Groups meet C^*-algebras, an appetizer

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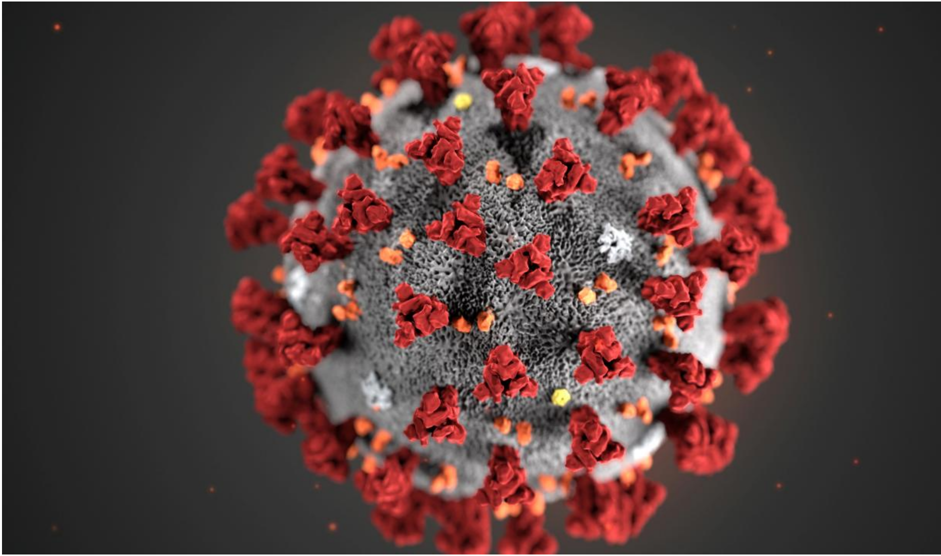
June 18th, 2020

- Conference Groups meet C*-algebras
- 7th Florianopolis - Münster - Ottawa Conference
- in honour of Siegfried Echterhoff's 60th birthday
- Münster, June 15-19, 2020



Happy Birthday Siegfried

- I will not present all the slides during the talk.

- I will post all of them on my homepage, but without the additional drawings or comments given during the talk.

Baby Example $\mathbb{C}$

- Let $f\colon V \to W$ be a $\mathbb{C}$-linear map between finite-dimensional complex vector spaces.
- Interesting numbers associated to f are $\dim_{\mathbb{C}}(\ker(f))$ and $\dim_{\mathbb{C}}(\operatorname{coker}(f))$.
- But they are not homotopy invariant notions.

- However the index, defined to be the integer

$$\text{index}(f) := \dim_{\mathbb{C}}(\text{coker}(f)) - \dim_{\mathbb{C}}(\ker(f)),$$

  is a homotopy invariant.

- Namely, the additivity of the dimension under exact sequences implies the formula

$$\text{index}(f) = \dim_{\mathbb{C}}(W) - \dim_{\mathbb{C}}(V).$$

- More generally, let C_* be a finite $\mathbb{C}$-chain-complex.

$$\cdots \to 0 \xrightarrow{c_{d+1}} C_d \xrightarrow{c_d} C_{d-1} \xrightarrow{c_{d-1}} \cdots C_1 \xrightarrow{c_1} C_0 \to 0 \to \cdots .$$

- Define its Euler characteristic (or index)

$$\chi(C_*) := \sum_i (-1)^i \cdot \dim_{\mathbb{C}}(H_i(C_*)).$$

- Since $H_n(C_*) = \ker(c_n)/\operatorname{im}(c_{n+1})$ is a $\mathbb{C}$-chain homotopy invariant, $\chi(C_*)$ is a chain homotopy invariant.
- We have

$$\chi(C_*) = \sum_i (-1)^i \cdot \dim_{\mathbb{C}}(C_i).$$

- Let X be a finite CW-complex.
- It comes with a finite $\mathbb{C}$-chain complex $C_*^c(X)$.
- Define its Euler characteristic $\chi(X)$ to be $\chi(C_*^c(X))$ which is a homotopy invariant of X.
- The formula above says

