polytope.** Let M be a compact oriented 3-manifold. Recall the definition in [96] of the *Thurston norm* $x_M(\phi)$ of a 3-manifold M and an element $\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}(\pi_1(M),\mathbb{Z})$:

$$x(\phi)– := \min\{\chi_-(F) \mid F \subset M \text{ properly embedded surface dual to } \phi\},$$

where, given a surface F with connected components $F_1, F_2, \ldots, F_k$, we define

$$\chi_-(F) \;=\; \sum_{i=1}^{k} \max\{-\chi(F_i), 0\}.$$

Thurston [96] showed that this defines a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$ which we also denote by x_M. In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(M;\mathbb{R})$

$$x_M(r \cdot \phi) \quad = \quad |r| \cdot x_M(\phi). \tag{1.3}$$

If $p\colon \widetilde{M} \to M$ is a finite covering with n sheets, then Gabai [36, Corollary 6.13] showed that

$$x_{\widetilde{M}}(p^*\phi) \quad = \quad n \cdot x_M(\phi). \tag{1.4}$$

If $F \to M \xrightarrow{p} S^1$ is a fiber bundle for a 3-manifold M and compact surface F, and $\phi \in H^1(M;\mathbb{Z})$ is given by $H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$, then by [96, Section 3] we have

$$x_M(\phi) \quad = \quad \begin{cases} -\chi(F), & \text{if } \chi(F) \leq 0; \\ 0, & \text{if } \chi(F) \geq 0. \end{cases} \tag{1.5}$$

We refer to

$$B_{x_M} \;:=\; \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leq 1\} \tag{1.6}$$

as the *Thurston norm ball*. In the sequel we will identify $H^1(M;\mathbb{R}) = H_1(M;\mathbb{R})^*$ and $V = V^{**}$ for a finite-dimensional real representation V by the obvious isomorphisms. Then there is a notion of a polytope dual to $B^*_{x_M}$, see [27, Subsection 3.5] and we define the *dual Thurston polytope*

$$T(M)^* := B^*_{x_M} \subset (H^1(M;\mathbb{R}))^* = H_1(M;\mathbb{R}). \tag{1.7}$$

Explicitly it is given by

$$T(M)^* = \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

Thurston [96, Theorem 2 on page 106 and first paragraph on page 107] has shown that $T(M)^*$ is an integral polytope, i.e, the convex hull of finitely many points in the integral lattice $H_1(M;\mathbb{Z})/\text{torsion} \subseteq H_1(M;\mathbb{R})$.

A *marking* for a polytope is a (possibly empty) subset of the set of its vertices. We conclude from Thurston [96, Theorem 5] that we can equip $T(M)^*$ with a marking such that $\phi \in H^1(M;\mathbb{R})$ is fibered if and only if it pairs maximally with a marked vertex, i.e., there exists a marked vertex v of $T(M)^*$, such that $\phi(v) > \phi(w)$ for any vertex $w \neq v$.

2. Brief survey on L^2-invariants

2.1. **Group von Neumann algebras.** Denote by $L^2(G)$ the Hilbert space $L^2(G)$ consisting of formal sums $\sum_{g\in G} \lambda_g \cdot g$ for complex numbers λ_g such that $\sum_{g\in G} |\lambda_g|^2 < \infty$. This is the same as the Hilbert space completion of the complex group ring $\mathbb{C}G$ with respect to the pre-Hilbert space structure for which G is an orthonormal basis. Note that left multiplication with elements in G induces an isometric G-action on $L^2(G)$. Given a Hilbert space H, denote by $\mathcal{B}(H)$ the C^*-algebra of bounded operators from H to itself, where the norm is the operator norm and the involution is given by taking adjoints.

Definition 2.1 (Group von Neumann algebra)**.** The *group von Neumann algebra* $\mathcal{N}(G)$ of the group G is defined as the algebra of G-equivariant bounded operators from $L^2(G)$ to $L^2(G)$

$$\mathcal{N}(G) \quad := \quad \mathcal{B}(L^2(G))^G.$$

In the sequel we will view the complex group ring $\mathbb{C}G$ as a subring of $\mathcal{N}(G)$ by the embedding of $\mathbb{C}$-algebras $\rho_r\colon \mathbb{C}G \to \mathcal{N}(G)$ which sends $g \in G$ to the G-equivariant operator $r_{g^{-1}}\colon L^2(G) \to L^2(G)$ given by right multiplication with g^{-1}.

Example 2.2 (The von Neumann algebra of a finite group)**.** If G is finite, then nothing happens, namely $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$.

Example 2.3 (The von Neumann algebra of $\mathbb{Z}^d$)**.** In general there is no concrete model for $\mathcal{N}(G)$. However, for $G = \mathbb{Z}^d$, there is the following illuminating model for the group von Neumann algebra $\mathcal{N}(\mathbb{Z}^d)$. Let $L^2(T^d)$ be the Hilbert space of equivalence classes of L^2-integrable complex-valued functions on the d-dimensional torus T^d, where two such functions are called equivalent if they differ only on a subset of measure zero. Define the ring $L^\infty(T^d)$ by equivalence classes of essentially bounded measurable functions $f\colon T^d \to \mathbb{C}$, where essentially bounded means that there is a constant $C > 0$ such that the set $\{x \in T^d \mid |f(x)| \geq C\}$ has measure zero. An element $(k_1,\dots,k_d)$ in $\mathbb{Z}^d$ acts isometrically on $L^2(T^d)$ by pointwise multiplication with the function $T^d \to \mathbb{C}$, which maps $(z_1,z_2,\dots,z_d)$ to $z_1^{k_1}\cdot\dots\cdot z_d^{k_d}$. The Fourier transform yields an isometric $\mathbb{Z}^d$-equivariant isomorphism $L^2(\mathbb{Z}^d) \xrightarrow{\cong} L^2(T^d)$. We conclude $\mathcal{N}(\mathbb{Z}^d) = \mathcal{B}(L^2(T^d))^{\mathbb{Z}^d}$. We obtain an isomorphism of C^*-algebras

$$L^\infty(T^d) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^d)$$

by sending $f \in L^\infty(T^d)$ to the $\mathbb{Z}^d$-equivariant operator $M_f\colon L^2(T^d) \to L^2(T^d),\ g \mapsto g\cdot f$, where $(g\cdot f)(x)$ is defined by $g(x)\cdot f(x)$.

2.2. **The von Neumann dimension.** An important feature of the group von Neumann algebra is its trace.

Definition 2.4 (Von Neumann trace)**.** The *von Neumann trace* on $\mathcal{N}(G)$ is defined by

$$\operatorname{tr}_{\mathcal{N}(G)}\colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e\rangle_{L^2(G)},$$

where $e \in G \subseteq L^2(G)$ is the unit element.

Definition 2.5 (Finitely generated Hilbert module)**.** A finitely generated *Hilbert $\mathcal{N}(G)$-module V* is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^r$ for some natural number r. A *morphism of Hilbert $\mathcal{N}(G)$-modules* $f\colon V \to W$ is a bounded G-equivariant operator.

Definition 2.6 (Von Neumann dimension)**.** Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a matrix $A = (a_{i,j}) \in M_{r,r}(\mathcal{N}(G))$ with $A^2 = A$ such that the image of the G-equivariant bounded operator $r_A^{(2)}\colon L^2(G)^r \to L^2(G)^r$ given by A is isometrically G-isomorphic to V. Define the *von Neumann dimension* of V by

$$\dim_{\mathcal{N}(G)}(V) \quad := \quad \sum_{i=1}^{r} \operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}) \quad \in \mathbb{R}^{\geq 0}.$$

The von Neumann dimension $\dim_{\mathcal{N}(G)}(V)$ depends only on the isomorphism class of the Hilbert $\mathcal{N}(G)$-module V but not on the choice of r and the matrix A. The von Neumann dimension $\dim_{\mathcal{N}(G)}$ is *faithful*, i.e. $\dim_{\mathcal{N}(G)}(V) = 0 \Leftrightarrow V = 0$ holds for any finitely generated Hilbert $\mathcal{N}(G)$-module V. It is weakly exact in the following sense, see [69, Theorem 1.12 on page 21].

Lemma 2.7. *Let $0 \to V_0 \xrightarrow{i} V_1 \xrightarrow{p} V_2 \to 0$ be a sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules. Suppose that it is weakly exact, i.e., i is injective, the closure of i is the kernel of p and the image of p is dense. Then*

$$\dim_{\mathcal{N}(G)}(V_1) = \dim_{\mathcal{N}(G)}(V_0) + \dim_{\mathcal{N}(G)}(V_0).$$

Example 2.8 (Von Neumann dimension for finite groups)**.** If G is finite, then $\dim_{\mathcal{N}(G)}(V)$ is $\frac{1}{|G|}$-times the complex dimension of the underlying complex vector space V.

Example 2.9 (Von Neumann dimension for $\mathbb{Z}^d$)**.** Let $X \subset T^d$ be any measurable set and $\chi_X \in L^\infty(T^d)$ be its characteristic function. Denote by $M_{\chi_X}\colon L^2(T^d) \to L^2(T^d)$ the $\mathbb{Z}^d$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^d)$-module with $\dim_{\mathcal{N}(\mathbb{Z}^d)}(V) = \operatorname{vol}(X)$.

2.3. **Weak isomorphisms.** A bounded G-equivariant operator $f\colon L^2(G)^r \to L^2(G)^s$ is called a *weak isomorphism* if and only if it is injective and its image is dense. If there exists a weak isomorphism $L^2(G)^r \to L^2(G)^s$, then we must have $r = s$ by Lemma 2.7. The following statements are equivalent for a bounded G-equivariant operator $f\colon L^2(G)^r \to L^2(G)^r$, see [69, Lemma 1.13 on page 23]:

(1) f is a weak isomorphism;
(2) Its adjoint f^* is a weak isomorphism;
(3) f is injective;
(4) f has dense image;
(5) The von Neumann dimension of the closure of the image of f is r.

2.4. **The Fuglede-Kadison determinant.**

Definition 2.10 (Spectral density function)**.** Let $f\colon V \to W$ be a morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules. Denote by $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator f^*f. Define the *spectral density function of f* by

$$F_f\colon \mathbb{R} \to \mathbb{R}^{\geq 0} \quad \lambda \mapsto \dim_{\mathcal{N}(G)}\bigl(\operatorname{im}(E_{\lambda^2}^{f^*f})\bigr) = \operatorname{tr}_{\mathcal{N}(G)}(E_{\lambda^2}^{f^*f}).$$

The spectral density function is monotone non-decreasing and right-continuous. We have $F(0) = \dim_{\mathcal{N}(G)}(\ker(f))$.

Example 2.11 (Spectral density function for finite groups)**.** Let G be finite and $f\colon U \to V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules, i.e., of finite-dimensional unitary G-representations. Then $F(f)$ is the right-continuous step function whose value at λ is the sum of the complex dimensions of the eigenspaces of f^*f for eigenvalues $\mu \le \lambda^2$ divided by the order of G, or, equivalently, the sum of the complex dimensions of the eigenspaces of $|f|$ for eigenvalues $\mu \le \lambda$ divided by the order of G.

Example 2.12 (Spectral density function for $\mathbb{Z}^d$)**.** Let $G = \mathbb{Z}^d$. In the sequel we use the notation and the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of Example 2.3. For $f \in L^\infty(T^d)$ the spectral density function $F(M_f)$ of $M_f\colon L^2(T^d) \to L^2(T^d)$ sends λ to the volume of the set $\{z \in T^d \mid |f(z)| \le \lambda\}$.

Definition 2.13 (Fuglede-Kadison determinant)**.** Let $f\colon V \to W$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. Let $F_f(\lambda)$ be the spectral density function of Definition 2.10 which is a monotone non-decreasing right-continuous function. Let dF be the unique measure on the Borel σ-algebra on $\mathbb{R}$ which satisfies $dF((a,b]) = F(b) - F(a)$ for $a < b$. Define the *Fuglede-Kadison determinant*

$$\det_{\mathcal{N}(G)}(f) \in \mathbb{R}^{\ge 0}$$

to be the positive real number

$$\det_{\mathcal{N}(G)}(f) = \exp\left(\int_{0+}^{\infty} \ln(\lambda)\; dF\right),$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda)\; dF$ converges to a real number, and to be 0 otherwise.

Note that in the definition above we do not require that the source and domain of f agree or that f is injective or that f is surjective. Our conventions imply that the Fulgede-Kadison operator of the zero operator $0\colon V \to W$ is 1.

