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VANISHING OF NIL-TERMS AND NEGATIVE K-THEORY FOR ADDITIVE CATEGORIES

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Abstract. We extend the notion of regular coherence from rings to additive categories and show that well-known consequences of regular coherence for rings also apply to additive categories. For instance the negative K-groups and all twisted Nil-groups vanish for an additive category, if it is regular coherent. This will be applied to nested sequences of additive categories, motivated by our ongoing project to determine the algebraic K-theory of the Hecke algebra of a reductive p-adic group.

1. Introduction

Background. The Bass-Heller-Swan Theorem gives isomorphisms

$$K_nR[t,t^{-1}] \cong K_{n-1}(R) \oplus K_n(R) \oplus \widetilde{\operatorname{Nil}}_{n-1}(R) \oplus \widetilde{\operatorname{Nil}}_{n-1}(R)$$

for K-theory of rings. For regular rings all Nil groups $\widetilde{\operatorname{Nil}}_n(R)$, $n \in \mathbb{Z}$ and negative K-groups $K_n(R)$, $n \in \mathbb{Z}_{<0}$ vanish and this simplifies the Bass-Heller-Swan formula. Waldhausen [22, 23] proved far reaching extensions of the Bass-Heller-Swan formula for other group rings. He also introduced regular coherence for rings and proved generalizations of the above vanishing results for regular coherent rings. Waldhausen's motivation was that some group rings are regular coherent (but not regular) and this allowed him to bootstrap K-theory computations for group rings. The Bass-Heller-Swan Theorem is also an important ingredient in K-theory computations via the Farrell-Jones conjecture. If R is regular, then so is $R[t,t^{-1}]$, but we do not know, whether the same inheritance statement holds for regular coherence. This is one reason, why we will not only concentrate on regular coherence here, but also on regularity.

The goal of this paper is to extend the notions of regularity and regular coherence from rings to additive categories and to extend the vanishing results in K-theory to additive categories. The basic strategy will be to embed a given additive category $\mathcal{A}$ in the category of $\mathbb{Z}\mathcal{A}$-modules. The latter category is abelian. This mimics the additive subcategory of finitely generated free R-modules of the abelian category of all R-modules and allows the extension of arguments and definitions from rings to additive categories. This is a standard construction and has been used for a long time, for example to define Noetherian additive categories and global dimension for additive categories.

We also extract intrinsic characterizations on the level of additive categories. For instance, we call a sequence $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ in $\mathcal{A}$ *exact at* A_1, if $f_1 \circ f_0 = 0$ and for every object A and morphism $g\colon A \to A_1$ with $f_1 \circ g = 0$ there exists a morphism $\overline{g}\colon A \to A_0$ with $f_0 \circ \overline{g} = g$, see Definition 5.9. We show in Lemma 6.6 (iv) that an idempotent complete additive category $\mathcal{A}$ is regular coherent, if and only if for

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every morphism $f_1\colon A_1 \to A_0$ we can find a sequence of finite length in $\mathcal{A}$

$$0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0,$$

which is exact at A_i for $i = 1, 2, \ldots, n$. It is *l-uniformly regular coherent* if the number n can be arranged to satisfy $n \le l$ for every morphisms f_1.

Our motivation. In our experience it is often more convenient to work with additive categories in place of rings in connection with K-theory. Sometimes a minor drawback is that results for K-theory of rings have not been fully developed for additive K-theory, although often they are really no more complicated. This paper takes care of the extension of regular coherence from rings to additive categories that we expect to be helpful.

More concretely, we rely on the present paper in our ongoing work aimed at the computation of the K-theory of Hecke algebras of reductive p-adic groups. There we apply regular coherence and the K-theory vanishing results to certain additive categories. Namely, we consider a decreasing nested sequence of additive subcategories $\mathcal{A}_* = \big(\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots\big)$, see Definition 13.1, and associate to it the additive category $\mathcal{S}(\mathcal{A}_*)$, called *sequence category*, and a certain quotient additive category $\mathcal{L}(\mathcal{A}_*)$, called *limit category*, see Definition 13.3. An object in $\mathcal{S}(\mathcal{A}_*)$ is a sequence $\underline{A} = (A_m)_{m \ge 0}$ of objects in $\mathcal{A}_0$ such that for every $l \in \mathbb{N}$ almost all ϕ_m lie in $\mathcal{A}_l$. A morphism $\underline{\phi}\colon \underline{A} \to \underline{A'}$ in $\mathcal{S}(\mathcal{A}_*)$ consists of a sequence of morphisms $\phi_m\colon A_m \to A'_m$ in $\mathcal{A}_0$ such that for every $l \in \mathbb{N}$ almost all ϕ_m lie in $\mathcal{A}_l$.[1]. If the system $\mathcal{A}_*$ is constant, i.e., $\mathcal{A}_m = \mathcal{A}_0$, then $\mathcal{S}(\mathcal{A}_*) = \prod_{m \in \mathbb{N}} \mathcal{A}_0$ and $\mathcal{L}(\mathcal{A}_*)$ is the quotient additive category $\prod_{m \in \mathbb{N}} \mathcal{A}_0 \big/ \bigoplus_{m \in \mathbb{N}} \mathcal{A}_0$.

Typically each $\mathcal{A}_m$ will be regular, but this does not imply that $\mathcal{S}(\mathcal{A}_*)$ or $\mathcal{L}(\mathcal{A}_*)$ is regular as well, see Remark 11.4. The problem is that the property Noetherian does not pass to infinite products of additive categories, see Example 13.15. Therefore we have to discard the condition Noetherian in our considerations.

Main results. As mentioned above we discuss various regularity properties, which are well-known for rings and extend them to additive categories. As long as we are concerned with the notion regular or Noetherian, we follow the standard proof for rings, which carry over to additive categories. This is done for the convenience of the reader.

As described above, we need to discard the property Noetherian and stick to regular coherence and the new notion of uniform regular coherence. These notions do pass to infinite products of additive categories, see Lemma 11.3, and more generally under a certain exactness condition about $\mathcal{A}_*$ to the additive categories $\mathcal{S}(\mathcal{A}_*)$ and $\mathcal{L}(\mathcal{S}_*)$, see Lemma 13.10. We remark that algebraic K-theory does commute with infinite products for additive categories, see [4] and also [7, Theorem 1.2], but not with infinite products of rings.

We will show the vanishing of twisted Nil-terms and of the negative K-theory for regular coherent additive categories in Sections 7 and Section 12.

The for us most valuable result is the technical Theorem 14.1, whose proof relies on the vanishing of twisted Nil-terms. It will be a key ingredient in our project to extend the K-theoretic Farrell-Jones Conjecture for discrete groups to reductive p-adic groups, notably, when we want to reduce the family of subgroups, which map with a compact kernel to $\mathbb{Z}$, to the family of compact open subgroups. For discrete groups there is a well-known similar reduction relying also on regularity conditions. However, in the discrete case it typically suffices to use regularity for

[1] These categories come in our situation from controlled algebra; typically control conditions get more restrictive with $m \to \infty$.

rings, while our approach to K-theory of reductive p-adic groups necessitates the use of the weaker regularity conditions introduced in the present paper.

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2. $\mathbb{Z}$-categories, additive categories, and idempotent completions

2.A. **$\mathbb{Z}$-categories.** A *$\mathbb{Z}$-category* is a category $\mathcal{A}$ such that for every two objects A and A' in $\mathcal{A}$ the set of morphisms $\operatorname{mor}_{\mathcal{A}}(A, A')$ has the structure of a $\mathbb{Z}$-module, for which composition is a $\mathbb{Z}$-bilinear map. Given a ring R, we denote by $\underline{R}$ the $\mathbb{Z}$-category with precisely one object, whose $\mathbb{Z}$-module of endomorphisms is given by R with its $\mathbb{Z}$-module structure and composition is given by the multiplication in R.

2.B. **Additive categories.** An *additive category* is a $\mathbb{Z}$-category such that for any two objects A_1 and A_2 there is a model for their direct sum, i.e., an object A together with morphisms $i_k\colon A_k \to A$ for $k = 1, 2$ such that for every object B in $\mathcal{A}$ the $\mathbb{Z}$-map

$$\operatorname{mor}_{\mathcal{A}}(A, B) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{A}}(A_1, B) \times \operatorname{mor}_{\mathcal{A}}(A_2, B), \quad f \mapsto (f \circ i_0, f \circ i_1)$$

is bijective.

Given a ring R, the category $R\text{-}\mathsf{MOD}_{\mathrm{fgf}}$ of finitely generated free left R-modules carries an obvious structure of an additive category.

An *equivalence* $F\colon \mathcal{A} \to \mathcal{A}'$ of $\mathbb{Z}$-categories or of additive categories respectively is a functor of $\mathbb{Z}$-categories or of additive categories respectively such that for all objects A_1, A_2 in $\mathcal{A}$ the induced map $F_{A_1,A_2}\colon \operatorname{mor}_{\mathcal{A}}(A_1, A_2) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{A}}(F(A_1), F(A_2))$ sending f to $F(f)$ is bijective, and for any object A' in $\mathcal{A}'$ there exists an object A in $\mathcal{A}$ such that $F(A)$ and A' are isomorphic in $\mathcal{A}'$. This is equivalent to the existence of a functor $F\colon \mathcal{A}' \to \mathcal{A}$ of $\mathbb{Z}$-categories or of additive categories respectively such that both composites $F \circ F'$ and $F' \circ F$ are naturally equivalent as such functors to the identity functors.

Given a $\mathbb{Z}$-category, let $\mathcal{A}_{\oplus}$ be the associated additive category, whose objects are finite tuples of objects in $\mathcal{A}$ and whose morphisms are given by matrices of morphisms in $\mathcal{A}$ (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [14, Section 1.3].

Let R be a ring. Then we can consider the additive category $\underline{R}_{\oplus}$. The obvious inclusion of additive categories

$$\theta_{\mathrm{fgf}}\colon \underline{R}_{\oplus} \xrightarrow{\simeq} R\text{-}\mathsf{MOD}_{\mathrm{fgf}} \tag{2.1}$$

is an equivalence of additive categories. Note that $\underline{R}_{\oplus}$ is small, in contrast to $R\text{-}\mathsf{MOD}_{\mathrm{fgf}}$.

2.C. **Idempotent completion.** Given an additive category $\mathcal{A}$, its *idempotent completion* $\operatorname{Idem}(\mathcal{A})$ is defined to be the following additive category. Objects are morphisms $p\colon A \to A$ in $\mathcal{A}$ satisfying $p \circ p = p$. A morphism f from $p_1\colon A_1 \to A_1$ to $p_2\colon A_2 \to A_2$ is a morphism $f\colon A_1 \to A_2$ in $\mathcal{A}$ satisfying $p_2 \circ f \circ p_1 = f$. The structure of an additive category on $\mathcal{A}$ induces the structure of an additive category on $\operatorname{Idem}(\mathcal{A})$ in the obvious way. The identity of an object (A,p) is given by the morphism $p\colon (A,p) \to (A,p)$. A functor of additive categories $F\colon \mathcal{A} \to \mathcal{A}'$ induces a functor $\operatorname{Idem}(F)\colon \operatorname{Idem}(\mathcal{A}) \to \operatorname{Idem}(\mathcal{A}')$ of additive categories by sending (A,p) to $(F(A), F(p))$.

There is a obvious embedding

$$\eta(\mathcal{A})\colon \mathcal{A} \to \operatorname{Idem}(\mathcal{A})$$

sending an object A to $\mathrm{id}_A\colon A \to A$ and a morphism $f\colon A \to B$ to the morphism given by f again. An additive category $\mathcal{A}$ is called *idempotent complete*, if $\eta(\mathcal{A})\colon \mathcal{A} \to \operatorname{Idem}(\mathcal{A})$ is an equivalence of additive categories, or, equivalently, if for every idempotent $p\colon A \to A$ in A there exists objects B and C and an isomorphism $f\colon A \xrightarrow{\cong} B \oplus C$ in $\mathcal{A}$ such that $f \circ p \circ f^{-1}\colon B \oplus C \to B \oplus C$ is given by $\begin{pmatrix} \mathrm{id}_B & 0 \\ 0 & 0 \end{pmatrix}$. The idempotent completion $\operatorname{Idem}(\mathcal{A})$ of an additive category $\mathcal{A}$ is idempotent complete.

For a ring R, let $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ be the additive category of finitely generated projective R-modules. We get an equivalence of additive categories $\operatorname{Idem}(R\text{-}\mathsf{MOD}_{\mathrm{fgf}}) \xrightarrow{\cong} R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ by sending an object (F,p) to $\operatorname{im}(p)$. It and the functor of (2.1) induce an equivalence of additive categories

$$\theta_{\mathrm{fgp}}\colon \operatorname{Idem}\big(\underline{R}_{\oplus}\big) \xrightarrow{\cong} R\text{-}\mathsf{MOD}_{\mathrm{fgp}}. \tag{2.2}$$

Note that $\operatorname{Idem}\big(\underline{R}_{\oplus}\big)$ is small, in contrast to $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$.

2.D. **Twisted finite Laurent category.** Let $\mathcal{A}$ be an additive category. Let $\Phi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

Definition 2.3 (Twisted finite Laurent category $\mathcal{A}_\Phi[t,t^{-1}]$)**.** Define the Φ-*twisted finite Laurent category* $\mathcal{A}_\Phi[t,t^{-1}]$ as follows. It has the same objects as $\mathcal{A}$. Given two objects A and B, a morphism $f\colon A \to B$ in $\mathcal{A}_\Phi[t,t^{-1}]$ is a formal sum $f = \sum_{i\in\mathbb{Z}} f_i \cdot t^i$, where $f_i\colon \Phi^i(A) \to B$ is a morphism in $\mathcal{A}$ from $\Phi^i(A)$ to B and only finitely many of the morphisms f_i are non-trivial. If $g = \sum_{j\in\mathbb{Z}} g_j \cdot t^j$ is a morphism in $\mathcal{A}_\Phi[t,t^{-1}]$ from B to C, we define the composite $g \circ f\colon A \to C$ by

$$g \circ f := \sum_{k\in\mathbb{Z}} \Bigg(\sum_{\substack{i,j\in\mathbb{Z},\\ i+j=k}} g_j \circ \Phi^j(f_i) \Bigg) \cdot t^k.$$

The direct sum and the structure of a $\mathbb{Z}$-module on the set of morphism from A to B in $\mathcal{A}_\Phi[t,t^{-1}]$ are defined in the obvious way using the corresponding structures of $\mathcal{A}$. We sometimes also write $\mathcal{A}_\Phi[\mathbb{Z}]$ instead of $\mathcal{A}_\Phi[t,t^{-1}]$.

Example 2.4. Let R be a ring with an automorphism $\phi\colon R \xrightarrow{\cong} R$ of rings. Let $R_\phi[t,t^{-1}]$ be the ring of ϕ-twisted finite Laurent series with coefficients in R. We obtain from ϕ an automorphism $\Phi\colon \underline{R} \xrightarrow{\cong} \underline{R}$ of $\mathbb{Z}$-categories. There is an obvious isomorphism of $\mathbb{Z}$-categories

$$\underline{R}_\Phi[t,t^{-1}] \xrightarrow{\cong} \underline{R_\phi[t,t^{-1}]}. \tag{2.5}$$

We obtain equivalences of additive categories

$$\begin{array}{rcl}(\underline{R}_{\oplus})_{\Phi}[t,t^{-1}] & \xrightarrow{\simeq} & R_{\phi}[t,t^{-1}]\text{-}\mathsf{MOD}_{\mathrm{fgf}};\\ \operatorname{Idem}\big((\underline{R}_{\oplus})_{\Phi}[t,t^{-1}]\big) & \xrightarrow{\simeq} & R_{\phi}[t,t^{-1}]\text{-}\mathsf{MOD}_{\mathrm{fgp}}.\end{array}$$

Definition 2.6 ($\mathcal{A}_{\Phi}[t]$ and $\mathcal{A}_{\Phi}[t^{-1}]$)**.** Let $\mathcal{A}_{\Phi}[t]$ and $\mathcal{A}_{\Phi}[t^{-1}]$ respectively be the additive subcategory of $\mathcal{A}_{\Phi}[t,t^{-1}]$, whose set of objects is the set of objects in $\mathcal{A}$ and whose morphism from A to B are given by finite formal Laurent series $\sum_{i\in\mathbb{Z}} f_i\cdot t^i$ with $f_i=0$ for $i<0$ and $i>0$ respectively.

3. The algebraic K-theory of $\mathbb{Z}$-categories

Let $\mathcal{A}$ be an additive category. One can interprete it as an exact category in the sense of Quillen or as a category with cofibrations and weak equivalence in the sense of Waldhausen and obtains the *connective algebraic K-theory spectrum* $\mathbf{K}(\mathcal{A})$ by the constructions due to Quillen [17] or Waldhausen [24]. A construction of the *non-connective K-theory spectrum* $\mathbf{K}^{\infty}(\mathcal{A})$ of an additive category can be found for instance in [13] or [16].

Definition 3.1 (Algebraic K-theory of $\mathbb{Z}$-categories)**.** We will define the *algebraic K-theory spectrum* $\mathbf{K}^{\infty}(\mathcal{A})$ of the $\mathbb{Z}$-category $\mathcal{A}$ to be the non-connective algebraic K-theory spectrum of the additive category $\mathcal{A}_{\oplus}$. Define for $n\in\mathbb{Z}$

$$K_n(\mathcal{A}) := \pi_n(\mathbf{K}^{\infty}(\mathcal{A})).$$

The connective algebraic K-theory spectrum $\mathbf{K}(\mathcal{A})$ is defined to be the connective algebraic K-theory spectrum of the additive category $\mathcal{A}_{\oplus}$.

If $\mathcal{A}$ is an additive category and $i(\mathcal{A})$ is the underlying $\mathbb{Z}$-category, then there is a canonical equivalence of additive categories $i(\mathcal{A})_{\oplus}\to\mathcal{A}$. Hence there are canonical weak homotopy equivalences $\mathbf{K}(i(\mathcal{A}))\to\mathbf{K}(\mathcal{A})$ and $\mathbf{K}^{\infty}(i(\mathcal{A}))\to\mathbf{K}^{\infty}(\mathcal{A})$.

A functor $F\colon\mathcal{A}\to\mathcal{A}'$ of $\mathbb{Z}$-categories induces a map of spectra

$$\mathbf{K}^{\infty}(F)\colon \mathbf{K}^{\infty}(\mathcal{A})\to\mathbf{K}^{\infty}(\mathcal{A}'). \tag{3.2}$$

We call a full additive subcategory $\mathcal{A}$ of $\mathcal{A}'$ *cofinal*, if for any object A' in $\mathcal{A}'$ there is an object A in $\mathcal{A}$ together with morphisms $i\colon A'\to A$ and $r\colon A'\to A$ satisfying $r\circ i=\mathrm{id}$.

Lemma 3.3. *Let $I\colon\mathcal{A}\to\mathcal{A}'$ be the inclusion of a full cofinal additive subcategory.*

(i) The induced map

$$\pi_n(\mathbf{K}(I))\colon \pi_n(\mathbf{K}(\mathcal{A}))\to\pi_n(\mathbf{K}(\mathcal{A}'))$$

is bijective for $n\geq 1$;

(ii) The induced map

$$\mathbf{K}^{\infty}(I)\colon\mathbf{K}^{\infty}(\mathcal{A})\to\mathbf{K}^{\infty}(\mathcal{A}')$$

is a weak homotopy equivalence.

Proof. (i) This is proved for $\mathcal{A}'=\operatorname{Idem}(\mathcal{A})$ in [21, Theorem A.9.1.]. Now the general case follows from the observation that $\operatorname{Idem}(\mathcal{A})\to\operatorname{Idem}(\mathcal{A}')$ is an equivalence of additive categories.

(ii) This follows from assertion (i) and [13, Corollary 3.7]. □

4. The Bass-Heller-Swan decomposition for additive categories

Denote by $\mathsf{Add\text{-}Cat}$ the category of additive categories. Let us consider the group $\mathbb{Z}$ as a groupoid with one object and denote by $\mathsf{Add\text{-}Cat}^{\mathbb{Z}}$ the category of functors $\mathbb{Z} \to \mathsf{Add\text{-}Cat}$, with natural transformations as morphisms. Note that an object of this category is a pair $(\mathcal{A}, \Phi)$ consisting of an additive category together with an automorphism $\Phi \colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories. We recall from [14, Theorem 0.1] using the notation of this paper here and in the sequel:

Theorem 4.1 (The Bass-Heller-Swan decomposition for non-connective K-theory of additive categories)**.** *Let $\Phi \colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.*

(i) There exists a weak homotopy equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$\mathbf{a}^\infty \vee \mathbf{b}_+^\infty \vee \mathbf{b}_-^\infty \colon \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t]) \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}])$$

where $\mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})}$ is the mapping torus of $\mathbf{K}^\infty(\Phi^{-1}) \colon \mathbf{K}^\infty(\mathcal{A}) \to \mathbf{K}^\infty(\mathcal{A})$ and $\mathbf{NK}^\infty(\mathcal{A}_\Phi[t^{\pm}])$ is the homotopy fiber of the map $\mathbf{K}^\infty(\mathcal{A}_\Phi[t^{-1}]) \to \mathbf{K}^\infty(\mathcal{A})$ given by evaluation $t = 0$;

(ii) There exist a functor $\mathbf{E}^\infty \colon \mathsf{Add\text{-}Cat}^{\mathbb{Z}} \to \mathsf{Spectra}$ and weak homotopy equivalences of spectra, natural in $(\mathcal{A}, \Phi)$,

$$\begin{aligned} \Omega\mathbf{NK}^\infty(\mathcal{A}_\Phi[t]) &\xleftarrow{\simeq} \mathbf{E}^\infty(\mathcal{A}, \Phi); \\ \mathbf{K}^\infty(\mathcal{A}) \vee \mathbf{E}^\infty(\mathcal{A}, \Phi) &\xrightarrow{\simeq} \mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}, \Phi), \end{aligned}$$

where $\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}, \Phi)$ is the non-connective K-theory of a certain Nil-category $\mathrm{Nil}(\mathcal{A}, \Phi)$.

Theorem 4.2 (Fundamental sequence of K-groups)**.** *Let $\mathcal{A}$ be an additive category. Then there exists for $n \in \mathbb{Z}$ a split exact sequence, natural in $\mathcal{A}$*

$$(4.3) \quad 0 \to K_n(\mathcal{A}) \xrightarrow{(k_+)_* \oplus -(k_-)_*} K_n(\mathcal{A}[t]) \oplus K_n(\mathcal{A}[t^{-1}]) \xrightarrow{(l_+)_* \oplus (l_-)_*} K_n(\mathcal{A}[t, t^{-1}]) \xrightarrow{\delta_n} K_{n-1}(\mathcal{A}) \to 0,$$

where $(k_+)_*$, $(k_-)_*$, $(l_+)_*$, and $(l_-)_*$ are induced by the obvious inclusions k_+, k_-, l_+, and l_- and δ_n is the composite of the inverse of the (untwisted) Bass-Heller-Swan isomorphism

$$K_n(\mathcal{A}) \oplus K_{n-1}(\mathcal{A}) \oplus NK_n(\mathcal{A}[t]) \oplus NK_n(\mathcal{A}[t^{-1}]) \xrightarrow{\cong} K_n(\mathcal{A}[t, t^{-1}]),$$

see Theorem 4.1, with the projection onto the summand $K_{n-1}(\mathcal{A})$.

Proof. This follows directly from the untwisted version of Theorem 4.1. □

There is also a version for the *connective K-theory spectrum* $\mathbf{K}$. Denote by $\mathsf{Add\text{-}Cat}_{ic} \subset \mathsf{Add\text{-}Cat}$ the full subcategory of idempotent complete additive categories.

Theorem 4.4 (The Bass-Heller-Swan decomposition for connective K-theory of additive categories)**.** *Let $\mathcal{A}$ be an additive category, which is idempotent complete. Let $\Phi \colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.*

(i) Then there is a weak equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$\mathbf{a} \vee \mathbf{b}_+ \vee \mathbf{b}_- \colon \mathbf{T}_{\mathbf{K}(\Phi^{-1})} \vee \mathbf{NK}(\mathcal{A}_\Phi[t]) \vee \mathbf{NK}(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\simeq} \mathbf{K}(\mathcal{A}_\Phi[t, t^{-1}])$$

where $\mathbf{T}_{\mathbf{K}(\Phi^{-1})}$ is the mapping torus of $\mathbf{K}(\Phi^{-1}) \colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ and $\mathbf{NK}(\mathcal{A}_\Phi[t^{\pm}])$ is the homotopy fiber of the map $\mathbf{K}(\mathcal{A}_\Phi[t^{-1}]) \to \mathbf{K}(\mathcal{A})$ given by evaluation $t = 0$;

(ii) *There exist a functor* $\mathbf{E}\colon (\mathsf{Add\text{-}Cat}_{ic})^{\mathbb{Z}} \to \mathsf{Spectra}$ *and weak homotopy equivalences of spectra, natural in* $(\mathcal{A}, \Phi)$*,*

$$\begin{array}{rcl} \Omega\mathbf{NK}(\mathcal{A}_\Phi[t]) & \xleftarrow{\simeq} & \mathbf{E}(\mathcal{A}, \Phi); \\ \mathbf{K}(\mathcal{A}) \vee \mathbf{E}(\mathcal{A}, \Phi) & \xrightarrow{\simeq} & \mathbf{K}(\mathrm{Nil}(\mathcal{A}, \Phi)), \end{array}$$

where $\mathbf{K}(\mathrm{Nil}(\mathcal{A}, \Phi))$ *is the connective* K*-theory of a certain Nil-category* $\mathrm{Nil}(\mathcal{A}, \Phi)$.

The purpose of the following sections is to find properties of $\mathcal{A}$, which imply for any automorphism Φ the vanishing of the Nil-terms above and are hopefully inherited by the passage from $\mathcal{A}$ to $\mathcal{A}[t, t^{-1}]$.

5. $\mathbb{Z}\mathcal{A}$-modules and the Yoneda embedding

5.a. **Basics about $\mathbb{Z}\mathcal{A}$-modules.** Let $\mathcal{A}$ be a $\mathbb{Z}$-category. We denote by $\mathbb{Z}\mathcal{A}$-MOD and MOD-$\mathbb{Z}\mathcal{A}$ respectively the abelian category of covariant or contravariant respectively functors of $\mathbb{Z}$-categories $\mathcal{A}$ to $\mathbb{Z}$-MOD. The abelian structure comes from the abelian structure in $\mathbb{Z}$-MOD. For instance, a sequence $F_0 \xrightarrow{T_1} F_1 \xrightarrow{T_2} F_2$ in MOD-$\mathbb{Z}\mathcal{A}$ is declared to be *exact*, if for each object $A \in \mathcal{A}$ the evaluation at A yields an exact sequence of $\mathbb{Z}$-modules $F_0(A) \xrightarrow{T_1(A)} F_1(A) \xrightarrow{T_2(A)} F_2(A)$. The cokernel and kernel of a morphism $T\colon F_0 \to F_1$ are defined by taking for each object $A \in \mathcal{A}$ the kernel or cokernel of the morphism $T(A)\colon F_0(A) \to F_1(A)$ in MOD-$\mathbb{Z}$.