$$\chi(X) = \sum_k (-1)^k \cdot |\{\text{cells of dimension } k\}|.$$

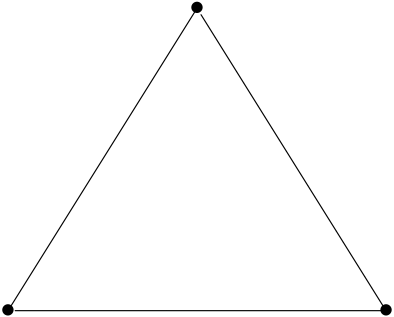
- There is the well-known formula from combinatorics

$$1 = \sum_{i=0}^{n} (-1)^i \cdot \binom{n+1}{i+1}.$$

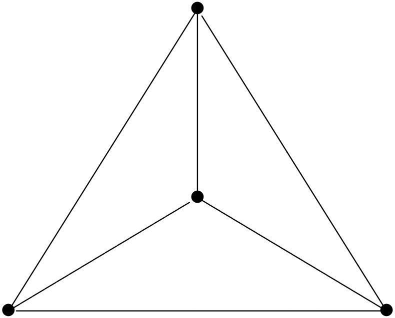
- We want to give a topological proof.
- Let Δ_n be the *n-dimensional simplex* which is the convex hull of the points $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$ $(0, 0, \dots, 1)$ in $\mathbb{R}^{n+1}$.
- Δ_0 is a point

- Δ_1 is the interval $[0, 1]$.

- Δ_2 is a (solid) triangle



- Δ_3 is the (solid) tetrahedron.



- Δ_n is a finite CW-complex whose numbers of i-cells is $\binom{n+1}{i+1}$.
- It is homotopy equivalent to Δ_0.
- Hence we get

$$1 = \chi(\Delta_0) = \chi(\Delta_n) = \sum_{i=0}^{n} (-1)^i \cdot |\{i - \text{cells}\}| = \sum_{i=0}^{n} (-1)^i \cdot \binom{n+1}{i+1}.$$

Passing to infinite dimensions

- For applications in analysis it is unrealistic to consider finite-dimensional vector spaces.
- An operator $T \colon H_0 \to H_1$ between Hilbert spaces (or pre-Hilbert spaces) is a continuous linear $\mathbb{C}$-map.
- It is called Fredholm operator if the dimension of its kernel is finite, its image is closed, and the dimension of its cokernel is finite.
- Its index is defined as above to be the integer

$$\operatorname{index}(f) = \dim_{\mathbb{C}}(\operatorname{coker}(f)) - \dim_{\mathbb{C}}(\ker(f)).$$

- This turns out to be a homotopy invariant and there is also a chain complex version.

- A basic example is the deRahm cochain complex $\Omega^*(M)$ associated to a smooth closed n-dimensional manifold M

$$\cdots \to 0 \to \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \cdots \Omega^n(M) \to 0 \to \cdots .$$

- It is a Fredholm cochain complex in the sense that $H^i(\Omega^*(M))$ is finite dimensional for every i.
- We cannot define its index on the cochain complex level, since each $\Omega^i(M)$ is infinite-dimensional, but can define its index to be the integer

$$\operatorname{index}(\Omega^*(M)) = \sum_i (-1)^i \cdot \dim_{\mathbb{C}}\bigl(H^i(\Omega^*(M))\bigr).$$