Example 2.14 (Fuglede-Kadison determinant for finite groups)**.** To illustrate this definition, we look at the example where G is finite. We essentially get the classical determinant $\det_{\mathbb{C}}$. Namely, let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the non-zero eigenvalues of f^*f with multiplicity μ_i. Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f,

$$\det_{\mathcal{N}(G)}(f) \quad = \quad \exp\left(\sum_{i=1}^{r} \frac{\mu_i}{|G|} \cdot \ln(\sqrt{\lambda_i})\right) \quad = \quad \prod_{i=1}^{r} \lambda_i^{\frac{\mu_i}{2\cdot|G|}} \quad = \quad \det_{\mathbb{C}}\big(\overline{f^*f}\big)^{\frac{1}{2\cdot|G|}},$$

where $\det_{\mathbb{C}}\big(\overline{f^*f}\big)$ is put to be 1 of f is the zero operator and hence $\overline{f^*f}$ is $\mathrm{id}\colon \{0\} \to \{0\}$. If $f\colon \mathbb{C}G^m \to \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{N}(G)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}.$$

Example 2.15 (Fuglede-Kadison determinant for $\mathcal{N}(\mathbb{Z}^d)$)**.** Let $G = \mathbb{Z}^d$. We use the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of Example 2.3. For $f \in L^\infty(T^d)$ we conclude from Example 2.12

$$\det_{\mathcal{N}(\mathbb{Z}^d)}\big(M_f\colon L^2(T^d) \to L^2(T^d)\big) = \exp\left(\int_{T^d} \ln(|f(z)|) \cdot \chi_{\{u \in T^d \mid f(u) \neq 0\}}\; \mathrm{dvol}_z\right)$$

using the convention $\exp(-\infty) = 0$.

Let $i\colon H \to G$ be an injective group homomorphism. Let V be a finitely generated Hilbert $\mathcal{N}(H)$-module. There is an obvious pre-Hilbert structure on $\mathbb{C}G \otimes_{\mathbb{C}H} V$ for which G acts by isometries since $\mathbb{C}G \otimes_{\mathbb{C}H} V$ as a complex vector space can be identified with $\bigoplus_{G/H} V$. Its Hilbert space completion is a finitely generated Hilbert $\mathcal{N}(G)$-module and denoted by i_*V. A morphism of finitely generated Hilbert $\mathcal{N}(H)$-modules $f\colon V \to W$ induces a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules $i_*f\colon i_*V \to i_*W$.

The following theorem can be found with proof in [69, Theorem 3.14 on page 128 and Lemma 3.15 (4) on page 129].

Theorem 2.16 (Fuglede-Kadison determinant)**.**

(1) Let $f\colon U \to V$ and $g\colon V \to W$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules such that f has dense image and g is injective. Then

$$\det_{\mathcal{N}(G)}(g \circ f) = \det_{\mathcal{N}(G)}(f) \cdot \det_{\mathcal{N}(G)}(g);$$

(2) Let $f_1\colon U_1 \to V_1$, $f_2\colon U_2 \to V_2$ and $f_3\colon U_2 \to V_1$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules such that f_1 has dense image and f_2 is injective. Then

$$\det_{\mathcal{N}(G)}\begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} = \det_{\mathcal{N}(G)}(f_1) \cdot \det_{\mathcal{N}(G)}(f_2);$$

(3) Let $f\colon U \to V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. Then

$$\det_{\mathcal{N}(G)}(f) = \det_{\mathcal{N}(G)}(f^*) = \sqrt{\det_{\mathcal{N}(G)}(f^*f)} = \sqrt{\det_{\mathcal{N}(G)}(ff^*)};$$

(4) Let $i\colon H \to G$ be the inclusion of a subgroup of finite index $[G:H]$. Let $i^*f\colon i^*U \to i^*V$ be the morphism of finitely generated Hilbert $\mathcal{N}(H)$-modules obtained from f by restriction. Then

$$\det_{\mathcal{N}(H)}(i^*f) \quad = \quad \det_{\mathcal{N}(G)}(f)^{[G:H]};$$

(5) Let $i\colon H \to G$ be an injective group homomorphism and let $f\colon U \to V$ be a morphism of finitely generated Hilbert $\mathcal{N}(H)$-modules. Then

$$\det_{\mathcal{N}(G)}(i_*f) = \det_{\mathcal{N}(H)}(f).$$

2.5. **L^2-Betti numbers and L^2-torsion of finite Hilbert $\mathcal{N}(G)$-chain complexes.** Let G be a group and let

$$\cdots 0 \to 0 \to C_n^{(2)} \xrightarrow{c_n^{(2)}} C_{n-1}^{(2)} \xrightarrow{c_{n-1}^{(2)}} \cdots \xrightarrow{c_2^{(2)}} C_1^{(2)} \xrightarrow{c_1^{(2)}} C_0^{(2)} \to 0 \to \cdots$$

be a finite $\mathcal{N}(G)$-chain complex $(C_*^{(2)}, c_*^{(2)})$, i.e., each $C_p^{(2)}$ is a finitely generated Hilbert $\mathcal{N}(G)$-module, each differential $c_p^{(2)}$ is a G-equivariant bounded operator and there is a natural number n such that $C_p^{(2)} = 0$ for $p < 0$ and for $p > n$. For $p \in \mathbb{N}$, we define the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_p^{(2)}(C_*^{(2)}) := \ker(c_p^{(2)})/\overline{\operatorname{im}(c_{p+1}^{(2)})}.$$

Note that we divide out the closure of the image of $c_{p+1}^{(2)}$ to ensure that we indeed obtain a finitely generated Hilbert $\mathcal{N}(G)$-module. Denote by

$$b_p^{(2)}(C_*^{(2)}) \quad := \quad \dim_{\mathcal{N}(G)}(H_p^{(2)}(C_*^{(2)})) \in \mathbb{R}^{\geq 0} \tag{2.17}$$

the *p-th L^2-Betti number* of $C_*^{(2)}$. We say that the complex $C_*^{(2)}$ is *L^2-acyclic* if all its L^2-Betti numbers vanish.

Define the pth *Laplace operator*

$$\Delta_p^{(2)} := c_{p+1}^{(2)} \circ (c_p^{(2)})^* + (c_{p-1}^{(2)})^* \circ c_p^{(2)}\colon C_p^{(2)} \to C_p^{(2)}.$$

Then $b_p^{(2)}(C_*^{(2)}) = \dim_{\mathcal{N}(G)}(\ker(\Delta_p^{(2)}))$, see [69, Lemma 1.18 on page 24]. Hence $b_p^{(2)}(C_*^{(2)})$ vanishes if and only if $\ker(\Delta_p^{(2)})$ is trivial, or, equivalently, $\Delta_p^{(2)}$ is a weak isomorphism.

We call $C_*^{(2)}$ of *determinant class* if $\det_{\mathcal{N}(G)}(c_p^{(2)}) > 0$ holds for every $p \in \mathbb{N}$. This is equivalent to the condition that $\det_{\mathcal{N}(G)}(\Delta_p^{(2)}) > 0$ holds for every $p \in \mathbb{N}$. If C_* is of determinant class, then we define the *L^2-torsion* of $C_*^{(2)}$ by

$$\rho^{(2)}(C_*^{(2)}) = \rho^{(2)}(C_*^{(2)};\mathcal{N}(G)) \quad := \quad -\sum_{p\in\mathbb{N}}(-1)^p \cdot \ln(\det_{\mathcal{N}(G)}(c_p^{(2)})). \tag{2.18}$$

This turns out to be the same as putting

$$\rho^{(2)}(C_*^{(2)}) \quad = \quad -\frac{1}{2}\cdot\sum_{p\in\mathbb{N}}(-1)^p \cdot p \cdot \ln(\det_{\mathcal{N}(G)}(\Delta_p^{(2)})). \tag{2.19}$$

2.6. **L^2-Betti numbers and L^2-torsion of finite based free chain complexes over group rings.** Let G be a group and let R be one of the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Let

$$\cdots 0 \to 0 \to C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \to 0 \to \cdots$$

be a finite based free RG-chain complex (C_*, c_*), i.e., each C_p is a finitely generated free RG-module equipped with a RG-basis, each differential is a RG-homomorphism and there is a natural number n such that $C_p = 0$ for $p < 0$ and for $p > n$. The basis induces on $C_p^{(2)} := L^2(G)\otimes_{RG}C_p$ the structure of a finitely generated Hilbert $\mathcal{N}(G)$-module in the obvious way. Note for the sequel that this structure is unchanged if we permute the basis elements or multiply one of the basis elements with $\pm g$ for some $g \in G$. The resulting differentials $c_p^{(2)} = \mathrm{id} \otimes c_p \colon C_i^{(2)} \to C_{i-1}^{(2)}$ are bounded G-equivariant operators. So we get a finite Hilbert $\mathcal{N}(G)$-chain complex $C_*^{(2)}$. For $p \in \mathbb{N}$, we define the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_p^{(2)}(C_*) := H_p^{(2)}(C_*^{(2)}).$$

Denote by

$$b_p^{(2)}(C_*) = b_p^{(2)}(C_*;\mathcal{N}(G)) \quad := \quad b_p^{(2)}(C_*^{(2)}) \in \mathbb{R}^{\geq 0} \tag{2.20}$$

the *p-th L^2-Betti number* of C_*. We say that the complex C_* is *L^2-acyclic* if all its L^2-Betti numbers vanish.

We call C_* of *determinant class* if $C_*^{(2)}$ is of determinant class. If C_* is of determinant class, then we define the *L^2-torsion* of C_* by

$$\rho^{(2)}(C_*) = \rho^{(2)}(C_*;\mathcal{N}(G)) \quad := \quad \rho^{(2)}(C_*^{(2)}). \tag{2.21}$$

2.7. **L^2-Betti numbers and L^2-torsion of regular coverings of finite CW-complexes.** Let G be a (discrete) group and X be a finite CW-complex. Let $G \to \overline{X} \to X$ be a G-principal bundle over X, or, equivalently, a normal covering with G a group of deck transformations. The cellular chain complex $C_*(\overline{X})$ of $\overline{X}$ with $\mathbb{Z}$-coefficients is a finite free $\mathbb{Z}G$-chain complex. If we choose an ordering on the set of cells of X, an orientation for each cell in X, and a lift of each cell in X to cell in $\overline{X}$, we obtain a $\mathbb{Z}G$-basis for $C_*(X)$ and we can consider the finite $\mathcal{N}(G)$-chain complex $C_*^{(2)}(\overline{X})$. One easily checks that $C_*^{(2)}(\overline{X})$ is independent of the choices above. Hence we can define the *pth L^2-Betti number*

$$b_p^{(2)}(\overline{X}) = b_p^{(2)}(\overline{X};\mathcal{N}(G)) \quad := \quad b_p^{(2)}(C_*^{(2)}(\overline{X})). \tag{2.22}$$

If $C_*^{(2)}(\overline{X})$ is of determinant class, we can also consider the *L^2-torsion*

$$\rho^{(2)}(\overline{X}) = \rho^{(2)}(\overline{X};\mathcal{N}(G)) \quad := \quad \rho^{(2)}(C_*^{(2)}(\overline{X})). \tag{2.23}$$

Let X be a finite (not necessarily connected) CW-complex. Let C be any of its path components. Let $\widetilde{C} \to C$ be the universal covering of C which is a $\pi_1(C)$-principal bundle. So $b_p^{(2)}(\widetilde{C})$ is defined. We put

$$b_p^{(2)}(\widetilde{X}) \quad := \quad \sum_{C \in \pi_0(X)} b_p^{(2)}(\widetilde{C}). \tag{2.24}$$

We call X *of determinant class* if $C_*^{(2)}(\widetilde{C})$ is of determinant class for every $C \in \pi_0(C)$. In this case we put

$$\rho^{(2)}(\widetilde{X}) \quad := \quad \sum_{C \in \pi_0(X)} \rho^{(2)}(\widetilde{C}). \tag{2.25}$$

We say that $\widetilde{X}$ is *L^2-acyclic* if $b_p^{(2)}(\widetilde{X})$ vanishes for all $p \in \mathbb{N}$. We say that $\widetilde{X}$ is *det-L^2-acyclic* if $\widetilde{X}$ is of determinant class and $b_p^{(2)}(\widetilde{X})$ vanishes for all $p \in \mathbb{N}$.

2.8. Basic properties of L^2-Betti numbers and L^2-torsion of universal coverings of finite CW-complexes.

Here is a list of basic properties of L^2-Betti numbers and L^2-torsion of universal coverings of finite CW-complexes. Note that the status of the Determinant Conjecture 3.10 will be reviewed in Remark 3.11. It is known to be true for a very large class of groups including sofic groups and fundamental groups of 3-manifolds.

(1) *(Simple) Homotopy invariance*, see [69, Theorem 1.35 (1) on page 37 and Theorem 3.96 (i) on page 163].

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes.

(a) Then
$$b_p^{(2)}(\widetilde{X}) = b_p^{(2)}(\widetilde{Y});$$

(b) Suppose that $\widetilde{X}$ or $\widetilde{Y}$ is det-L^2-acyclic. Then both $\widetilde{X}$ and $\widetilde{Y}$ are det-L^2-acyclic;

(c) Suppose that f is a simple homotopy equivalence or that f is a homotopy equivalence and $\pi_1(X)$ satisfies the Determinant Conjecture 3.10. Assume that $\widetilde{X}$ and $\widetilde{Y}$ are det-L^2-acyclic. Then
$$\rho^{(2)}(\widetilde{Y}) = \rho^{(2)}(\widetilde{X});$$

(2) *Euler-Poincaré formula*, see [69, Theorem 1.35 (2) on page 37].

We get for a finite CW-complex X
$$\chi(X) = \sum_{p \in \mathbb{N}} (-1)^p \cdot b_p^{(2)}(\widetilde{X});$$

(3) *Sum formula*, see [69, Theorem 3.96 (2) on page 164].