In the sequel $\mathbb{Z}\mathcal{A}$-module means contravariant $\mathbb{Z}\mathcal{A}$-module, unless specified explicitly differently.

Given an object A in $\mathcal{A}$, we obtain an object $\mathrm{mor}_{\mathcal{A}}(?, A)$ in MOD-$\mathbb{Z}\mathcal{A}$ by assigning to an object B the $\mathbb{Z}$-module $\mathrm{mor}_{\mathcal{A}}(B, A)$ and to a morphism $g\colon B_0 \to B_1$ the $\mathbb{Z}$-homomorphism $g^*\colon \mathrm{mor}_{\mathcal{A}}(B_1, A) \to \mathrm{mor}_{\mathcal{A}}(B_0, A)$ given by precomposition with g.

The elementary proof of the following lemma is left to the reader.

Lemma 5.1 (Yoneda Lemma)**.** *For each object* A *in* $\mathcal{A}$ *and each object* M *in* MOD-$\mathbb{Z}\mathcal{A}$*, we obtain an isomorphism of* $\mathbb{Z}$*-modules*

$$\mathrm{mor}_{\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}}(\mathrm{mor}_{\mathcal{A}}(?, A), M(?)) \xrightarrow{\cong} M(A), \quad T \mapsto T(A)(\mathrm{id}_A).$$

We call a $\mathbb{Z}\mathcal{A}$-module M *free*, if it is isomorphic as $\mathbb{Z}\mathcal{A}$-module to $\bigoplus_I \mathrm{mor}_{\mathcal{A}}(?, A_i)$ for some collection of objects $\{A_i \mid i \in I\}$ in $\mathcal{A}$ for some index set I. A $\mathbb{Z}\mathcal{A}$-module M is called *projective*, if for any epimorphism $p\colon N_0 \to N_1$ of $\mathbb{Z}\mathcal{A}$-modules and morphism $f\colon M \to N_1$ there is a morphism $\overline{f}\colon M \to N_0$ with $p \circ \overline{f} = f$. A $\mathbb{Z}\mathcal{A}$-module M is *finitely generated*, if there exists a collection of objects $\{A_j \mid j \in J\}$ in $\mathcal{A}$ for some finite index set J and an epimorphism of $\mathbb{Z}\mathcal{A}$-modules $\bigoplus_{j \in J} \mathrm{mor}_{\mathcal{A}}(?, A_j) \to M$. Equivalently, M is finitely generated, if there exists a finite collection of objects $\{A_j \mid j \in J\}$ in $\mathcal{A}$ together with elements $x_j \in M(A_j)$ such that for any object A and any $x \in M(A)$ there are morphisms $\varphi\colon A \to A_j$ such that $x = \sum_j M(\varphi_j)(x_j)$. (The x_j are the images of id_{A_j} under the above epimorphism.) Given a collection of objects $\{A_i \mid i \in I\}$ in $\mathcal{A}$ for some index set I, the free $\mathbb{Z}\mathcal{A}$-module $\bigoplus_I \mathrm{mor}_{\mathcal{A}}(?, A_i)$ is finitely generated, if and only if I is finite. A $\mathbb{Z}\mathcal{A}$-module M is *finitely presented*, if there are finitely generated free $\mathbb{Z}\mathcal{A}$-modules F_1 and F_0 and an exact sequence $F_1 \to F_0 \to M \to 0$. We say that a $\mathbb{Z}\mathcal{A}$-module *has projective dimension* $\leq d$, denoted by $\mathrm{pdim}_{\mathbb{Z}\mathcal{A}}(M) \leq d$, for a natural number d, if there exists an exact sequence $0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such that each $\mathbb{Z}\mathcal{A}$-module P_i is projective. If we replace projective by free, we get an equivalent definition, if $d \geq 1$. We call a $\mathbb{Z}\mathcal{A}$-module *of type* FL or *of type* FP respectively, if there exists an exact sequence of finite length $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that each $\mathbb{Z}\mathcal{A}$-module F_i is finitely generated free or finitely generated projective respectively.

Remark 5.2. Note the setting in this paper is different from the one appearing in [8], since here a $\mathbb{Z}\mathcal{A}$-module M satisfies $M(f+g) = M(f) + M(g)$ for two morphisms $f, g\colon A \to B$, which is not required in [8]. Nevertheless many of the arguments in [8] carry over to the setting of this paper because of the Yoneda Lemma 5.1, which replaces the corresponding Yoneda Lemma in [8, Subsection 9.16 on page 167].

However, the next result has no analogue in the setting of [8].

Lemma 5.3. *Let $\mathcal{A}$ be an additive category. For a $\mathbb{Z}\mathcal{A}$-module M and objects $A_1, A_2, \ldots, A_n$, we obtain a natural isomorphism*

$$\bigoplus_{i=1}^{n} M(\mathrm{pr}_i)\colon \bigoplus_{i=1}^{n} M(A_i) \xrightarrow{\cong} M\biggl(\bigoplus_{i=1}^{n} A_i\biggr),$$

where $\mathrm{pr}_j\colon \bigoplus_{i=1}^{n} A_i \to A_j$ is the canonical projection for $j = 1, 2 \ldots, n$.

Proof. One easily checks, using the fact that the functor M is compatible with the $\mathbb{Z}$-module structures on the morphisms, that the inverse is given

$$M\biggl(\bigoplus_{i=1}^{n} A_i\biggr) \to \bigoplus_{i=1}^{n} M(A_i), \quad x \mapsto \bigl(M(k_i)(x)\bigr)_i,$$

where $k_j\colon A_j \to \bigoplus_{i=1}^{n} A_i$ is the inclusion of the j-the summand for $j = 1, 2 \ldots, n$. □

Lemma 5.4. *Let $\mathcal{A}$ be a $\mathbb{Z}$-category.*

(i) Every free $\mathbb{Z}\mathcal{A}$-module is projective;

(ii) Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $\mathbb{Z}\mathcal{A}$-modules. If both M and M'' are free or projective respectively, then M' is free or projective respectively;

(iii) Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $\mathbb{Z}\mathcal{A}$-modules. If two of the $\mathbb{Z}\mathcal{A}$-modules M, M' and M'' are of type FL *or* FP *respectively, then all three are of type* FL *or* FP *respectively;*

(iv) Let C_* be a projective $\mathbb{Z}\mathcal{A}$-chain complex i.e., a $\mathbb{Z}\mathcal{A}$-chain complex, all whose chain modules C_n are projective. Then the following assertions are equivalent:

(a) Consider a natural number d. Let $B_d(C_*)$ be the image of $c_{d+1}\colon C_{d+1} \to C_d$ and $j\colon B_d(C) \to C_d$ be the inclusion. There is a $\mathbb{Z}\mathcal{A}$-submodule $C_d^{\perp}$ such that for the inclusion $i\colon C_d^{\perp} \to C_d$ the map $i \oplus j\colon C_d^{\perp} \oplus B_d(C_*) \to C_d$ is an isomorphism. Moreover, the following chain map from a d-dimensional projective $\mathbb{Z}\mathcal{A}$-chain complex to C_* is a $\mathbb{Z}\mathcal{A}$-chain homotopy equivalence

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C_d^{\perp} & \xrightarrow{c_d \circ i} & C_{d-1} & \xrightarrow{c_{d-1}} & \cdots & \xrightarrow{c_1} & C_0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}_{C_{d-1}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{C_0}} \\
\cdots & \xrightarrow{c_{d+3}} & C_{d+2} & \xrightarrow{c_{d+2}} & C_{d+1} & \xrightarrow{c_{d+1}} & C_d & \xrightarrow{c_d} & C_{d-1} & \xrightarrow{c_{d-1}} & \cdots & \xrightarrow{c_1} & C_0;
\end{array}$$

(b) C_* is $\mathbb{Z}\mathcal{A}$-chain homotopy equivalent to a d-dimensional projective $\mathbb{Z}\mathcal{A}$-chain complex;

(c) C_* is dominated by d-dimensional projective $\mathbb{Z}\mathcal{A}$-chain complex D_*, i.e., there are $\mathbb{Z}\mathcal{A}$-chain maps $i\colon C_* \to D_*$ and $r_*\colon D_* \to C_*$ satisfying $r_* \circ i_* \simeq \mathrm{id}_{C_*}$;

(d) $B_d(C_*)$ *is a direct summand in* C_d *and* $H_i(C_*) = 0$ *for* $i \geq d+1$*;*

(e) $H^{d+1}_{\mathbb{Z}\mathcal{A}}(C_*; M) := H^{d+1}(\hom_{\mathbb{Z}\mathcal{A}}(C_*, M))$ *vanishes for every* $\mathbb{Z}\mathcal{A}$*-module* M *and* $H_i(C_*) = 0$ *for all* $i \geq d+1$*;*

(v) Let $0 \to M \to M' \to M'' \to 0$ *be an exact sequence of* $\mathbb{Z}\mathcal{A}$*-modules.*
If $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M), \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M'') \leq d$*, then* $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M') \leq d$*;*
If $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M), \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M') \leq d$*, then* $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M'') \leq d+1$*;*
If $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M') \leq d$*,* $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M'') \leq d+1$*, then* $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M) \leq d$*;*

(vi) Suppose that $\mathcal{A}$ *is an additive category. For two objects* A_0 *and* A_1 *in* $\mathcal{A}$ *together with a choice of a direct sum* $i_k\colon A_k \to A_0 \oplus A_1$ *for* $k = 0, 1$*, the induced* $\mathbb{Z}$*-map*

$$i_{0*} \oplus i_{1*}\colon \operatorname{mor}_{\mathcal{A}}(?, A_0) \oplus \operatorname{mor}_{\mathcal{A}}(?, A_1) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{A}}(?, A_0 \oplus A_1)$$

is an isomorphism. In particular each finitely generated free $\mathbb{Z}\mathcal{A}$*-module is isomorphic to* $\mathbb{Z}\mathcal{A}$*-module of the shape* $\operatorname{mor}_{\mathcal{A}}(?, A)$ *for an appropriate object* A *in* $\mathcal{A}$*.*

Proof. (i) This follows from the Yoneda Lemma 5.1.

(ii) This is obviously true.

(iii) The proof is analogous to the one of [8, Lemma 11.6 on page 216].

(iv) The proof is analogous to the one of [8, Proposition 11.10 on page 221].

(v) This follows from (iv) for the projective dimension using the long exact (co)homology sequence associated to a short exact sequence of (co)chain complexes, since every $\mathbb{Z}\mathcal{A}$-module has a free resolution by the Yoneda Lemma 5.1.

(vi) This is obvious and hence the proof of Lemma 5.4 is finished. □

If M and N are $\mathbb{Z}\mathcal{A}$-modules, then $\hom_{\mathbb{Z}\mathcal{A}}(M, N)$ is the $\mathbb{Z}$-module of $\mathbb{Z}\mathcal{A}$-homomorphisms $M \to N$. Given a contravariant or covariant $\mathbb{Z}\mathcal{A}$-module M and a $\mathbb{Z}$-module T, we obtain a covariant or contravariant $\mathbb{Z}\mathcal{A}$-module $\hom_{\mathbb{Z}}(M, T)$ by sending an object A to $\hom_{\mathbb{Z}}(M(A), T)$. Given a contravariant $\mathcal{A}$-module M and covariant $\mathbb{Z}\mathcal{A}$-module N, their *tensor product* $M \otimes_{\mathbb{Z}\mathcal{A}} N$ is the $\mathbb{Z}$-module given by $\bigoplus_{A \in \operatorname{ob}(\mathcal{A})} M(A) \otimes_{\mathbb{Z}} N(A)/T$. Here T is the $\mathbb{Z}$-submodule of $\bigoplus_{A \in \operatorname{ob}(\mathcal{A})} M(A) \otimes_{\mathbb{Z}} N(A)$ generated by elements of the form $mf \otimes n - m \otimes fn$ for a morphism $f\colon A \to B$ in $\mathcal{A}$, $m \in M(A)$ and $n \in N(B)$, where $mf := M(f)(m)$ and $fn = N(f)(n)$. It is characterized by the property that for any $\mathbb{Z}$-module T, there are natural adjunction isomorphisms

$$\hom_{\mathbb{Z}}(M \otimes_{\mathbb{Z}\mathcal{A}} N, T) \xrightarrow{\cong} \hom_{\mathbb{Z}\mathcal{A}}(M, \hom_{\mathbb{Z}}(N, T)); \tag{5.5}$$

$$\hom_{\mathbb{Z}}(M \otimes_{\mathbb{Z}\mathcal{A}} N, T) \xrightarrow{\cong} \hom_{\mathbb{Z}\mathcal{A}}(N, \hom_{\mathbb{Z}}(M, T)). \tag{5.6}$$

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of $\mathbb{Z}$-categories. Then the restriction functor

$$F^*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{B} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$$

is given by precomposition with F. The induction functor

$$F_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{B}$$

sends a contravariant $\mathbb{Z}\mathcal{A}$-module M to $M(?) \otimes_{\mathbb{Z}\mathcal{A}} \operatorname{mor}_{\mathcal{B}}(??, F(?))$. We get for a $\mathbb{Z}\mathcal{B}$-module an identification $F^*N = \hom_{\mathbb{Z}\mathcal{B}}(\operatorname{mor}_{\mathcal{A}}(??, F(?)), N(??))$ from the Yoneda Lemma 5.1. We conclude from (5.5)

$$\hom_{\mathbb{Z}\mathcal{B}}(F_*M, N) \xrightarrow{\cong} \hom_{\mathbb{Z}\mathcal{A}}(M, F^*N) \tag{5.7}$$

for a $\mathbb{Z}\mathcal{A}$-module M and a $\mathbb{Z}\mathcal{B}$-module N. The counit $\beta(N)\colon F_*F^*(N) \to N$ of the adjunction (5.7) is the adjoint of id_{F^*N} and sends the equivalence class of $x \otimes f$ for $x \in N(F(A))$ and $f \in \operatorname{mor}_{\mathcal{B}}(B, F(A))$ to $xf = N(f)(x)$. The unit $\alpha(M)\colon M \to$

$F^*F_*(M)$ is the adjoint of id_{F_*M} and sends $x \in M(A)$ to the equivalence class of $x \otimes \mathrm{id}_{F(A)}$.

The functor F^* is flat. The functor F_* is compatible with direct sums over arbitrary index sets, is right exact, see [25, Theorem 2.6.1. on page 51], and $F_* \operatorname{mor}_{\mathbb{Z}\mathcal{A}}(?, C)$ is $\mathbb{Z}\mathcal{B}$-isomorphic to $\operatorname{mor}_{\mathbb{Z}\mathcal{B}}(?, F(C))$. In particular F_* respect the properties finitely generated, free, and projective.

5.B. **The Yoneda embedding.** The *Yoneda embedding* is the following covariant functor

$$\iota\colon \mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}. \tag{5.8}$$

It sends an object A to $\iota(A) = \operatorname{mor}_{\mathcal{A}}(?, A)$ and a morphism $f\colon A_0 \to A_1$ to the transformation $\iota(f)\colon \operatorname{mor}_{\mathcal{A}}(?, A_0) \to \operatorname{mor}_{\mathcal{A}}(?, A_1)$ given by composition with f. Let $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathcal{A}}$ be the full subcategory of $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ consisting of $\mathbb{Z}\mathcal{A}$-modules $\operatorname{mor}_{\mathcal{A}}(?, A)$ for any object A in $\mathcal{A}$. Let $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}$ be the full subcategory of $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ consisting of finitely generated free $\mathbb{Z}\mathcal{A}$-modules.

Definition 5.9. Let $\mathcal{A}$ be a $\mathbb{Z}$-category. We call a sequence $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ in $\mathcal{A}$ *exact at* A_1, if $f_1 \circ f_0 = 0$ and, for every object A and morphism $g\colon A \to A_1$ with $f_1 \circ g = 0$, there exists a morphism $\overline{g}\colon A \to A_0$ with $f_0 \circ \overline{g} = g$.

Lemma 5.10. *If $\mathcal{A}$ is a $\mathbb{Z}$-category, the Yoneda embedding* (5.8) *induces an equivalence of $\mathbb{Z}$-categories denoted by the same symbol*

$$\iota\colon \mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathcal{A}}.$$

If $\mathcal{A}$ is an additive category, the Yoneda embedding (5.8) *induces an equivalence of additive categories denoted by the same symbol*

$$\iota\colon \mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}.$$

Both functors are faithfully flat.

Proof. This follows directly from the Yoneda Lemma 5.1 and Lemma 5.4 (vi). □

The gain of Lemma 5.10 is that we have embedded $\mathcal{A}$ as a full subcategory of the abelian category $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ and we can now do certain standard homological constructions in $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$, which a priori make no sense in $\mathcal{A}$.

The elementary proof of the following lemma based on Lemma 5.10 is left to the reader.

Lemma 5.11. *An additive category $\mathcal{A}$ is idempotent complete, if and only if every finitely generated projective $\mathbb{Z}\mathcal{A}$-module is a finitely generated free $\mathbb{Z}\mathcal{A}$-module.*

6. Regularity properties of additive categories

6.A. **Definition of regularity properties in terms of the Yoneda embedding.** Recall the following standard ring theoretic notions:

Definition 6.1 (Regularity properties of rings)**.** Let R be a ring and let l be a natural number.

(i) We call R *Noetherian*, if any R-submodule of a finitely generated R-module is again finitely generated;
(ii) We call R *regular coherent*, if every finitely presented R-module M is of type FP;
(iii) We call R *l-uniformly regular coherent*, if every finitely presented R-module M admits an l-dimensional finite projective resolution, i.e., there exist an exact sequence $0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ such that each P_i is finitely generated projective;

(iv) We call R *von Neumann regular*, if for any element $r \in R$ there exists an element $s \in R$ with $r = rsr$;
(v) We call R *regular*, if it is Noetherian and regular coherent;
(vi) We call R *l-uniformly regular*, if it is Noetherian and l-uniformly regular coherent;
(vii) We say that R has global dimension $\leq l$, if each R-module M has projective dimension $\leq l$.

The notion von Neumann regular should not be confused with the notion regular. It stems from operator theory. A ring is von Neumann regular, if and only if it is 0-uniformly regular coherent. For more information about von Neumann regular rings, see for instance [9, Subsection 8.2.2 on pages 325-327].

Let $\mathcal{A}$ be an additive category. Then we define analogously:

Definition 6.2 (Regularity properties of additive categories)**.** Let $\mathcal{A}$ be an additive category and let l be a natural number.
(i) We call $\mathcal{A}$ *Noetherian* if the category $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ is Noetherian in the sense that any $\mathbb{Z}\mathcal{A}$-submodule of a finitely generated $\mathbb{Z}\mathcal{A}$-module is again finitely generated, see [15, p.18][2];
(ii) We call $\mathcal{A}$ *regular coherent*, if every finitely presented $\mathbb{Z}\mathcal{A}$-module M is of type FP;
(iii) We call $\mathcal{A}$ *l-uniformly regular coherent*, if every finitely presented $\mathbb{Z}\mathcal{A}$-module M possesses an l-dimensional finite projective resolution, i.e., there exist an exact sequence $0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$ such that each P_i is finitely generated projective;
(iv) We call $\mathcal{A}$ *regular*, if it is Noetherian and regular coherent;
(v) We call $\mathcal{A}$ *l-uniformly regular*, if is Noetherian and l-uniformly regular coherent;
(vi) We say that $\mathcal{A}$ has global dimension $\leq l$, if each $\mathbb{Z}\mathcal{A}$-module M has projective dimension $\leq l$, see [15, page 42].

6.B. **The definitions of the regularity properties for rings and additive categories are compatible.**

Lemma 6.3. *Let R be a ring. The functor*

$$F\colon R\text{-}\mathsf{MOD} \to \mathsf{MOD}\text{-}\mathbb{Z}\underline{R}_{\oplus}$$

sending M to $\hom_R(\theta_{\mathrm{fgf}}(-), M)$ is an equivalence of additive categories, is faithfully flat, and respects each of the properties finitely generated, free and projective, where the equivalence θ_{fgf} has been defined in (2.1)

Proof. In the sequel we denote by $[n]$ the n-fold direct sum in $\underline{R}_{\oplus}$ of the unique object in $\underline{R}$. Notice that $\theta([n]) = R^n$. Define

$$G\colon \mathsf{MOD}\text{-}\mathbb{Z}\underline{R}_{\oplus} \to R\text{-}\mathsf{MOD}$$

by sending M to $M(\theta(1))$. There is a natural equivalence $G \circ F \to \mathrm{id}_{R\text{-}\mathsf{MOD}}$ of functors of additive categories, its value on the R-module M is given by evaluating at $1 \in R = \theta([1])$,

$$G \circ F(M) = \hom_R(\theta([1]), M) \xrightarrow{\cong} M.$$

Next we construct an equivalence $S\colon F \circ G \to \mathrm{id}_{R\text{-}\mathsf{MOD}}$ of functors of additive categories. For a $\mathbb{Z}\mathcal{A}$-module N and objects $A_1, \ldots, A_n$, we obtain from Lemma 5.3

[2]In [15, page 18] this is called left Noetherian; one obtains right Noetherian by working with $\mathbb{Z}\mathcal{A}$-MOD in place of MOD-$\mathbb{Z}\mathcal{A}$

a natural isomorphism

$$\bigoplus_{i=1}^{n} N(\mathrm{pr}_i)\colon \bigoplus_{i=1}^{n} N(A_i) \xrightarrow{\cong} N\biggl(\bigoplus_{i=1}^{n} A_i\biggr),$$

where $\mathrm{pr}_j\colon \bigoplus_{i=1}^{n} A_i \to A_j$ is the canonical projection for $j = 1, 2 \ldots, n$.

Recall that $[n]$ is the n-fold direct sum of copies of $[1]$, in other words, we have an identification $[n] = \bigoplus_{i=1}^{n}[1]$. It induces an isomorphism

$$\bigoplus_{i=1}^{n} \theta([1]) \xrightarrow{\cong} \theta([n]).$$

Given an object $[n]$ in $\underline{R}_{\oplus}$ and an R-module M, we define $S(M)([n])$ by the following composite of R-isomorphisms

$$\begin{aligned} F \circ G(M)([n]) = \hom_R\bigl(\theta([n]), M(\theta(1))\bigr) &\xrightarrow{\cong} \hom_R\biggl(\bigoplus_{k=1}^{n} \theta([1]), M(\theta(1))\biggr) \\ \xrightarrow{\cong} \bigoplus_{k=1}^{n} \hom_R\bigl(\theta([1]), M(\theta(1))\bigr) &= \bigoplus_{k=1}^{n} \hom_R\bigl(R, M(\theta(1))\bigr) \\ \xrightarrow{\cong} \bigoplus_{k=1}^{n} M(\theta(1)) &\xrightarrow{\cong} M\biggl(\bigoplus_{i=1}^{n} \theta([1])\biggr) = M(\theta([n])). \end{aligned}$$

The functor F is faithfully exact, since for any object $[n]$ in $\underline{R}_{\oplus}$ there is an R-isomorphism $\bigoplus_{i=1}^{n} M \xrightarrow{\cong} F(M)([n])$, natural in M. Since F is compatible with direct sums over arbitrary index sets and sends R to $\hom_R(\theta(-), R) = \mathrm{mor}_{\underline{R}_{\oplus}}(?, [1])$, it respects the properties finitely generated, free and projective. □

The following lemma implies in particular that the inclusion $i\colon \mathcal{A} \to \mathrm{Idem}(\mathcal{A})$ induces equivalences

$$\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A} \underset{i^*}{\overset{i_*}{\rightleftarrows}} \mathsf{MOD}\text{-}\mathbb{Z}\,\mathrm{Idem}(\mathcal{A}) .$$

Lemma 6.4. *Let $i\colon \mathcal{A} \to \mathcal{A}'$ be a inclusion of an additive subcategory $\mathcal{A}$ of the additive subcategory $\mathcal{A}'$, which is full and cofinal, for instance $\mathcal{A} \to \mathcal{A}' = \mathrm{Idem}(\mathcal{A})$. Then:*

(i) If M is a $\mathbb{Z}\mathcal{A}$-module, then the adjoint

$$\alpha(M)\colon M \xrightarrow{\cong} i^* i_* M$$

of id_{i_*M} under the adjunction (5.7) *is an isomorphism of $\mathbb{Z}\mathcal{A}$-modules, natural in M;*

(ii) The restriction functor $i^*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}' \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ is faithfully flat. It sends a finitely generated $\mathbb{Z}\mathcal{A}'$-module to a finitely generated $\mathbb{Z}\mathcal{A}$-module and a projective $\mathbb{Z}\mathcal{A}'$-module to a projective $\mathbb{Z}\mathcal{A}$-module;

(iii) The induction functor $i_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}'$ is faithfully flat. It sends a finitely generated $\mathbb{Z}\mathcal{A}$-module to a finitely generated $\mathbb{Z}\mathcal{A}'$-module and a projective $\mathbb{Z}\mathcal{A}$-module to a projective $\mathbb{Z}\mathcal{A}'$-module;

(iv) If M' is a $\mathbb{Z}\mathcal{A}'$-module, then the adjoint

$$\beta(M')\colon i_* i^* M' \xrightarrow{\cong} M'$$

of $\mathrm{id}_{i^*M'}$ under the adjunction (5.7) *is an isomorphism of $\mathbb{Z}\mathcal{A}'$-modules, natural in M';*

(v) $\mathcal{A}$ is Noetherian, if and only $\mathcal{A}'$ is Noetherian;

(vi) The category $\mathcal{A}$ is regular coherent or l-uniformly regular coherent respectively, if and only if $\mathcal{A}'$ is regular coherent or l-uniformly regular coherent;

(vii) The category $\mathcal{A}$ is of global dimension $\leq l$, if and only if $\mathcal{A}'$ is of global dimension $\leq l$.

Proof. (i) This follows from the Yoneda-Lemma 5.1, namely, an inverse of $\alpha(M)$ is given by

$$i^*i_*M = M(?) \otimes_{\mathbb{Z}\mathcal{A}} \operatorname{mor}_{\mathbb{Z}\mathcal{A}'}(i(?'), i(?)) = M(?) \otimes_{\mathbb{Z}\mathcal{A}} \operatorname{mor}_{\mathbb{Z}\mathcal{A}}(?', ?) \xrightarrow{\cong} M(?'),$$
$$x \otimes \phi \mapsto x\phi = M(\phi)(x).$$

(ii) Obviously i^* is flat.