- The famous deRham Theorem says that $H^i(\Omega^*(M))$ agrees with the i-th cellular cohomology of M with $\mathbb{C}$-coefficients and hence in particular we get a kind of Index Theorem

$$\operatorname{index}(\Omega^*(M)) = \chi(M).$$

- Another important example is the Dirac operator D on a closed Spin manifold M.
- One can assign to it its analytic index $\operatorname{index}(D) \in \mathbb{Z}$.
- One can assign to a closed Spin manifold M by topological methods an (a priori rational) number, its $\widehat{A}$-genus $\widehat{A}(M)$.
- The famous Atiyah-Singer Index Theorem says in particular

$$\operatorname{index}(D) = \widehat{A}(M).$$

- The Atiyah-Singer Index Theorem was motivated and reproves the Hirzebruch Signature Theorem

$$\operatorname{sign}(M) = \widehat{L}(M).$$

Taking the action of a finite group into account

- Let G be a finite group.
- Let V and W be finite-dimensional complex vector spaces with linear G-actions.
- Let $f\colon V \to W$ be a linear G-map.
- We want to improve our results from the beginning by a more sophisticated "counting".

Definition (Projective class group $K_0(R)$)

The projective class group $K_0(R)$ of a ring R is defined to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.
- If F is a field, e.g., $F = \mathbb{C}$, the dimension induces an isomorphism $K_0(F) \xrightarrow{\cong} \mathbb{Z}$.

- If R is any ring and G is any (discrete) group, the group ring RG is the R-algebra, whose underlying R-module is the free R-module generated by G and whose multiplication comes from the group structure.

- An element $x \in RG$ is a formal sum $\sum_{g \in G} r_g \cdot g$ such that only finitely many of the coefficients $r_g \in R$ are different from zero.

- The multiplication comes from the tautological formula $g \cdot h = g \cdot h$, more precisely

$$\left(\sum_{g \in G} r_g \cdot g\right) \cdot \left(\sum_{g \in G} s_g \cdot g\right) := \sum_{g \in G} \left(\sum_{h,k \in G, hk=g} r_h s_k\right) \cdot g.$$

- A RG-module P is the same as G-representation with coefficients in R, i.e., a R-modul P together with a G-action by R-linear maps.

- If G is finite, a finitely generated projective $\mathbb{C}G$-module is the same as a finite-dimensional complex G-representation, and we get an identification of $K_0(\mathbb{C}G)$ and the complex representation ring $R_{\mathbb{C}}(G)$.
- We get from Wedderburn's Theorem

$$\mathbb{C}G \cong \prod_V M_{n(V)}(\mathbb{C})$$

where V runs through the isomorphism classes of irreducible G-representations.
- This implies

$$R_{\mathbb{C}}(G) \cong K_0(\mathbb{C}G) \cong \prod_V \mathbb{Z} \cong \mathbb{Z}^c$$

where c is the number of isomorphism classes of irreducible G-representations which agrees with the number of conjugacy classes of elements in G.

- Now we will count in $R_{\mathbb{C}}(G)$ instead of $\mathbb{Z}$.
- We go back to the G-equivariant linear map $f\colon V \to W$ of finite-dimensional G-representations.
- We define $\text{index}^G(f) = [\ker(f)] - [\text{cok}(f)] \in R_{\mathbb{C}}(G)$.
- We get in $R_{\mathbb{C}}(G)$
$$\text{index}^G(f) = [W] - [V]$$
and hence $\text{index}^G(f)$ is a G-homotopy invariant.

Definition (G-CW-complex)

Let G be a topological group. A G-CW-complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

- Let G be a finite group and let X be a G-CW-complex which is finite, e.g., build by finitely many G-cells, or, equivalently, X is compact.
- Define the G-Euler characteristic of X

$$\chi^G(X) = \sum_i (-1)^i \cdot [H_i(X)] \quad \in R_{\mathbb{C}}(G) = K_0(\mathbb{C}G).$$

- We get by the same proof as before the equality in $R_{\mathbb{C}}(G) = K_0(\mathbb{C}G)$

$$\chi^G(X) = \sum_c (-1)^{n(c)} \cdot [\mathbb{C}[G/H_c]]$$

where c runs through the equivariant cells $c = G/H_c \times D^{n(c)}$.

- Also the index theorems mentioned above carry over in this fashion.