Consider the pushout of finite CW-complexes such that j_1 is an inclusion of CW-complexes, j_2 is cellular and X inherits its CW-complex structure from X_0, X_1 and X_2

$$\begin{tikzcd} X_0 \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "i_1"] \\ X_2 \arrow[r, "i_2"'] & X. \end{tikzcd}$$

Assume that for $k = 0, 1, 2$ the map $\pi_1(i_k, x)\colon \pi_1(X_k, x) \to \pi_1(X, j_k(x))$ induced by the obvious map $i_k\colon X_k \to X$ is injective for all base points x in X_k.

(a) If $\widetilde{X_0}$, $\widetilde{X_1}$, and $\widetilde{X_2}$ are L^2-acyclic, then $\widetilde{X}$ is L^2-acyclic;

(b) If $\widetilde{X_0}$, $\widetilde{X_1}$, and $\widetilde{X_2}$ are det-L^2-acyclic, then $\widetilde{X}$ is det-L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0});$$

(4) *Poincaré duality*, see [69, Theorem 1.35 (3) on page 37 and Theorem 3.96 (3) on page 164].

Let M be a closed manifold of dimension n

(a) Then

$$b_p^{(2)}(\widetilde{M}) = b_{n-p}^{(2)}(\widetilde{M});$$

(b) Suppose that n is even and $\widetilde{M}$ is det-L^2-acyclic. Then

$$\rho^{(2)}(\widetilde{M}) = 0;$$

(5) *Product formula*, see [69, Theorem 1.35 (4) on page 37 and Theorem 3.96 (4) on page 164].

Let X and Y be finite CW-complexes.

(a) Then

$$b_p^{(2)}(\widetilde{X \times Y}) = \sum_{i,j \in \mathbb{N}, p=i+j} b_i^{(2)}(\widetilde{X}) \cdot b_j^{(2)}(\widetilde{Y});$$

(b) Suppose that $\widetilde{X}$ is det-L^2-acyclic. Then $\widetilde{X \times Y}$ is det-L^2-acyclic and

$$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\widetilde{X});$$

(6) *Multiplicativity*, see [69, Theorem 1.35 (9) on page 38 and Theorem 3.96 (3) on page 164].

Let $X \to Y$ be a finite covering of finite CW-complexes with d sheets.

(a) Then

$$b_p(\widetilde{X}) = d \cdot b_p(\widetilde{Y});$$

(b) Then $\widetilde{X}$ is det-L^2-acyclic if and only if $\widetilde{Y}$ is det-L^2-acyclic, and in this case

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y});$$

(7) *Determinant class.*

If $\pi_1(C)$ satisfies the Determinant Conjecture 3.10 for each component C of the finite CW-complex X, then $\widetilde{X}$ is of determinant class;

(8) *0th L^2-Betti number*, see [69, Theorem 1.35 (8) on page 38].

If X is a connected finite CW-complex with fundamental group π, then

$$b_0^{(2)}(\widetilde{X}) = \begin{cases} \frac{1}{|\pi|} & \text{if } \pi \text{ is finite;} \\ 0 & \text{otherwise;} \end{cases}$$

(9) *Fibration formula*, see [69, Lemma 1.41 on page 45 and Corollary 3.103 on page 166].

(a) Let $p\colon E \to B$ a fibration such that B is a connected finite CW-complex and the fiber is homotopy equivalent to a finite CW-complex Z. Suppose that for every $b \in B$ and $x \in F_b := p^{-1}(b)$ the inclusion $p^{-1}(b) \to E$ induces an injection on the fundamental groups $\pi_1(F_b, x) \to \pi_1(E, x)$, and that Z is L^2-acyclic.

Then E is homotopy equivalent to a finite CW-complex X which is L^2-acyclic;

(b) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be locally trivial fiber bundle of finite CW-complexes. Suppose B is connected, that the map $\pi_1(F,x) \to \pi_1(E,i(x))$ is bijective for every base point $x \in F$, and that $\widetilde{F}$ is det-L^2-acyclic.
Then $\widetilde{E}$ is det-L^2-acyclic and

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F});$$

(10) *S^1-actions*, see [69, Theorem 1.40 on page 43 and Theorem 3.105 on page 168].
Let X be a connected compact S^1-CW-complex, for instance a closed smooth manifold with smooth S^1-action. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Then $\widetilde{X}$ is det-L^2-acyclic and $\rho^{(2)}(\widetilde{M})$ vanishes;

(11) *Aspherical spaces*, see [69, Theorem 3.111 on page 171 and Theorem 3.113 on page 172].

(a) Let M be an aspherical closed smooth manifold with a smooth S^1-action. Then the conditions appearing in assertion (10) are satisfied and hence $\widetilde{M}$ is det-L^2-acyclic and $\rho^{(2)}(\widetilde{X})$ vanishes;
(b) If X is an aspherical finite CW-complex whose fundamental group contains an infinite, elementary amenable, and normal subgroup, then $\widetilde{X}$ is det-L^2-acyclic and $\rho^{(2)}(\widetilde{X})$ vanishes;

(12) *Mapping tori*, see [69, Theorem 1.39 on page 42].
Let $f\colon X \to X$ be a self homotopy equivalence of a finite CW-complex. Denote by T_f its mapping torus.
(a) Then $\widetilde{T_f}$ is L^2-acyclic;
(b) If $\widetilde{X}$ is det-L^2-acyclic, then $\rho^{(2)}(\widetilde{T_f})$ vanishes;

(13) *Hyperbolic manifolds*, see [43], [69, Theorem 1.39 on page 42].
Let M be a hyperbolic closed manifold M of dimension n.
(a) If n is odd, $\widetilde{M}$ is det-L^2-acyclic;
(b) Suppose $n = 2m$ is even. Then $b_p^{(2)}(\widetilde{M})$ vanishes for $p \neq m$, and we have $(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0$;
(c) For every number m there exists an explicit constant $C_m > 0$ with the following property: If M is a hyperbolic closed manifold of dimension $(2m+1)$ with volume $\mathrm{vol}(M)$, then

$$\rho^{(2)}(\widetilde{M}) = (-1)^m \cdot C_m \cdot \mathrm{vol}(M).$$

We have $C_1 = \frac{1}{6\pi}$. The number $\pi^m \cdot C_m$ is always rational;

(14) *Approximation of L^2-Betti numbers by classical Betti numbers*, see [68],[69, Chapter 13].
Let X be a connected finite CW-complex with fundamental group $G = \pi_1(X)$. Suppose that G comes with a descending chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is normal in G, the index $[G : G_i]$ is finite and we have $\bigcap_{i \geq 0} G_i = \{1\}$. Let $b_p(G_i\backslash\widetilde{X})$ be the p-th Betti number of the finite CW-complex $G_i\backslash\widetilde{X}$.
Then $G_i\backslash\overline{X} \to X$ is a finite $[G : G_i]$-sheeted covering and we have

$$b_p^{(2)}(\widetilde{X}) = \lim_{i\to\infty} \frac{b_p(G_i\backslash\widetilde{X})}{[G : G_i]}.$$

There is also version, where the subgroups are not necessarily normal, see Farber [24].

L^2-Betti numbers were originally defined by Atiyah [4]. The definition of L^2-torsion in the analytic setting goes back to Lott [66] and Mathai [76] , and in the topological setting to Lück-Rothenberg [74].

For more information about L^2-invariants we refer for instance to [48, 65, 69].

3. Some open conjectures about L^2-invariants

We briefly review some prominent and open conjectures about L^2-invariants.

3.1. The Atiyah Conjecture.

Conjecture 3.1 (Atiyah Conjecture)**.** *We say that a torsionfree group G satisfies the* Atiyah Conjecture *if for any matrix $A \in M_{m,n}(\mathbb{Q}G)$ the von Neumann dimension* $\dim_{\mathcal{N}(G)}(\ker(r_A))$ *of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A is an integer.*

The Atiyah Conjecture can also be formulated for any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ and matrices $A \in M_{m,n}(FG)$ and for any group with a bound on the order of its finite subgroups. However, we only need and therefore consider in this paper the case, where $F = \mathbb{Q}$ and G is torsionfree.

Definition 3.2 (Admissible 3-manifold)**.** A 3-manifold is called *admissible* if it is connected, orientable, compact and irreducible, its boundary is empty or a disjoint union of tori, its fundamental group is infinite, and it is not homeomorphic to $S^1 \times D^2$.

For some information about the proof and in particular of references in the literature we refer to [28, Theorem 3.2] except for assertion (5) which is due to Jaikin-Zapirain and Lopez-Alvarez [46, Proposition 6.5]. A group is called *locally indicable* if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Examples are torsionfree one-relator groups.

Theorem 3.3 (Status of the Atiyah Conjecture)**.**

1. *If the torsionfree group G satisfies the Atiyah Conjecture, see Conjecture 3.1, then also each of its subgroups satisfies the Atiyah Conjecture;*
2. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that G is a torsionfree group which belongs to $\mathcal{C}$.*
   Then G satisfies the Atiyah Conjecture;
3. *Let G be an infinite group that is the fundamental group of an admissible 3-manifold M which is not a closed graph manifold. Then G is torsionfree and belongs to $\mathcal{C}$. In particular G satisfies the Atiyah Conjecture;*
4. *Let $\mathcal{D}$ be the smallest class of groups such that*
   - *The trivial group belongs to $\mathcal{D}$;*
   - *If $p\colon G \to A$ is an epimorphism of a torsionfree group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}$ for every finite group $B \subset A$, then $G \in \mathcal{D}$;*
   - *$\mathcal{D}$ is closed under taking subgroups;*
   - *$\mathcal{D}$ is closed under colimits and inverse limits over directed systems.*

   If the group G belongs to $\mathcal{D}$, then G is torsionfree and the Atiyah Conjecture holds for G.

   The class $\mathcal{D}$ is closed under direct sums, direct products and free products. Every residually torsionfree elementary amenable group belongs to $\mathcal{D}$;

(5) A locally indicable group satisfies the Atiyah Conjecture. More generally, if $1 \to H \to G \to Q \to 1$ is an extension of groups, H satisfies the Atyiah Conjecture and Q is locally indicable, then G satisfies the Atyiah Conjecture.

Remark 3.4. Let G be a finitely presented torsionfree group. Then G satisfies the Atiyah Conjecture if and only if the pth L^2-Betti number $b_p^{(2)}(\widetilde{M})$ is an integer for every $p \geq 0$ and every closed manifold M with $\pi_1(M) \cong G$.

Remark 3.5 (Analytic version of L^2-Betti numbers)**.** One can define the L^2-Betti number $b_p^{(2)}(\widetilde{M})$ of a closed Riemannian manifold M by the analytic expression

$$b_p^{(2)}(\widetilde{M}) \quad = \quad \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_p}(x,x))\, \mathrm{dvol}\,. \tag{3.6}$$

Here $e^{-t\Delta_p}(x,y)$ denotes the heat kernel for p-forms on the universal covering $\widetilde{M}$ and $\operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_p}(x,x))$ is its trace, and $\mathcal{F}$ is a *fundamental domain* for the $\pi_1(M)$-action on $\widetilde{M}$, see for instance [4, Proposition 4.16 on page 63] or [69, Section 1.3.2]. In view of expression (3.6) it is rather surprizing that this should always be an integer if the fundamental group is torsionfree.

Note that any non-negative real number occurs as the von Neumann dimension of the kernel of some morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules if G contains an element of infinite order. So it is crucial that the matrices appearing in the Atiyah Conjecture 3.3 live already over the group ring.

Associated to the von Neumann algebra $\mathcal{N}(G)$ is the algebra of affiliated operators $\mathcal{U}(G)$ which contains $\mathcal{N}(G)$. It can be defined analytically or just as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-zero divisors. Now one can consider the so called division closure $\mathcal{D}(G)$ of $\mathbb{Q}G$ in $\mathcal{U}(G)$.

The proof of the following is based on ideas of Peter Linnell from [61] which have been explained in detail and a little bit extended in [69, Lemma 10.39 on page 10.39 and Chapter 10] and [86], see also [29, Theorem 3.8].

Theorem 3.7 (Main properties of $\mathcal{D}(G)$)**.** *Let G be a torsionfree group.*

(1) The group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}(G)$ is a skew field;

(2) Suppose that G satisfies the Atiyah Conjecture. Let C_* be a $\mathbb{Q}G$-chain complex whose chain-modules are finitely generated projective. Then we get for all $n \geq 0$

$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*\big) = \dim_{\mathcal{D}(G)}\big(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big).$$

In particular $b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*\big)$ is an integer.

Theorem 3.7 shows that the Atyiah Conjecture is related to the question whether for a torsionfree group G the group ring $\mathbb{Q}G$ can be embedded into a skew field, see for instance [42].

There is a program of Linnell [61] to prove the Atyiah Conjecture which is discussed in details for instance in [69, Theorem 10.38 on page 387 and Section 10.3] and [86]. This shows that one has at least some ideas why the Atyiah Conjecture is true and that the Atyiah Conjecture is related to some deep ring theory and to algebraic K-theory, notably to projective class groups. This connection to ring theory has been explained and exploited for instance in [45, 46], where the division closure is replaced by the $*$-regular closure.

For more information about the Atyiah Conjecture we refer for instance to [69, Chapter 10].