Consider a sequence of $\mathbb{Z}\mathcal{A}'$-modules $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_1$ such that restriction with i yields the exact sequence of $\mathbb{Z}\mathcal{A}$-modules $i^*M_0 \xrightarrow{i^*f_0} i^*M_1 \xrightarrow{i^*f_1} i^*M_1$. We have to show for any object A' in $\mathcal{A}'$ that the sequence of R-modules $M_0(A') \xrightarrow{f_0(A')} M_1(A') \xrightarrow{f_1(A')} M_1(A')$ is exact. Since $\mathcal{A}$ is by assumption cofinal in $\mathcal{A}'$, we can find an object A in $\mathcal{A}$ and and morphisms $j\colon A' \to i(A)$ and $r\colon i(A) \to A'$ in $\mathcal{A}'$ satisfying $r \circ i = \operatorname{id}_{A'}$. We obtain the following commutative diagram of R-modules

$$\begin{array}{ccccc}
M_0(A') & \xrightarrow{f_0(A')} & M_1(A') & \xrightarrow{f_1(A')} & M_2(A') \\
\downarrow{\scriptstyle M_0(j)} & & \downarrow{\scriptstyle M_1(j)} & & \downarrow{\scriptstyle M_2(j)} \\
M_0(i(A)) & \xrightarrow{f_0(i(A))} & M_1(i(A)) & \xrightarrow{f_1(i(A))} & M_2(i(A)) \\
\downarrow{\scriptstyle M_0(r)} & & \downarrow{\scriptstyle M_1(r)} & & \downarrow{\scriptstyle M_2(r)} \\
M_0(A') & \xrightarrow{f_0(A')} & M_1(A') & \xrightarrow{f_1(A')} & M_2(A')
\end{array}$$

such that the composite of the two vertical arrows appearing in each of the three columns is the identity. Since the middle horizontal sequence is exact, an easy diagram chase shows that the upper horizontal sequence is exact. This shows that i^* is faithfully flat.

Consider an object A' in $\mathcal{A}'$. Since $\mathcal{A}$ is by assumption cofinal in $\mathcal{A}$, we can find an object A in $\mathcal{A}$ and and morphism $j\colon A' \to i(A)$ and $q\colon i(A) \to A'$ in $\mathcal{A}'$ satisfying $q \circ j = \operatorname{id}_{A'}$. Composition with q and j yield maps of $\mathbb{Z}\mathcal{A}'$-modules $J\colon \operatorname{mor}_{\mathcal{A}'}(?', A') \to \operatorname{mor}_{\mathcal{A}'}(?', i(A))$ and $Q\colon \operatorname{mor}_{\mathcal{A}'}(?', i(A)) \to \operatorname{mor}_{\mathcal{A}'}(?', A')$ satisfying $Q \circ J = \operatorname{id}_{\operatorname{mor}_{\mathcal{A}'}(?', A')}$. If we apply i^*, we obtain homomorphisms of $\mathbb{Z}\mathcal{A}'$-modules $i^*J\colon i^*\operatorname{mor}_{\mathcal{A}'}(?', A') \to i^*\operatorname{mor}_{\mathcal{A}'}(?', i(A))$ and $i^*Q\colon i^*\operatorname{mor}_{\mathcal{A}'}(?', i(A)) \to i^*\operatorname{mor}_{\mathcal{A}'}(?', A')$ satisfying $i^*Q \circ i^*J = \operatorname{id}_{i^*\operatorname{mor}_{\mathcal{A}'}(?', A')}$. Since $i^*\operatorname{mor}_{\mathcal{A}'}(?', i(A)) = \operatorname{mor}_{\mathcal{A}'}(i(?'), i(A)) = \operatorname{mor}_{\mathcal{A}}(?', A)$, the $\mathbb{Z}\mathcal{A}$-module $i^*\operatorname{mor}_{\mathcal{A}'}(?', A')$ is a direct summand in $\operatorname{mor}_{\mathcal{A}}(?', A)$ and hence a finitely generated projective $\mathbb{Z}\mathcal{A}$-module.

Let M' be a finitely generated $\mathbb{Z}\mathcal{A}'$-module. Fix an epimorphism $\operatorname{mor}_{\mathcal{A}}(?', A') \to M'$ for some object A' in $\mathcal{A}'$. We conclude that the $\mathbb{Z}\mathcal{A}$-module i^*M is a quotient of $\operatorname{mor}_{\mathcal{A}}(?, A)$ for some object A in $\mathcal{A}$ and hence finitely generated. Hence i^* respects the property finitely generated.

Let P be a projective $\mathbb{Z}\mathcal{A}'$-module. Then we can find a collection of objects $\{A'_k \mid k \in K\}$ together with an epimorphism $\bigoplus_{k\in K} \operatorname{mor}_{\mathcal{A}'}(?, A'_k) \to P$ by the Yoneda Lemma 5.1. Since P is projective, P is a direct summand in $\bigoplus_{k\in K} \operatorname{mor}_{\mathcal{A}'}(?, A'_k)$. This implies that i^*P is a direct summand in the direct sum $\bigoplus_{k\in K} i^*\operatorname{mor}_{\mathcal{A}'}(?, A'_i)$ of projective $\mathbb{Z}\mathcal{A}$-modules and hence itself a projective $\mathbb{Z}\mathcal{A}$-module. Hence i^* respects the property projective.

(iii) The faithful flatness follows from assertions (i) and (ii). Since $i_*\operatorname{mor}_{\mathcal{A}}(?, A) =$

$\mathrm{mor}_{\mathcal{A}'}(?, i(A))$ holds for any object A in $\mathcal{A}$, the functor i_* respects the properties finitely generated and projective.

(iv) We begin with the case $M = \mathrm{mor}_{\mathcal{A}'}(?', i(A)) = i_* \mathrm{mor}_{\mathcal{A}'}(?, A)$ for some object A in $\mathcal{A}$. Then the claim follows from assertion (i) applied to the $\mathbb{Z}\mathcal{A}$-module $\mathrm{mor}_{\mathcal{A}'}(?, A)$, since in this case $\beta(M) = i_*\alpha(M)$. Consider an object A' in $\mathcal{A}'$. Since $\mathcal{A}$ is by assumption cofinal in $\mathcal{A}$, we can find an object A in $\mathcal{A}$ and and morphism $j\colon A' \to i(A)$ and $q\colon i(A) \to A'$ in $\mathcal{A}'$ satisfying $q \circ j = \mathrm{id}_{A'}$. Composition with q and j yield maps of $\mathbb{Z}\mathcal{A}'$-modules $J\colon \mathrm{mor}_{\mathcal{A}'}(?', A') \to \mathrm{mor}_{\mathcal{A}'}(?', i(A))$ and $Q\colon \mathrm{mor}_{\mathcal{A}'}(?', i(A)) \to \mathrm{mor}_{\mathcal{A}'}(?', A')$ satisfying $Q \circ J = \mathrm{id}_{\mathrm{mor}_{\mathcal{A}'}(?', A')}$. Hence we get a commutative diagram of $\mathbb{Z}\mathcal{A}'$-modules

$$\begin{array}{ccc}
i_*i^* \mathrm{mor}_{\mathcal{A}'}(?', A') & \xrightarrow{\beta(\mathrm{mor}_{\mathcal{A}'}(?', A'))} & \mathrm{mor}_{\mathcal{A}'}(?', A') \\
\downarrow{\scriptstyle i_*i^*J} & & \downarrow{\scriptstyle J} \\
i_*i^* \mathrm{mor}_{\mathcal{A}'}(?', i(A)) & \xrightarrow{\beta(\mathrm{mor}_{\mathcal{A}'}(?', i(A)))} & \mathrm{mor}_{\mathcal{A}'}(?', i(A)) \\
\downarrow{\scriptstyle i_*i^*Q} & & \downarrow{\scriptstyle Q} \\
i_*i^* \mathrm{mor}_{\mathcal{A}'}(?', A') & \xrightarrow{\beta(\mathrm{mor}_{\mathcal{A}'}(?', A'))} & \mathrm{mor}_{\mathcal{A}'}(?', A')
\end{array}$$

such that the composite of the two vertical maps in each of the two columns is the identity and the middle arrow is an isomorphism. Hence the upper arrow is an isomorphism.

For any $\mathbb{Z}\mathcal{A}'$-module M' we can find a collection of objects $\{A'_k \mid k \in K\}$ in $\mathcal{A}'$ together with an epimorphism $f_0\colon F_0 := \bigoplus_{k\in K} \mathrm{mor}_{\mathcal{A}'}(?', A'_k) \to M'$ by the Yoneda Lemma 5.1. Repeating this construction for $\ker(f_0)$ instead of M, we obtain another collection $\{A''_l \mid l \in L\}$ of objects in $\mathcal{A}'$ together with a map $f_1\colon F_1 := \bigoplus_{l\in L} i^* \mathrm{mor}_{\mathcal{A}'}(?, A'_l) \to F_0$ whose image is $\ker(f_1)$. We obtain from assertions (ii) and (iii) a commutative diagram of $\mathbb{Z}\mathcal{A}'$-modules with exact rows

$$\begin{array}{ccccccc}
i_*i^*F_1 & \xrightarrow{i_*i^*f_1} & i_*i^*F_0 & \xrightarrow{i_*i^*f_0} & i_*i^*M' & \longrightarrow & 0 \\
\downarrow{\scriptstyle \beta(F_1)} & & \downarrow{\scriptstyle \beta(F_0)} & & \downarrow{\scriptstyle \beta(M)} & & \\
F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M' & \longrightarrow & 0.
\end{array}$$

Since β is compatible with direct sums over arbitrary index sets, the maps $\beta(F_1)$ and $\beta(F_0)$ are isomorphisms. Hence $\beta(M')$ is an isomorphism.

(v), (vi) and (vii) They follow now directly from assertions (i), (ii), (iii) and (iv). □

We conclude from Lemma 6.3 and Lemma 6.4 (v), (vi), and (vii).

Corollary 6.5. *Let R be a ring and let l be a natural number. Then the following assertions are equivalent:*

(i) The ring R is Noetherian, regular coherent, l-uniformly regular coherent, regular, uniformly l-regular, or of global dimension $\leq l$ in the sense of Definition 6.1 respectively;

(ii) The additive category $\underline{R}_\oplus$ is Noetherian, regular coherent, l-uniformly regular coherent, regular, uniformly l-regular, or of global dimension $\leq d$ in the sense of Definition 6.2 respectively;

(iii) The additive category $\mathrm{Idem}(\underline{R}_\oplus)$ is Noetherian, regular coherent, l-uniformly regular coherent, regular, uniformly l-regular, or of global dimension $\leq l$ in the sense of Definition 6.2 respectively.

6.C. **Intrinsic definitions of the regularity properties.** One can give an intrinsic definition of the regularity properties above without referring to the Yoneda embedding. The situation is quite nice for regular coherent and l-uniformly regular coherent for an idempotent complete additive category as as explained below.

Lemma 6.6 (Intrinsic Reformulation of regular coherent)**.** *Let $\mathcal{A}$ be an idempotent complete additive category.*

(i) Let $l \geq 2$ be a natural number. Then $\mathcal{A}$ is l-uniformly regular coherent, if and only if for every morphism $f_1 \colon A_1 \to A_0$ we can find a sequence of length l in $\mathcal{A}$

$$0 \to A_l \xrightarrow{f_l} A_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact at A_i for $i = 1, 2, \ldots, n$;

(ii) $\mathcal{A}$ is 1-uniformly regular coherent, if and only if for every morphism $f \colon A_1 \to A_0$ we can find a factorization $A_1 \xrightarrow{f_1} B \xrightarrow{f_0} A_0$ of f such that f_1 is surjective and f_0 is injective;

(iii) The following assertions are equivalent:

(a) $\mathcal{A}$ is 0-uniformly regular coherent;

(b) For every morphism $f_1 \colon A_1 \to A_0$ there exists a morphism $f_0 \colon A_0 \to A_{-1}$ such that $A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{-1} \to 0$ is exact;

(c) For every morphism $f \colon A_1 \to A_0$ there exists a morphism $g \colon A_0 \to A_1$ satisfying $f \circ g \circ f = f$;

(iv) $\mathcal{A}$ is regular coherent, if and only if for every morphism $f_1 \colon A_1 \to A_0$ we can find a sequence of finite length in $\mathcal{A}$

$$0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact at A_i for $i = 1, 2, \ldots, n$.

Proof. (i) it suffices to prove that the following statements are equivalent:

(a) For any morphisms $f_1 \colon P_1 \to P_0$ of finitely generated projective $\mathbb{Z}\mathcal{A}$-modules, we can find finitely generated projective $\mathbb{Z}\mathcal{A}$-modules $P_2, P_3, \ldots, P_l$ and an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to P_l \xrightarrow{f_l} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0;$$

(b) For any finitely presented $\mathbb{Z}\mathcal{A}$-module M, there exists finitely generated projective $\mathbb{Z}\mathcal{A}$-modules $P_0, P_1, \ldots, P_l$ and an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to P_l \xrightarrow{f_l} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0.$$

The implication implication (b) $\implies$ (a) is obvious, since $\operatorname{cok}(f_1)$ is a finitely presented $\mathbb{Z}\mathcal{A}$-module. It remains to prove the implication (a) $\implies$ (b). Let $f_1 \colon P_1 \to P_0$ be a $\mathbb{Z}\mathcal{A}$-homomorphism of finitely generated projective $\mathbb{Z}\mathcal{A}$-modules. By assumption we can find an exact sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to Q_l \xrightarrow{c_l} Q_{l-1} \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_2} Q_1 \xrightarrow{c_1} Q_0 \xrightarrow{c_0} \operatorname{cok}(f_1) \to 0.$$

Let P_* be the 1-dimensional $\mathbb{Z}\mathcal{A}$-chain complex whose first differential is f_1. Let Q_* be the l-dimensional $\mathbb{Z}\mathcal{A}$-chain complex whose ith chain module is Q_i for $0 \leq i \leq l$ and whose ith differential is $c_i \colon Q_i \to Q_{i-1}$ for $1 \leq i \leq l$. One easily constructs a $\mathbb{Z}\mathcal{A}$-chain map $u_* \colon P_* \to Q_*$ such that $H_0(u_*)$ is an isomorphism. Let $\operatorname{cone}(u_*)$ be the mapping cone. We conclude $H_i(\operatorname{cone}(u_*)) = 0$ for $i \neq 2$ from the long exact homology sequence associated to the exact sequence $0 \to P_* \xrightarrow{i_*} \operatorname{cyl}(u_*) \xrightarrow{p_*} \operatorname{cone}(u_*) \to 0$ and the fact that the canonical projection $q_* \colon \operatorname{cyl}(u_*) \to Q_*$ is a $\mathbb{Z}\mathcal{A}$-chain homotopy equivalence with $q_* \circ i_* = u_*$. Let $D_* \subseteq \operatorname{cone}(u_*)$ be the $\mathbb{Z}\mathcal{A}$-subchain complex, whose i-th chain module is $\operatorname{cone}(u_*)$ for $i \geq 3$, the kernel of the second differential of $\operatorname{cone}(u_*)$ for $i = 2$ and $\{0\}$ for $i = 0, 1$. Then D_i is finitely generated projective for $i \geq 0$ and the inclusion $k_*\colon D_* \to \operatorname{cone}(u_*)$ induces isomorphisms on homology groups. Define the $\mathbb{Z}\mathcal{A}$-chain complex C_* by the pullback

$$\begin{array}{ccc} C_* & \xrightarrow{\overline{p_*}} & D_* \\ \Big\downarrow{\scriptstyle \overline{k_*}} & & \Big\downarrow{\scriptstyle k_*} \\ \operatorname{cyl}(u_*) & \xrightarrow{p_*} & \operatorname{cone}(u_*) \end{array}$$

This can be extended to a commutative diagram of $\mathbb{Z}\mathcal{A}$-chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_* & \xrightarrow{\overline{i_*}} & C_* & \xrightarrow{\overline{p_*}} & D_* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \overline{k_*}} & & \Big\downarrow{\scriptstyle k_*} & & \\ 0 & \longrightarrow & P_* & \xrightarrow{i_*} & \operatorname{cyl}(u_*) & \xrightarrow{p_*} & \operatorname{cone}(u_*) & \longrightarrow & 0 \end{array}$$

Then C_* is an l-dimensional $\mathbb{Z}\mathcal{A}$-chain complex whose $\mathbb{Z}\mathcal{A}$-chain modules are finitely generated projective. Since $D_i = 0$ for $i = 0, 1$, we can identify $P_1 = C_1$ and $P_0 = C_0$ and the first differentials of P_* and C_*. Since k_* induces isomorphisms on homology, the same is true for $\overline{k_*}$. Hence C_* yields the desired extension of f_1 to an exact sequence

$$0 \to C_l \to C_{l-1} \to \cdots \to C_2 \to P_1 \xrightarrow{f_1} P_0$$

This finishes the proof of assertion (i).

(ii) Suppose that $\mathcal{A}$ is 1-uniformly regular coherent. Consider a morphism $f\colon A_1 \to A_0$. Let M be the finitely presented $\mathbb{Z}\mathcal{A}$-module given by the cokernel of the $\mathbb{Z}\mathcal{A}$-homomorphism $\iota(f)\colon \iota(A_1) \to \iota(A_0)$. By assumption we can find an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ of $\mathbb{Z}\mathcal{A}$-modules, where P_1 and P_0 are finitely generated projective. We conclude from Lemma 5.4 (iv) that the image of $\iota(f)$ is finitely generated projective. Hence we obtain a factorization of $\iota(f)$ as a composite $\iota(f)\colon \iota(A_1) \xrightarrow{f_1'} \operatorname{im}(\iota(f)) \xrightarrow{f_0'} \iota(A_0)$ such that $\operatorname{im}(\iota(f))$ is a finitely generated projective $\mathbb{Z}\mathcal{A}$-module, f_1' is surjective, and f_0' is injective. We conclude from Lemma 5.10 and Lemma 5.11 that $\operatorname{im}(f)$ can be identified with $\iota(B)$ for some object B in $\mathcal{A}$ and there are morphisms $f_1\colon A_1 \to B$ and $f_0\colon B \to A_1$ such that $f_1' = \iota(f_1)$ and $f_0' = \iota(f_1)$. Moreover, f_1 is surjective, f_0 is injective and $f = f_0 \circ f_1$.

Suppose that for every morphism $f\colon A_1 \to A_0$ we can find a factorization $A_1 \xrightarrow{f_1} B \xrightarrow{f_0} A_0$ of f such that f_1 is surjective and f_0 is injective. Consider any finitely presented $\mathbb{Z}\mathcal{A}$-module M. We conclude from Lemma 5.10 that there is a morphism $f\colon A_1 \to A_0$ in $\mathcal{A}$ and a morphism $p\colon \iota(A_0) \to M$ of $\mathbb{Z}\mathcal{A}$-modules such that the sequence $\iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{p} M \to 0$ is exact. Choose a factorization $f = f_1 \circ f_0$ such that f_1 is surjective and f_0 is injective. Let B be the domain of f_1. We conclude from Lemma 5.10 that we obtain a short exact sequence $0 \to \iota(B) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{p} M \to 0$. This is a 1-dimensional finite projection $\mathbb{Z}\mathcal{A}$-resolution of M. This finishes the proof of assertion (ii).

(iii) We first show (iii)a $\implies$ (iii)c. Consider a morphism $f\colon A_1 \to A_0$. Let M be the finitely presented $\mathbb{Z}\mathcal{A}$-module given by the cokernel of $\iota(f)\colon \iota(A_1) \to \iota(A_0)$. We obtain an exact sequence of $\mathbb{Z}\mathcal{A}$-modules $\iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{p} M \to 0$. By assumption M is a finitely generated projective $\mathbb{Z}\mathcal{A}$-module. Let $\iota(f)\colon \iota(A_1) \xrightarrow{q} \operatorname{im}(\iota(f)) \xrightarrow{j} \iota(A_0)$ be the obvious factorization of $\iota(f)$. Since M projective, $\operatorname{im}(f)$

is a direct summand in $\iota(A_0)$. We conclude from Lemma 5.10 and Lemma 5.11 that we can identify $\operatorname{im}(\iota(f))$ with $\iota(B)$ for an appropriate object B in $\mathcal{A}$ and can find morphisms $r\colon A_0 \to B$ and $s\colon B \to A_1$ in $\mathcal{A}$ such that $\iota(r) \circ j = \operatorname{id}_{\iota(B)}$ and $q \circ \iota(s) = \operatorname{id}_{\iota(B)}$. Define $g\colon A_0 \to A_1$ by $g = s \circ r$. One easily checks that $\iota(f) \circ \iota(g) \circ \iota(f) = \iota(f)$. Hence $f \circ g \circ f = f$.

Next we show (iii)c $\implies$ (iii)b. Let $f\colon A_1 \to A_0$ be a morphism in $\mathcal{A}$. Choose a morphism $h\colon A_0 \to A_1$ with $f \circ h \circ f = f$. Then $f \circ h\colon A_0 \to A_0$ is an idempotent. Since $\mathcal{A}$ is idempotent complete, we can find objects A_{-1} and $A_{-1}^{\perp}$ and an isomorphism $u\colon A_0 \xrightarrow{\cong} A_{-1} \oplus A_{-1}^{\perp}$ in $\mathcal{A}$ such that $u \circ (\operatorname{id}_{A_0} - f \circ h) \circ u^{-1}$ is $\begin{pmatrix} \operatorname{id}_{A_{-1}} & 0 \\ 0 & 0 \end{pmatrix}$. Define $g\colon A_0 \to A_{-1}$ by the composite $A_0 \xrightarrow{u} A_{-1} \oplus A_{-1}^{\perp} \xrightarrow{\operatorname{pr}_{A_{-1}}} A_{-1}$. One easily checks that the sequence $A_1 \xrightarrow{f_1} A_0 \xrightarrow{g} A_{-1} \to 0$ is exact.

Finally we show (iii)b $\implies$ (iii)a. Consider a finitely presented $\mathbb{Z}\mathcal{A}$-module M. We conclude from Lemma 5.10 and that we an find a morphism $f_1\colon A_1 \to A_0$ together with an exact sequence of $\mathbb{Z}\mathcal{A}$-modules $\iota(A_1) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{p} M \to 0$. Choose a morphism $f_0\colon A_0 \to A_{-1}$ such that the sequence $A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{-1} \to 0$ is exact in $\mathcal{A}$. Then we obtain an exact sequence of $\mathbb{Z}\mathcal{A}$-modules $\iota(A_1) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{\iota(f_0)} \iota(A_{-1}) \to 0$ by Lemma 5.10. This implies that M is $\mathbb{Z}\mathcal{A}$-isomorphic to $\iota(A_{-1})$ and hence finitely generated projective. This finishes the proof of assertion (iii).

(iv) This follows from assertion (i). This finishes the proof of Lemma 6.6. □

Next we deal with the property Noetherian. Consider two morphisms $f\colon A \to B$ and $f'\colon A' \to B$. We write $f \subseteq f'$ if there exists a morphism $g\colon A \to A'$ with $f = f' \circ g$. Obviously we have

(6.7) $$f \subseteq f' \iff \operatorname{im}(f_*\colon \operatorname{mor}_{\mathcal{A}}(?, A) \to \operatorname{mor}_{\mathcal{A}}(?, B)) \subseteq \operatorname{im}(f'_*\colon \operatorname{mor}_{\mathcal{A}}(?, A') \to \operatorname{mor}_{\mathcal{A}}(?, B)).$$

Lemma 6.8 (Intrinsic Reformulation of Noetherian)**.** *Let $\mathcal{A}$ be an additive category. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ is Noetherian;
(ii) Each object A has the following property: Consider a sequence of morphisms $f_n\colon A_n \to A$ with fixed target A and $f_n \subseteq f_{n+1}$ for $n \geq 0$. Then there exists n_0 such that $f_n \subseteq f_{n_0}$ holds for all $n \in \mathbb{N}$ with $n \geq n_0$.

Proof. Let N be a finitely generated $\mathbb{Z}\mathcal{A}$-module and $M \subseteq N$ a $\mathbb{Z}\mathcal{A}$-submodule. Then there exists an object A in $\mathcal{A}$ together with an epimorphism of $\mathbb{Z}\mathcal{A}$-module $u\colon \operatorname{mor}_{\mathcal{A}}(?, A) \to M$, see Lemma 5.3. Obviously M is finitely generated if $u^{-1}(M)$ is finitely generated. Hence $\mathcal{A}$ is Noetherian if and only if for any object A in $\mathcal{A}$ the $\mathbb{Z}\mathcal{A}$-module $\operatorname{mor}_{\mathcal{A}}(?, A)$ is Noetherian, i.e., any $\mathbb{Z}\mathcal{A}$-submodule M of $\operatorname{mor}_{\mathcal{A}}(?, A)$ is finitely generated. By the usual argument $\operatorname{mor}_{\mathcal{A}}(?, A)$ is Noetherian if and only if for any nested sequence $M_0 \subseteq M_1 \subseteq M_2 \subset M_3 \subseteq \cdots$ of finitely generated $\mathbb{Z}\mathcal{A}$-submodules of $\operatorname{mor}_{\mathcal{A}}(?, A)$ there exists a natural number n_0 such that $M_{n_0} \subseteq M_n$ holds, see for instance [18, Proposition 0.2.17 on page 18].

Consider $\mathbb{Z}\mathcal{A}$-submodules $M_0, M_1, M_2, \ldots$ of $\operatorname{mor}_{\mathcal{A}}(?, A)$. We can find for every natural number n an object A_n together with an epimorphism $\operatorname{mor}_{\mathbb{Z}\mathcal{A}}(?, A_n) \to M_n$, see Lemma 5.3. By the Yoneda Lemma 5.1 there is a morphism $f_n\colon A_n \to A$ such that the image of $(f_n)_*\colon \operatorname{mor}_{\mathcal{A}}(?, A_n) \to \operatorname{mor}_{\mathcal{A}}(?, A)$ is M_n. Hence we get $M_m \subseteq M_n$ if and and only if $(f_m) \subseteq (f_n)$

holds. Now Lemma 6.8 follows. □

Lemma 6.9. *Let $\mathcal{A}$ be a full additive subcategory of the additive category $\mathcal{B}$. If $\mathcal{B}$ is Noetherian, 0-uniformly regular coherent, or 0-uniformly regular, then $\mathcal{A}$ has the same property.*

Proof. This follows from Lemma 6.4 (vi), Lemma 6.6 (iii) and Lemma 6.8. □

7. Vanishing of Nil-terms

7.a. **Nil-categories.** The next definition is taken from [14, Definition 7.1].

Definition 7.1 (Nilpotent morphisms and Nil-categories)**.** Let $\mathcal{A}$ be an additive category and Φ be an automorphism of $\mathcal{A}$.

(i) A morphism $f\colon \Phi(A) \to A$ of $\mathcal{A}$ is called Φ-*nilpotent*, if for some $n \geq 1$, the n-fold composite

$$f^{(n)} := f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f)\colon \Phi^n(A) \to A.$$

is trivial;

(ii) The category $\operatorname{Nil}(\mathcal{A}, \Phi)$ has as objects pairs (A, ϕ) where $\phi\colon \Phi(A) \to A$ is a Φ-nilpotent morphism in $\mathcal{A}$. A morphism from (A, ϕ) to (B, μ) is a morphism $u\colon A \to B$ in $\mathcal{A}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \Phi(A) & \xrightarrow{\phi} & A \\ \Big\downarrow{\scriptstyle \Phi(u)} & & \Big\downarrow{\scriptstyle u} \\ \Phi(B) & \xrightarrow{\mu} & B. \end{array}$$

The category $\operatorname{Nil}(\mathcal{A}, \Phi)$ inherits the structure of an exact category from $\mathcal{A}$, a sequence in $\operatorname{Nil}(\mathcal{A}, \Phi)$ is declared to be exact if the underlying sequence in $\mathcal{A}$ is split exact.