The passage to infinite groups and the role of the group C^*-algebra

- There are many reasons why one would like to consider also infinite groups.
- One reason is that the fundamental group π of a closed smooth manifold M is infinite and that one would like to carry out the analogues of some of the previous constructions for its universal covering $\widetilde{M}$ taking the π-action into account.
- This causes formidable problems concerning the analysis and one is forced to replace the complex group ring by certain larger completions.

- Let G be (countable discrete) group and $L^2(G)$ be the associated Hilbert space. One obtains an embedding

$$\mathbb{C}G \subseteq \mathcal{B}(L^2(G), L^2(G))$$

into the algebra of bounded G-operators $L^2(G) \to L^2(G)$ by the regular representation sending $g \in G$ to the operator $R_g \colon L^2(G) \to L^2(G),\ x \mapsto xg^{-1}$.

- We can define a string of subalgebras of $\mathcal{B}(L^2(G), L^2(G))$ by completing $\mathbb{C}G$ with respect to specific topologies or norms.

$$\mathbb{C}G \subseteq \mathcal{F}(G) \subseteq L^1(G) \subseteq C_r^*(G) \subseteq \mathcal{N}(G) = \mathcal{B}(L^2(G), L^2(G))^G \subseteq \mathcal{B}(L^2(G), L^2(G))$$

- $L^1(G)$, $C_r^*(G)$, and $\mathcal{N}(G)$ are Banach algebras with involutions, i.e., normed complete complex vector spaces with the structure of a $\mathbb{C}$-algebra satisfying $||A \cdot B|| \leq ||A|| \cdot ||B||$ and an isometric involution $*$.
- $C_r^*(G)$ and $\mathcal{N}(G)$ are C^*-algebras, i.e., Banach algebras with involution satisfying the so called C^*-identity $||xx^*|| = ||x||^2$.
- If $G = \mathbb{Z}$, one can find nice models, which is in general not possible

$$\begin{aligned} \mathcal{F}(\mathbb{Z}) &= \mathcal{O}(\mathbb{C} \setminus \{0\}); \\ C_r^*(\mathbb{Z}) &= C(S^1); \\ \mathcal{N}(\mathbb{Z}) &= L^\infty(S^1). \end{aligned}$$

- One can think of the notion of a C^*-algebra as a non-commutative space since any commutative C^*-algebra is of the form $C(X)$ equipped with the supremums norm for a compact Hausdorff space X.

- A very important invariant of a C^*-algebra A is its topological K-theory which assigns to A a $\mathbb{Z}$-graded abelian group $K_*(A)$.
- $K_0(A)$ is the projective class group and is independent of the topological structure on A;
- $K_n(A)$ is $\pi_n(GL(A))$ for $n \geq 1$ and does depend of the topological structure on A;
- Topological K-theory satisfies Bott periodicity: $K_n(A) \cong K_{n+2}(A)$;
- If X is a compact Hausdorff space, then $K_n(C(X))$ agrees with the classical topological K-theory $K^n(X)$ defined in terms of complex vector bundles over X.

- There has been tremendous progress in the classification of C^*-algebras in the sense that certain classes of C^*-algebras can be classified by their topological K-theory.
- A lot of major contributions to the construction and classification of C^*-algebras are due to mathematicians from Münster, notably Joachim Cuntz, Wilhelm Winter, and Xin Li.
- However, members of these classes are simple and nuclear, and group C^*-algebras do not have this property in general.

- The topological K-groups $K_*(C_r^*(G))$ are the natural recipients for indices of G-equivariant operators acting on proper cocompact smooth Riemannian G-manifolds by isometries.
- An example is the Dirac operator $\widetilde{D}$ acting on the universal covering $\widetilde{M}$ of an n-dimensional closed Spin manifold M with fundamental group π.
- We can assign to it its C^*-index

$$\mathrm{index}_{C_r^*(\pi)}(M) = \mathrm{index}_{C_r^*(\pi)}(\widetilde{D}) \in K_n(C_r^*(\pi)).$$

and also its variant over $\mathbb{R}$ instead of $\mathbb{C}$

$$\mathrm{index}_{C_r^*(\pi;\mathbb{R})}(M) = \mathrm{index}_{C_r^*(\pi;\mathbb{R})}(\widetilde{D}_{\mathbb{R}}) \in KO_n(C_r^*(\pi;\mathbb{R})).$$

- Next we illustrate its significance.