3.2. The Singer Conjecture.

Conjecture 3.8 (Singer Conjecture). *If M is an aspherical closed manifold, then*

$$b_p^{(2)}(\widetilde{M}) = 0 \quad \textit{if } 2p \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature of even dimension $\dim(M) = 2m$*, then*

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

Note that the equality $(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M})$ appearing in the Singer Conjecture 3.8 above follows from the the Euler-Poincaré formula $\chi(M) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{M})$. Obviously Singer Conjecture 3.8 implies the following conjecture in the cases, where M is aspherical or has negative sectional curvature.

Conjecture 3.9 (Hopf Conjecture). *If M is an aspherical closed manifold of even dimension* $\dim(M) = 2m$*, then*

$$(-1)^m \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension $\dim(M) = 2m$ *with sectional curvature* $\sec(M)$*, then*

$$\begin{array}{rclcrcl}
(-1)^m \cdot \chi(M) & > & 0 & \textit{if} & \sec(M) & < & 0; \\
(-1)^m \cdot \chi(M) & \geq & 0 & \textit{if} & \sec(M) & \leq & 0; \\
\chi(M) & = & 0 & \textit{if} & \sec(M) & = & 0; \\
\chi(M) & \geq & 0 & \textit{if} & \sec(M) & \geq & 0; \\
\chi(M) & > & 0 & \textit{if} & \sec(M) & > & 0.
\end{array}$$

In original versions of the Singer Conjecture 3.8 and the Hopf Conjecture 3.9 the condition aspherical closed manifolds was replaced by the condition closed Riemannian manifold with non-positive sectional curvature. Note that a closed Riemannian manifold with non-positive sectional curvature is aspherical by Hadamard's Theorem.

Note that the Singer Conjecture 3.9 is consistent with the Atiyah Conjecture in the sense that it predicts that the L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ for an aspherical closed manifold M are all integers.

The *action dimension* of a discrete group G is the smallest dimension of a contractible manifold that admits a proper action of G. This notion and its relation to the Singer Conjecture is explained in [6, 81].

In contrast to the Atyiah Conjecture the evidence for the Singer Conjecture 3.8 comes from computations only and no good strategy is known for a potential proof. In some sense Poincaré duality and the L^2-conditions seems to force the L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ of an aspherical closed manifold to concentrate in the middle dimension. One may wonder what happens if we replace M by an aspherical finite Poincaré complex in the Singer Conjecture 3.8. There are counterexamples to the Singer Conjecture 3.8 if one weakens aspherical to rationally aspherical, see [5, Theorem 4].

For more information about the Singer Conjecture and its status we refer for instance to [21, Conjecture 2], [69, Chapter 11], and [92].

3.3. The Determinant Conjecture.

Conjecture 3.10 (Determinant Conjecture for a group G). *For any matrix $A \in M_{r,s}(\mathbb{Z}G)$, the Fuglede-Kadison determinant of the morphism of Hilbert modules $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$ given by right multiplication with A satisfies*

$$\det{}_{\mathcal{N}(G)}^{(2)}\big(r_A^{(2)}\big) \geq 1.$$

Remark 3.11 (Status of the Determinant Conjecture)**.** We will want to assume that the Determinant Conjecture 3.10 is true because then the condition of determinant class is automatically satisfied. This is an acceptable condition since the Determinant Conjecture 3.10 is known for a very large class of groups. Namely, the following is known, see [22, Theorem 5], [69, Section 13.2], [87, Theorem 1.21]. Let $\mathcal{F}$ be the class of groups for which the Determinant Conjecture 3.10 is true. Then:

(1) Amenable quotient
Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{F}$ and the quotient G/H is amenable. Then $G \in \mathcal{F}$;
(2) Colimits
If $G = \operatorname{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I (with not necessarily injective structure maps) and each G_i belongs to $\mathcal{F}$, then G belongs to $\mathcal{F}$;
(3) Inverse limits
If $G = \lim_{i \in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{F}$, then G belongs to $\mathcal{F}$;
(4) Subgroups
If H is isomorphic to a subgroup of a group G with $G \in \mathcal{F}$, then $H \in \mathcal{F}$;
(5) Quotients with finite kernel
Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. If K is finite and G belongs to $\mathcal{F}$, then Q belongs to $\mathcal{F}$;
(6) Sofic groups belong to $\mathcal{F}$;
(7) The fundamental group of a 3-manifold belongs to $\mathcal{F}$.

The class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and is closed under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups and fundamental groups of 3-manifolds. One expects that there exists non-sofic groups but no example is known. More information about sofic groups can be found for instance in [23] and [82].

For more information about the Determinant Conjecture we refer for instance to [69, Chapter 13].

3.4. **Approximation Conjecture for L^2-Betti numbers.** Let G be a group together with an *exhausting normal inverse system of subgroups* $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i \in I} G_i = \{1\}$. If I is given by the natural numbers, this boils down to a nested sequence of normal subgroups of G

$$G = G_0 \supset G_1 \supseteq G_2 \supseteq \cdots$$

satisfying $\bigcap_{n \geq 1} G_n = \{1\}$.

Notation 3.12 (Inverse systems and matrices)**.** Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$. Given a matrix $A \in M_{r,s}(RG)$, let $A[i] \in M_{r,s}(R[G/G_i])$ be the matrix obtained from A by applying elementwise the ring homomorphism $RG \to R[G/G_i]$ induced by the projection $G \to G/G_i$. Let $r_A \colon RG^r \to RG^s$ and $r_{A[i]} \colon R[G/G_i]^r \to R[G/G_i]^s$ be the RG- and $R[G/G_i]$-homomorphisms given by right multiplication with A and $A[i]$. Let $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$ and $r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s$ be the morphisms of Hilbert $\mathcal{N}(G)$- and Hilbert $\mathcal{N}(G/G_i)$-modules given by right multiplication with A and $A[i]$.

Conjecture 3.13 (Approximation Conjecture for L^2-Betti numbers)**.** *A group G together with an exhausting normal inverse system of subgroups $\{G_i \mid i \in I\}$ satisfies the* Approximation Conjecture for L^2-Betti numbers *if one of the following equivalent conditions holds:*

(1) *Matrix version*

Let $A \in M_{r,s}(\mathbb{Q}G)$ be a matrix. Then

$$\begin{aligned}\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr)\bigr) \\ = \lim_{i\in I} \dim_{\mathcal{N}(G/G_i)}\bigl(\ker\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr)\bigr);\end{aligned}$$

(2) *CW-complex version*

Consider normal covering $p \colon \overline{X} \xrightarrow{p} X$ with G as group of deck transformation over a CW-complex X of finite type. Put $\overline{X}[i] := G_i\backslash\overline{X}$. Then we get a normal covering $p[i] \colon \overline{X}[i] \to X$ with G/G_i as group of deck transformation and

$$b_p^{(2)}(\overline{X};\mathcal{N}(G)) \quad = \quad \lim_{i\in I} b_p^{(2)}(\overline{X}[i];\mathcal{N}(G/G_i)).$$

The two conditions appearing in Conjecture 3.13 are equivalent by [69, Lemma 13.4 on page 455].

Theorem 3.14 (The Determinant Conjecture implies the Approximation Conjecture for L^2-Betti numbers)**.** *If for each $i \in I$ the quotient G/G_i satisfies the Determinant Conjecture 3.10, then the conclusion of the Approximation Conjecture 3.13 holds for $\{G_i \mid i \in I\}$.*

Proof. See [69, Theorem 13.3 (1) on page 454] and [87]. □

Suppose that each quotient G/G_i is finite. Then we rediscover (14) appearing in Subsection 2.8 from Remark 3.11 and Theorem 3.14.

For more information about the Approximation Conjecture for L^2-Betti numbers 3.13 we refer for instance to [69, Chapter 13] and [87, Conjecture 1.10].

3.5. **Approximation Conjectures for Fuglede-Kadison determinants and L^2-torsion.** Next we turn to Fuglede-Kadison determinants and L^2-torsion.

3.5.1. *Approximation Conjecture for Fuglede-Kadison determinants.*

Conjecture 3.15 (Approximation Conjecture for Fuglede-Kadison determinants)**.** *A group G together with an exhausting normal inverse system of subgroups $\{G_i \mid i \in I\}$ satisfies the* Approximation Conjecture for Fuglede-Kadison determinants *if for any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant*

$$\begin{aligned}\det_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr) \quad &> \quad 0;\\ \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr) \quad &> \quad 0,\end{aligned}$$

and

$$\begin{aligned}\det_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr) \\ = \lim_{i\in I} \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr),\end{aligned}$$

where the existence of the limit above is part of the claim.

Remark 3.16 ($\mathbb{Q}$-coefficients are necessary)**.** Recall that the Atiyah Conjecture may be true if we consider matrices over the complex group ring instead of the rational group ring. Conjecture 3.15 does not hold if one replaces $\mathbb{Q}$ by $\mathbb{C}$ by the following result appearing in [69, Example 13.69 on page 481].

There exists a sequence of integers $2 \le n_1 < n_2 < n_3 < \cdots$ and a real number s such that for $G = \mathbb{Z}$ and $G_i = n_i \cdot \mathbb{Z}$ and the $(1,1)$-matrix A given by the element $z - \exp(2\pi i s)$ in $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ we get for all $i \ge 1$

$$\begin{aligned} \ln\bigl(\det^{(2)}_{\mathcal{N}(G)}(r_A^{(2)})\bigr) &= 0; \\ \frac{\ln\bigl(\det(r^{(2)}_{A[i]})\bigr)}{[G:G_i]} &\le -1/2. \end{aligned}$$

A strategy for the proof of Conjecture 3.15 is discussed in [71, Section 17], see also [57]. The uniform integrability condition appearing in [71, Theorem 16.3 (v) and Remark 16.13] seems to play a key role. It would be automatically satisfied if one has a uniform estimate on the spectral density functions of the intermediate stages for $i \in I$. Roughly speaking, the spectrum has to be uniformly thin at zero for $r^{(2)}_{A[i]}$ each $i \in I$. The crudest way to guarantee this condition is to require a uniform gap at zero in the spectrum for $r^{(2)}_{A[i]}$ each $i \in I$, see [71, Lemma 16.14 and Remark 16.15].

3.5.2. *The chain complex version.*

Notation 3.17 (Inverse systems and chain complexes)**.** Let C_* be a finite based free $\mathbb{Q}G$-chain complex. In the sequel we denote by $C[i]_*$ the $\mathbb{Q}[G/G_i]$-chain complex $\mathbb{Q}[G/G_i] \otimes_{\mathbb{Q}G} C_*$, by $C_*^{(2)}$ the finite Hilbert $\mathcal{N}(G)$-chain complex $L^2(G) \otimes_{\mathbb{Q}G} C_*$, and by $C[i]_*^{(2)}$ the finite Hilbert $\mathcal{N}(G/G_i)$-chain complex $L^2(G/G_i) \otimes_{\mathbb{Q}[G/G_i]} C[i]_*$. The $\mathbb{Q}G$-basis for C_* induces a $\mathbb{Q}[G/G_i]$-basis for $C[i]_*$ and Hilbert space structures on $C_*^{(2)}$ and $C[i]_*^{(2)}$ using the standard Hilbert structure on $L^2(G)$ and $L^2(G/G_i)$. We emphasize that in the sequel after fixing a $\mathbb{Q}G$-basis for C_* the $\mathbb{Q}[G/G_i]$-basis for $C_*[i]$ and the Hilbert structures on $C_*^{(2)}$ and $C[i]_*^{(2)}$ have to be chosen in this particular way.

Denote by

$$\rho^{(2)}(C_*) := -\sum_{p \ge 0} (-1)^p \cdot \ln\bigl(\det_{\mathcal{N}(G)}(c_p^{(2)})\bigr); \tag{3.18}$$

$$\rho^{(2)}(C[i]_*) := -\sum_{p \ge 0} (-1)^p \cdot \ln\bigl(\det_{\mathcal{N}(G/G_i)}(c[i]_p^{(2)})\bigr), \tag{3.19}$$

their L^2*-torsion* over $\mathcal{N}(G)$ and $\mathcal{N}(G/G_i)$ respectively, provided that C_* and $C[i]_*$ are of determinant class.

We have the following chain complex version of Conjecture 3.15 which is obviously equivalent to Conjecture 3.15

Conjecture 3.20 (Approximation Conjecture for L^2-torsion of chain complexes)**.** *A group G together with an exhausting normal inverse system $\{G_i \mid i \in I\}$ satisfies the* Approximation Conjecture for L^2-torsion of chain complexes *if the finite based free $\mathbb{Q}G$-chain complex C_* and $C[i]_*$ are of determinant class and we have*

$$\rho^{(2)}(C_*) = \lim_{i \in I} \rho^{(2)}(C[i]_*).$$

3.5.3. *Analytic L^2-torsion.* Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Denote by $\overline{M}[i]$ the Riemannian manifold obtained from $\overline{M}$ by dividing out the G_i-action. The Riemannian metric on $\overline{M}[i]$ is induced by the one on M. There is an obvious proper free cocompact isometric G/G_i-action on $\overline{M}[i]$ induced by the given G-action on $\overline{M}$. Note that $M = \overline{M}/G$ is a closed Riemannian manifold and we get a G-covering $\overline{M} \to M$ and a G/G_i-covering $\overline{M}[i] \to M$ for the closed Riemannian manifold $M = \overline{M}/G$, which are compatible with the Riemannian metrics. Denote by

$$\rho_{\mathrm{an}}^{(2)}(\overline{M};\mathcal{N}(G)) \in \mathbb{R}; \tag{3.21}$$

$$\rho_{\mathrm{an}}^{(2)}(\overline{M}[i];\mathcal{N}(G/G_i)) \in \mathbb{R}, \tag{3.22}$$

their *analytic L^2-torsion* over $\mathcal{N}(G)$ and $\mathcal{N}(G/G_i)$ respectively, provided that $\overline{M}$ and $\overline{M}[i]$ are of determinant class. For the notion of analytic L^2-torsion we refer for instance to [69, Chapter 3]. Burghelea-Friedlander-Kappeler-McDonald [18] have shown that the analytic L^2-torsion agrees with the L^2-torsion defined in terms of the cellular chain complex in (2.23) in the L^2-acyclic case.