Let $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ be an automorphism of an additive category $\mathcal{A}$. It induces an automorphism ${\Phi^{-1}}^*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A} \xrightarrow{\cong} \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ of abelian categories by precomposition with $\Phi^{-1}\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$. It sends $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}$ to itself, since ${\Phi^{-1}}^* \operatorname{mor}_{\mathcal{A}}(?, A)$ is isomorphic to $\operatorname{mor}_{\mathcal{A}}(?, \Phi(A))$. Thus we obtain an automorphism of additive categories ${\Phi^{-1}}^*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}} \xrightarrow{\cong} \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}$

Lemma 7.2. *There is an equivalence of exact categories*

$$\iota\colon \operatorname{Nil}(\mathcal{A}; \Phi) \xrightarrow{\simeq} \operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}; {\Phi^{-1}}^*)$$

Proof. The desired functors ι sends an object (A, f) in $\operatorname{Nil}(\mathcal{A}; \phi)$ given by a morphism $f\colon \Phi(A) \to A$ to the object in $\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}; {\Phi^{-1}}^*)$ given by the composite

$${\Phi^{-1}}^* \operatorname{mor}_{\mathcal{A}}(?, A) = \operatorname{mor}_{\mathcal{A}}(\Phi^{-1}(?), A) \xrightarrow{\Phi} \operatorname{mor}_{\mathcal{A}}(?, \Phi(A)) \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f)} \operatorname{mor}_{\mathcal{A}}(?, A).$$

A morphism $u\colon (A, f) \to (A', f')$ in $\operatorname{Nil}(\mathcal{A}; \Phi)$, which given by a morphism $u\colon A \to A'$ in $\mathcal{A}$ satisfying $f' \circ \Phi(u) = u \circ f$, is sent to the morphism in $\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}; {\Phi^{-1}}^*)$ given by the morphism $u_*\colon \operatorname{mor}_{\mathcal{A}}(?, A) \to \operatorname{mor}_{\mathcal{A}}(?, A')$. It defines indeed a morphism from $\iota(A, f)$ to $\iota(A', f')$ by the commutativity of the following diagram

$$\begin{array}{ccccc} \operatorname{mor}_{\mathcal{A}}(\Phi^{-1}(?), A) & \xrightarrow{\Phi} & \operatorname{mor}_{\mathcal{A}}(?, \Phi(A)) & \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f)} & \operatorname{mor}_{\mathcal{A}}(?, A) \\ \Big\downarrow{\scriptstyle \operatorname{mor}_{\mathcal{A}}(\Phi^{-1}(?), u)} & & \Big\downarrow{\scriptstyle \operatorname{mor}_{\mathcal{A}}(?, \Phi(u))} & & \Big\downarrow{\scriptstyle \operatorname{mor}_{\mathcal{A}}(?, u)} \\ \operatorname{mor}_{\mathcal{A}}(\Phi^{-1}(?), A') & \xrightarrow{\Phi} & \operatorname{mor}_{\mathcal{A}}(?, \Phi(A')) & \xrightarrow{\operatorname{mor}_{\mathcal{A}}(?, f')} & \operatorname{mor}_{\mathcal{A}}(?, A') \end{array}$$

It is an equivalence of additive categories by Lemma 5.10. □

7.B. Connective K-theory.

Lemma 7.3. *Let $\mathcal{A}$ be an idempotent complete additive category. Suppose that $\mathcal{A}$ is regular coherent. Let $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ be any automorphism of additive categories. Denote by $J\colon \mathcal{A} \to \operatorname{Nil}(\mathcal{A}, \phi)$ the inclusion sending an object A to the object $(A, 0)$.*

Then the induced map on connective K-theory

$$\mathbf{K}(J)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\operatorname{Nil}(\mathcal{A}, \Phi))$$

is a weak homotopy equivalence.

Proof. We abbreviate $\Psi = \Phi^{-1^*}$ We have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\quad J \quad} & \operatorname{Nil}(\mathcal{A}, \Phi) \\
\Big\downarrow \iota & & \Big\downarrow \iota \\
\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}} & \xrightarrow[J]{} & \operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}, \Psi)
\end{array}$$

where the vertical arrows are equivalences of exact categories given by Yoneda embeddings, see Lemma 5.10 and Lemma 7.2, and the lower horizontal arrow is the obvious analogue of the upper horizontal arrow. Hence it suffices to show that the map

$$\mathbf{K}(J)\colon \mathbf{K}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}) \to \mathbf{K}(\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}, \Psi))$$

is a weak homotopy equivalence.

Denote by $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}$ the full subcategory of $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ consisting of $\mathbb{Z}\mathcal{A}$-modules which are of type FL, i.e., possess a finite dimensional resolution by finitely generated free $\mathbb{Z}\mathcal{A}$-modules.

Consider the following commutative diagram

$$\begin{array}{ccc}
\mathbf{K}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}) & \longrightarrow & \mathbf{K}(\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}, \Psi)) \\
\Big\downarrow & & \Big\downarrow \\
\mathbf{K}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}) & \longrightarrow & \mathbf{K}(\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}, \Psi))
\end{array} \tag{7.4}$$

where all arrows are induced by the obvious inclusions of categories.

The left vertical arrow in the diagram (7.4) is a weak homotopy equivalence by the Resolution Theorem, see [20, Theorem 4.6 on page 41].

Next we show that the lower horizontal arrow in the diagram (7.4) is a weak homotopy equivalence. Consider an object (M, f) in $\operatorname{Nil}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}, \Psi)$.

Recall that nilpotent means that for some natural number $n \geq 0$ the composite

$$f^{(n)}\colon \Psi^n(M) \xrightarrow{\Psi^{n-1}(f)} \Psi^{n-1}(M) \xrightarrow{\Psi^{n-2}(f)} \cdots \xrightarrow{\Psi(f)} \Psi(M) \xrightarrow{f} M$$

is trivial. We get a filtration of (M,f) by subobjects

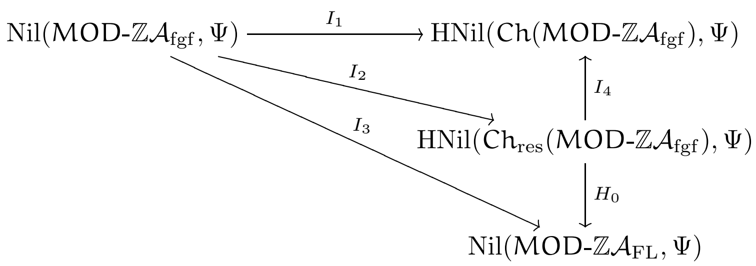
$$\begin{aligned}
(M, f) \supseteq (\operatorname{im}(f), f|_{\Psi(\operatorname{im}(f))}) &\supseteq (\operatorname{im}(f^{(2)}), f|_{\Psi(\operatorname{im}(f^{(2)}))}) \\
&\supseteq \cdots \supseteq (\operatorname{im}(f^{(n-1)}), f|_{\Psi(\operatorname{im}(f^{(n-1)}))}) \\
&\supseteq (\operatorname{im}(f^{(n)}), f|_{\Psi(\operatorname{im}(f^{(n)}))}) = (\{0\}, \operatorname{id}_{\{0\}}).
\end{aligned}$$

Here we consider $\Psi(\operatorname{im}(f^{(i)}))$ as a $\mathbb{Z}\mathcal{A}$-submodule of $\Psi(M)$ by the injective map $\Psi(\operatorname{im}(f^{(i)})) \to \Psi(M)$, which is obtained by applying Ψ to the inclusion $\operatorname{im}(f^{(i)}) \to M$. We get exact sequences of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to \operatorname{im}(f^{(i)}) \to M \to M/\operatorname{im}(f^{(i)}) \to 0;$$
$$0 \to \operatorname{im}(f^{i+1}) \to \operatorname{im}(f^{i}) \to \operatorname{im}(f^{i})/\operatorname{im}(f^{i+1}) \to 0.$$

Since M is finitely presented and $\operatorname{im}(f^i)$ is finitely generated, $M/\operatorname{im}(f^{(i)})$ is finitely presented. Since $\mathcal{A}$ is regular coherent and idempotent complete by assumption, M and $M/\operatorname{im}(f^{(i)})$ for all i are of type FL. We conclude by induction over $i = 0, 1, \ldots$ from Lemma 5.4 (iii) that $\operatorname{im}(f^{(i)})$ and $\operatorname{im}(f^{(i)})/\operatorname{im}(f^{(i+1)})$ belong to $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}$ again. The quotient of $(\operatorname{im}(f^{(i)}), f|_{\Psi(\operatorname{im}(f^{(i)}))})$ by $(\operatorname{im}(f^{(i+1)}), f|_{\Psi(\operatorname{im}(f^{(i+1)}))})$ is given by $(\operatorname{im}(f^{(i)})/\operatorname{im}(f^{(i+1)}), 0)$, and hence belongs to $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{FL}}$ for all i. Now the lower horizontal arrow in diagram (7.4) is a weak homotopy equivalence by the Devissage Theorem, see [20, Theorem 4.8 on page 42].

Next we show that the right vertical arrow in the diagram (7.4) induces split injections on homotopy groups. For this purpose we consider the following commutative diagram of exact categories



The category $\mathrm{HNil}(\mathsf{Ch}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}), \Psi)$ is given by finite-dimensional chain complexes C_* over $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}$ (with $C_i = 0$ for $i \leq -1$) together with chain maps $\phi\colon C_* \to C_*$, which are homotopy nilpotent, and $\mathrm{HNil}(\mathsf{Ch}_{\mathrm{res}}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}), \Psi)$ is the full subcategory of $\mathrm{HNil}(\mathsf{Ch}(\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathrm{fgf}}), \Psi)$ consisting of those chain complexes, for which $H_i(C_*) = 0$ for $i \geq 1$. The maps I_k for $k = 1, 2, 3, 4$ are the obvious inclusions, the functor H_0 is given by taking the zeroth homology group. The upper horizontal arrow induces a weak homotopy equivalence on connective K-theory by [14, page 173]. The functor H_0 induces a weak homotopy equivalence on connective K-theory by the Approximation Theorem of Waldhausen, see for instance [14, Theorem 4.18]. Hence the map induced by I_3 on connective K-theory, which is the right vertical arrow in the diagram (7.4), induces split injections on homotopy groups.

We conclude that all arrows appearing in the diagram (7.4) induce weak homotopy equivalences on connective algebraic K-theory. This finishes the proof of Lemma 7.3. ☐

Theorem 7.5 (The connective K-theory of additive categories)**.** *Let $\mathcal{A}$ be an additive category, which is idempotent complete and regular coherent. Consider any automorphism $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories.*

Then we get a map of connective spectra

$$\mathbf{a}\colon \mathbf{T}_{\mathbf{K}(\Phi^{-1})} \to \mathbf{K}(\mathcal{A}_\Phi[t, t^{-1}])$$

such that $\pi_n(\mathbf{a})$ is bijective for $n \geq 1$.

Proof. This follows from Theorem 4.4, since Lemma 7.3 implies $\pi_n(\mathbf{E}(R, \Phi)) = 0$ for $n \geq 0$ and hence $\pi_n(\mathbf{NK}(\mathcal{A}_\Phi[t])) = \pi_n(\mathbf{NK}(\mathcal{A}_\Phi[t^{-1}])) = 0$ for all $n \geq 1$. ☐

We will need later the following consequence of Lemma 7.3, where we can drop the assumption that $\mathcal{A}$ is idempotent complete.

Lemma 7.6. *Let $\mathcal{A}$ be an additive category. Suppose that $\mathcal{A}$ is regular coherent. Let $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ be any automorphism of additive categories. Denote by $J\colon \mathcal{A} \to \mathrm{Nil}(\mathcal{A}, \phi)$ the inclusion sending an object A to the object $(A, 0)$.*

Then the induced map

$$\pi_n(\mathbf{K}(J))\colon \pi_n(\mathbf{K}(\mathcal{A})) \to \pi_n(\mathbf{K}(\mathrm{Nil}(\mathcal{A},\Phi)))$$

is bijective for $n \geq 1$.

Proof. We have the obvious commutative diagram coming from the inclusion $\mathcal{A} \to \mathrm{Idem}(\mathcal{A})$.

$$\begin{array}{ccc} \pi_n(\mathbf{K}(\mathcal{A})) & \longrightarrow & \pi_n(\mathbf{K}(\mathrm{Nil}(\mathcal{A},\Phi))) \\ \downarrow & & \downarrow \\ \pi_n(\mathbf{K}(\mathrm{Idem}(\mathcal{A}))) & \longrightarrow & \pi_n(\mathbf{K}(\mathrm{Nil}(\mathrm{Idem}(\mathcal{A})),\mathrm{Idem}(\Phi))). \end{array}$$

The left vertical arrow is bijective for $n \geq 1$ by Lemma 3.3 (i). The lower horizontal arrow is bijective for $n \geq 1$ by Lemma 7.3, since $\mathrm{Idem}(\mathcal{A})$ is regular coherent by Lemma 6.4 (vi). Hence we have to show that the right vertical arrow is bijective for $n \geq 1$. For this purpose it suffices to show because of Lemma 3.3 (i) that $\mathrm{Nil}(\mathcal{A},\Phi)$ is a cofinal full subcategory of $\mathrm{Nil}(\mathrm{Idem}(\mathcal{A}),\mathrm{Idem}(\Phi))$. This follows from the fact that $\mathcal{A}$ is a cofinal full subcategory of $\mathrm{Idem}(\mathcal{A})$. □

7.C. **Non-connective K-theory.** In the sequel define $\mathcal{A}[\mathbb{Z}^m]$ inductively over m by $\mathcal{A}[\mathbb{Z}^m] := \mathcal{A}[\mathbb{Z}^{m-1}]_{\mathrm{id}}[t,t^{-1}]$, where $\mathcal{A}[\mathbb{Z}^{m-1}]_{\mathrm{id}}[t,t^{-1}]$ is the (untwisted) finite Laurent category associated to $\mathcal{A}[\mathbb{Z}^{m-1}]$ and the automorphism given by the identity, see Subsection 2.D.

Lemma 7.7. *Let $\mathcal{A}$ be an additive category. Suppose that $\mathcal{A}[\mathbb{Z}^m]$ is regular coherent for every $m \geq 0$. Consider any automorphism $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories. Denote by $J\colon \mathcal{A} \to \mathrm{Nil}(\mathcal{A},\Phi)$ the inclusion sending an object A to the object $(A,0)$.*

Then the induced map on non-connective K-theory

$$\mathbf{K}^\infty(J)\colon \mathbf{K}^\infty(\mathcal{A}) \to \mathbf{K}^\infty_{\mathrm{Nil}}(\mathrm{Nil}(\mathcal{A},\Phi))$$

is a weak homotopy equivalence.

Proof. Fix $n \in \mathbb{Z}$. We have to show that $\pi_n(\mathbf{K}^\infty(J))$ is bijective. This follows from Lemma 7.6 for $n \geq 1$ and is proved in general as follows.

From the definitions and the construction in [13, Section 6], one obtains for every $n \in \mathbb{Z}$ a commutative diagram

$$\begin{array}{ccc} \pi_n(\mathbf{K}^\infty(\mathcal{A})) & \longrightarrow & \pi_n(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A},\Phi)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \pi_{n+1}(\mathbf{K}^\infty(\mathcal{A}[\mathbb{Z}])) & \longrightarrow & \pi_{n+1}(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}[\mathbb{Z}],\Phi[\mathbb{Z}])) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ \pi_n(\mathbf{K}^\infty(\mathcal{A})) & \longrightarrow & \pi_n(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A},\Phi)) \end{array}$$

where $r \circ i = \mathrm{id}$ and $j \circ s = \mathrm{id}$ and these maps are part of the corresponding (untwisted) Bass-Heller-Swan decompositions. Iterating this, one obtains for every

$m \geq 0$ a commutative diagram

$$\begin{array}{ccc}
\pi_n(\mathbf{K}^\infty(\mathcal{A})) & \longrightarrow & \pi_n(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}, \Phi)) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\
\pi_{n+m}(\mathbf{K}^\infty(\mathcal{A}[\mathbb{Z}^m])) & \longrightarrow & \pi_{n+m}(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}[\mathbb{Z}^m], \Phi[\mathbb{Z}^m])) \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\
\pi_n(\mathbf{K}^\infty(\mathcal{A})) & \longrightarrow & \pi_n(\mathbf{K}^\infty_{\mathrm{Nil}}(\mathcal{A}, \Phi))
\end{array}$$

where $r \circ i = \mathrm{id}$ and $j \circ s = \mathrm{id}$ holds. Now choose m such that $n+m \geq 1$ holds. Then the middle horizontal arrow can be identified by construction with its connective version

$$\pi_{n+m}(\mathbf{K}(\mathcal{A}[\mathbb{Z}^m])) \to \pi_{n+m}(\mathbf{K}(\mathrm{Nil}(\mathcal{A}[\mathbb{Z}^m], \Phi[\mathbb{Z}^m]))).$$

Since this map is a bijection by Lemma 7.3 the upper horizontal arrow is a retract of an isomorphism and hence itself an isomorphism. □

Theorem 7.8 (The non-connective K-theory of additive categories)**.** *Let $\mathcal{A}$ be an additive category. Suppose that $\mathcal{A}[\mathbb{Z}^m]$ is regular coherent for every $m \geq 0$. Consider any automorphism $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories.*

Then we get a weak homotopy equivalence of non-connective spectra

$$\mathbf{a}^\infty\colon \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}]).$$

Proof. This follows from Theorem 4.1, since Lemma 7.7 implies $\pi_n(\mathbf{E}^\infty(R, \Phi)) = 0$ and hence $\pi_n(\mathbf{NK}^\infty(R_\Phi[t])) = \pi_n(\mathbf{NK}^\infty(\mathcal{A}_\Phi[t^{-1}])) = 0$ for all $n \in \mathbb{Z}$. □

8. Noetherian additive categories

Theorem 8.1 (Hilbert Basis Theorem for additive categories)**.**

Consider an additive category $\mathcal{A}$ together with an automorphism $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$.

(i) If the additive category $\mathcal{A}$ is Noetherian, then the additive categories $\mathcal{A}_\Phi[t]$, $\mathcal{A}_\Phi[t^{-1}]$, and $\mathcal{A}_\Phi[t, t^{-1}]$ are Noetherian;

(ii) If the additive category $\mathcal{A}_\Phi[t]$ is Noetherian, then the additive category $\mathcal{A}_\Phi[t, t^{-1}]$ is Noetherian.

Proof. (i) We only treat $\mathcal{A}_\Phi[t]$, the proof for $\mathcal{A}_\Phi[t^{-1}]$ is analogous. For $\mathcal{A}_\Phi[t, t^{-1}]$ the claim will follow then from (ii).

We translate the usual proof of the Hilbert Basis Theorem for rings to additive categories. Consider a finitely generated $\mathbb{Z}\mathcal{A}_\Phi[t]$-module N and a $\mathbb{Z}\mathcal{A}_\Phi[t]$-submodule $M \subseteq N$. We have to show that M is finitely generated. Lemma 5.4 (vi) implies that there is an epimorphisms $\phi\colon \mathrm{mor}_{\mathcal{A}_\Phi[t]}(?, A) \to N$ for some object A. If $\phi^{-1}(M)$ is finitely generated, then M is finitely generated, since f induces an epimorphism $f^{-1}(M) \to M$. Hence we can assume without loss of generality $N = \mathrm{mor}_{\mathcal{A}_\Phi[t]}(?, A)$.

Fix an object Z in $\mathcal{A}$. Consider a non-trivial element $f\colon Z \to A$ in $N(Z)$. We can write it as a finite sum $\sum_{k=0}^{d(f)} f_k \cdot t^k$, where $f_k\colon \Phi^k(Z) \to A$ is a morphism in $\mathcal{A}$ and $f_{d(f)} \neq 0$. We call the natural number $d(f)$ the *degree* of f and $R(f) = f_{d(f)}\colon \Phi^{d(f)}(Z) \to A$ the *leading coefficient* of f. We put $d(0\colon Z \to A) = -\infty$ and $R(0\colon Z \to A) = 0$.

We define now I_d as the $\mathbb{Z}\mathcal{A}$-submodule of $\mathrm{mor}_\mathcal{A}(?, A)$ that is generated by all $R(f)$ with $f \in M(Z)$ and $d(f) = d$ for some object Z from $\mathcal{A}$. We have $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ and define I to be the $\mathbb{Z}\mathcal{A}$-submodule $\bigcup_{d \geq 0} I_d$. As $\mathcal{A}$ is by assumption Noetherian, I and all the I_d are finitely generated. Therefore we find a finite collection of morphisms $f_i \in M(Z_i) \subseteq \mathrm{mor}_{\mathcal{A}_\Phi[t]}(Z_i, \mathcal{A})$ such that the $R(f_i)$ generate I. We abbreviate $d_i := d(f_i)$. Since each f_i lies in of the I_d-s, we can find a natural number d_0 such that $I = I_{d_0} = I_d$ holds for $d \geq d_0$. Hence we can also arrange for the f_i to have the following property: for each d the $R(f_i)$ with $d_i \leq d$ generate I_d. We record that $R(f_i) \in I_{d_i}(\Phi^{d_i}(Z_i)) \subseteq \operatorname{mor}_{\mathcal{A}}(\Phi^{d_i}(Z_i), A)$.

We will show that the f_i generate M. Let $f \in M(Z)$, $f \neq 0$. We abbreviate $d := d(f)$. We have $R(f) \in I_d(\Phi^d(Z)) \subseteq \operatorname{mor}_{\mathcal{A}}(\Phi^d(Z), A)$. We can write

$$R(f) = \sum_i R(f_i) \circ \varphi_i$$

with $\varphi_i \in \operatorname{mor}_{\mathcal{A}}(\Phi^d(Z), \Phi^{d_i}(Z_i)$ and $\varphi_i = 0$ whenever $d(f_i) > d(f)$. Set

$$\tilde{\varphi}_i := \Phi^{-d_i}(\varphi_i) \cdot t^{d-d_i} \in \operatorname{mor}_{\mathcal{A}_\phi[t]}(Z, Z_i).$$

Then

$$R\Big(\sum_i f_i \circ \tilde{\varphi}_i\Big) = \sum_i R(f_i) \circ \Phi^{d_i}(\Phi^{-d_i}(\varphi_i)) = \sum_i R(f_i) \circ \varphi_i.$$

Thus $d(f - \sum_i f_i \circ \tilde{\varphi}) < d$. Now we can repeat the argument for $f' := f - \sum_i f_i \circ \tilde{\varphi}$. By induction on $d(f)$ we now find that f belongs to the submodule of $\operatorname{mor}_{\mathcal{A}_\phi[t](?,A)}$ generated by the f_i. Hence M is a finitely generated $\mathbb{Z}\mathcal{A}_\Phi[t]$-module.

(ii) It suffices to show for a $\mathcal{A}_\Phi[t,t^{-1}]$-submodule M of $\operatorname{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(?, A)$ that M is finitely generated as $\mathcal{A}_\Phi[t,t^{-1}]$-module. For $Z \in \mathcal{A}$ we have $\operatorname{mor}_{\mathcal{A}_\Phi[t]}(Z, A) \subseteq \operatorname{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(Z, A)$. We define the $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M' by

$$M'(Z) := M(Z) \cap \operatorname{mor}_{\mathcal{A}_\Phi[t]}(Z, A).$$

Since $\mathcal{A}_\Phi[t]$ is Noetherian, we find a finite collection of morphisms $f_i \in M'(Z_i) \subseteq M(Z_i) \subseteq \operatorname{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(Z_i, A)$ that generate M' as an $\mathbb{Z}\mathcal{A}_\Phi[t]$-module. We claim that the f_i also generate M as an $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module. Let $f \in M(Z) \subseteq \operatorname{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(Z, A)$. For $d \geq 0$ we have $\operatorname{id}_Z \cdot t^d \in \operatorname{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(\Phi^{-d}(Z), Z)$. For sufficiently large d we have $f \circ (\operatorname{id}_Z \cdot t^d) \in \operatorname{mor}_{\mathcal{A}_\Phi[t]}(\Phi^{-d}(Z), A) \cap M(\Phi^{-d}(Z)) = M'(\Phi^{-d}(Z))$. Thus $f \circ (\operatorname{id}_Z \cdot t^d)$ belongs to the $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-submodule of M generated by the f_i. As $(\operatorname{id}_Z \cdot t^d)$ is an isomorphism in $\mathcal{A}_\Phi[t,t^{-1}]$, f also belongs to the $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-submodule of M generated by the f_i. □

9. Additive categories with finite global dimension

Let $\Phi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of the additive category $\mathcal{A}$. Let $\overline{\Phi}\colon \mathcal{A}_\phi[t] \to \mathcal{A}_\Phi[t]$ be the automorphism of additive categories induced by Φ, which sends the morphisms $\sum_{k=0}^\infty f_k \cdot t^k\colon A \to B$ to the morphism $\sum_{k=0}^\infty \Phi(f_k) \cdot t^k\colon \Phi(A) \to \Phi(B)$. Denote by $i\colon \mathcal{A} \to \mathcal{A}_\Phi[t]$ the inclusion sending $f\colon A \to B$ to $(f \cdot t^0)\colon A \to B$. Obviously we have $\overline{\Phi} \circ i = i \circ \Phi$.