- A Bott manifold is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8;
- We fix such a choice. (The particular choice does not matter.)
- We have

$$\operatorname{ind}_{C_r^*(\pi;\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(\pi;\mathbb{R})}(M \times B).$$

- If M carries a Riemannian metric with positive scalar curvature, then the index $\operatorname{index}_{C_r^*(\pi;\mathbb{R})}(M) \in KO_n(C_r^*(\pi;\mathbb{R}))$ must vanish by the Bochner-Lichnerowicz formula.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin-manifold of dimension $n \geq 5$. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\operatorname{index}_{C_r^*(\pi;\mathbb{R})}(M) = 0 \quad \in KO_n(C_r^*(\pi;\mathbb{R})).$$

- The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.
- The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0$.
- Schick(1998) has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau for $G = \mathbb{Z}^4 \times \mathbb{Z}/3$.
- It is not known whether the unstable version is true or false for finite fundamental groups.

Classifying space for proper G-actions

Definition (Classifying space for proper G-actions)

A model for the classifying space for proper G-actions is a G-CW-complex $\underline{E}G$ which has the following properties:

- All isotropy groups of $\underline{E}G$ are finite;
- For every finite subgroup $H \subseteq G$ the H-fixed point set $\underline{E}G^H$ is weakly contractible.

- There always exists a model for $\underline{E}G$
- For every proper G-CW-complex X there is up to G-homotopy precisely one G-map $X \to \underline{E}G$.
- Two models for $\underline{E}G$ are G-homotopy equivalent.

- These spaces play a central role in equivariant homotopy theory over infinite groups.
- The space $\underline{E}G$ have often very nice geometric models and capture much more information about a group G and its geometry than EG if G is not torsionfree.
- Here is a list of examples

group	space
hyperbolic group	Rips complex
Mapping class group	Teichmüller space
$\mathrm{Out}(F_n)$	Outer space
lattice L in a connected Lie group G	G/K

- Other nice models come from appropriate actions of a group G on trees or manifolds with non-negative sectional curvature.

The Baum-Connes Conjecture

- There is a G-homology theory K_*^G which assigns to every G-CW-complex X a $\mathbb{Z}$-graded abelian group $K_*^G(X)$.
- We have for finite $H \subseteq G$

$$K_n(G/H) = \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

- There is for every $n \in \mathbb{Z}$ an assembly map given essentially by taking C^*-indices of operators

$$K_n^G(\underline{E}G) \to K_n(C_r^*(G)).$$

Conjecture (Baum-Connes Conjecture)

A group G satisfies the Baum-Connes Conjecture if the assembly map is bijective for every $n \in \mathbb{Z}$.

- The Baum-Connes Conjecture is one of the most important conjectures about group C^*-algebras.
- It has many consequences for the theory of C^*-algebras, also for ones which do not come from groups.
- It also has a lot of consequences for questions about groups, geometry, and topology.
- For instance it implies the famous Novikov Conjecture about the homotopy invariance of higher signatures.
- It implies the Stable Gromov-Lawson-Rosenberg Conjecture as proved by Stolz.

- The Baum-Connes Conjecture (and its version with coefficients) is known for a large class of groups including groups having the Haagerup property and hyperbolic groups. This is due to Higson-Kasparov and Lafforgue.
- There is a long list of mathematicians who made substantial contributions to the Baum-Connes Conjecture.
- Permanence properties for the Baum-Connes Conjecture (with coefficients) have been established by Chabert-Echterhoff.
- The Baum-Connes Conjecture has been proved by Chabert-Echterhoff-Nest for almost connected second countable Hausdorff groups.
- The Baum-Connes Conjecture is open for $SL_n(\mathbb{Z})$ for $n \geq 3$.