We will not discuss the condition of determinant class here and in the sequel. This is not necessary if each G_i satisfies the Determinant Conjecture 3.10, which is true for a very large class of groups, see Remark 3.11.

Conjecture 3.23 (Approximation Conjecture for analytic L^2-torsion)**.** *Consider a group G together with an exhausting normal inverse system $\{G_i \mid i \in I\}$. Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Then $\overline{M}$ and $\overline{M}[i]$ are of determinant class and*

$$\rho_{\mathrm{an}}^{(2)}(\overline{M};\mathcal{N}(G)) = \lim_{i\in I} \rho_{\mathrm{an}}^{(2)}(M[i];\mathcal{N}(G/G_i)).$$

Remark 3.24. The conjectures above imply a positive answer to [20, Question 21] and [69, Question 13.52 on page 478 and Question 13.73 on page 483]. They also would settle [54, Problem 4.4 and Problem 6.4] and [55, Conjecture 3.5]. One may wonder whether it is related to the Volume Conjecture due to Kashaev [51] and H. and J. Murakami [79, Conjecture 5.1 on page 102].

The proof of the following result can be found in [71, Section 16]. It reduces in the weakly acyclic case Conjecture 3.23 to Conjecture 3.15.

Theorem 3.25. *Consider a group G together with an exhausting normal inverse system $\{G_i \mid i \in I\}$. Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Suppose that $b_p^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ for all $p \geq 0$. Assume that the Approximation Conjecture for L^2-torsion of chain complexes 3.20 (or, equivalently, Conjecture 3.15) holds for G.*

Then Conjecture 3.23 holds for M, i.e., $\overline{M}$ and $\overline{M}[i]$ are of determinant class and

$$\rho_{\mathrm{an}}^{(2)}(\overline{M};\mathcal{N}(G)) = \lim_{i\in I} \rho_{\mathrm{an}}^{(2)}(\overline{M}[i];\mathcal{N}(G/G_i)).$$

Note that in Theorem 3.25 we are not assuming that $b_p^{(2)}(\overline{M}[i];\mathcal{N}(G/G_i))$ vanishes for all $p \geq 0$ and $i \in I$.

It is conceivable that Theorem 3.25 remains true if we drop the assumption that $b_p^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for all $p \geq 0$, but our present proof works only under this assumption, see [71, Remark 16.2].

More information about the conjectures above can be found in [71, Section 15 – 17].

3.6. **Homological growth and L^2-torsion.** Denote by $H_n(X;\mathbb{Z})$ the singular homology with integer coefficients. If X is a compact manifold or a finite CW-complex, then $H_n(X;\mathbb{Z})$ is a finitely generated group and hence its torsion part $\mathrm{tors}(H_n(X;\mathbb{Z}))$ is a finite abelian group.

Conjecture 3.26 (Homological growth and L^2-torsion for aspherical manifolds)**.** *Let M be an aspherical closed manifold of dimension d with fundamental group $G = \pi_1(M)$. Consider a nested sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ of normal subgroups* *of G of finite index $[G:G_i]$ satisfying $\bigcap_{i=0}^\infty G_i = \{1\}$. Let $M[i] = \widetilde{M}/G_i \to M$ be the $[G:G_i]$-sheeted covering of M associated to $G_i \subseteq G$.*

Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}(H_n(M[i];\mathbb{Z}))\right|\right)}{[G:G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}(H_n(M[i];\mathbb{Z}))\right|\right)}{[G:G_i]} = (-1)^n \cdot \rho_{\mathrm{an}}^{(2)}(\widetilde{M}) \geq 0.$$

Recall that $\rho_{\mathrm{an}}^{(2)}(\widetilde{M}) = \rho^{(2)}(\widetilde{M})$ holds, if $\widetilde{M}$ is L^2-acyclic and that the Singer Conjecture implies for an aspherical closed manifold of odd dimension that $\widetilde{M}$ is L^2-acyclic. Moreover, since G appearing in Conjecture 3.26 is residually finite, the condition of determinant class is automatically satisfied for $\widetilde{M}$.

One may wonder what happens if we replace M by an aspherical finite Poincaré complex in Conjecture 3.26.

Conjecture 3.26 is known to be true in the case that G contains a normal infinite elementary amenable subgroup or admits a non-trivial S^1-action, see [70]. However, to the author's knowledge there is no hyperbolic 3-manifold for which Conjecture 3.26 is known to be true.

Conjecture 3.26 is attributed to Bergeron-Venkatesh [12]. They allow only locally symmetric spaces for M. They also consider the case of twisting with a finite-dimensional integral representation. Further discussions about this conjecture can be found for instance in [3, Section 7.5.1], [11], and [17].

The relation between Conjecture 3.15 and Conjecture 3.26 is discussed in [71, Section 9 and 10].

The chain complex version Conjecture 3.26 is stated in [71, Conjeture 7.12]. We at least explain what it says for 1-dimensional chain complexes, or, equivalently, matrices. Here it is important to work over the integral group ring.

Conjecture 3.27 (Approximating Fuglede-Kadison determinants by homology)**.** *Consider a nested sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ of normal subgroups of G of finite index $[G:G_i]$ satisfying $\bigcap_{i=0}^\infty G_i = \{1\}$. Consider $A \in M_{r,r}(\mathbb{Z}G)$.*

Then we get using Notation 3.12 for $R = \mathbb{Z}$

$$\det{}_{\mathcal{N}(G)}^{(2)}(r_A^{(2)}) \quad = \quad \lim_{i\to\infty} \left|\operatorname{tors}(\operatorname{coker}(r_{A[i]}))\right|^{1/[G:G_i]}.$$

Recall that for Conjecture 3.15 we could formulate a good condition, namely the uniform integrability condition, which implies its validity. Nothing like this is known for Conjecture 3.27. The only infinite group for which Conjecture 3.27 is known to be true is $\mathbb{Z}$. The proof indicates that some deep number theory may enter in a potential proof of Conjecture 3.27. Note that for Conjecture 3.27 it is crucial that the matrix A lives over the integral group ring, whereas for Conjecture 3.15 it suffices that A lives over the rational group ring, see Remark 3.16.

3.7. **L^2-invariants and the simplicial volume.** We briefly recall the definition of the *simplicial volume*.

Let X be a topological space and let $C_*^{\mathrm{sing}}(X;\mathbb{R})$ be its singular chain complex with real coefficients. Recall that a singular p-simplex of X is a continuous map $\sigma\colon \Delta_p \to X$, where here Δ_p denotes the standard p-simplex (and not the Laplace operator). Let $S_p(X)$ be the set of all singular p-simplices. Then $C_p^{\mathrm{sing}}(X;\mathbb{R})$ is the real vector space with $S_p(X)$ as basis. The p-th differential ∂_p sends the element σ given by a p-simplex $\sigma\colon \Delta_p \to X$ to $\sum_{i=0}^p (-1)^i \cdot \sigma \circ s_i$, where $s_i\colon \Delta_{p-1} \to \Delta_p$

is the i-th face map. Define the L^1-*norm* of an element $x \in C_p^{\mathrm{sing}}(X;\mathbb{R})$, which is given by the (finite) sum $\sum_{\sigma \in S_p(X)} \lambda_\sigma \cdot \sigma$, by

$$||x||_1 \quad := \quad \sum_\sigma |\lambda_\sigma|.$$

We define the L^1-*seminorm* of an element y in the p-th singular homology $H_p^{\mathrm{sing}}(X;\mathbb{R}) := H_p(C_*^{\mathrm{sing}}(X;\mathbb{R}))$ by

$$||y||_1 \quad := \quad \inf\{||x||_1 \mid x \in C_p^{\mathrm{sing}}(X;\mathbb{R}), \partial_p(x) = 0, y = [x]\}.$$

Notice that $||y||_1$ defines only a seminorm on $H_p^{\mathrm{sing}}(X;\mathbb{R})$, it is possible that $||y||_1 = 0$ but $y \neq 0$. The next definition is taken from [38, page 8].

Definition 3.28 (Simplicial volume)**.** Let M be a closed connected orientable manifold of dimension n. Define its *simplicial volume* to be the non-negative real number

$$||M|| := ||j([M])||_1 \ \in \mathbb{R}^{\geq 0}$$

for any choice of fundamental class $[M] \in H_n^{\mathrm{sing}}(M;\mathbb{Z})$ and $j\colon H_n^{\mathrm{sing}}(M;\mathbb{Z}) \to H_n^{\mathrm{sing}}(M;\mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$.

There is the following interesting but poorly understood conjecture relating the simplicial volume and L^2-invariants for aspherical orientable closed manifolds, see [69, Chapter 14.1].

Conjecture 3.29 (Simplicial volume and L^2-invariants)**.** *Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then $\widetilde{M}$ is of determinant class and*

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad \text{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

For more information about this conjecture we refer for instance to [32], [69, Chapter 14], [88]. It has been verified by computations if M is a locally symmetric space, if M is 3-manifold, if M carries a non-trivial S^1-action, or $\pi_1(M)$ is elementary amenable. But no strategy for a potential proof is known to the author.

4. The computation of L^2-Betti numbers and L^2-torsion of 3-manifolds

4.1. **L^2-Betti numbers of 3-manifolds.** The following theorem is taken from [67, Theorem 0.1].

Theorem 4.1 (L^2-Betti numbers of 3-manifolds)**.** *Let M be the connected sum $M_1 \# \ldots \# M_r$ of compact connected orientable prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ are given by*

$$\begin{aligned} b_0^{(2)}(\widetilde{M}) &= 0; \\ b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^r \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right| - \chi(M); \\ b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^r \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\ b_3^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

In particular, $\widetilde{M}$ *is* L^2*-acyclic if and only if* M *is homotopy equivalent, or, equivalently, homeomorphic, to* $\mathbb{RP}^3\#\mathbb{RP}^3$ *or a prime 3-manifold with infinite fundamental group whose boundary is empty or a union of tori.*

4.2. **L^2-torsion of 3-manifolds.** Finally we state the values for the L^2-torsion, see [75, Theorem 0.6].

Theorem 4.2 (L^2-torsion of 3-manifolds)**.** *Let* M *be a compact connected orientable prime* 3*-manifold with infinite fundamental group such that the boundary of* M *is empty or a disjoint union of incompressible tori. Let* M_1*,* M_2*, ...,* M_r *be the hyperbolic pieces. (They all have finite volume* [78, Theorem B on page 52]*.)*

Then $\widetilde{M}$ *is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{M}) = -\frac{1}{6\pi} \cdot \sum_{i=1}^{r} \operatorname{vol}(M_i).$$

In particular, $\rho^{(2)}(\widetilde{M})$ *is* 0 *if and and only if* M *is* $S^1 \times S^2$ *or is a graph manifold.*

5. The status of the conjectures about L^2-invariants for 3-manifolds

5.1. **The Atiyah Conjecture.** The fundamental group of an admissible 3-manifold M that is not a closed graph manifold, is torsionfree and satisfies the Atiyah Conjecture 3.1, see Theorem 3.3 (3).

5.2. **The Singer Conjecture.** Every aspherical closed 3-manifold satisfies the Singer Conjecture 3.8 by Theorem 4.2 since an aspherical closed 3-manifold is irreducible and has infinite fundamental group.

5.3. **The Determinant Conjecture.** If G is the fundamental group of a compact 3-manifold, then G is residually finite and hence satisfies the Determinant Conjecture 3.10 by Remark 3.11.

5.4. **Homological growth and L^2-torsion for aspherical manifolds.** The Conjecture 3.26 about homological growth and L^2-torsion for aspherical manifolds, is wide open. To the authors knowledge, there is no hyperbolic 3-manifold, where it is known to be true. Already this case would be very interesting.

Namely, suppose that M is a closed hyperbolic 3-manifold. Then $\rho_{\mathrm{an}}(\widetilde{M})$ is known to be $-\frac{1}{6\pi} \cdot \operatorname{vol}(M)$, by Theorem 4.2 and hence Conjecture 3.26 predicts

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}(H_1(G_i))\right|\right)}{[G:G_i]} = \frac{1}{6\pi} \cdot \operatorname{vol}(M).$$

Since the volume is always positive, the equation above implies that $|\operatorname{tors}(H_1(G_i))|$ grows exponentially in $[G : G_i]$. Some evidence for Conjecture 3.26 for closed hyperbolic 3-manifolds is given in Sun [94, Corollary 1,6], where it is shown that for any finitely generated abelian group A, and any closed hyperbolic 3-manifold M, there exists a finite cover N of M, such that A is a direct summand of $H_1(N;\mathbb{Z})$.