9.a. **The characteristic sequence.** Consider a $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M. Let

$$e\colon i_* i^* M \to M$$

be the $\mathbb{Z}\mathcal{A}_\Phi[t]$-morphism, which is the adjoint of the $\mathbb{Z}\mathcal{A}$-homomorphism $\operatorname{id}\colon i^*M \to i^*M$ under the adjunction (5.7). We get for every object A in $\mathcal{A}$ a morphism $\operatorname{id}_{\Phi(A)} \cdot t\colon A \to \Phi(A)$ in $\mathcal{A}_\Phi[t]$. It induces a $\mathbb{Z}\mathcal{A}_\Phi[t]$-morphism $M(\operatorname{id}_{\phi(A)} \cdot t)\colon M(\Phi(A)) \to M(A)$. Since for a morphisms $u\colon A \to B$ in $\mathcal{A}$ we have

$$\begin{aligned}(\operatorname{id}_{\phi(B)} \cdot t) \circ i(u) = (\operatorname{id}_{\phi(B)} \cdot t) \circ (u \cdot t^0) = \Phi(u) \cdot t \\ = (\Phi(u) \cdot t^0) \circ (\operatorname{id}_{\Phi(A)} \cdot t) = i(\Phi(u)) \circ (\operatorname{id}_{\Phi(A)} \cdot t),\end{aligned}$$

we obtain a morphism of $\mathbb{Z}\mathcal{A}$-modules

$$\alpha'\colon \Phi^* i^* M \xrightarrow{\cong} i^* M. \tag{9.1}$$

By applying i_* we obtain a morphism of $\mathbb{Z}\mathcal{A}_\Phi[t]$-modules

$$\alpha\colon i_*\Phi^*i^*M \to i_*i^*M.$$

The morphism $\mathrm{id}_{\Phi(A)}\cdot t\colon A \to \Phi(A)$ in $\mathcal{A}_\Phi[t]$ induces also a $\mathbb{Z}$-map

$$\beta(A)\colon i_*\Phi^*i^*M(A) = i_*i^*M(\Phi(A)) \to i_*i^*M(A).$$

Since for any morphism $v = \sum_{k=0}^\infty f_k \cdot t^k\colon A \to B$ in $\mathcal{A}_\Phi[t]$ we have

$$\begin{aligned}
(\mathrm{id}_{\Phi(B)}\cdot t)\circ v &= (\mathrm{id}_{\Phi(B)}\cdot t)\circ\left(\sum_{k=0}^\infty f_k\cdot t^k\right)\\
&= \sum_{k=0}^\infty(\mathrm{id}_{\Phi(B)}\cdot t)\circ(f_k\cdot t^k)\\
&= \sum_{k=0}^\infty \Phi(f_k)\cdot t^{k+1}\\
&= \sum_{k=0}^\infty(\Phi(f_k)\cdot t^k)\circ(\mathrm{id}_{\Phi(A)}\cdot t)\\
&= \overline{\Phi}\left(\sum_{k=0}^\infty f_k\cdot t^k\right)\circ(\mathrm{id}_{\Phi(A)}\cdot t)\\
&= \overline{\Phi}(v)\circ(\mathrm{id}_{\Phi(A)}\cdot t),
\end{aligned}$$

we get a $\mathbb{Z}\mathcal{A}_\Phi[t]$-homomorphism denoted by

$$\beta\colon i_*\Phi^*i^*M \to i_*i^*M.$$

Define the so called *characteristic sequence* of $\mathbb{Z}\mathcal{A}_\phi[t]$-modules by

$$0 \to i_*\Phi^*i^*M \xrightarrow{\alpha-\beta} i_*i^*M \xrightarrow{e} M \to 0. \tag{9.2}$$

Given an object $A \in \mathcal{A}$, $(\alpha-\beta)(A)$ is explicitly given by

$$\begin{aligned}
&M(\Phi(?))\otimes_{\mathbb{Z}\mathcal{A}}\mathrm{mor}_{\mathcal{A}_\phi[t]}(A,?) \to M(?)\otimes_{\mathbb{Z}\mathcal{A}}\mathrm{mor}_{\mathcal{A}_\phi[t]}(A,?),\\
&\qquad x\otimes(f_k\cdot t^k\colon A\to ?) \mapsto M(\mathrm{id}_{\Phi(?)}\cdot t\colon ?\to\Phi(?))(x)\otimes(f_k\cdot t^k\colon A\to ?)\\
&\qquad\qquad\qquad\qquad - x\otimes(\Phi(f_k)\cdot t^{k+1}\colon A\to\Phi(?)),
\end{aligned}$$

and $e(A)$ is explicitly given by

$$M(?)\otimes_{\mathbb{Z}\mathcal{A}}\mathrm{mor}_{\mathcal{A}_\phi[t]}(A,?) \to M(A), \quad x\otimes(u\colon A\to ?)\mapsto M(u)(x) = xu.$$

Lemma 9.3. *The characteristic sequence* (9.2) *is natural in M and exact.*

Proof. It is obviously natural in M. To prove exactness, it suffices to prove the exactness of the sequence of $\mathbb{Z}\mathcal{A}$-modules

$$0 \to i^*i_*\Phi^*i^*M \xrightarrow{\alpha-\beta} i^*i_*i^*M \xrightarrow{e} i^*M \to 0. \tag{9.4}$$

Let N be a $\mathbb{Z}\mathcal{A}$-module. We obtain a $\mathbb{Z}\mathcal{A}$-isomorphism

$$S(N)\colon \bigoplus_{k=0}^\infty \Phi^k(N) \xrightarrow{\cong} i^*i_*N, \tag{9.5}$$

which is defined for an object A in $\mathcal{A}$ by the $\mathbb{Z}$-isomorphism

$$S(N)(A)\colon \bigoplus_{k=0}^\infty N(\Phi^k(A)) \xrightarrow{\cong} i^*i_*N(A) = N(?)\otimes_{\mathbb{Z}\mathcal{A}}\mathrm{mor}_{\mathcal{A}_\Phi t}(i(A),?)$$

sending $(x_k)_{k\geq 0}$ to $\sum_{k=0}^{\infty} x \otimes \big(\mathrm{id}_{\Phi^k(A)} \cdot t^k \colon A \to \Phi^k(A)\big)$. The inverse of $S(N)(A)$ sends $y \otimes \Big(\sum_{k=0}^{\infty} f_k \cdot t^k \colon A \to ?\big)$ to $\sum_{k=0}^{\infty} N(f_k)(y)$. Applying this to $N = i^*\overline{\Phi}^* M = \Phi^* i^* M$ and $N = i^*M$, we get identifications

$$i^* i_* i^* \overline{\Phi}^* M = \bigoplus_{k=1}^{\infty} (\Phi^k)^* i^* M;$$
$$i^* i_* i^* M = \bigoplus_{k=0}^{\infty} (\Phi^k)^* i^* M.$$

Consider natural numbers m and n with $m \geq n$. For an object A let the map $s_{m,n}(A) \colon (\overline{\Phi}^m)^* M(A) \to (\overline{\Phi}^n)^* M(A)$ be the map obtained by applying M to the morphism $\mathrm{id}_{\Phi^m(A)} \cdot t^{m-n} \colon \Phi^n(A) \to \Phi^m(A)$ in $\mathcal{A}_\Phi[t]$. This yields a map of $\mathbb{Z}\mathcal{A}$-modules

$$s_{m,n} \colon (\Phi^m)^* i^* M \to (\Phi^n)^* i^* M.$$

Under these identifications the $\mathbb{Z}\mathcal{A}$-sequence (9.4) becomes the sequence

$$0 \to \bigoplus_{m=1}^{\infty} (\Phi^m)^* i^* M \xrightarrow{\begin{pmatrix} -s_{1,0} & 0 & 0 & 0 & \cdots \\ \mathrm{id} & -s_{2,1} & 0 & 0 & \cdots \\ 0 & \mathrm{id} & -s_{3,2} & 0 & \cdots \\ 0 & 0 & \mathrm{id} & -s_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{n=0}^{\infty} (\Phi^n)^* i^* M \xrightarrow{\begin{pmatrix} \mathrm{id} & s_{1,0} & s_{2,0} & \cdots \end{pmatrix}} i^* M \to 0.$$

Since $s_{m,n} \circ s_{l,m} = s_{l,n}$ for $l \geq m \geq n$ and $s_{m,m} = \mathrm{id}$ hold, this sequence is split exact, with a splitting given by

$$\bigoplus_{m=1}^{\infty} (\Phi^m)^* M \xleftarrow{\begin{pmatrix} 0 & \mathrm{id} & s_{2,1} & s_{3,1} & s_{4,1} & \cdots \\ 0 & 0 & \mathrm{id} & s_{3,2} & s_{4,2} & \cdots \\ 0 & 0 & 0 & \mathrm{id} & s_{4,3} & \cdots \\ 0 & 0 & 0 & 0 & \mathrm{id} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \bigoplus_{n=0}^{\infty} (\Phi^n)^* M \xleftarrow{\begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}} M.$$

□

9.B. **Localization.**

Definition 9.6 (Local module)**.** We call a $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M *local*, if for any object A in $\mathcal{A}$ and any natural number $k \in \mathbb{N}$ the map

$$M(\mathrm{id}_{\Phi^k(A)} \cdot t^k) \colon M(A) \to M(\Phi^k(A))$$

induced by the morphism $\mathrm{id}_{\Phi^k(A)} \cdot t^k \colon A \to \Phi^k(A)$ in $\mathcal{A}_\Phi[t]$ is bijective.

Let $j \colon \mathcal{A}_\Phi[t] \to \mathcal{A}_\Phi[t, t^{-1}]$ be the inclusion.

Lemma 9.7. *A $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M is local, if and only if there is a $\mathbb{Z}\mathcal{A}_\Phi[t, t^{-1}]$-module N such that M and j^*N are isomorphic as $\mathbb{Z}\mathcal{A}_\Phi[t]$-modules.*

Proof. Since the morphism $\mathrm{id}_{\Phi^k(A)} \cdot t^k \colon A \to \Phi^k(A)$ in $\mathcal{A}_\Phi[t]$ becomes invertible when considered in $\mathcal{A}_\Phi[t, t^{-1}]$, a $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M is local, if there is a $\mathbb{Z}\mathcal{A}_\Phi[t, t^{-1}]$-module N such that M and j^*N are isomorphic as $\mathbb{Z}\mathcal{A}_\Phi[t, t^{-1}]$-modules.

Now consider a local $\mathbb{Z}\mathcal{A}_\Phi[t]$-module M. We have to explain how the $\mathbb{Z}\mathcal{A}_\Phi[t]$-structure extends to a $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-structure. Consider a morphism $u\colon A \to B$ in $\mathcal{A}_\Phi[t,t^{-1}]$. Then we can choose a natural number m such that the composite $A \xrightarrow{u} B \xrightarrow{\mathrm{id}_{\Phi^m(B)} \cdot t^m} \Phi^m(B)$ is a morphism in $\mathcal{A}_\Phi[t]$. Hence we have the $\mathbb{Z}$-map $M((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\colon M(\Phi^m(B)) \to M(A)$. Since M is local, the $\mathbb{Z}$-map $M(\mathrm{id}_{\Phi^m(B)} \cdot t^m)\colon M(\Phi^m(B)) \to M(B)$ is a isomorphism. Now define

$$M(u)\colon M(B) \xrightarrow{M(\mathrm{id}_{\Phi^m(B)} \cdot t^m)^{-1}} M(\Phi^m(B)) \xrightarrow{M((\mathrm{id}_{\Phi^m(B)} \cdot t^m)\circ u)} M(A).$$

We leave it to the reader to check that the definition of $M(u)$ is independent of the choice of m and that we obtain the desired $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-structure on M extending the given $\mathbb{Z}\mathcal{A}_\Phi[t]$-structure. □

Let M be a $\mathbb{Z}\mathcal{A}_\Phi[t]$-module. We want to assign to it a $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module $S^{-1}M$ as follows. Consider an object A in $\mathcal{A}$. Define the abelian group

$$S^{-1}M(A) := \{(l,x) \mid l \in \mathbb{Z}, x \in M(\Phi^l(A))\}/\sim$$

for the equivalence relation $\sim$, where (l_0,x_0) and (l_1,x_1) are equivalent, if and only if there is an integer $l \in \mathbb{Z}$ with $l \leq l_0, l_1$ such that the elements $M(\mathrm{id}_{\Phi^{l_0}(A)} \cdot t^{l_0-l})(x_0)$ and $M(\mathrm{id}_{\Phi^{l_1}(A)} \cdot t^{l_1-l})(x_0)$ of $M(\Phi^l(A))$ agree. Given a morphism $u\colon A \to B$ in $\mathcal{A}_\Phi[t,t^{-1}]$, we can choose a natural number m such that the composite $A \xrightarrow{u} B \xrightarrow{\mathrm{id}_{\Phi^m(B)} \cdot t^m} \Phi^m(B)$ is a morphism in $\mathcal{A}_\Phi[t]$. Define $S^{-1}(M)(u)\colon S^{-1}(M)(B) \to S^{-1}M(A)$ by sending $[l,x]$ to the class of $\bigl(l-m, M\bigl(\overline{\Phi}^{l-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\bigr)(x)\bigr)$. This is independent of the choice of the representative of $[l,x]$, since we get for the different representative $(l-1, M(\mathrm{id}_{\Phi^l(B)} \cdot t)(x))$

$$\begin{aligned}
&S^{-1}(M)([l-1, M(\mathrm{id}_{\Phi^l(B)} \cdot t)(x)]) \\
&\quad= \bigl[l-1-m, M\bigl(\overline{\Phi}^{l-1-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\bigr) \circ M(\mathrm{id}_{\Phi^l(B)} \cdot t)(x)\bigr] \\
&\quad= \bigl[l-1-m, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t) \circ \overline{\Phi}^{l-1-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-1-m, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t) \circ (\mathrm{id}_{\Phi^{l-1}(B)} \cdot t^m) \circ \overline{\Phi}^{l-1-m}(u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-1-m, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t^m \circ \mathrm{id}_{\phi^{l-m}(B)} \cdot t) \circ \overline{\Phi}^{l-1-m}(u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-1-m, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t^m) \circ \overline{\Phi}^{l-m}(u)\bigr)(x)\bigr] \\
&\quad= S^{-1}(M)([l,x]).
\end{aligned}$$

This is independent of the choice of m by the following calculation

$$\begin{aligned}
&\bigl[l-(m+1), M\bigl(\overline{\Phi}^{l-(m+1)}((\mathrm{id}_{\Phi^{m+1}(B)} \cdot t^{m+1}) \circ u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-(m+1), M\bigl(\overline{\Phi}^{l-(m+1)}(\mathrm{id}_{\Phi^{m+1}(B)} \cdot t^{m+1}) \circ \overline{\Phi}^{l-(m+1)}(u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-(m+1), M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t^{m+1}) \circ \overline{\Phi}^{l-(m+1)}(u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m-1, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t^m) \circ (\mathrm{id}_{\Phi^{l-m}(B)} \cdot t) \circ \overline{\Phi}^{l-m-1}(u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m-1, M\bigl((\mathrm{id}_{\Phi^l(B)} \cdot t^m) \circ \overline{\Phi}^{l-m}(u) \circ (\mathrm{id}_{\Phi^{l-m}(B)} \cdot t)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m-1, M\bigl(\overline{\Phi}^{l-m}(\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ \overline{\Phi}^{l-m}(u) \circ (\mathrm{id}_{\Phi^{l-m}(B)} \cdot t)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m-1, M\bigl(\overline{\Phi}^{l-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u) \circ (\mathrm{id}_{\Phi^{l-m}(B)} \cdot t)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m-1, M(\mathrm{id}_{\phi^{l-m}(B)} \cdot t) \circ M\bigl(\overline{\Phi}^{l-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\bigr)(x)\bigr] \\
&\quad= \bigl[l-m, M\bigl(\overline{\Phi}^{l-m}((\mathrm{id}_{\Phi^m(B)} \cdot t^m) \circ u)\bigr)(x)\bigr].
\end{aligned}$$

We leave it to the reader to check that $S^{-1}M(v \circ u) = S^{-1}M(u) \circ S^{-1}M(v)$ holds for any two composable morphisms $u\colon A \to B$ and $v\colon B \to C$ in $\mathcal{A}_\Phi[t,t^{-1}]$ and $S^{-1}M(\mathrm{id}_A) = \mathrm{id}_{S^{-1}M(A)}$ holds for any object A in $\mathcal{A}$. Note that the $\mathcal{A}_\phi[t]$-module $j^*S^{-1}M$ is local by Lemma 9.7

There is a natural map of $\mathcal{A}_\Phi[t]$-modules

$$I\colon M \to j^*S^{-1}M,$$

which is given for an object A of $\mathcal{A}$ by the map $I(A)\colon M(A) \to S^{-1}M(A)$ sending x to $(0,x)$. We claim that I is *a localization* in the sense that for any local $\mathcal{A}_\phi[t]$-module N and any $\mathcal{A}_\phi[t]$-homomorphism $f\colon M \to N$ there exists precisely one $\mathcal{A}_\phi[t]$-homomorphism $S^{-1}f\colon S^{-1}M \to N$.

Firstly we explain that there is at most one such map $S^{-1}f$ with these properties. Namely, consider an object $A \in \mathcal{A}$ and an element $[m,x] \in S^{-1}(M)(A)$. If $m \geq 0$, then we compute

$$\begin{aligned} S^{-1}f(A)([m,x]) &= S^{-1}(A)([0, M(\mathrm{id}_{\Phi^m(A)} \cdot t^m)(x)]) \\ &= S^{-1}(A) \circ I(A) \circ M(\mathrm{id}_{\Phi^m(A)} \cdot t^m)(x) \\ &= f(A) \circ M(\mathrm{id}_{\Phi^m(A)} \cdot t^m)(x). \end{aligned} \tag{9.8}$$

Suppose $m \leq 0$. Since we have $S^{-1}(M)(\mathrm{id}_A \cdot t^{-m})([m,x]) = [0,x]$, we compute for $[m,x] \in S^{-1}M(A)$

$$\begin{aligned} & S^{-1}(N)(\mathrm{id}_A \cdot t^{-m}) \circ S^{-1}f(A)([m,x]) \\ &= S^{-1}f(\Phi^m(A)) \circ S^{-1}(M)(\mathrm{id}_A \cdot t^{-m})([m,x]) \\ &= S^{-1}f(\Phi^m(A))([0,x]) \\ &= S^{-1}f(\Phi^m(A)) \circ I(A)(x) \\ &= f(\Phi^m(A))(x). \end{aligned}$$

Since the locality of N implies that $S^{-1}(N)(\mathrm{id}_{\phi^m(A)} \cdot t^m)$ is an isomorphism, we conclude

$$S^{-1}f(A)([m,x]) = S^{-1}(N)(\mathrm{id}_A \cdot t^{-m})^{-1} \circ f(\Phi^m(A))(x). \tag{9.9}$$

Hence $S^{-1}f(A)$ is determined by the equations (9.8) and (9.9). We leave it to the reader to check that it makes sense to define the desired $\mathbb{Z}_{\mathcal{A}}[t]$-homomorphism $S^{-1}f(A)$ by the equations (9.8) and (9.9).

The adjoint of $I\colon M \to j^*S^{-1}M$ under the adjunction (5.7) is denoted by

$$\alpha\colon j_*M \to S^{-1}M. \tag{9.10}$$

The adjoint of id_{j_*M} under the adjunction (5.7) is the $\mathbb{Z}\mathcal{A}_\Phi[t]$-homomorphism

$$\lambda\colon M \to j^*j_*M, \tag{9.11}$$

which is explicitly given by $M(??) \to \mathrm{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(??,?) \otimes_{\mathbb{Z}\mathcal{A}_\Phi[t]} M(?)$ sending $u \in M(??)$ to $\mathrm{id}_{??} \otimes u$. Given an $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module N, the adjoint of id_{j^*N} under the adjunction (5.7) is the $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-homomorphism

$$\rho\colon j_*j^*N \to N, \tag{9.12}$$

which is explicitly given by $N(?) \otimes_{\mathbb{Z}\mathcal{A}_\Phi[t]} \mathrm{mor}_{\mathcal{A}_\Phi[t,t^{-1}]}(??,?) \to N(??)$ sending $x \otimes u$ to $N(u)(x) = xu$.

Lemma 9.13. *(i) The $\mathbb{Z}\mathcal{A}_\Phi[t]$-homomorphism $\lambda\colon M \to j^*j_*M$ of* (9.11) *is a localization;*

(ii) The $\mathbb{Z}\mathcal{A}_\phi[t,t^{-1}]$-homomorphism $\alpha\colon j_*M \to S^{-1}M$ of (9.10) *is an isomorphism, which is natural in M;*

(iii) Let N be a $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module. Then the $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-map $\rho\colon j_*j^*N \to N$ of (9.12) *is an isomorphism.*

Proof. (i) Let $f\colon M \to N$ be a $\mathbb{Z}\mathcal{A}_\Phi[t]$-map with a local $\mathbb{Z}\mathcal{A}_\Phi[t]$-module as target. Because of Lemma 9.7 there is a $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module N' and a $\mathbb{Z}\mathcal{A}_\Phi[t]$-isomorphism $u\colon N \to j^*N'$. Let the $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-map $v_0\colon j_*M \to N$ be the adjoint of $u \circ f$ under the adjunction (5.7). Because of the naturality of the adjunction (5.7) we get for the composite $\overline{f}\colon j^*j_*M \xrightarrow{j^*v_0} j^*N' \xrightarrow{u^{-1}} N$ that $\overline{f} \circ \lambda = f$ holds. We conclude that $\overline{f}$ is uniquely determined by $\overline{f} \circ \lambda = f$ from the explicite description of λ and from the fact that for any morphism $u\colon A \to B$ in $\mathcal{A}_\Phi[t,t^{-1}]$ there is a natural number such that the composite of u with $\mathrm{id}_{\Phi^m(B)} \cdot t^m\colon \Phi(M) \to \Phi^m(B)$ lies in $\mathcal{A}_\Phi[t]$.

(ii) Obviously α is natural in M. The naturality of the adjunction (5.7) implies

$$j^*\alpha \circ \lambda = I.$$

Since both $I\colon M \to j^*S^{-1}M$ and $\lambda\colon M \to j^*j_*M$ a localizations, $j^*\alpha$ and hence α are bijective.

(iii) It suffices to show that $j^*\rho\colon j^*j_*j^*N \to j^*N$ is bijective. Assertion (i) applied to j^*N and the naturality of the adjunction (5.7) imply that $j^*\rho\colon j^*N \to j^*j_*j^*N$ is a localization. Since $\mathrm{id}_{j^*N}\colon j^*N \to j^*N$ is a localization, $j^*\rho$ is an isomorphism. □

Lemma 9.14. *The functor $j_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\phi[t] \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\phi[t,t^{-1}]$ is flat.*

Proof. Because of the adjunction (5.7) the functor j_* is right exact by a general argument, see [25, Theorem 2.6.1. on page 51]. Hence it remains to show that for an injective $\mathbb{Z}\mathcal{A}_\phi[t]$-map $i\colon M \to N$ the $\mathbb{Z}\mathcal{A}_\phi[t,t^{-1}]$-map $j_*i\colon j_*M \to j_*N$ is injective. In view of Lemma 9.13 (ii) it suffices to show that $S^{-1}i\colon S^{-1}M \to S^{-1}N$ is injective. Consider an object A in $\mathcal{A}$ and an element $[l,x]$ in the kernel of $S^{-1}(i)(A)$. Since $S^{-1}i([l,x]) = [l, i(\Phi^l(A))(x)]$, there is a natural number $m \le l$ such that $N(\mathrm{id}_{\Phi^l(A)} \cdot t^{m-l})(i(\Phi^l(A))(x)) = 0$. Since $N(\mathrm{id}_{\Phi^l(A)} \cdot t^{m-l}) \circ i(\Phi^l(A)) = i(\Phi^{m-l}(A)) \circ M(\mathrm{id}_{\Phi^l(A)} \cdot t^{m-l})$ and $i(\Phi^{m-l}(A))$ is by assumption injective, $M(\mathrm{id}_{\Phi^l(A)} \cdot t^{m-l})(x) = 0$. This implies $[l,x] = 0$. □

9.C. **Global dimension.** Recall that an additive category $\mathcal{A}$ has *global dimension* $\le d$, if the abelian category $\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ has global dimension $\le d$, i.e., if every $\mathbb{Z}\mathcal{A}$-module has a projective d-dimensional resolution, see Definition 6.2 (vi).

Theorem 9.15 (Global dimension and the passage from $\mathcal{A}$ to $\mathcal{A}_\Phi[t]$)**.** *Let $\mathcal{A}$ be an additive category $\mathcal{A}$ and $\Phi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.*

- *(i) Let M be a $\mathbb{Z}\mathcal{A}_\Phi[t]$-module. If $\mathrm{pdim}_{\mathcal{A}}(i^*M) \le d$, then $\mathrm{pdim}_{\mathcal{A}[t]}(M) \le d+1$;*
- *(ii) If $\mathcal{A}$ has global dimension $\le d$, then $\mathcal{A}_\Phi[t]$ has global dimension $\le (d+1)$.*

Theorem 9.15 is a version of the Hilbert syzygy theorem. Its the proof is not much different from the classical syzygy Theorem for rings. For a more general version see [15, Corollary 31.1 on page 119].

Proof of Theorem 9.15. Obviously $i^*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\Phi[t] \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}$ is faithfully flat and is compatible with direct sums over arbitrary index sets. Next we show that i^* sends projective $\mathbb{Z}\mathcal{A}_\phi[t]$-modules to projective $\mathbb{Z}\mathcal{A}$-modules. It suffices to show that $i^*\operatorname{mor}_{\mathcal{A}_\Phi[t]}(?,A) \cong i^*i_*\operatorname{mor}_{\mathcal{A}}(?,A)$ is free as a $\mathbb{Z}\mathcal{A}$-module for any object A. This follows from the $\mathbb{Z}\mathcal{A}$-isomorphism (9.5), since $(\Phi^k)^*\operatorname{mor}_{\mathcal{A}}(?,A) \cong \operatorname{mor}_{\mathcal{A}}(?,\Phi^{-k}(A))$.

The functor $i_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A} \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\Phi[t]$ is compatible with direct sums over arbitrary index sets, is right exact and sends $\operatorname{mor}_{\mathcal{A}}(?,A)$ to $\operatorname{mor}_{\mathcal{A}_\Phi[t]}(?,A)$. In particular i_* respects the properties finitely generated, free, and projective. Next we want to show that i_* is faithfully flat. For this purpose it suffices to show that $i^* \circ i_*$ is faithfully flat. This is obvious since $i^* \circ i_*$ is the functor sending a morphism $f\colon M \to N$ to the morphism $\bigoplus_{k\in\mathbb{N}}(\Phi^k)^*(f)\colon \bigoplus_{k\in\mathbb{N}}(\Phi^k)^*(M) \to \bigoplus_{k\in\mathbb{N}}(\Phi^k)^*(f)$ under the identification (9.5).

Now consider a $\mathbb{Z}_\Phi[t]$-module M with $\operatorname{pdim}_{\mathcal{A}}(i^*M) \le d$. Since the $\mathbb{Z}\mathcal{A}$-modules i^*M and Φ^*i^*M are isomorphic, see (9.1), we get $\operatorname{pdim}_{\mathcal{A}}(\phi^*i^*M) \le d$. Since i_* is faithfully flat and respects projective modules, we conclude $\operatorname{pdim}_{\mathcal{A}_\Phi[t]}(i_*i^*M) \le d$ and $\operatorname{pdim}_{\mathcal{A}_\Phi[t]}(i_*\Phi^*i^*M) \le d$. Now Lemma 5.4 (v) and Lemma 9.3 together imply $\operatorname{pdim}_{\mathcal{A}_\Phi[t]}(M) \le (d+1)$.

(ii) This follows directly from assertion (i). □

Theorem 9.16 (Global dimension and the passage from $\mathcal{A}_\phi[t]$ to $\mathcal{A}_\Phi[t,t^{-1}]$)**.** *Let $\mathcal{A}$ be an additive category and let $\Phi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.*

(i) Let M be a $\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]$-module. If we have $\operatorname{pdim}_{\mathcal{A}[t]}(j^*M) \le d$, then we get $\operatorname{pdim}_{\mathcal{A}[t,t^{-1}]}(M) \le d$;

(ii) If $\mathcal{A}_\Phi[t]$ has global dimension $\le d$, then $\mathcal{A}_\Phi[t,t^{-1}]$ has global dimension $\le d$.

Proof. (i) Let M be a $\mathbb{Z}[\mathcal{A}]_\Phi[t,t^{-1}]$-module satisfying $\operatorname{pdim}_{\mathcal{A}[t]}(j^*M) \le d$. The functor $j_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\phi[t] \to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_\phi[t,t^{-1}]$ is flat by Lemma 9.14. Since it respects the property projective, we get $\operatorname{pdim}_{\mathcal{A}_\Phi[t,t^{-1}]}(j_*j^*M) \le d$. Lemma 9.13 (iii) implies $\operatorname{pdim}_{\mathcal{A}[t,t^{-1}]}(M) \le d$.