Computations based on the Baum-Connes Conjecture

- Most computations of $K_n(C_r^*(G))$ are based on the Baum-Connes Conjecture since the source of the assembly map is much more accessible than the target.
- This comes from certain techniques from equivariant homotopy theory, e.g. equivariant Atiyah-Hirzebruch spectral sequence, p-chain spectral sequence, or good models for $\underline{E}G$ coming from geometry.
- Rationally $K_n^G(\underline{E}G)$ is rather well understood due to equivariant Chern characters, see Baum-Connes, Lück
- Integrally calculations can only be done in special cases, no general pattern is in sight and actually not expected.

- Such calculations are interesting in their own right. Often they have interesting consequences for questions and the classification of certain C^*-algebras, which are not necessarily themselves group C^*-algebras but in some sense connected, thanks to the meanwhile well-established classification of certain classes of C^*-algebras by their topological K-theory.

- An example is the classification of certain C^*-algebras, which Cuntz assigned to the ring of integers in number fields, by Li-Lück. It turned out that these C^*-algebras do not capture much from the number theory. This led Cuntz, Deninger, Li to the insight that one has to take certain dynamical systems into account.

- Another application of such computations to questions about C^*-algebras is the analysis of the structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$ and the tracial Rokhlin property by Echterhoff-Lück-Phillipps-Walter.
- Recall that Schick disproved the unstable Gromov-Lawson-Rosenberg Conjecture for $G = \mathbb{Z}^4 \times \mathbb{Z}/3$.
- On the other hand Davis-Lück proved it for certain semi-direct products $G = \mathbb{Z}^4 \rtimes \mathbb{Z}/3$ based on calculating $KO(C_r^*(\mathbb{Z}^4 \rtimes \mathbb{Z}/3))$.
- This shows that the class of groups, for which the unstable version holds, is not closed under extensions.

- The following computation is due to Langer-Lück.
- Consider the extension of groups $1 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z}/m \to 1$ such that the conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$.
- We obtain an isomorphism

$$\omega_1 \colon K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(\Gamma \backslash \underline{E}\Gamma).$$

- Restriction with the inclusion $k \colon \mathbb{Z}^n \to \Gamma$ induces an isomorphism

$$k^* \colon K_1(C_r^*(\Gamma)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}.$$

- Let $\mathcal{M}$ be the set of conjugacy classes of maximal finite subgroups of Γ.
- There is an exact sequence

$$0 \to \bigoplus_{(M)\in\mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M)\in\mathcal{M}} i_M} K_0(C_r^*(\Gamma)) \xrightarrow{\omega_0} K_0(\Gamma\backslash\underline{E}\Gamma) \to 0,$$

  where $\widetilde{R}_{\mathbb{C}}(M)$ is the kernel of the map $R_{\mathbb{C}}(M) \to \mathbb{Z}$ sending the class $[V]$ of a complex M-representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$ and the map i_M comes from the inclusion $M \to \Gamma$ and the identification $R_{\mathbb{C}}(M) = K_0(C_r^*(M))$.
- We have

$$K_i(C_r^*(\Gamma)) \cong \mathbb{Z}^{s_i}$$

  where

$$s_i = \begin{cases} \left(\sum_{(M)\in\mathcal{M}}(|M|-1)\right) + \sum_{l\in\mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l}\mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ even;} \\ \sum_{l\in\mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l+1}\mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ odd.} \end{cases}$$