Bergeron-Sengun-Venkatesh [11] consider the equality above for arithmetic hyperbolic 3-manifolds and relate it to a conjecture about classes in the second integral homology.

Some numerical evidence for the equality above is given in Sengun [90].

The inequality

$$\limsup_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}(H_1(G_i))\right|\right)}{[G:G_i]} \leq \frac{1}{6\pi} \cdot \operatorname{vol}(M)$$

is proved by Thang [59] for a compact connected orientable irreducible 3-manifold M with infinite fundamental group and empty or toroidal boundary.

5.5. **L^2-invariants and the simplicial volume.** Define the positive real number v_3 to be the supremum of the volumes of all n-dimensional geodesic simplices, i.e., the convex hull of $(n+1)$ points in general position, in the n-dimensional hyperbolic space $\mathbb{H}^3$. If M is an admissible 3-manifold, then one gets from [75, Theorem 0.6], [93], and [95], see [69, Theorem 14.18 on page 490]

$$||M|| = \frac{-6\pi}{v_3} \cdot \rho^{(2)}(\widetilde{M}).$$

In particular, $\rho^{(2)}(\widetilde{M}) = 0$ if and only if $||M|| = 0$. Hence Conjecture 3.29 is true in dimension 3.

It is not true for odd $n \geq 9$ that there exists a dimension constant C_n such that for an aspherical orientable closed manifold M of dimension n we have $\rho^{(2)}(\widetilde{M}) = C_n \cdot ||M||$, see [69, Theorem 14.38 on page 498].

There are variants of the simplicial volume, namely, the notion of the *integral foliated simplicial volume*, see [39, page 305f], [88], or [32, Section 2], and of the *stable integral simplicial volume*, see [32, page 709]. The integral foliated simplicial volume gives an upper bound on the torsion growth for an oriented closed manifold, i.e, an upper bound on $\limsup_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}(H_n(M[i];\mathbb{Z}))\right|\right)}{[G:G_i]}$ in the situation of Conjecture 3.26, see [32, Theorem 1.6]. There are the open questions whether for an aspherical oriented closed manifold the simplicial volume and the integral foliated simplicial volume agree and whether for an aspherical oriented closed manifold with residually finite fundamental group the integral foliated simplicial volume and the stable integral simplicial volume agree, see [32, Question 1.2 and Question 1.3]. The stable integral simplicial volume and the simplicial volume agree for aspherical oriented closed 3-manifolds, see [26, Theorem 1].

6. Twisting L^2-invariants with finite-dimensional representations

In general one would like to twist L^2-Betti numbers and L^2-torsion with a finite-dimensional representation. In this section we discuss the general case and the technical difficulties and potential applications. The case, where the representation is a 1-dimensional real representation, is much easier, since then all the technical problems have been solved, and is very interesting for 3-manifolds. It is treated in Section 7 and a reader may directly pass to Section 7.

A strategy to do this is discussed in [72]. Consider a group G and a d-dimensional complex G-representation V. Consider a $\mathbb{C}G$-homomorphism $f\colon \mathbb{C}G^m \to \mathbb{C}G^n$. Choose a $\mathbb{C}$-basis B for the underlying complex vector spaces V. (No compatibility conditions with the G-actions are required for this basis.) Then one can define a new $\mathbb{C}G$-homomorphism $\eta^G_{V,B}(f)\colon \mathbb{C}G^{md} \to \mathbb{C}G^{nd}$, see Remark 6.8. By applying $L^2(G) \otimes_{\mathbb{C}G} -$, we obtain a bounded G-equivariant operator $\eta^{(2)}_{V,B}(f)\colon L^2(G)^{md} \to L^2(G)^{nd}$. Analogously, we can assign to a finite based free $\mathbb{C}G$-chain complex C_* a finite Hilbert $\mathcal{N}(G)$-chain complex $\eta^{(2)}_{V,B}(C_*)$.

One important question is what the relationship of the L^2-Betti numbers of $\eta^{(2)}_{V,B}(C_*)$ and of $C^{(2)}_*$ are. The hope is that

$$b_p^{(2)}(\eta^{(2)}_{V,B}(C_*)) = \dim_{\mathbb{C}}(V) \cdot b_p^{(2)}(C^{(2)}_*) \tag{6.1}$$

holds. This has interesting consequences for the behaviour of L^2-Betti numbers under fibrations, see [72, Section 5.2]. The answer to Question 6.1 is positive if G is a torsionfree elementary amenable group, see [72, Lemma 5.2].

For L^2-torsion the following question is crucial.

Question 6.2. *Suppose that* $C_*^{(2)}$ *is* det-L^2*-acyclic. Is then* $\eta^{(2)}_{V,B}(C_*)$ det-L^2-*acyclic?*

Remark 6.3 (L^2-torsion and character varieties)**.** Let G be a group and denote by $R(G, \mathrm{GL}_n(\mathbb{C}))$ the character variety given by group homomorphisms $u\colon G \to \mathrm{GL}_n(\mathbb{C})$. Let C_* be a det-L^2-acyclic $\mathbb{C}G$-chain complex. It may come from the cellular $\mathbb{Z}G$-chain complex $C_*(\widetilde{X})$ for an appropriate det-L^2-acyclic CW-complex X with $G \cong \pi_1(X)$, for instance from a closed aspherical manifold X of odd dimension with $G \cong \pi_1(X)$.

If the answer to Question 6.2 is positive, one could study the interesting function from the character variety $R(G, \mathrm{GL}_n(\mathbb{C}))$, which assign to such $u \in R(G, \mathrm{GL}_n(\mathbb{C}))$ the L^2-torsion of $\eta^{(2)}_{V_u,B_u}(C_*)$ for V_u the n-dimensional complex G-representation with the obvious basis B associated to u. One may ask whether the function on the character variety is continuous. This problem is in general wide open, but solved in some special case as we will see below. We will describe some special cases, where this type of function leads to interesting results.

If $\eta^{(2)}_{V_u,B_u}(C_*)$ has a gap at the spectrum at zero, then obviously the L^2-torsion of $\eta^{(2)}_{V_u,B_u}(C_*)$ is well-defined. Moreover the function sending $v \in R(G, \mathrm{GL}_n(\mathbb{C}))$ to the L^2-torsion of $\eta^{(2)}_{V_u,B_u}(C_*)$ is well-defined and continuous in neighborhood of u. This follows form the continuity of the Fulgede-Kadison determinant for invertible matrices over the group von Neumann algebra with respect to the norm topology, see [19, Theorem 1.10 (d)], [33, Theorem 1 (3)], or, [72, Lemma 9.14]. This is studied in more detail for a hyperbolic 3-manifold M with empty or incompressible torus boundary and the canonical holonomy representation $h\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ by Bénard-Raimbault [10]. They actually show that this function is real analytic near h.

Remark 6.4 (Twisting L^2-torsion for det-L^2-acyclic finite CW-complexes)**.** Of course it is interesting to study for a det-L^2-acyclic finite CW-complex its twisted L^2-torsion $\rho^{(2)}(X; V, B) = \rho^{(2)}(\eta^{(2)}_{V,B}(C_*(\widetilde{X})))$ for a finite dimensional complex $\pi_1(X)$-representation V with a basis B for its underlying vector space. The basic properties including the independence of the choice of B are discussed in [72, Theorem 6.7].

Unfortunately, deciding Question 6.2 seems to be very hard. The only case, where one knows that the answer is positive, is the one, where G is finitely generated residually finite, V is a $\mathbb{Z}^d$-representation and V is viewed as G-representation by a group homomorphism from G or $\pi_1(X)$ to $\mathbb{Z}^d$, see [72, Theorem 6.7]. There are interesting results in this setting as we see below, for instance if V is 1-dimensional.

Remark 6.5 (Unitary representations)**.** If the representation is unitary, then (6.1) is true and the answer to Question 6.2 is positive, Moreover, we have $\rho^{(2)}(\eta^{(2)}_{V,B}(C_*)) = \dim_{\mathbb{C}}(V) \cdot \rho^{(2)}(C_*^{(2)})$ and hence the twisting has no interesting effect, see [72, Theorem 3.1]. Hence it is crucial to consider not necessarily unitary representations.

All these problems are related to the following question. Define the *regular Fuglede-Kadison determinant* of a morphism $f\colon U \to U$ of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\det^r_{\mathcal{N}(G)}(f) \quad := \quad \begin{cases} \det_{\mathcal{N}(\Gamma)}(f) & \text{if } f \text{ is injective and of determinant class;} \\ 0 & \text{otherwise,} \end{cases} \tag{6.6}$$

One should not confuse the Fuglede-Kadison determinant $\det_{\mathcal{N}(G)}(f)$ and the regular Fuglede-Kadison determinant $\det^r_{\mathcal{N}(G)}(f)$ of a morphism $f\colon U \to V$ of finitely generated Hilbert $\mathcal{N}(G)$-modules, see [72, Remark 8.9].

For an element $x = \sum_{g \in G} \lambda_g \cdot g$ in $\mathbb{C}G$ define its support $\operatorname{supp}_G(x)$ to be the finite subset $\{g \in G \mid \lambda_g \neq 0\}$ of G. For a matrix $A = (a_{i,j}) \in M(m, n; \mathbb{C}G)$ define its *support* to be the finite subset $\bigcup_{i,j} \operatorname{supp}_G(a_i, j)$ of G. The following question is taken from [72, Question 9.11].

Question 6.7 (Continuity of the regular determinant)**.** *Let G be a group for which there exists a natural number d, such that the order of any finite subgroup $H \subseteq G$ is bounded by d, e.g., G is torsionfree. Let $S \subseteq G$ be a finite subset. Put $\mathbb{C}[n, S] := \{A \in M_{n,n}(\mathbb{C}G) \mid \operatorname{supp}_G(A) \subseteq S\}$ and equip it with the standard topology coming from the structure of a finite-dimensional complex vector space.*

(1) Is the function given by the regular Fuglede-Kadison determinant

$$\mathbb{C}[n, S] \to [0, \infty], \quad A \mapsto \det^r_{\mathcal{N}(G)}(r_A^{(2)} : L^2(G)^n \to L^2(G)^n)$$

continuous?

(2) Consider $A \in \mathbb{C}[S]$ such that $r_A^{(2)} : L^2(G)^n \to L^2(G)^n$ is a weak isomorphism of determinant class. Does there exist an open neighbourhood U of A in $\mathbb{C}[S]$ such that for every element $B \in U$ also $r_B^{(2)} : L^2(G)^n \to L^2(G)^n$ is a weak isomorphism of determinant class?

We mention that the answer to this question is known to be negative for some finitely presented groups for which there is no bound on the order of its finite subgroups, see [72, Remark 9.12]. Moreover, one cannot discard the condition about the existence of the finite set S, see [72, Remark 9.13]. The answer is positive if G is finitely generated abelian. It is possible that the answer is always positive for a torsionfree finitely generated group G.

Remark 6.8 (Basic idea of the construction of $\eta^G_{V,B}$)**.** The basic idea is the following. Let M and V be $\mathbb{C}G$-modules. Denote by $(M \otimes_{\mathbb{C}} V)_1$ the $\mathbb{C}G$-module whose underlying vector space is $M \otimes_{\mathbb{C}} V$ and on which $g \in G$ acts only on the first factor, i.e., $g(u \otimes v) = gu \otimes v$. Denote by $(M \otimes_{\mathbb{C}} V)_d$ the $\mathbb{C}G$-module whose underlying vector space is $M \otimes_{\mathbb{C}} V$ and on which $g \in G$ acts diagonally, i.e., $g(u \otimes v) = gu \otimes gv$. Note that $(M \otimes_{\mathbb{C}} V)_1$ is independent of the G-action on V and $\mathbb{C}G$-isomorphic to the direct sum of $\dim_{\mathbb{C}}(V)$ copies of M, whereas $(\mathbb{C}G \otimes_{\mathbb{C}} M)_d$ does depend on the G-action on M. We obtain a $\mathbb{C}G$-isomorphism

$$\xi_V(M) : (M \otimes_{\mathbb{C}} V)_1 \xrightarrow{\cong} (M \otimes_{\mathbb{C}} V)_d, \quad g \otimes v \mapsto g \otimes gv,$$

whose inverse sends $g \otimes v$ to $g \otimes g^{-1}v$. Given a $\mathbb{C}G$-homomorphism $f : \mathbb{C}G^m \to \mathbb{C}G^n$, we obtain a $\mathbb{C}G$-homomorphism $(f \otimes_{\mathbb{C}} \mathrm{id}_V)_d : (\mathbb{C}G^m \otimes_{\mathbb{C}} V)_d \to (\mathbb{C}G^n \otimes_{\mathbb{C}} V)_d$. If V is a d-dimensional complex representation which comes with a basis for the underlying complex vector space, we obtain an identification $(\mathbb{C}G^m \otimes_{\mathbb{C}} V)_1 = \mathbb{C}G^{md}$ and we define $\eta^G_{V,B}(f)$ by requiring that the following diagram commutes

$$\begin{tikzcd}
\mathbb{C}G^{md} = (\mathbb{C}G^m \otimes_{\mathbb{C}} V)_1 \arrow[r, "\xi_V(\mathbb{C}G^m)"] \arrow[d, "\eta^G_{V,B}(f)"'] & (\mathbb{C}G^m \otimes_{\mathbb{C}} V)_d \arrow[d, "f \otimes_{\mathbb{C}} \mathrm{id}_V"] \\
\mathbb{C}G^{nd} = (\mathbb{C}G^n \otimes_{\mathbb{C}} V)_1 \arrow[r, "\xi_V(\mathbb{C}G^n)"'] & (\mathbb{C}G^n \otimes_{\mathbb{C}} V)_d
\end{tikzcd}$$

More details can be found in [72, Section 1 and 2].