(ii) This follows from assertion (i). □

10. Regular additive categories

Regularity for additive categories $\mathcal{A}$ requires finite resolutions of finitely presented modules, but not for arbitrary modules. In particular, regularity has no consequence for global dimension and we cannot use Theorem 9.15 in the following result.

Theorem 10.1 (Regularity and the passage from $\mathcal{A}$ to $\mathcal{A}_\Phi[t]$)**.** *Let $\mathcal{A}$ be an additive category $\mathcal{A}$ and let $\Phi\colon \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories. Let l be a natural number.*

(i) Suppose that $\mathcal{A}$ is regular or l-uniformly regular respectively. Then $\mathcal{A}_\Phi[t]$ is regular or $(l+2)$-uniformly regular respectively;

(ii) Suppose that $\mathcal{A}[t]$ is regular or l-uniformly regular respectively. Then $\mathcal{A}_\Phi[t,t^{-1}]$ is regular or l-uniformly regular respectively.

Proof. (i) We know already that $\mathcal{A}_{\phi[t]}$ is Noetherian because of Theorem 8.1 (i). Let M be a finitely generated $\mathcal{A}_\phi[t]$-module. We have to show that it has a finitely generated projective resolution, which is finite-dimensional or $(l+1)$-dimensional. Since $\mathcal{A}_\phi[t]$ is Noetherian, there exists a finitely generated projective resolution of M, which may be infinite-dimensional. We conclude from Theorem 5.4 (iv) that it suffices to show the projective dimension of M is finite or bounded by $(l+1)$ respectively. As M is finitely generated, we find a finite collection of elements $x_j \in M(Z_j)$ with objects Z_j from $\mathcal{A}$ such that the x_j generate M as an $\mathbb{Z}\mathcal{A}_\Phi[t]$-module. For $d \ge 0$ consider the morphism $\operatorname{id}_{Z_j} \cdot t^d\colon \Phi^{-d}(Z_j) \to Z_j$ in $\mathcal{A}_\Phi[t]$ and set $x_j[d] := M(\operatorname{id}_{Z_j} \cdot t^d)(x_j) \in M(\Phi^{-d}(Z_j))$. Let M_n be the $\mathbb{Z}\mathcal{A}$-submodule of i^*M generated by all $x_j[d]$ with $d \le n$. We obtain an increasing subsequence $M_0 \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of $\mathbb{Z}\mathcal{A}$-submodules of i^*M with $i^*M = \bigcup_{n\ge 0} M_n$. Let $T_n\colon i^*M \to \Phi^*i^*M$ be the following $\mathbb{Z}\mathcal{A}$-morphism. For an object Z from $\mathcal{A}$ consider $\operatorname{id}_{\Phi^n(Z)} \cdot t^n \in \operatorname{mor}_{\mathcal{A}_\Phi[t]}(Z, \Phi^n(Z))$ and define $T_Z\colon i^*M(Z) = M(Z) \to \Phi^*i^*M(Z) = M(\Phi(Z))$ to be $M(\operatorname{id}_{\Phi^n(Z)} \cdot t^n)$. Let $\operatorname{pr}_n\colon (\Phi^n)^*(M_n) \to (\Phi^n)^*(M_n)/(\Phi^n)^*(M_{n-1})$ be the projection. The composite

$$f_n\colon M_0 \xrightarrow{\overline{T_n}} (\Phi^n)^*(M_n) \xrightarrow{\operatorname{pr}_n} (\Phi^n)^*(M_n)/(\Phi^n)^*(M_{n-1})$$

is surjective and we write K_n for its kernel. We obtain an increasing sequence of $\mathcal{A}$-submodules $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of M_0. Since $\mathcal{A}$ is Noetherian and M_0 is finitely generated, there exists an integer $n_0 \geq 1$ such that $K_n = K_{n_0}$ holds for $n \geq n_0$. Define $g_n \colon (\Phi^{n_0})^* M_{n_0}/(\Phi^{n_0})^* M_{n_0-1} \to (\Phi^n)^* M_n/(\Phi^n)^* M_{n-1}$ for $n \geq n_0$ to be the map induced by $\Phi^*(T_{n-n_0})$ for $n \geq n_0$. We obtain for every natural number n with $n \geq n_0$ a commutative diagram of $\mathbb{Z}\mathcal{A}$-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{n_0} & \longrightarrow & M_0 & \xrightarrow{f_{n_0}} & (\Phi^{n_0})^* M_{n_0}/(\Phi^{n_0})^* M_{n_0-1} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \mathrm{id}_{M_0} & & \downarrow g_n & & \\ 0 & \longrightarrow & K_n & \longrightarrow & M_0 & \xrightarrow{f_n} & (\Phi^n)^* M_n/(\Phi^n)^* M_{n-1} & \longrightarrow & 0 \end{array}$$

Hence g_n is an isomorphism of $\mathbb{Z}\mathcal{A}$-module $n \geq n_0$. As Φ^* is an isomorphism we have

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}} M_n/M_{n-1} = \operatorname{pdim}_{\mathbb{Z}\mathcal{A}} (\Phi^n)^*(M_n/M_{n-1}) = \operatorname{pdim}_{\mathbb{Z}\mathcal{A}} (\Phi^n)^* M_n/(\Phi^n)^* M_{n-1}).$$

Thus for $n \geq n_0$ we have $\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_n/M_{n-1}) = \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_{n_0}/M_{n_0-1})$. We have the short exact sequence $0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0$ and hence get from Lemma 5.4 (v)

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_n) \leq \sup\{\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_{n-1}), \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_n/M_{n-1})\}.$$

This implies by induction over $n \geq n_0$

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_n) \leq \sup\{\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_{n_0-1}), \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_{n_0}/M_{n_0-1})\}.$$

Put

$$D := \sup\bigl\{\sup\{\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_k) \mid k = 0, 1, \ldots, n_0 - 1\}, \operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_{n_0}/M_{n_0-1})\bigr\}.$$

Note that $D < \infty$, if $\mathcal{A}$ is regular, and $D \leq l$, if $\mathcal{A}$ is uniformly l-regular. We get

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}\left(\bigoplus_{n \in \mathbb{N}} M_n\right) \leq \sup\{\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(M_n) \mid n \geq 0\} \leq D.$$

We have the short exact sequence of $\mathcal{A}$-modules

$$0 \to \bigoplus_{n \in \mathbb{N}} M_n \to \bigoplus_{n \in \mathbb{N}} M_n \to i^* M \to 0,$$

where the first map is given by $(x_n)_{n \geq 0} \mapsto (x_0, x_1 - x_0.x_2, -x_1, \ldots)$ and the second by $(x_n)_{n \geq 0} \mapsto \sum_{n \geq 0} x_n$. We conclude from Lemma 5.4 (v)

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}}(i^* M) \leq D + 1.$$

Now Theorem 9.15 (i) implies

$$\operatorname{pdim}_{\mathbb{Z}\mathcal{A}_\Phi[t]}(M) \leq D + 2.$$

This finishes the proof if assertion (i).

(ii) We know already that $\mathcal{A}_\phi[t, t^{-1}]$ is Noetherian because of Theorem 8.1 (ii). Let M be a finitely generated $\mathcal{A}_\phi[t, t^{-1}]$-module. We can find a finitely generated free $\mathbb{Z}\mathcal{A}_\Phi[t]$-module F_0 and a free $\mathbb{Z}\mathcal{A}_\Phi[t]$-module F_1 together with an exact sequence of $\mathbb{Z}\mathcal{A}_\Phi[t, t^{-1}]$-modules $j_* \mathrm{F}_0 \xrightarrow{f} j_* F_1 \xrightarrow{e} M \to 0$. Here we write j for the inclusion $\mathcal{A}_\Phi[t] \to \mathcal{A}_\phi[t, t^{-1}]$. By composing f with an appropriate automorphism of $j_* F_0$ one can arrange that $f = j_* g$ for some $\mathbb{Z}\mathcal{A}_\Phi[t]$-homomorphism $g \colon F_0 \to F_1$. The cokernel of g is a finitely generated $\mathbb{Z}\mathcal{A}_\Phi[t]$-module N and there is an obvious exact sequence of $\mathbb{Z}\mathcal{A}_\Phi[t]$-modules $F_1 \xrightarrow{g} F_1 \to N \to 0$. Since the functor j_* is flat by Lemma 9.14 and respects the property projective, we obtain an $\mathbb{Z}\mathcal{A}_\Phi[t, t^{-1}]$-isomorphism $j_* N \xrightarrow{\cong} M$ and have $\dim_{\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]}(j_* N) \leq \dim_{\mathbb{Z}\mathcal{A}_\Phi[t]}(N)$. Hence we get $\dim_{\mathbb{Z}\mathcal{A}_\Phi[t,t^{-1}]}(M) \leq \dim_{\mathbb{Z}\mathcal{A}_\Phi[t]}(N)$. This finishes the proof of Theorem 10.1. $\square$

Remark 10.2. We do not know whether Theorem 10.1 remains true if we replace regular by regular coherent. To our knowledge it is an open problem, whether for a regular coherent ring R the rings $R[t]$ or $R[t, t^{-1}]$ are regular coherent again.

11. Directed union and infinite products of additive categories

A functor of additive categories $F\colon \mathcal{A} \to \mathcal{B}$ is called *flat*, if for every exact sequence $A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2$ in $\mathcal{A}$ the sequence in $F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(p)} F(A_2)$ in $\mathcal{B}$ is exact. It is called *faithfully flat*, if a sequence $A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2$ in $\mathcal{A}$ is exact, if and only if the sequence in $F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(p)} F(A_2)$ in $\mathcal{B}$ is exact.

Lemma 11.1. *Let $i\colon \mathcal{A} \to \mathcal{A}'$ and $j\colon \mathcal{B} \to \mathcal{B}'$ be inclusions of cofinal full additive subcategories. Suppose that the following diagram of functors of additive categories commutes*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{A}' & \xrightarrow[F']{} & \mathcal{B}' \end{array}$$

Then

(i) *The inclusion $i\colon \mathcal{A} \to \mathcal{A}'$ is faithfully flat;*
(ii) *F is flat or faithfully flat respectively if and only if F' is flat or faithfully flat respectively.*

Proof. We first show that F' is exact or faithfully exact respectively, provided that F is exact or faithfully exact respectively.

Consider morphisms $f'\colon A_0' \to A_1'$ and $g'\colon A_1' \to A_2'$ in $\mathcal{A}$. Choose objects A_k in $\mathcal{A}$ and morphisms $i_k\colon A_k' \to A_k$ and $r_k\colon A_k \to A_k'$ in $\mathcal{A}'$ satisfying $r_k \circ i_k = \mathrm{id}_{A_k}$ for $k = 0, 1, 2$. Define $f\colon A_0 \to A_1$ and $g\colon A_1 \to A_2$ by $f = i_1 \circ f' \circ r_0$ and $g = i_2 \circ g' \circ r_1$. Then the following diagram of morphisms in $\mathcal{A}'$ commutes

$$\begin{array}{ccccc} A_0' & \xrightarrow{f'} & A_1' & \xrightarrow{g'} & A_2' \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2 \oplus 0} \\ A_0 & \xrightarrow{f} & A_1 & \xrightarrow{g \oplus (\mathrm{id}_{A_1} - i_1 \circ r_1)} & A_2 \oplus A_1 \\ {\scriptstyle r_0}\downarrow & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2 \oplus 0} \\ A_0' & \xrightarrow{f'} & A_1' & \xrightarrow{g'} & A_2'. \end{array}$$

Next we check that the middle row is exact in $\mathcal{A}$, if and only if the upper row is exact in $\mathcal{A}'$. Suppose that the middle row is exact in $\mathcal{A}$. Consider a morphism $v'\colon B' \to A_1'$ in $\mathcal{A}'$ such that $g' \circ v' = 0$. Choose an object B in $\mathcal{A}$ and maps $j\colon B' \to B$ and $s\colon B \to B'$ with $s \circ j = \mathrm{id}_{B'}$. Then we have the morphism $i_1 \circ v' \circ s\colon B \to A_1$ whose composite with $g \oplus (\mathrm{id}_{A_1} - i_1 \circ r_1)\colon A_1 \to A_2 \oplus A_1$ is zero. Hence we can find a morphism $u_0\colon B \to A_0$ with $f \circ u_0 = i_1 \circ v' \circ s$. Define $u'\colon B' \to A_0'$ by the composite $r_0 \circ u_0 \circ j$. One easily checks that $f' \circ u' = v'$. Hence the upper row is exact in $\mathcal{A}'$.

Suppose that the upper row is exact in $\mathcal{A}'$. Consider a morphism $v\colon B \to A_1$ in $\mathcal{A}$ such that $g \oplus (\mathrm{id}_{A_1} - i_1 \circ r_1) \circ v = 0$. Then $g \circ v = 0$ and $v = i_1 \circ r_1 \circ v$. We conclude

$$g' \circ (r_1 \circ v) = r_2 \circ i_2 \circ g' \circ r_1 \circ v = r_2 \circ g \circ i_1 \circ r_1 \circ v = r_2 \circ g \circ v = r_2 \circ 0 = 0.$$

Since the upper row is exact, we can find $u'\colon B \to A_0'$ satisfying $f' \circ u_0' = r_1 \circ v$. Define $u\colon B \to A_0$ by $i_0 \circ u'$. Then

$$f \circ u = f \circ i_0 \circ u' = i_1 \circ f' \circ u' = i_1 \circ r_1 \circ v = v.$$

Hence the middle row is exact.

If we apply F' and put $i_k' = F'(i_k)$ and $r_k' = F'(r_k)$, we get $r_k' \circ i_k' = \mathrm{id}_{F'(A_k')}$ and the commutative diagram

$$\begin{array}{ccccc}
F'(A_0') & \xrightarrow{F'(f')} & F'(A_1') & \xrightarrow{F'(g')} & F'(A_2') \\
\big\downarrow{\scriptstyle i_0'} & & \big\downarrow{\scriptstyle i_1'} & & \big\downarrow{\scriptstyle i_2' \oplus 0} \\
F(A_0) & \xrightarrow{F(f)} & F(A_1) & \xrightarrow{F(g)\oplus(\mathrm{id}_{F(A_1)} - i_1' \circ r_1')} & F'(A_2) \oplus F(A_1) \\
\big\downarrow{\scriptstyle r_0'} & & \big\downarrow{\scriptstyle r_1'} & & \big\downarrow{\scriptstyle r_2 \oplus 0} \\
F'(A_0') & \xrightarrow{F'(f')} & F'(A_1') & \xrightarrow{F'(g')} & F'(A_2')
\end{array}$$

and, by the same argument as above, the middle row is exact in $\mathcal{B}$, if and only if the upper row is exact in $\mathcal{B}'$. We conclude that the functor F' is exact or faithfully exact respectively, provided that F is exact or faithfully exact respectively.

Since both $\mathrm{id}_{\mathcal{A}}$ and $\mathrm{id}_{\mathcal{B}}$ are faithfully flat, this special case shows that both $i\colon \mathcal{A} \to \mathcal{A}'$ and $j\colon \mathcal{B} \to \mathcal{B}'$ are faithfully flat.

Suppose that F' is flat or faithfully flat respectively. Then $j \circ F = F' \circ i$ is flat or faithfully flat respectively. This implies that F is flat or faithfully flat respectively. This finishes the proof of Lemma 11.1. □

Lemma 11.2. *Let $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ be the directed union of additive subcategories $\mathcal{A}_i$ for an arbitrary directed set I.*

- *(i) The idempotent completion* $\mathrm{Idem}(\mathcal{A})$ *is the directed union of the idempotent completions* $\mathrm{Idem}(\mathcal{A}_i)$*;*
- *(ii) Consider $l \geq 1$.*
  Suppose that $\mathcal{A}_i$ is regular coherent or l-uniformly regular coherent respectively for every $i \in I$ and for every $i, j \in I$ with $i \leq j$ the inclusion $\mathcal{A}_i \to \mathcal{A}_j$ is flat. Then the inclusion $\mathrm{Idem}(\mathcal{A}_i) \to \mathrm{Idem}(\mathcal{A}_j)$ *is flat for every $i, j \in I$ with $i \leq j$ and both $\mathcal{A}$ and* $\mathrm{Idem}(\mathcal{A})$ *are regular coherent or l-uniformly regular coherent respectively;*
- *(iii) Suppose that $\mathcal{A}_i$ is 0-uniformly regular coherent respectively for every $i \in I$. Then both $\mathcal{A}$ and* $\mathrm{Idem}(\mathcal{A})$ *are 0-uniformly regular coherent respectively.*

Proof. (i) This is obvious.

(ii) If the inclusion $\mathcal{A}_i \to \mathcal{A}_j$ is flat, then also the inclusion $\mathrm{Idem}(\mathcal{A}_i) \to \mathrm{Idem}(\mathcal{A}_j)$ is flat by Lemma 11.1. In view of Lemma 6.4 (vi) and assertion (i), we can assume without loss of generality that each $\mathcal{A}_i$ and $\mathcal{A}$ are idempotent complete. Hence we can use the criterion for regular coherent given in Lemma 6.6 in the sequel. We treat only the case $l \geq 2$, the case $l = 1$ is proved analogously

Consider a morphism $f_1\colon A_1 \to A_0$ in $\mathcal{A}$. Choose an index i such that f_1 belongs to $\mathcal{A}_i$. Then we can find a sequence of morphisms

$$0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is in $\mathcal{A}_i$ exact at A_k for $k = 1, 2, \ldots, n$. It remains to show that this sequence is exact at $\mathcal{A}$ at A_k for $k = 1, 2, \ldots, n$. Fix $k \in \{1, 2, \ldots, n\}$. It remains to show for any object $A \in \mathcal{A}$ and morphism $g\colon A \to A_k$ with $f_k \circ g = 0$ that there exists a morphism $\overline{g}\colon A \to A_{k+1}$ with $f_{k+1} \circ \overline{g} = g$. We can choose $j \in I$ with $i \leq j$ such that g belongs to $\mathcal{A}_j$. Since $A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1}$ is exact in $\mathcal{A}_i$, we conclude from the assumptions that it is also exact in $\mathcal{A}_j$ and hence we can construct the desired lift $\overline{g}$ already in $\mathcal{A}_j$.

(iii) In view of Lemma 6.4 (vi) and assertion (i), we can assume without loss of generality that each $\mathcal{A}_i$ and $\mathcal{A}$ are idempotent complete. Now the claim follows from the equivalence (iii)a $\Longleftrightarrow$ (iii)c appearing in Lemma 6.6 (iii). $\square$

Lemma 11.3. *Let l be a natural number. Let $\mathcal{A} = \{\mathcal{A}_i \mid i \in I\}$ be a collection of additive categories $\mathcal{A}_i$ for an arbitrary index set I. Let l be a natural number.*

(i) Suppose that each $\mathcal{A}_i$ is l-uniformly regular coherent. Then $\prod_{i\in I} \operatorname{Idem}(\mathcal{A}_i)$ is l-uniformly regular coherent;

(ii) Suppose that each $\mathcal{A}_i$ is l-uniformly regular coherent, l-uniformly regular, regular coherent, regular, or Noetherian. Then $\bigoplus_{i\in I} \mathcal{A}_i$ has the same property.

Proof. Obviously $\prod_{i\in I} \mathcal{A}_i$ inherits the structure of an additive category. Recall that $\bigoplus_{i\in I} \mathcal{A}_i$ is the full additive subcategory of $\prod_{i\in I} \mathcal{A}_i$ consisting of those objects $A_i \mid i \in I\}$, for which only finitely many of the objects A_i are different from zero. Obviously

$$\begin{aligned} \operatorname{Idem}(\bigoplus_{i\in I} \mathcal{A}_i) &\cong \bigoplus_{i\in I} \operatorname{Idem}(\mathcal{A}_i); \\ \operatorname{Idem}(\prod_{i\in I} \mathcal{A}_i) &\cong \prod_{i\in I} \operatorname{Idem}(\mathcal{A}_i). \end{aligned}$$

Lemma 6.6 implies that $\bigoplus_{i\in I} \operatorname{Idem}(\mathcal{A}_i)$ and $\prod_{i\in I} \operatorname{Idem}(\mathcal{A}_i)$ are l-uniformly regular coherent, if each $\operatorname{Idem}(\mathcal{A}_i)$ is l-uniformly regular coherent. We conclude from Lemma 6.4 (vi) that $\bigoplus_{i\in I} \mathcal{A}_i$ and $\prod_{i\in I} \mathcal{A}_i$ are l-uniformly regular coherent, if each $\mathcal{A}_i$ is l-uniformly regular coherent.

Analogously one shows that $\bigoplus_{i\in I} \mathcal{A}_i$ is regular coherent if each $\mathcal{A}_i$ is regular coherent.

Next we show that $\bigoplus_{i\in I} \mathcal{A}_i$ is Noetherian if each $\mathcal{A}_i$ is Noetherian. Consider any object A in $\bigoplus_{i\in I} \mathcal{A}_i$. Choose a finite index set $J \subseteq I$ such that A belongs to $\bigoplus_{i\in J} \mathcal{A}_i$. Let $\operatorname{pr}\colon \bigoplus_{i\in I} \mathcal{A}_i \to \bigoplus_{i\in J} \mathcal{A}_i$ be the projection. Consider two morphism $f_0\colon A_0 \to A$ and $f_1\colon A_1 \to A$ in $\bigoplus_{i\in I} \mathcal{A}_i$. Then $f_0 = f_1$ holds if and only if $\operatorname{pr}(f_0) = \operatorname{pr}(f_1)$ holds. Hence $f_0 \subseteq f_1$ holds in $\bigoplus_{i\in I} \mathcal{A}_i$ if and only if $\operatorname{pr}(f_0) \subseteq \operatorname{pr}(f_1)$ holds in $\bigoplus_{i\in J} \mathcal{A}_i$. We conclude from Lemma 6.8 that it suffices to show that $\bigoplus_{i\in J} \mathcal{A}_i$ is Noetherian for any finite subset $J \subseteq I$. But this is an easy consequence of Lemma 6.8 again. $\square$

Lemma 11.3 (i) will be generalized in Lemma 13.10.

Remark 11.4 (Advantage of the notion l-uniformly regular coherent)**.** The decisive advantage of the notion l-uniformly regular coherent is that it satisfies both Lemma 11.2 and Lemma 11.3. Lemma 11.2 and Lemma 11.3 (i) are not true, if one replaces l-uniformly regular coherent by any of the properties regular coherent, l-uniformly regular, regular or Noetherian, unless I is finite.

12. Vanishing of negative K-groups

Theorem 12.1 (Vanishing of negative K-groups)**.** *Let $\mathcal{A}$ be an additive category, such that $\mathcal{A}[t_1, t_2, \ldots, t_m]$ is regular coherent for every $m \geq 0$.*

Then $K_n(\mathcal{A}) = 0$ holds for all $n \leq -1$.

Proof. For an additive category $\mathcal{B}$ define $G_0'(\mathbb{Z}\mathcal{B})$ to be the abelian group with isomorphism classes $[M]$ of finitely presented $\mathbb{Z}\mathcal{B}$-modules M as generators such that for each exact sequence of finitely presented $\mathbb{Z}\mathcal{B}$-modules $0 \to M_0 \to M_1 \to M_2 \to 0$

we have the relation $[M_0]-[M_1]+[M_2]=0$. Define $K_0(\mathbb{Z}\mathcal{B})$ analogously but with finitely presented replaced by finitely generated projective. A functor of additive categories $F\colon \mathcal{B}\to\mathcal{B}'$ induces a homomorphism $F_*\colon K_0(\mathbb{Z}\mathcal{B})\to K_0(\mathbb{Z}\mathcal{B}')$ by sending $[M]$ to $[F_*M]$. It induces a homomorphism $F_*\colon G_0'(\mathbb{Z}\mathcal{B})\to G_0'(\mathbb{Z}\mathcal{B}')$ by sending $[M]$ to $[F_*M]$, if $F_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathcal{B}}\to \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}_{\mathcal{B}'}$ is flat. There is the forgetful functor $U\colon K_0(\mathbb{Z}\mathcal{B})\to G_0'(\mathbb{Z}\mathcal{B}')$. If $\mathcal{B}$ is regular coherent, then U is a bijection by the Resolution Theorem, see [20, Theorem 4.6 on page 41]. The Yoneda embedding induces an isomorphism $K_0(\mathcal{B})\xrightarrow{\cong}K_0(\mathbb{Z}\mathcal{B})$, natural in $\mathcal{B}$.

Suppose $\mathcal{A}[t]$ is regular coherent. We show that $\mathcal{A}[t,t^{-1}]$ is regular coherent and $K_{-1}(\mathcal{A})=0$. The functor $j_*\colon \mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}[t]\to\mathsf{MOD}\text{-}\mathbb{Z}\mathcal{A}[t,t^{-1}]$ is flat by Lemma 9.14. Let M_* be a finitely presented $\mathbb{Z}\mathcal{A}[t,t^{-1}]$-module. Then we can find a morphism $f\colon A\to A'$ in $\mathcal{A}[t,t^{-1}]$ together with an exact sequence of $\mathbb{Z}\mathcal{A}[t,t^{1-}]$-modules

$$\mathrm{mor}_{\mathcal{A}[t,t^{-1}]}(?,A)\xrightarrow{f_*}\mathrm{mor}_{\mathcal{A}[t,t^{-1}]}(?,A')\to M\to 0.$$

Choose a natural number s and a morphism $g\colon A\to A'$ in $\mathcal{A}[t]$ such that $(\mathrm{id}_{A'}\cdot t^s)\circ f=g$ holds in $\mathcal{A}[t,t^{-1}]$. Since $\mathrm{id}_{A'}\cdot t^s\colon A'\xrightarrow{\cong}A'$ is an isomorphism in $\mathcal{A}[t,t^{-1}]$, we obtain an exact sequence of $\mathbb{Z}\mathcal{A}[t,t^{-1}]$-modules

$$j_*(\mathrm{mor}_{\mathcal{A}[t]}(?,A))\xrightarrow{j_*(g_*)}j_*(\mathrm{mor}_{\mathcal{A}[t]}(?,A'))\to M\to 0.$$

Let N be the finitely presented $\mathbb{Z}\mathcal{A}[t]$-module, which is the cokernel of the $\mathbb{Z}\mathcal{A}[t]$-homomorphism $g_*\colon\ \mathrm{mor}_{\mathcal{A}[t]}(?,A)\to\mathrm{mor}_{\mathcal{A}[t]}(?,A')$. Since j_* is flat and in particular right exact, we obtain an isomorphism of finitely presented $\mathbb{Z}\mathcal{A}[t,t^{-1}]$-modules $j_*N\xrightarrow{\cong}M$. This implies that the homomorphism $j_*\colon G_0'(\mathbb{Z}\mathcal{A}[t])\to G_0'(\mathbb{Z}\mathcal{A}[t,t^{-1}])$ is surjective and that $\mathcal{A}[t,t^{-1}]$ is regular coherent since $\mathcal{A}[t]$ is regular coherent by assumption.