- If m is even, then $s_1 = 0$ and

$$K_1(C_r^*(\Gamma)) \cong \{0\}.$$

- If m is a prime p, then

$$s_i = \begin{cases} p^k \cdot (p-1) + \frac{2^n+p-1}{2p} + \frac{p^{k-1}\cdot(p-1)}{2} & p \neq 2 \text{ and } i \text{ even;} \\ \frac{2^n+p-1}{2p} - \frac{p^{k-1}\cdot(p-1)}{2} & p \neq 2 \text{ and } i \text{ odd;} \\ 3 \cdot 2^{k-1} & p = 2 \text{ and } i \text{ even;} \\ 0 & p = 2 \text{ and } i \text{ odd.} \end{cases}$$

- The group Γ is a crystallographic group. The computation of the topological K-theory of the group C^*-algebra seems to be out of reach for crystallographic groups in general.

- The proof of the results above is surprisingly complicated.
- It is based on computations of the group homology of $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ by Langer-Lück.
- They prove a conjecture of Adem-Ge-Pan-Petrosyan which says that the associated Lyndon-Hochschild-Serre spectral sequence collapses in the strongest sense, in the special case that the conjugation action of $\mathbb{Z}/m$ of $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$;
- Moreover, they use generalizations of the Atiyah-Segal Completion Theorem for finite groups to infinite groups due to Lück-Oliver.
- Interestingly, the conjecture of Adem-Ge-Pan-Petrosyan is disproved in general by Langer-Lück.

How much does $K_*(C_r^*(G))$ tells us about G?

- The answer is, roughly speaking, not much.
- One can compute the topological K-theory of the group C^*-algebra for certain classes of groups and it turns out in many cases that the result does only depend on a few invariants of the group. (This is of course good news from the computational point of view.)
- If G is a finite abelian group, then $K_*(C_r^*(G))$ is $\mathbb{Z}^{|G|}$ in even dimensions and $\{0\}$ in odd dimensions and hence depends only on the order $|G|$ of G.
- This phenomenon can be confirmed for instance for one-relator groups, right-angled Artin groups and right-angled Coxeter groups, where complete calculations are possible. Also the computations above for $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ support this.

- There is a trace homomorphism

$$\mathrm{tr}\colon K_0(C_r^*(G)) \to \mathbb{R}.$$

- If the group G contains an element of order n, then $1/n$ is in the image $\mathrm{im}(\mathrm{tr})$.
- In particular G is torsionfree only if $\mathrm{im}(\mathrm{tr}) = \mathbb{Z}$.
- Suppose that G satisfies the Baum-Connes Conjecture. Then:
  - G is torsionfree if and only if $\mathrm{im}(\mathrm{tr}) = \mathbb{Z}$.
  - G contains non-trivial p-torsion, if and only if $1/p \in \mathrm{im}(\mathrm{tr})$.
  - Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be the ring obtained from $\mathbb{Z}$ by inverting the orders of all finite subgroups of G. Then $\mathrm{im}(\mathrm{tr}) \subseteq \Lambda$. This follows from Λ-valued Chern character of Lück.
- There is a group G such that any non-trivial finite subgroup is isomorphic to $\mathbb{Z}/3$, but $1/9$ is contained in $\mathrm{im}(\mathrm{tr})$, see Roy.

How much does $C_r^*(G)$ tells us about G?

- Also here the answer seems to be not much but of course more than $K_*(C_r^*(G))$. Here are some positive or negative results.
- Two finite abelian groups have isomorphic complex group rings. However, two finite abelian groups are isomorphic if and only if their rational group rings are isomorphic.
- The quaternion group and the dihedral group of order eight have isomorphic complex group rings.
- Actually, Hertweck gave in 2001 a counterexample to the conjecture that two finite groups are isomorphic if and only if their integral group rings are isomorphic.

- Two finitely generated free groups are isomorphic if and only if the group C^*-algebras are isomorphic. This is a famous unsolved problem for group von Neumann algebras. Actually, one can read off the rank of a free group from the topological K-theory of its group C^*-algebra.
- A group G is amenable if and only if its group C^*-algebra $C_r^*(G)$ is nuclear.