Some information about the equality of analytic and topological torsion for the twisted versions can be found for instance in [98].

7. Twisting L^2-invariants with a homomorphism to $\mathbb{R}$

Consider a finite connected CW-complex X and an element $\phi \in H^1(X;\mathbb{R}) = \hom(\pi_1(X),\mathbb{R})$. We call two functions $f_0, f_1 \colon \mathbb{R}^{>0} \to \mathbb{R}$ *equivalent* if there exists an element $r \in \mathbb{R}$ such that $f_0(t) - f_1(t) = r \cdot \ln(t)$ holds for all $t \in \mathbb{R}^{>0}$. In the sequel function $\mathbb{R}^{>0} \to \mathbb{R}$ is often to be understood as an equivalence class of functions $\mathbb{R}^{>0} \to \mathbb{R}$. One has to interprete some statements to be for one and hence all representatives and equality of functions means the equality of their equivalence classes.

Assumption 7.1. We will assume that

- The finite CW-complex X is det-L^2-acyclic;
- Its fundamental group $\pi_1(X)$ is residually finite;
- Its fundamental group $\pi_1(X)$ satisfies the Farrell-Jones Conjecture for $\mathbb{Z}G$.

Remark 7.2 (Assumption 7.1)**.** The reader does not need to know what the K-theoretic Farrell-Jones Conjecture for $\mathbb{Z}G$ is, it can be used as a black box. The reader should have in mind that it is known for a large class of groups, e.g., hyperbolic groups, CAT(0)-groups, solvable groups, lattices in almost connected Lie groups, fundamental groups of 3-manifolds and passes to subgroups, finite direct products, free products, and colimits of directed systems of groups (with arbitrary structure maps). For more information we refer for instance to [7, 8, 9, 25, 49, 73, 99].

In particular Assumption 7.1 is satisfied if X is an admissible 3-manifold.

Then from the construction of Section 6 we get a well-defined (equivalence class of) function $\mathbb{R}^{>0} \to \mathbb{R}$, denoted by

$$\overline{\rho}^{(2)}(\widetilde{X};\phi)\colon \mathbb{R}^{>0} \to \mathbb{R} \tag{7.3}$$

and called *reduced twisted L^2-torsion function*. It sends $t \in \mathbb{R}^{>0}$ to $\rho^{(2)}(\widetilde{X};\mathbb{C}_t)$ for the complex representation with underlying complex vector space $\mathbb{C}$ on which $g \in \pi_1(X)$ acts by multiplication with the real number $t^{\phi(g)}$.

If X is a finite not necessarily connected CW-complex, we require that Assumption 7.1 holds for each component C of X and we define

$$\overline{\rho}^{(2)}(\widetilde{X};\phi) = \sum_{C \in \pi_0(X)} \overline{\rho}^{(2)}(\widetilde{C};\phi|_C).$$

Theorem 7.4 (Properties of the twisted L^2-torsion function)**.** *Let X be a finite CW-complex which satisfies Assumption 7.1 and comes with an element $\phi \in H^1(X;\mathbb{R})$.*

(1) Well-definedness
The function $\overline{\rho}^{(2)}(\widetilde{X};\phi)$ is well-defined;

(2) Logarithmic estimate
There exist constants $C \geq 0$ and $D \geq 0$, such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \overline{\rho}^{(2)}(\widetilde{X};\phi)(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \overline{\rho}^{(2)}(\widetilde{X};\phi)(t) \leq C \cdot \ln(t) + D;$$

(3) G-homotopy invariance
Let Y be a finite CW-complex and let $f\colon Y \to X$ be a G-homotopy equivalence. Denote by $f^*\phi \in H^1(Y;\mathbb{R})$ the image of ϕ under the isomorphism $H^1(f;\mathbb{R})\colon H^1(X;\mathbb{R}) \xrightarrow{\cong} H^1(Y;\mathbb{R})$.

Then Y satisfies Assumption 7.1 with respect to $f^*\phi$ and we get

$$\overline{\rho}^{(2)}(Y;f^*\phi) = \rho^{(2)}(X;\phi);$$

(4) Sum formula
Consider a cellular pushout of finite CW-complexes

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

where i_1 is cellular, i_0 is an inclusion of CW-complexes and X has the obvious CW-structure coming from the ones on X_0, X_1 and X_2. Suppose that for $i = 0, 1, 2$ the map j_i is π_1-injective, i.e., for any choice of bases point $x_i \in X_i$ the induced map $\pi_1(j_i, x_i) \colon \pi_1(X_i, x_i) \to \pi_1(X, j_i(x_i))$ is injective. Suppose we are given elements $\phi_i \in H^1(X_i; \mathbb{R})$ and $\phi \in H^1(X; \mathbb{R})$ such that $j_i^*(\phi) = \phi_i$ holds for $i = 0, 1, 2$. Assume that X_i for $i = 0, 1, 2$ and X satisfy Assumption 7.1.

Then we get

$$\overline{\rho}^{(2)}(\widetilde{X}; \phi) = \overline{\rho}^{(2)}(\widetilde{X_1}; \phi_1) + \overline{\rho}^{(2)}(\widetilde{X_2}; \phi_2) - \overline{\rho}^{(2)}(\widetilde{X_0}; \phi_0);$$

(5) Product formula
Let Y be a finite connected CW-complex such that $\pi_1(Y)$ is residually finite. Consider an element $\phi' \in H^1(X \times Y; \mathbb{R})$ such that ϕ is the image of ϕ' under the map $H^1(X \times Y; \mathbb{R}) \to H^1(X; \mathbb{R})$ induced by the inclusion $X \to X \times Y$, $x \mapsto (x, y)$ for any choice of base point $y \in Y$. Suppose that X satisfies Assumption 7.1.

Then $X \times Y$ satisfies Assumption 7.1 with respect to ϕ' and we get

$$\overline{\rho}^{(2)}(\widetilde{X \times Y}; \phi') = \chi(Y) \cdot \overline{\rho}^{(2)}(\widetilde{X}; \phi);$$

(6) Poincaré duality
Let X be a finite orientable n-dimensional Poincaré complex, e.g., a closed orientable manifold of dimension n without boundary. Then

$$\overline{\rho}^{(2)}(\widetilde{X}; \phi)(t) = (-1)^{n+1} \cdot \overline{\rho}^{(2)}(\widetilde{X}; \phi)(t^{-1});$$

(7) Finite coverings
Let $p \colon Y \to X$ be a d-sheeted covering. Then Y satisfies Assumption 7.1 with respect to $p^*\phi$ and we get

$$\overline{\rho}^{(2)}(\widetilde{Y}; p^*\phi_X) = d \cdot \overline{\rho}^{(2)}(\widetilde{X}; \phi);$$

(8) Scaling ϕ
Let $r \in \mathbb{R}$ be a real number. Then

$$\rho^{(2)}(X; r \cdot \phi)(t) = \rho^{(2)}(X; \phi)(t^r).$$

(9) Value for $t = 0$.
The value $\overline{\rho}^{(2)}(\widetilde{X}; \phi)(0)$ is the L^2-torsion $\widetilde{\rho}(\widetilde{X})$.

Definition 7.5 (Degree of an equivalence class of functions $\mathbb{R}^{>0} \to \mathbb{R}$)**.** Let $\overline{\rho}$ be an equivalence class of functions $\mathbb{R}^{>0} \to \mathbb{R}$. Let ρ be a representative. Assume that $\liminf_{t \to 0+} \frac{\rho(t)}{\ln(t)} \in \mathbb{R}$ and $\limsup_{t \to \infty} \frac{\rho(t)}{\ln(t)} \in \mathbb{R}$.

Then define the *degree at zero* and the *degree at infinity* of ρ to be the real numbers

$$\begin{aligned} \deg_0(\rho) &:= \liminf_{t \to 0+} \frac{\rho(t)}{\ln(t)}; \\ \deg_\infty(\rho) &:= \limsup_{t \to \infty} \frac{\rho(t)}{\ln(t)}. \end{aligned}$$

Define the *degree* of $\overline{\rho}$ to be the real number

$$\deg(\overline{\rho}) \quad := \quad \deg_{\infty}(\rho) - \deg_0(\rho) = \limsup_{t \to \infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t \to 0+} \frac{\rho(t)}{\ln(t)}.$$

Thus we can assign to a finite CW-complex X satisfying Assumption 7.1 and $\phi \in H^1(X;\mathbb{R})$ its *degree*

$$\deg(X;\phi) := \deg(\overline{\rho}^{(2)}(\widetilde{X};\phi)) \in \mathbb{R}. \tag{7.6}$$

This is a new invariant with high potential although it is very hard to compute. We will be able to relate the degree to the Thurston norm for an admissible 3-manifold in the next Section 8.

Conjecture 7.7. *The reduced twisted L^2-torsion function $\overline{\rho}^{(2)}(\widetilde{X};\phi)\colon \mathbb{R}^{>0} \to \mathbb{R}$ is continuous.*

Moreover, the $\liminf$ *and* $\limsup$ *terms appearing in Definition 7.5 are actually limits* $\lim$.

Conjecture 7.7 has been proved for admissible 3-manifolds X by Liu [63, Theorem 1.2], where also multiplicative convexity is shown.

Moreover, for an admissible 3-manifold X the degree defines a continuous function on $H^1(X;\mathbb{R})$, see [63, Theorem 6.1]. We conjecture that this is true for every finite CW-complex X which satisfies Assumption 7.1.

Many of the results of this section are inspired by classical results on the Mahler measure, see [13, 14], which is the same as the Fuglede-Kadison determinant in the special case $G = \mathbb{Z}^d$, see [69, Example 3.13 on page 128 and (3.23) on page 136].

8. The degree of the reduced twisted L^2-torsion function and the Thurston norm

.

The following result was proved independently by Friedl-Lück[29, Theorem 0.1] and by Liu [63, Theorem 1.2]. The proofs depend on the facts that both the Thurston Geometrization Conjecture and the Virtually Fibering Conjecture are true, see Subsection 1.3 and 1.4.

Theorem 8.1. *Let M be an admissible* 3*-manifold in the sense of Definition 3.2. Then we get for any element $\phi \in H^1(M;\mathbb{Q})$ that*

$$\deg(M;\phi) = -x_M(\phi),$$

where the degree $\deg(M;\phi) := \deg(\overline{\rho}^{(2)}(\widetilde{M};\phi))$ *has been defined in Section 7 and $x_M(\phi)$ is the Thurston norm, see Subsection 1.7.*

Actually, Friedl-Lück [29, Theorem 5.1] get a much more general result, where one can consider not only the universal covering but appropriate G-coverings $G \to \overline{M} \to M$ and get estimates for the L^2-function for all times $t \in (0,\infty)$ which imply the equality of the degree and the Thurston norm.

9. The universal L^2-torsion and the Thurston polytope

9.1. **The weak Whitehead group.** Next we assign to a group G the weak K_1-groups $K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$ and the weak Whitehead group $\mathrm{Wh}^w(G)$, which are variations on the corresponding classical groups.

Definition 9.1 ($K_1^w(\mathbb{Z}G)$)**.** Define the *weak K_1-group*

$$K_1^w(\mathbb{Z}G)$$

to be the abelian group defined in terms of generators and relations as follows. Generators $[f]$ are given by of $\mathbb{Z}G$-endomorphisms $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ for $n \in \mathbb{Z}, n \geq 0$ such that the induced bounded G-equivariant operator $f^{(2)}\colon L^2(G)^n \to L^2(G)^n$ is a weak isomorphism of finite Hilbert $\mathcal{N}(G)$-modules. If $f_1, f_2\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ are $\mathbb{Z}G$-endomorphisms such that $f_1^{(2)}$ and $f_2^{(2)}$ are weak isomorphisms, then we require the relation

$$[f_2 \circ f_1] = [f_1] + [f_2].$$

If $f_0\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$, $f_2\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ and $f_1\colon \mathbb{Z}G^n \to \mathbb{Z}G^m$ are $\mathbb{Z}G$-maps such that $f_0^{(2)}$ and $f_2^{(2)}$ are weak isomorphisms, then we get for the $\mathbb{Z}G$-map

$$f = \begin{pmatrix} f_0 & f_1 \\ 0 & f_2 \end{pmatrix} \colon \mathbb{Z}G^{m+n} = \mathbb{Z}G^m \oplus \mathbb{Z}G^n \to \mathbb{Z}G^m \oplus \mathbb{Z}G^n$$

the relation

$$[f] = [f_0] + [f_2].$$

Let

$$\widetilde{K}_1^w(\mathbb{Z}G)$$

be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup generated by the element $[-\operatorname{id}\colon \mathbb{Z}G \to \mathbb{Z}G]$. This is the same as the cokernel of the obvious composite $K_1(\mathbb{Z}) \to K_1^w(\mathbb{Z}G)$. Define the *weak Whitehead group* of G

$$\operatorname{Wh}^w(G)$$

to be the cokernel of the homomorphism

$$\{\sigma \cdot g \mid \sigma \in \{\pm 1\}, g \in G\} \to K_1^w(\mathbb{Z}G), \quad \sigma \cdot g \mapsto [r_{\sigma \cdot g}\colon \mathbb{Z}G \to \mathbb{Z}G].$$

These groups are in general much larger than their classical analogues. For example, we have $\operatorname{Wh}(\mathbb{Z}) = 0$ and $\operatorname{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^\times/\{\pm z^n \mid n \in \mathbb{Z}\}$. More generally, if G is torsionfree, the Farrell-Jones Conjecture implies $\operatorname{Wh}(G) = \{0\}$ and we have the following result taken from [60, Theorem 0.1].