Hence we obtain a commutative diagram

$$\begin{array}{ccc}
K_0(\mathcal{A}[t]) & \longrightarrow & K_0(\mathcal{A}[t,t^{-1}]) \\
\cong\big\downarrow & & \big\downarrow\cong \\
K_0(\mathbb{Z}\mathcal{A}[t]) & \longrightarrow & K_0(\mathbb{Z}\mathcal{A}[t,t^{-1}]) \\
\cong\big\downarrow & & \big\downarrow\cong \\
G_0'(\mathbb{Z}\mathcal{A}[t]) & \longrightarrow & G_0'(\mathbb{Z}\mathcal{A}[t,t^{-1}]),
\end{array}$$

whose vertical arrows are bijections and whose lowermost horizontal arrow is surjective. Hence the uppermost horizontal arrow is surjective. We conclude from Theorem 4.2 that $K_{-1}(\mathcal{A})$ vanishes.

Next we show by induction for $n=1,2\ldots$ that $K_{-m}(\mathcal{A})$ vanishes for $m=1,2,\ldots,n$. The induction beginning $n=1$ has been taken care of above. The induction step from $n\ge 1$ to $n+1$ is done as follows. One shows using the claim above by induction for $i=1,2,\ldots,n$ that $\mathcal{A}[\mathbb{Z}^i][t_{i+1},\ldots,t_{n+1}]$ is regular coherent. In particular $\mathcal{A}[\mathbb{Z}^n][t_{n+1}]$ is regular coherent.

We conclude from the n-times iterated Bass-Heller-Swan isomorphism, see Theorem 4.1 that $K_{-n-1}(\mathcal{A})$ is a direct summand in $K_{-1}(\mathcal{A}[\mathbb{Z}^n])$. Hence it suffices to show that $K_{-1}(\mathcal{A}[\mathbb{Z}^n])$ is trivial. This follows from the induction beginning applied to $\mathcal{A}[\mathbb{Z}^n]$. □

We conclude from Theorem 10.1 and Theorem 12.1

Corollary 12.2 (Vanishing of negative K-groups of regular additive categories)**.** *Let $\mathcal{A}$ be an additive category which is regular.*

Then $K_n(\mathcal{A})=0$ holds for all $n\le -1$.

Remark 12.3. As noted in Lemma 5.10 the Yoneda embedding identifies $\mathcal{A}$ with the category of finitely generated free $\mathbb{Z}\mathcal{A}$-modules. If $\mathcal{A}$ is Noetherian, then the category of finitely generated $\mathbb{Z}\mathcal{A}$-modules is abelian. If it is in addition regular coherent (i.e., if $\mathcal{A}$ is regular), then $\mathcal{A}$ is derived equivalent to this abelian category. Schlichting showed in [19, Theorem 6 on page 117] that K_{-1} of abelian categories is trivial. It follows that for regular $\mathcal{A}$ we can obtain Theorem 12.1 from Schlichting's result. Similarly, Corollary 12.2 can alternatively be deduced from [19, Theorem 7 on page 118].

13. Nested sequences, the associated categories, and their K-theory

13.a. Nested sequences and the associated categories.

Definition 13.1 (Nested sequences of additive categories)**.** A *nested sequence of additive categories* $\mathcal{A}_*$ is a decreasing sequence of additive subcategories

$$\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots .$$

We have two notions of morphisms.

Definition 13.2 ((Pro-)morphisms of nested sequences of additive categories)**.** A *morphism of nested sequences of additive categories* $F\colon \mathcal{A}_* \to \mathcal{A}'_*$ is a sequence of functors of additive categories $F_m\colon \mathcal{A}_m \to \mathcal{A}'_m$ for $m \in \mathbb{N}$ such that F_m restricted to $\mathcal{A}_{m+1}$ is F_{m+1}.

A *pro-morphism of nested sequences of additive categories* $F\colon \mathcal{A}_* \to \mathcal{A}'_*$ is a functor of additive categories $F\colon \mathcal{A}_0 \xrightarrow{\cong} \mathcal{A}'_0$ such that there is a function $N\colon \mathbb{N} \to \mathbb{N}$ with the property that for every $m, n \in \mathbb{N}$ with $m \geq N(n)$ we have $F(\mathcal{A}_m) \subseteq \mathcal{A}'_n$.

Obviously a morphism F_* defines a pro-morphisms by taking F_0, but not every pro-morphism comes from a morphism in this way. The composite of two morphisms and of two pro-morphisms is defined in the obvious way.

Note that a pro-automorphism $\phi\colon \mathcal{A}_* \to \mathcal{A}_*$ is the same as an automorphism of additive categories $\Phi\colon \mathcal{A}_0 \xrightarrow{\cong} \mathcal{A}_0$ such that there is a function $N\colon \mathbb{N} \to \mathbb{N}$ with the property that for every $m, n \in \mathbb{N}$ with $m \geq N(n)$ we have $\Phi(\mathcal{A}_m) \subseteq \mathcal{A}_n$ and $\mathcal{A}_m \subseteq \Phi(\mathcal{A}_n)$.

Definition 13.3 (The sequence category $\mathcal{S}(\mathcal{A}_*)$ and the limit category $\mathcal{L}(\mathcal{A}_*)$)**.** Define the additive category $\mathcal{S}(\mathcal{A}_*)$, called *sequence category*, associated to the nested sequence of additive categories $\mathcal{A}_*$ as follows:

- An object in $\mathcal{S}(\mathcal{A}_*)$ is a sequence $\underline{A} = (A_m)_{m\in\mathbb{N}}$ of objects in $\mathcal{A}_0$ such that there exists a function (depending on $\underline{A}$) $L\colon \mathbb{N} \to \mathbb{N}$ with the property that A_m belongs to $\mathcal{A}_l$ for $l, m \in \mathbb{N}$ with $m \geq L(l)$;
- A morphism $\underline{\varphi}\colon \underline{A} \to \underline{A'}$ in $\mathcal{S}(\mathcal{A}_*)$ consists of a sequence of morphisms $\varphi_m\colon A_m \to A'_m$ in $\mathcal{A}_0$ for $m \in \mathbb{N}$ such that there exists a function $L\colon \mathbb{N} \to \mathbb{N}$ with the property that $\varphi_m\colon A_m \to A'_m$ belongs to $\mathcal{A}_l$ for $m, l \in \mathbb{N}$ with $m \geq L(l)$;
- Composition and the structure of an additive category on $\mathcal{S}(\mathcal{A}_*)$ comes from the corresponding structure on $\mathcal{A}_0$.

Let $\mathcal{T}(\mathcal{A}_*)$ be the full subcategory of $\mathcal{S}(\mathcal{A}_*)$ consisting of objects $\underline{A}$, for which there exists a natural number M (depending on $\underline{A}$) with $A_m = 0$ for $m \geq M$.

The additive category $\mathcal{L}(\mathcal{A}_*)$, called *limit category*, is defined to be the additive quotient category $\mathcal{S}(\mathcal{A}_*)/\mathcal{T}(\mathcal{A}_*)$. Recall that for a full additive subcategory $\mathcal{U} \subseteq \mathcal{B}$ of an additive category $\mathcal{B}$ the additive quotient category $\mathcal{B}/\mathcal{U}$ has the same objects as $\mathcal{B}$ and that morphisms in $\mathcal{B}/\mathcal{U}$ are equivalence classes of morphisms in $\mathcal{B}$, where two morphisms φ, φ' are identified, if their difference $\varphi - \varphi'$ can be factorized through an object of $\mathcal{U}$. For $\mathcal{L}(\mathcal{A}_*)$ this means that morphisms φ, φ' from $\mathcal{S}(\mathcal{A}_*)$ are identified in $\mathcal{L}(\mathcal{A}_*)$, if and only if $\varphi_m = \varphi'_m$ for all but finitely many m.

Obviously $\mathcal{S}(\mathcal{A}_*)$ is an additive subcategory of $\prod_{m\in\mathbb{N}} \mathcal{A}_0$ and is equal to it, if the nested sequence is constant, i.e., $\mathcal{A}_0 = \mathcal{A}_m$ for $m \in \mathbb{N}$. Obviously $\mathcal{T}(\mathcal{A}_*)$ can be identified with $\bigoplus_{m\in\mathbb{N}} \mathcal{A}_m$. Controlled categories appear for instance in proofs of the Farrell-Jones Conjecture. In for us important cases controlled categories correspond to nested sequences of additive categories, where one should think of the control condition to become sharper the larger the index m gets.

For the notion of a Karoubi filtration and the associated weak homotopy fibration sequence we refer for instance to [3], and [6, Definition 5.4] or [11, Section 12.2]. One easily checks

Lemma 13.4. *The inclusion $\mathcal{T}(\mathcal{A}_*) \subseteq \mathcal{S}(\mathcal{A}_*)$ is a Karoubi filtration and we have the weak homotopy fibration sequence*

$$\mathbf{K}^\infty(\mathcal{T}(\mathcal{A}_*)) \to \mathbf{K}^\infty(\mathcal{S}(\mathcal{A}_*)) \to \mathbf{K}^\infty(\mathcal{L}(\mathcal{A}_*)).$$

Let $F\colon \mathcal{A}_* \to \mathcal{A}'_*$ be a pro-morphisms. It induces functors of additive categories

$$\mathcal{S}(F)\colon \mathcal{S}(\mathcal{A}_*) \to \mathcal{S}(\mathcal{A}'_*); \tag{13.5}$$

$$\mathcal{T}(F)\colon \mathcal{T}(\mathcal{A}_*) \to \mathcal{T}(\mathcal{A}'_*); \tag{13.6}$$

$$\mathcal{L}(F)\colon \mathcal{L}(\mathcal{A}_*) \to \mathcal{L}(\mathcal{A}'_*), \tag{13.7}$$

as follows. We begin with $\mathcal{S}(F)$. It sends an object $\underline{A} = (A_m)_{m\in\mathbb{N}}$ to the object $\mathcal{S}(F)(\underline{A}) = (F(A_m))_{m\in\mathbb{N}}$. We have to check that the latter collection defines an element in $\mathcal{S}(\mathcal{A}'_*)$. Recall that there is a function $L\colon \mathbb{N} \to \mathbb{N}$ with the property that A_m belongs to $\mathcal{A}_l$ for $l, m \in \mathbb{N}$ with $m \geq L(l)$ and that there is a function $N\colon \mathbb{N} \to \mathbb{N}$ with the property that for every $m, n \in \mathbb{N}$ with $m \geq N(n)$ we have $F(\mathcal{A}_m) \subseteq \mathcal{A}_n$. Consider the function $L \circ N\colon \mathbb{N} \to \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m \geq L \circ N(n)$ we conclude $A_m \in \mathcal{A}_{N(n)}$ and hence $F(A_m) \in F(\mathcal{A}_{N(n)}) \subseteq \mathcal{A}_n$. Hence $(F(A_m))_{m\in\mathbb{N}}$ is a well-defined object in $\mathcal{S}(\mathcal{A}_*)$.

Given a morphism $\underline{\varphi}\colon \underline{A} \to \underline{A'}$ in $\mathcal{S}(\mathcal{A}_*)$, we can define $\mathcal{S}(F)(\underline{\varphi})\colon \mathcal{S}(F)(\underline{A}) \to \mathcal{S}(F)(\underline{A'})$ in $\mathcal{S}(\mathcal{A}'_*)$ by the collection $(F(\varphi_m))_{m\in\mathbb{N}}$. Obviously $\mathcal{S}(F' \circ F) = \mathcal{S}(F') \circ \mathcal{S}(F)$ and $\mathcal{S}(\mathrm{id}_{\underline{A}}) = \mathrm{id}_{\underline{A}}$ holds. By construction $\mathcal{S}(F)$ induces a functor of additive categories $\mathcal{T}(F)\colon \mathcal{T}(\mathcal{A}_*) \to \mathcal{T}(\mathcal{A}'_*)$. By passing to the quotients we also get a functor of additive categories $\mathcal{L}(F)\colon \mathcal{L}(\mathcal{A}_*) \to \mathcal{L}(\mathcal{A}'_*)$.

Note that a pro-automorphism $\Phi\colon \mathcal{A}_* \to \mathcal{A}_*$ induces automorphisms of additive categories

$$\begin{aligned} \mathcal{S}(\Phi)\colon \mathcal{S}(\mathcal{A}_*) &\xrightarrow{\cong} \mathcal{S}(\mathcal{A}_*); \\ \mathcal{T}(\Phi)\colon \mathcal{T}(\mathcal{A}_*) &\xrightarrow{\cong} \mathcal{T}(\mathcal{A}_*); \\ \mathcal{L}(\Phi)\colon \mathcal{L}(\mathcal{A}_*) &\xrightarrow{\cong} \mathcal{L}(\mathcal{A}_*). \end{aligned}$$

Definition 13.8. We call a function $I\colon \mathbb{N} \to \mathbb{N}$ *admissible*, if it has the following properties

$$\begin{array}{rcll} I(m) & \leq & m & \text{for } m \in \mathbb{N}; \\ I(m) & \leq & I(m+1) & \text{for } m \in \mathbb{N}; \\ \lim_{m\to\infty} I(m) & = & \infty. & \end{array}$$

Let $\mathcal{I}$ be the set of admissible functions. It becomes a directed set by defining for $I, J \in \mathcal{I}$

$$I \leq J \iff I(m) \geq J(m) \text{ for all } m \in \mathbb{N}.$$

Note that $\mathcal{I}$ is indeed directed. For $I, J \in \mathcal{I}$ define $K\colon \mathbb{N} \to \mathbb{N}$ by $K(m) = \min\{I(m), J(m)\}$. Then $K \in \mathcal{I}$ and $I, J \leq K$ holds.

Lemma 13.9. *(i) Let $\underline{\phi}$ be a morphism in $\mathcal{S}(\mathcal{A}_*)$. Then there exists an admissible function $I \in \mathcal{I}$ such that $\phi_m \in \mathcal{A}_{I(m)}$ holds for all $m \in \mathbb{N}$;*

(ii) Let $\underline{\phi}$ be a sequence of morphisms $\phi_m \colon A_m \to A'_m$ in A_0. Suppose that there exists an admissible function $I \in \mathcal{I}$ such that $\phi_m \in \mathcal{A}_{I(m)}$ holds for all $m \in \mathbb{N}$. Then $\underline{\phi}$ belongs to $\mathcal{S}(\mathcal{A}_*)$.

Proof. (i) Choose a function $L \colon \mathbb{N} \to \mathbb{N}$ such that ϕ_m belongs to $\mathcal{A}_l$ if $m \geq L(l)$. Define a new function $I' \colon \mathbb{N} \to \mathbb{N}$ by

$$I'(m) = \max\{i \in \{0,1,\ldots,m\} \mid \phi_m \in \mathcal{A}_i\}.$$

It satisfies

$$\begin{array}{lcll} I'(m) & \leq & m & \text{for } m \in \mathbb{N}; \\ l & \leq & I'(m) & \text{for } l, m \in \mathbb{N}, m \geq L(l), m \geq l; \\ \phi_m & \in & \mathcal{A}_{I'(m)} & \text{for } m \in \mathbb{N}. \end{array}$$

Define the function $I \colon \mathbb{N} \to \mathbb{N}$ by

$$I(m) = \min\{I'(j) \mid j \in \mathbb{N}, m \leq j\}.$$

Then we get for all $n \in \mathbb{N}$

$$\begin{array}{lcll} I(m) & \leq & m & \text{for } m \in \mathbb{N}; \\ I(m) & \leq & I(m+1) & \text{for } m \in \mathbb{N}; \\ l & \leq & I(m) & \text{for } l, m, \in \mathbb{N}, m \geq L(l), m \geq l; \\ \phi_m & \in & \mathcal{A}_{I(m)} & \text{for } m \in \mathbb{N}. \end{array}$$

The first three properties imply $I \in \mathcal{I}$.

(ii) Suppose that there exists $I \in \mathcal{I}$ satisfying $\phi_m \in \mathcal{A}_{I(m)}$ for all $m \in \mathbb{N}$. Define the desired function $L \colon \mathbb{N} \to \mathbb{N}$ by $L(l) = \min\{m \in \mathbb{N} \mid l \leq I(m)\}$. ☐

13.B. Uniform regular coherence.

Lemma 13.10. *Consider the nested sequence $\mathcal{A}_*$ of additive categories $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots$. Suppose that for the natural number $l \geq 2$ each of the additive categories $\mathcal{A}_m$ is l-uniformly regular coherent and that the inclusion $\mathcal{A}_m \to \mathcal{A}_{m+1}$ is flat for $m \in \mathbb{N}$.*

Then $\mathcal{S}(\mathcal{A}_*)$ and $\mathcal{L}(\mathcal{A}_*)$ are l-uniformly regular coherent.

Proof. We first treat $\mathcal{S}(\mathcal{A}_*)$. Let $\underline{\phi}^1 \colon \underline{A}^1 \to \underline{A}^0$ be a morphism in $\mathcal{S}(\mathcal{A}_*)$. Because of Lemma 13.9 (i) we can choose $I \in \mathcal{I}$ with $A^1_m, A^0_m, \phi^1_m \in \mathcal{A}_{I(m)}$. By assumption we can find for each $m \in \mathbb{N}$ an exact sequence

$$0 \to A^l_m \xrightarrow{\phi^l_m} A^{l-1}_m \xrightarrow{\phi^l_m} \cdots \xrightarrow{\phi^2_m} A^1_m \xrightarrow{\phi^1_1} A^0_m$$

in $\mathcal{A}_{I(m)}$. We conclude from Lemma 13.9 (ii) that the collection of these sequences for $m = 0, 1, 2 \ldots$ defines a sequence in $\mathcal{S}(\mathcal{A}_*)$

$$0 \to \underline{A}^l \xrightarrow{\underline{\phi}^l} \underline{A}^{l-1} \xrightarrow{\underline{\phi}^{l-1}} \cdots \xrightarrow{\underline{\phi}^2} \underline{A}^1 \xrightarrow{\underline{\phi}^1} \underline{A}^0. \tag{13.11}$$

Finally we show that the sequence (13.11) is exact as a sequence in $\mathcal{S}(\mathcal{A}_*)$. We have to solve for every $j \in \{1, \ldots, l\}$ the following lifting problem in $\mathcal{S}(\mathcal{A}_*)$.

$$\begin{array}{ccccc} \underline{A}^{j+1} & \xrightarrow{\underline{\phi}^{j+1}} & \underline{A}^{j} & \xrightarrow{\underline{\phi}^{j}} & \underline{A}^{j-1} \\ & \nwarrow {\scriptstyle \underline{\nu}} & \uparrow {\scriptstyle \underline{\mu}} & \nearrow {\scriptstyle 0} & \\ & & \underline{B}. & & \end{array} \tag{13.12}$$

Because of Lemma 13.9 (i) we can choose $J \in \mathcal{I}$ such $B_m, \mu_m \in \mathcal{A}_{J(m)}$ holds. Choose $K \in \mathcal{I}$ with $I, J \leq K$. Now consider the following lifting problem in $\mathcal{A}_{K(m)}$

(13.13)
$$\begin{array}{ccccc} A_m^{j+1} & \xrightarrow{\phi_m^{j+1}} & A_m^j & \xrightarrow{\phi_m^j} & A_m^{j-1} \\ & \nwarrow^{\nu_m} & \uparrow{\scriptstyle \mu_m} & \nearrow_{0} & \\ & & B_m. & & \end{array}$$

As the inclusion $\mathcal{A}_{I(m)} \to \mathcal{A}_{K(m)}$ is flat by assumption, and the sequence $A_m^{j+1} \xrightarrow{\phi_m^{j+1}} A_m^j \xrightarrow{\phi_m^j} A_m^{j-1}$ is by construction exact at A_m^j, when considered in $\mathcal{A}_{I(m)}$, it is exact at A_m^j, when considered in $\mathcal{A}_{K(m)}$. Hence (13.13) has a solution $\nu_m \colon B_m \to A^{j+1}$ when considered in $\mathcal{A}_{K(m)}$. We conclude from Lemma 13.9 (ii) that the collection of the morphisms ν_m yields a morphism $\underline{\nu} \colon \underline{B} \to \underline{A}^{j+1}$ in $\mathcal{S}(\mathcal{A}_*)$. Therefore $\overline{\nu}$ is a solution to the lifting problem (13.12) in $\mathcal{S}(\mathcal{A}_*)$. We conclude that (13.11) is an exact sequence in $\mathcal{S}(\mathcal{A}_*)$. This finishes the proof of Lemma 13.10 for $\mathcal{S}(\mathcal{A}_*)$.

The proof for $\mathcal{L}(\mathcal{A}_*)$ is the following modification of the one for $\mathcal{S}(\mathcal{A}_*)$. Let $\overline{\underline{\phi}^1} \colon \underline{A}^1 \to \underline{A}^0$ be a morphism in $\mathcal{L}(\mathcal{A}_*)$. Choose a representative $\underline{\phi}^1 \colon \underline{A}^1 \to \underline{A}^0$ in $\mathcal{S}(\mathcal{A}_*)$. Now one proceeds as above and constructs the sequence (13.11) in $\mathcal{S}(\mathcal{A}_*)$. However, instead of solving the lifting problem (13.12) in $\mathcal{S}(\mathcal{A}_*)$ we have to solve the lifting problem

(13.14)
$$\begin{array}{ccccc} \underline{A}^{j+1} & \xrightarrow{\overline{\underline{\phi}^{j+1}}} & \underline{A}^j & \xrightarrow{\overline{\underline{\phi}^j}} & \underline{A}^{j-1} \\ & \nwarrow^{\underline{\nu}} & \uparrow{\scriptstyle \overline{\underline{\mu}}} & \nearrow_{0} & \\ & & \underline{B} & & \end{array}$$

in $\mathcal{L}(\mathcal{A}_*)$. Choose a representative $\underline{\mu}$ for $\overline{\underline{\mu}}$. There is a natural number M such that $\phi_m^j \circ \mu_m = 0$ holds for $m \geq M$. We can change the representative $\underline{\mu}$ by putting $\mu_m = 0$ for $m < M$ and by leaving μ_m unchanged for $m \geq M$. Now we choose a solution ν_m to the lifting problem (13.13) in $\mathcal{A}_{K(m)}$ for $m \geq M$. Put $\nu_m = 0$ for $m < M$. Then we get a morphism $\underline{\nu} \colon \underline{B} \to \underline{A}^{j+1}$ in $\mathcal{S}(\mathcal{A}_*)$ such that its class $\overline{\underline{\nu}} \colon \underline{B} \to \underline{A}^{j+1}$ in $\mathcal{L}(\mathcal{A}_*)$ is a solution to the lifting problem (13.14). This finishes the proof of Lemma 13.10. ☐

Example 13.15 (The property Noetherian does not pass to the sequence category)**.** The analogue of Lemma 13.10 for the properties Noetherian, regular, or l-uniformly regular instead of uniformly l-regular coherent does not hold as the following example shows. Suppose that none of the $\mathcal{A}_m$ is the trivial additive category. Consider an object $\underline{A}$ of $\mathcal{S}(\mathcal{A}_*)$ such that $A_m \neq \{0\}$ for $m \in \mathbb{N}$, and the $\mathbb{Z}\mathcal{S}(\mathcal{A}_*)$-module

$$F = \operatorname{mor}_{\mathcal{S}(\mathcal{A}_*)}(\underline{?}, \underline{A}).$$

Define a $\mathbb{Z}\mathcal{S}(\mathcal{A}_*)$-submodule V of F by

$$V(?) = \{\underline{\phi} \in F(?) \mid \exists M(\underline{\phi}) \in \mathbb{N} \text{ with } \phi_m = 0 \text{ for } m \geq M(\underline{\phi})\}.$$

Suppose that there exists an object $\underline{B}$ and an epimorphism $f \colon \operatorname{mor}_{\mathcal{S}(\mathcal{A}_*)}(\underline{?}, \underline{B}) \to V$. If we write $f(\mathrm{id}_{\underline{B}}) = \underline{\psi} \in V(\underline{B})$, then there must be a natural number M with $\psi_m = 0$ for $m \geq M$. This implies that for any $\underline{\phi} \in V(?)$ we have $\phi_m = 0$ for $m \geq M$. This is a contradiction, since M does not depend on $\underline{\phi}$. Hence V is not finitely generated and $\mathcal{S}(\mathcal{A}_*)$ is not Noetherian. This construction yields also a counterexample for $\mathcal{L}(\mathcal{A}_*)$.

13.c. **The algebraic K-theory of the sequence categories $\mathcal{S}(\mathcal{A}_*)$, $\mathcal{T}(\mathcal{A}_*)$ and $\mathcal{L}(\mathcal{A}_*)$.** Given $I \in \mathcal{I}$, define a subcategory $\mathcal{S}(\mathcal{A}_*)_I$ of $\mathcal{S}(\mathcal{A}_*)$ as follows. An object $\underline{A}$ in $\mathcal{S}(\mathcal{A}_*)$ belongs to $\mathcal{S}(\mathcal{A}_*)_I$, if $A_m \in \mathcal{A}_{I(m)}$ holds for $m \in \mathbb{N}$. A morphism $\underline{\phi}\colon \underline{A} \to \underline{T}$ in $\mathcal{S}(\mathcal{A}_*)$ belongs to $\mathcal{S}(\mathcal{A}_*)_I$, if and only if $\phi_m \in \mathcal{A}_{I(m)}$ holds for $m \in \mathbb{N}$.

Next we show

$$\mathcal{S}(\mathcal{A}_*)_I \subseteq \mathcal{S}(\mathcal{A}_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \tag{13.16}$$

$$\mathcal{S}(\mathcal{A}_*) = \bigcup_{I \in \mathcal{I}} \mathcal{S}(\mathcal{A}_*)_I. \tag{13.17}$$

The first equation is obvious since $\mathcal{A}_i \subseteq \mathcal{A}_j$ for $i \geq j$. The second follows from Lemma 13.9.

Define analogously the subcategory $\mathcal{T}(\mathcal{A}_*)_I$ of $\mathcal{T}(\mathcal{A}_*)$. Then we get

$$\mathcal{T}(\mathcal{A}_*)_I \subseteq \mathcal{T}(\mathcal{A}_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \tag{13.18}$$

$$\mathcal{T}(\mathcal{A}_*) = \bigcup_{I \in \mathcal{I}} \mathcal{T}(\mathcal{A}_*)_I. \tag{13.19}$$

One easily checks that the inclusion $\mathcal{T}(\mathcal{A}_*)_I \subseteq \mathcal{S}(\mathcal{A}_*)_I$ is Karoubi filtration. Define

$$\mathcal{L}(\mathcal{A}_*)_I = \mathcal{S}(\mathcal{A}_*)_I / \mathcal{T}(\mathcal{A}_*)_I. \tag{13.20}$$

Then we get

$$\mathcal{L}(\mathcal{A}_*)_I \subseteq \mathcal{L}(\mathcal{A}_*)_J \quad \text{for } I, J \in \mathcal{I}, I \leq J; \tag{13.21}$$

$$\mathcal{L}(\mathcal{A}_*) = \bigcup_{I \in \mathcal{I}} \mathcal{L}(\mathcal{A}_*)_I. \tag{13.22}$$

Lemma 13.23.