Theorem 9.2 ($K_1^w(G)$ and units in $\mathcal{D}(G)$)**.** *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let G be a torsionfree group which belongs to $\mathcal{C}$.*

Then the division closure $\mathcal{D}(G)$ of $\mathbb{Q}G$ in $\mathcal{U}(G)$ is a skew field and there are isomorphisms of abelian groups

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

9.2. **The universal L^2-torsion.** Given an L^2-acyclic finite based free $\mathbb{Z}G$-chain complex C_*, Friedl-Lück [27, Definition 1.7] assign to it its *universal L^2-torsion,*

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(G). \tag{9.3}$$

It is characterized by the universal properties that

$$\rho_u^{(2)}\big(0 \to \mathbb{Z}G \xrightarrow{\pm \operatorname{id}} \mathbb{Z}G \to 0\big) = 0$$

and that for any short based exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of L^2-acyclic finite based free $\mathbb{Z}G$-chain complexes we get $\rho_u^{(2)}(D_*) = \rho_u^{(2)}(C_*) + \rho_u^{(2)}(E_*)$, as explained in [27, Definition 1.16]. If X is a det-L^2-acyclic finite CW-complex with fundamental group $\pi = \pi_1(X)$, it defines an element

$$\rho_u^{(2)}(\widetilde{X}) \in \operatorname{Wh}^w(\pi) \tag{9.4}$$

determined by $\rho_u^{(2)}(C_*(\widetilde{X}))$, where $C_*(\widetilde{X})$ is the cellular $\mathbb{Z}\pi$-chain complex of the universal covering $\widetilde{X}$ of X.

The basic properties of these invariants including homotopy invariance, sum formula, product formula, and Poincaré duality are collected in [27, Theorem 2.11]. One can show for a finitely presented group G, for which there exists at least one L^2-acyclic finite connected CW-complex X with $\pi_1(X) \cong G$, that every element in $\mathrm{Wh}^w(G)$ can be realized as $\rho_u^{(2)}(\widetilde{Y})$ for some L^2-acyclic finite connected CW-complex Y with $G \cong \pi_1(Y)$, see [27, Lemma 2.8].

The point of this new invariant is that it encompasses many other well-known invariants such as the reduced twisted L^2-torsion function (which is sometimes also called L^2-Alexander torsion), as explained in [27, Introduction]. We next illustrate this by considering the dual Thurston polytope of an admissible 3-manifold.

9.3. **Polytopes.** A *polytope* in a finite-dimensional real vector space V is a subset which is the convex hull of a finite subset of V. An element p in a polytope is called *extreme* if the implication $p = \frac{q_1}{2} + \frac{q_2}{2} \Longrightarrow q_1 = q_2 = p$ holds for all elements q_1 and q_2 in the polytope. Denote by $\mathrm{Ext}(P)$ the set of extreme points of P. If P is the convex hull of the finite set S, then $\mathrm{Ext}(P) \subseteq S$ and P is the convex hull of $\mathrm{Ext}(P)$. The *Minkowski sum* of two polytopes P_1 and P_2 is defined to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p \in P_2\}.$$

It is the convex hull of the set $\{p_1 + p_2 \mid p_1 \in \mathrm{Ext}(P_1), p_2 \in \mathrm{Ext}(P_2)\}$.

Let H be a finitely generated free abelian group. We obtain a finite-dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} H$. An *integral polytope* in $\mathbb{R} \otimes_{\mathbb{Z}} H$ is a polytope such that $\mathrm{Ext}(P)$ is contained in H, where we consider H as a lattice in $\mathbb{R} \otimes_{\mathbb{Z}} H$ by the standard embedding $H \to \mathbb{R} \otimes_{\mathbb{Z}} H,\ h \mapsto 1 \otimes h$. The Minkowski sum of two integral polytopes is again an integral polytope. Hence the integral polytopes form an abelian monoid under the Minkowski sum with the integral polytope $\{0\}$ as neutral element.

Definition 9.5 (Grothendieck group of integral polytopes)**.** Let $\mathcal{P}_{\mathbb{Z}}(H)$ be the abelian group given by the Grothendieck construction applied to the abelian monoid of integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum.

Notice that for polytopes P_0, P_1 and Q in a finite-dimensional real vector space we have the implication $P_0 + Q = P_1 + Q \Longrightarrow P_0 = P_1$, see [85, Lemma 2]. Hence elements in $\mathcal{P}_{\mathbb{Z}}(H)$ are given by formal differences $[P] - [Q]$ for integral polytopes P and Q in $\mathbb{R} \otimes_{\mathbb{Z}} H$ and we have $[P_0] - [Q_0] = [P_1] - [Q_1] \Longleftrightarrow P_0 + Q_1 = P_1 + Q_0$.

There is an obvious homomorphism of abelian groups $i\colon H \to \mathcal{P}_{\mathbb{Z}}(H)$ which sends $h \in H$ to the class of the polytope $\{h\}$. Denote its cokernel by

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \quad = \quad \mathrm{coker}\big(i\colon H \to \mathcal{P}_{\mathbb{Z}}(H)\big). \tag{9.6}$$

Put differently, in $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ two polytopes are identified if they are obtained by translation with some element in the lattice H from one another.

Example 9.7. An integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}$ is given by an interval $[m, n]$ for integers m, n with $m \leq n$. The Minkowski sum becomes $[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$. One easily checks that one obtains isomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^2 \quad [[m, n]] \mapsto (n - m, m); \tag{9.8}$$

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \quad [[m, n]] \mapsto n - m. \tag{9.9}$$

Given a homomorphism of finitely generated abelian groups $f\colon H \to H'$, we can assign to an integral polytope $P \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H$ an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H'$ by the image of P under $\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f \colon \mathbb{R} \otimes_{\mathbb{Z}} H \to \mathbb{R} \otimes_{\mathbb{Z}} H'$ and thus we obtain homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f) \colon \mathcal{P}_{\mathbb{Z}}(H) \to \mathcal{P}_{\mathbb{Z}}(H'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)]; \tag{9.10}$$

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(f) \colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H'). \tag{9.11}$$

The elementary proof of the next lemma can be found in [27, Lemma 3.8].

Lemma 9.12. *Let H be a finitely generated free abelian group. Then:*

(1) The homomorphism

$$\xi \colon \mathcal{P}_{\mathbb{Z}}(H) \to \prod_{\phi \in \hom_{\mathbb{Z}}(H,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad [P] - [Q] \mapsto \big(\mathcal{P}_{\mathbb{Z}}(\phi)([P] - [Q])\big)_{\phi}$$

is injective;

(2) The canonical short sequence of abelian groups

$$0 \to H \xrightarrow{i} \mathcal{P}_{\mathbb{Z}}(H) \xrightarrow{\mathrm{pr}} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \to 0$$

is split exact;

(3) The abelian groups $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ are free. They are finitely generated free if and only if $H \cong \mathbb{Z}$.

Explicit bases of the free abelian groups $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ are constructed by Funke [34].

9.4. **The polytope homomorphism and the L^2-torsion polytope.** Given a torsionfree group G that satisfies the Atiyah Conjecture, the *polytope homomorphism*

$$\mathbb{P} \colon \mathrm{Wh}^w(\mathbb{Z}G) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) \tag{9.13}$$

is constructed in [27, Section 3.2 3.2].

Definition 9.14 (L^2-torsion polytope)**.** Let X is an L^2-acyclic finite CW-complex such that $\pi_1(X)$ is torsionfree and satisfies the Atiyah Conjecture. The L^2*-torsion polytope*

$$P(\widetilde{X}) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f)$$

is defined to be the negative of the image of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{X})$ defined in Subsection 9.2 under the polytope homomorphism (9.13).

Note that we abuse language here a little bit, the L^2-torsion polytope is a formal difference of integral polytopes and not itself a polytope.

9.5. **The dual Thurston polytope and the L^2-torsion polytope.** Of particular interest is the composition of the universal torsion with the polytope homomorphism. For example let M be an admissible 3-manifold that is not a closed graph manifold. Then we obtain a well-defined element

$$P(\widetilde{M}) := \mathbb{P}(\rho_u(\widetilde{M})) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f).$$

Recall from Theorem 3.3 3 the fundamental group of an admissible 3-manifold M satisfies the Atiyah Conjecture and hence its L^2-torsion polytope $P(\widetilde{X}) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f)$ is defined. The next result is taken from [27, Theorem 3.7].

Theorem 9.15 (The dual Thurston polytope and the L^2-torsion polytope)**.** *Let M be an admissible 3-manifold. Then*

$$[T(M)^*] = 2 \cdot P(\widetilde{M}) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f).$$

10. Profinite completion of the fundamental group of a 3-manifold

We can associate to a (discrete) group *its profinite completion* defined as

$$\widehat{G} \quad := \quad \operatorname{invlim}_N G/N \tag{10.1}$$

where N runs through normal subgroups N of G with finite index $[G : N]$. The inverse limit $\widehat{G}$ is a compact, totally disconnected group. The canonical group homomorphism $i\colon G \to \widehat{G}$ has dense image.

Definition 10.2 (Profinitely rigid)**.** A finitely generated residually finite group G is *profinitely rigid* if for every finitely generated residually finite group K with $\widehat{K} \cong \widehat{G}$ we have $K \cong G$.

It makes no difference whether $\widehat{K} \cong \widehat{G}$ means abstract isomorphism of groups or topological group isomorphism, see Nikolov and Segal [80, Theorem 1.1].

Recall that an admissible 3-manifold N is topologically rigid in the sense that any other 3-manifold N with $\pi_1(N) \cong \pi_1(M)$ is homeomorphic to M, see Subsection 1.5. So one may ask whether for two admissible 3-manifolds the following three assertions are equivalent

- $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$;
- $\pi_1(M) \cong \pi_1(N)$;
- M and N are homeomorphic.

To the author's knowledge profinite rigidity of fundamental groups of hyperbolic closed 3-manifolds, even among themselves, is an open question. Examples of hyperbolic closed 3-manifolds, whose fundamental groups are profinite rigid in the absolute sense, are constructed in [16]. A weaker but still open problem is the following which is equivalent to [48, Conjecture 6.33].

Conjecture 10.3 (Volume and profinitely rigidity)**.** *Let M and N be admissible closed 3-manifolds. Then $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$ implies $\rho^{(2)}(\widetilde{M}) = \rho^{(2)}(\widetilde{N})$.*

Recall that for two hyperbolic 3-manifolds we have $\rho^{(2)}(\widetilde{M}) = \rho^{(2)}(\widetilde{M}) \iff \operatorname{vol}(M) = \operatorname{vol}(N)$, see Theorem 4.2, and there are up to diffeomorphism only finitely many hyperbolic 3-manifolds with the same volume.

Liu [64, Theorem 1.1] has shown that among the class of finitely generated 3-manifold groups, every finite-volume hyperbolic 3-manifold group is profinitely almost rigid, where a group G is *profinitely almost rigid* among a class of groups $\mathcal{C}$, if there exist finitely many groups in $\mathcal{C}$, such that any group in $\mathcal{C}$ that is profinitely isomorphic to G is isomorphic to one of those groups. Seifert manifolds and graph manifolds have been treated in [100, 101].

The proof of the following result can be found in [48, Satz 6.34].

Theorem 10.4. *If Conjecture 3.26 holds in dimension 3, Conjecture 10.3 is true.*

The first L^2-Betti number is profinite among finitely presented residually finite groups, i.e., the first L^2-Betti numbers of two finitely presented residually finite groups agree if the profinite completion of these two groups are isomorphic, see [15, Corollary 3.3]. Higher L^2-Betti numbers are not profinite rigid among finitely presented residually finite groups, see [50, Theorem 1].

For more information about profinite rigidity we refer for instance to [48, Section 6.7].

11. Miscellaneous

We can also use this approach to assign formal differences of polytopes to many other groups, e.g., free-by-cyclic groups and two-generator one-relator groups.

These examples are discussed in more details in [30], where further references to the literature can be found.

Finally let G be any group that admits a finite model for BG and that satisfies the Atiyah Conjecture and let $f\colon G \to G$ be a monomorphism. Then we can associate to this monomorphism the polytope invariant of the corresponding ascending HNN-extension. If $G = F_2$ is the free group on two generators this polytope invariant has been studied by Funke-Kielak [35]. We hope that this invariant of monomorphisms of groups will have other interesting applications.

Bieri-Neumann-Strebel invariants are related to polytopes and L^2-invariants, see for instance [52].

There are further interesting connections between L^2-invariants and group theory, orbit equivalence and von Neumann algebras which we cannot cover here, see for instance [69].

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