(i) *We get for $I, J \in \mathcal{I}$ with $I \leq J$ functors*

$$\begin{aligned} \mathcal{S}(A_*)_I &\to \mathcal{L}(A_*)_J; \\ \mathcal{S}(A_*)_I &\to \mathcal{L}(A_*), \end{aligned}$$

and analogously for $\mathcal{T}$ and $\mathcal{L}$;

(ii) *The functors appearing in assertion (i) induce weak homotopy equivalences, natural in $\mathcal{A}_*$*

$$\begin{aligned} \operatorname{hocolim}_{I \in \mathcal{I}} \mathbf{K}^\infty(\mathcal{S}(\mathcal{A}_*)_I) &\xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{S}(\mathcal{A}_*)); \\ \operatorname{hocolim}_{I \in \mathcal{I}} \mathbf{K}^\infty(\mathcal{T}(\mathcal{A}_*)_I) &\xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{T}(\mathcal{A}_*)); \\ \operatorname{hocolim}_{I \in \mathcal{I}} \mathbf{K}^\infty(\mathcal{L}(\mathcal{A}_*)_I) &\xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{L}(\mathcal{A}_*)). \end{aligned}$$

Proof. (i) The desired functors come from (13.16), (13.18), and (13.21).

(ii) This follows from of (13.17) (13.19), (13.22) and [13, Corollary 7.2]. □

Given $I \in \mathcal{I}$, we define $\bigoplus_{m \in \mathbb{N}} \mathcal{A}_{I(m)}$ to be the full subcategory of $\prod_{m \in \mathbb{N}} \mathcal{A}_{I(m)}$ consisting of those objects $\underline{A}$ for which there exists a natural number M (depending on $\underline{A}$) satisfying $A_m = 0$ for $m \geq M$. Let $\left(\prod_{m \in \mathbb{N}} \mathcal{A}_{I(m)}\right) / \left(\bigoplus_{m \in \mathbb{N}} \mathcal{A}_{I(m)}\right)$ be the quotient additive category.

Lemma 13.24. *Fix $I \in \mathcal{I}$. There are weak homotopy equivalences, natural in $\mathcal{A}_*$,*

$$\begin{aligned} \mathbf{K}^\infty\Big(\prod_{m \in \mathbb{N}} \mathcal{A}_{I(m)}\Big) &\xrightarrow{\simeq} \prod_{m \in \mathbb{N}} \mathbf{K}^\infty(\mathcal{A}_{I(m)}); \\ \bigvee_{m \in \mathbb{N}} \mathbf{K}^\infty(\mathcal{A}_{I(m)}) &\xrightarrow{\simeq} \mathbf{K}^\infty\Big(\bigoplus_{m \in \mathbb{N}} \mathcal{A}_{I(m)}\Big), \end{aligned}$$

and

$$\operatorname{hocofib}\big(\mathbf{K}^\infty\big(\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\to\mathbf{K}^\infty\big(\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\big) \xrightarrow{\simeq} \mathbf{K}^\infty\left(\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\Big/\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\right).$$

Proof. The first one is weak homotopy equivalence by [4], see also [7, Theorem 1.2], since the non-connective algebraic K-theory spectrum is indeed an Ω-spectrum. The second one is a weak homotopy equivalence, since $\bigoplus_{m\in\mathbb{N}}\mathcal{A}_m$ is the union of the subcategories $\bigoplus_{m=0}^n\mathcal{A}_i$ and hence we get a weak homotopy equivalence

$$\operatorname{hocolim}_{n\to\infty}\mathbf{K}^\infty\big(\bigoplus_{m=0}^{n}\mathcal{A}_{I(m)}\big)\xrightarrow{\simeq}\mathbf{K}^\infty\big(\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big),$$

and the natural map

$$\bigvee_{m=0}^{n}\mathbf{K}(\mathcal{A}_{I(m)})\xrightarrow{\simeq}\mathbf{K}^\infty\big(\bigoplus_{m=0}^{n}\mathcal{A}_{I(m)}\big)$$

is a weak homotopy equivalence. The third map is a weak homotopy equivalence, since the inclusion $\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\subseteq\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}$ is a Karoubi filtration. □

Lemma 13.25. *Given $I\in\mathcal{I}$, there are weak homotopy equivalences, natural in $\mathcal{A}_*$,*

$$\begin{aligned}
\mathbf{K}^\infty\big(\mathcal{S}(\mathcal{A}_*)_I\big) &\xrightarrow{\simeq} \prod_{m\in\mathbb{N}}\mathbf{K}^\infty\big(\mathcal{A}_{I(m)}\big);\\
\bigvee_{m\in\mathbb{N}}\mathbf{K}^\infty(\mathcal{A}_{I(m)}) &\xrightarrow{\simeq} \mathbf{K}^\infty\big(\mathcal{T}(\mathcal{A}_*)_I\big);\\
\operatorname{hocofib}\big(\mathbf{K}^\infty\big(\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\to\mathbf{K}^\infty\big(\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\big) &\xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{L}(\mathcal{A})_I).
\end{aligned}$$

Proof. The are obvious identifications

$$\begin{aligned}
\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)} &= \mathcal{S}(\mathcal{A}_*)_I;\\
\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)} &= \mathcal{T}(\mathcal{A}_*)_I;\\
\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\Big/\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)} &= \mathcal{L}(\mathcal{A}_*)_I.
\end{aligned}$$

Now the claim follows from Lema 13.24. □

As a consequence of Lemma 13.23 (ii) and Lemma 13.25 we get

Lemma 13.26 (K-groups of $\mathcal{S}(\mathcal{A}_*)$, $\mathcal{T}(\mathcal{A}_*)$, and $\mathcal{L}(\mathcal{A}_*)$)**.** *There are zigzags of weak homotopy equivalences of spectra, natural in $\mathcal{A}_*$*

$$\begin{aligned}
\operatorname{hocolim}_{i\in\mathcal{I}}\prod_{m\in\mathbb{N}}\mathbf{K}^\infty\big(\mathcal{A}_{I(m)}\big) &\overset{\simeq}{\leftrightarrows} \mathbf{K}^\infty\big(\mathcal{S}(\mathcal{A}_*)\big);\\
\operatorname{hocolim}_{i\in\mathcal{I}}\bigvee_{m\in\mathbb{N}}\mathbf{K}^\infty(\mathcal{A}_{I(m)}) &\overset{\simeq}{\leftrightarrows} \mathbf{K}^\infty\big(\mathcal{T}(\mathcal{A}_*)\big);\\
\operatorname{hocolim}_{i\in\mathcal{I}}\left(\operatorname{hocofib}\big(\mathbf{K}^\infty\big(\bigoplus_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\to\mathbf{K}^\infty\big(\prod_{m\in\mathbb{N}}\mathcal{A}_{I(m)}\big)\big)\right) &\overset{\simeq}{\leftrightarrows} \mathbf{K}^\infty(\mathcal{L}(\mathcal{A})).
\end{aligned}$$

In particular we get from Lemma 13.26 for every $n \in \mathbb{Z}$ an isomorphism

$$\operatorname{colim}_{i \in I}\left(\prod_{m \in \mathbb{N}} K_n(A_{I(m)}) \Big/ \bigoplus_{m \in \mathbb{N}} K_n(A_{I(m)})\right) \quad \cong \quad K_n(\mathcal{L}(\mathcal{A}_*)). \tag{13.27}$$

14. The main technical result

14.A. **The statement of the main technical result.** Fix a natural number r and a nested sequence of additive categories $\mathcal{A}_*$. We have defined the additive category $\mathcal{L}(\mathcal{A}_*)$ in Definition 13.3. Suppose that each $\mathcal{A}_*$ comes with a $\mathbb{Z}^r$-action $\Psi\colon \mathbb{Z}^r \to \text{pro-aut}(\mathcal{A}_*)$ by pro-automorphisms in the sense of Definition 13.2. Then we obtain a $\mathbb{Z}^r$-action $\Phi\colon \mathbb{Z}^r \to \operatorname{aut}(\mathcal{L}(\mathcal{A}_*))$ on $\mathcal{L}(\mathcal{A}_*)$ see (13.7). We obtain a covariant functor, see for instance [1, Section 9],

$$\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}\colon \mathsf{Or}(\mathbb{Z}^r) \to \mathsf{Spectra}.$$

It determines a $\mathbb{Z}^r$-homology theory $H_n^{\mathbb{Z}^r}(-, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)})$ with the property that for every subgroup $H \subseteq \mathbb{Z}^r$ and $n \in \mathbb{Z}$ we have the natural isomorphisms

$$H_n^{\mathbb{Z}^r}(\mathbb{Z}^r/H, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) \xrightarrow{\cong} K_n(\mathcal{L}(\mathcal{A}_*) \rtimes_{\Phi|_H} H)$$

as explained for instance in [1, Section 9]. The nested sequence of additive categories $\mathcal{A}_*$ yields for any natural number d another nested sequence of additive categories $\mathcal{A}_*[\mathbb{Z}^d]$ by $\mathcal{A}_0[\mathbb{Z}^d] \supseteq \mathcal{A}_1[\mathbb{Z}^d] \supseteq \mathcal{A}_2[\mathbb{Z}^d] \supseteq \cdots$, where $\mathcal{A}_m[\mathbb{Z}]$ is the untwisted case of Definition 2.3 and we define inductively $\mathcal{A}_m[\mathbb{Z}^d] = (\mathcal{A}_m[\mathbb{Z}^{d-1}])[\mathbb{Z}]$. Moreover, $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ inherits a $\mathbb{Z}^r$-action $\Phi[\mathbb{Z}^d]\colon \mathbb{Z}^r \to \operatorname{aut}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))$.

The main result of this section is

Theorem 14.1. *Suppose:*

(i) *For every natural number d there exists a natural number $l(d)$ such that for any natural number m the additive category $\mathcal{A}_m[\mathbb{Z}^d]$ is $l(d)$-uniformly regular coherent;*

(ii) *The inclusion $\mathcal{A}_{m+1}[\mathbb{Z}^d] \to \mathcal{A}_m[\mathbb{Z}^d]$ is exact for any natural numbers d and m.*

Then the map induced by the projection $E\mathbb{Z}^r \to \mathrm{pt}$

$$H_n^{\mathbb{Z}^r}(E\mathbb{Z}^r; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) \to H_n^{\mathbb{Z}^r}(\mathrm{pt}; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) = K_n(\mathcal{L}(\mathcal{A}_*) \rtimes_{\phi} \mathbb{Z}^r)$$

is bijective for all $n \in \mathbb{Z}$.

14.B. **Reduction to the case $r = 1$.**

Lemma 14.2. *If Theorem 14.1 holds for $r = 1$, it is true for all $r \geq 1$.*

Proof. The Farrell-Jones Conjecture is known to be true for $\mathbb{Z}^r$ and implies that the map induced by the projection $E_{\mathcal{V}\mathrm{cyc}}(\mathbb{Z}^r) \to \mathrm{pt}$

$$H_n^{\mathbb{Z}^r}(E_{\mathcal{V}\mathrm{cyc}}(\mathbb{Z}^r); \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) \to H_n^{\mathbb{Z}^r}(\mathrm{pt}; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)})$$

is an isomorphism for all $n \in \mathbb{Z}$, where $E_{\mathcal{V}\mathrm{cyc}}(\mathbb{Z}^r)$ is the classifying space of the family $\mathcal{V}\mathrm{cyc}$ of virtually cyclic subgroups of G, see for instance [10]. We also have the map induced by the up to $\mathbb{Z}^r$-homotopy unique $\mathbb{Z}^r$-map $E\mathbb{Z}^r \to E_{\mathcal{V}\mathrm{cyc}}(\mathbb{Z}^r)$.

$$H_n^{\mathbb{Z}^r}(E\mathbb{Z}^r; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) \to H_n^{\mathbb{Z}^r}(E_{\mathcal{V}\mathrm{cyc}}(\mathbb{Z}^r); \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}). \tag{14.3}$$

Obviously it suffices to show that (14.3) is bijective for all $n \in \mathbb{Z}$. By the Transitivity Principle, see for instance [12, Theorem 65 on page 742], this boils down to show that for any non-trivial virtually cyclic subgroup V of $\mathbb{Z}^m$ the map induced by the projection $EV \to \mathrm{pt}$

$$H_n^V(EV; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}|_V) \to H_n^V(\mathrm{pt}; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}|_V)$$

is bijective for all $n \in \mathbb{Z}$. Since any non-trivial virtually cyclic subgroup V of $\mathbb{Z}^r$ is isomorphic $\mathbb{Z}$, we have reduced the proof of Theorem 14.1 to the special case $r = 1$. ☐

14.C. **Strategy of proof of Theorem 14.1.** We first explain, why the proof of Theorem 14.1 is more difficult than we had expected and the reader may anticipate. There are are the following reasons.

- There is no proof of the Devissage Theorem for non-connective K-theory.

  We have used the Devissage Theorem for connective K-theory in the proof of Lemma 7.3. If Devissage would hold also in the non-connective setting, the proof of Lemma 7.3 could be extended to the non-connective K-theory spectrum and hence Lemma 7.7 and Theorem 7.8 would still be true, if we replace the assumption that $\mathcal{A}[\mathbb{Z}^m]$ is regular coherent for every $m \geq 0$ by the weaker assumption that that $\mathcal{A}$ is regular coherent.

  Then the stronger version of Theorem 14.1, where one demands the conditions only for $d = 0$, would follow from the version of Theorem 7.8 mentioned above, and Lemma 13.10, and Lemma 14.2.

  Since the known proofs of the Devissage Theorem for non-connective K-theory make strong assumptions on the underlying categories, which are not at all true in our case, we cannot argue like this. The consequence is that we need to make regularity assumptions about $\mathcal{A}_m[\mathbb{Z}^d]$ for all $d \geq 0$ and not only for $d = 0$.
- If the additive category $\mathcal{A}$ is regular coherent, it is unknown whether $\mathcal{A}[\mathbb{Z}]$ is regular coherent.

  It is an open well-known problem, whether for a regular coherent ring R the ring $R[\mathbb{Z}^d]$ is again regular coherent. Hence we do not know, whether for a regular coherent additive category $\mathcal{A}$, the additive category $\mathcal{A}[\mathbb{Z}^d]$ is regular coherent again.
- The additive category $\mathcal{L}(\mathcal{A}_*)$ is not Noetherian and hence not regular.

  We have shown that for a regular additive category $\mathcal{A}$, the additive category $\mathcal{A}[\mathbb{Z}^d]$ is regular again, see Theorem 10.1. However, the additive category $\mathcal{L}(\mathcal{A}_*)$ is not Noetherian and hence not regular, see Example 13.15.

  Comment 1 (by A.): Den nächsten bullet-point habe ich umgeschrieben.
- The canonical inclusion $\mathcal{L}(\mathcal{A}_*)[\mathbb{Z}^d] \to \mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ is not an equivalence of additive categories for $d \geq 1$[3].

  The assumptions on the $\mathcal{A}_m$ imply that $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ is regular coherent, but give us no information $\mathcal{L}(\mathcal{A}_*)[\mathbb{Z}^d]$. For this reason we cannot use the fact (coming from the Bass-Heller-Swan decomposition) that $K_n(\mathcal{L}(\mathcal{A}_*))$ is a direct summand of $K_{n+d}(\mathcal{L}(\mathcal{A}_*)[\mathbb{Z}^d])$ to deduce Theorem 14.1 for all n from the case $n >> 0$. On the other hand for $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ we do not have a Bass-Heller-Swan decomposition[4]. But we will exhibit $K_n(\mathcal{L}(\mathcal{A}_*))$ also as a direct summand of $K_{n+d}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))$ and this will be the main work in the remainder of this paper. To this end we will check that enough of the arguments used in [14] to construct the Bass-Heller-Swan decomposition can also be applied to $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ and yields the desired direct summand.

[3] The inclusion is bijective on objects, but not on morphisms.

[4] It seems not obvious how a Bass-Heller-Swan decomposition for $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ should look like and what the Nil terms should be.

Comment 2 (by A.): Den folgenden Absatz "Our strategy ... yields Theorem 14.41." würde ich nun weglassen, da es inhaltlich schon im letzten bullet-point der obigen Liste abgehandelt wird.

Our strategy is still to reduce the non-connective case to the connective case by writing the desired isomorphism for the $(n-d)$th K-groups for $\mathcal{A}_*$ as a retract of the desired isomorphism for the nth K-groups for $\mathcal{A}_*[\mathbb{Z}^d]$ for $n \geq 2$ and any $d \geq 0$. We do not need and are not seeking to prove in this setting a Bass-Heller Swan decomposition. Actually, it is unlikely that it holds in this setting, where we want to express the K-theory of $\mathcal{L}(\mathcal{A}_*[t,t^{-1}])_{\mathcal{S}(\Phi[t,t^{-1}])}[\mathbb{Z}]$ in terms of the one of $\mathcal{L}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}]$. Our construction of the retraction above is motivated, but more complicated than the one appearing in [14] and finally yields Theorem 14.41.

14.D. **Twisted Laurent categories.** Recall the notion of a (twisted) finite Laurent category from Definition 2.3. Given a sequences of additive subcategories $\mathcal{A}_*$, define the new sequences of additive subcategories $\mathcal{A}_*[t^{\pm}]$ and $\mathcal{A}_*[t,t^{\pm}]$ by replacing $\mathcal{A}_m$ by the associated untwisted finite Laurent categories $\mathcal{A}_m[t^{\pm}]$ and $\mathcal{A}_m[t,t^{-1}]$. A pro-automorphism $\Phi\colon \mathcal{A}_* \to \mathcal{A}_*$ defines pro-automorphisms $\Phi[t^{\pm}]\colon \mathcal{A}_*[t^{\pm}] \to \mathcal{A}_*[t^{\pm}]$ and $\Phi[t,t^{-1}]\colon \mathcal{A}_*[t,t^{-1}] \to \mathcal{A}_*[t,t^{-1}]$. Define

$$\begin{aligned} \mathcal{C}(\Phi) &= \mathcal{S}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}];\\ \mathcal{C}[t^{\pm}](\Phi) &= \mathcal{S}(\mathcal{A}_*[t^{\pm}])_{\mathcal{S}(\Phi[t^{\pm}])}[\mathbb{Z}];\\ \mathcal{C}[t,t^{-1}](\Phi) &= \mathcal{S}(\mathcal{A}_*[t,t^{-1}])_{\mathcal{S}(\Phi[t,t^{-1}])}[\mathbb{Z}]. \end{aligned}$$

The passage from Φ to $\mathcal{C}(\Phi)$, $\mathcal{C}[t^{\pm}](\Phi)$, and $\mathcal{C}[t,t^{-1}](\Phi)$ is natural. Often we omit Φ from the notation. The set of objects of the categories $\mathcal{C}(\Phi)$, $\mathcal{C}[t^{\pm}](\Phi)$, and $\mathcal{C}[t,t^{-1}](\Phi)$ can and will be identified with the set of objects of $\mathcal{S}(\mathcal{A}_*)$ and hence are independent of Φ.

Comment 3 (by A.): Den folgenden Absatz "Note that ... with finite products" würde ich ersatzlos streichen. Den Punkt haben wir schon am Ende der Liste in Subsection 14.C gemacht. Note that $\mathcal{C}[t^{\pm}](\Phi)$ does not agree with $\mathcal{C}(\Phi)[t^{\pm}]$ and that $\mathcal{C}[t,t^{-1}](\Phi)$ does not agree with $\mathcal{C}(\Phi)[t,t^{-1}]$ as illustrated by the following example. Suppose that $\mathcal{A}_*$ is constant, i.e. $\mathcal{A}_0 = \mathcal{A}_m$ for all $m \in \mathbb{N}$. Then a pro-automorphism $\Phi\colon \mathcal{A}_* \to \mathcal{A}_*$ is just an automorphism $\Phi\colon \mathcal{A}_0 \xrightarrow{\cong} \mathcal{A}_0$ and

$$\begin{aligned} \mathcal{C}[t,t^{-1}](\Phi) &= \left(\prod_{m\in\mathbb{N}} \mathcal{A}_0[t,t^{-1}]\right)_{\prod_{m\in\mathbb{N}}\Phi[t,t^{-1}]}[\mathbb{Z}];\\ \mathcal{C}(\Phi)[t,t^{-1}] &= \left(\left(\prod_{m\in\mathbb{N}} \mathcal{A}_0\right)_{\prod_{m\in\mathbb{N}}\Phi}[\mathbb{Z}]\right)[t,t^{-1}]. \end{aligned}$$

These two categories have the same set of objects. The category $\mathcal{C}(\Phi)[t,t^{-1}]$ can be viewed as a subcategory of $\mathcal{C}[t,t^{-1}](\Phi)$ with the same set of objects, but has fewer morphisms essentially, since the passage $\mathcal{A}_0 \to \mathcal{A}_0[t,t^{-1}]$ does not commute with infinite products.

14.E. **Induction functors.** Next we define a commutative square of additive categories

(14.4)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i_+} & \mathcal{C}[t] \\ {\scriptstyle i_-}\downarrow & \searrow^{i_0} & \downarrow{\scriptstyle j_+} \\ \mathcal{C}[t^{-1}] & \xrightarrow[j_-]{} & \mathcal{C}[t,t^{-1}] \end{array}$$

where the functors i_0, i_+, i_-, j_-, and j_+ induce the identity on the set of objects. For each additive category $\mathcal{A}_m$.we have a diagram of functors of untwisted Laurent categories, where all functors are the canoncial ones, see [14, Section 1.4],

$$\begin{array}{ccc} \mathcal{A}_m & \xrightarrow{i_+(\mathcal{A}_m)} & \mathcal{A}_m[t] \\ {\scriptstyle i_-(\mathcal{A}_m)}\downarrow & \searrow^{i_0(\mathcal{A}_m)} & \downarrow{\scriptstyle j_+(\mathcal{A}_m)} \\ \mathcal{A}_m[t^{-1}] & \xrightarrow[j_-(\mathcal{A}_m)]{} & \mathcal{A}_m[t,t^{-1}]. \end{array}$$

This construction is natural with respect to the inclusions $\mathcal{A}_{m+1} \to \mathcal{A}_m$. Hence we obtain a commutative diagram of additive categories

$$\begin{array}{ccc} \mathcal{S}(\mathcal{A}_*) & \xrightarrow{\mathcal{S}(i_+(\mathcal{A}_*))} & \mathcal{S}(\mathcal{A}_*[t]) \\ {\scriptstyle \mathcal{S}(i_-(\mathcal{A}_*))}\downarrow & \searrow^{\mathcal{S}(i_0(\mathcal{A}_*))} & \downarrow{\scriptstyle \mathcal{S}(j_+(\mathcal{A}_*))} \\ \mathcal{S}(\mathcal{A}_*)[t^{-1}] & \xrightarrow[\mathcal{S}(j_-(\mathcal{A}_*))]{} & \mathcal{S}(\mathcal{A}_*[t,t^{-1}]). \end{array}$$

Since it is compatible with the automorphisms of additive categories $\mathcal{S}(\Phi)$ of $\mathcal{S}(\mathcal{A}_*)$, $\mathcal{S}(\Phi[t^{\pm}])$ of $\mathcal{S}(\mathcal{A}_*[t^{\pm}])$, and $\mathcal{S}(\Phi[t,t^{-1}])$ of $\mathcal{S}(\mathcal{A}_*[t,t^{-1}])$, it yield the desired diagram (14.4) in the obvious way.

Note that the diagram diagram (14.4) is natural in $\Phi\colon \mathcal{A}_* \to \mathcal{A}_*$.

14.F. **Formally adjoining infinite direct sums.** In [14, Section 1.3] a functorial extension $\mathcal{A} \subseteq \mathcal{A}^{\kappa}$ is constructed for every additive category $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{A}^{\kappa}$ is an inclusion of additive categories and in $\mathcal{A}^{\kappa}$ the direct sums over a collection of objects over a countable set is defined. Now define for a nested sequence of additive categories a functor of additive categories $\mathcal{A}_*$ given by $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots$ the nested sequence of additive categories $\mathcal{A}_*^{\kappa}$ by $\mathcal{A}_0^{\kappa} \supseteq \mathcal{A}_1^{\kappa} \supseteq \mathcal{A}_2^{\kappa} \supseteq \cdots$. The given pro-automorphism $\phi\colon \mathcal{A}_* \to \mathcal{A}_*$ yields a pro-automorphism $\phi^{\kappa}\colon \mathcal{A}_*^{\kappa} \to \mathcal{A}_*^{\kappa}$ by passing to $\phi^{\kappa}\colon \mathcal{A}_0^{\kappa} \to \mathcal{A}_0^{\kappa}$. Define

$$\begin{array}{rcl} \mathcal{C}^{\kappa} & = & \mathcal{S}(\mathcal{A}_*^{\kappa})_{\mathcal{S}(\Phi^{\kappa})}[\mathbb{Z}]; \\ \mathcal{C}[t^{\pm}]^{\kappa} & = & \mathcal{S}(\mathcal{A}_*[t^{\pm}]^{\kappa})_{\mathcal{S}(\Phi[t^{\pm}]^{\kappa})}[\mathbb{Z}]; \\ \mathcal{C}[t,t^{-1}]^{\kappa} & = & \mathcal{S}(\mathcal{A}_*[t,t^{-1}]^{\kappa})_{\mathcal{S}(\Phi^{\kappa}[t,t^{-1}]^{\kappa})}[\mathbb{Z}]. \end{array}$$

Note that we are working with $\mathcal{A}_m[t^{\pm}]^{\kappa}$, which is different from $(\mathcal{A}_m^{\kappa})[t^{\pm}]$. The same comment applies to $\mathcal{A}_m[t,t^{-1}]^{\kappa}$, which is different from $(\mathcal{A}_m^{\kappa})[t,t^{-1}]$.

The diagram (14.4) extends in the obvious way to the diagram, natural in Φ.

(14.5)
$$\begin{array}{ccc} \mathcal{C}^{\kappa} & \xrightarrow{i_+^{\kappa}} & \mathcal{C}[t]^{\kappa} \\ {\scriptstyle i_-^{\kappa}}\downarrow & \searrow^{i_0^{\kappa}} & \downarrow{\scriptstyle j_+^{\kappa}} \\ \mathcal{C}[t^{-1}]^{\kappa} & \xrightarrow[j_-^{\kappa}]{} & \mathcal{C}[t,t^{-1}]^{\kappa}. \end{array}$$

14.G. **Restriction functors.** In [14, Section 1.5] restriction functors are defined for each additive category $\mathcal{A}_m$

$$\begin{aligned} i^0(\mathcal{A}_m)\colon \mathcal{A}_m[t,t^{-1}]^\kappa &\to \mathcal{A}_m^\kappa; \\ i^\pm(\mathcal{A}_m)\colon \mathcal{A}_m[t^{\pm 1}]^\kappa &\to \mathcal{A}_m^\kappa, \end{aligned}$$

such that they come with adjunctions $(i_0(\mathcal{A}_m)^\kappa, i^0(\mathcal{A}_m))$, $(i_+(\mathcal{A}_m)^\kappa, i^+(\mathcal{A}_m))$, and $(i_-(\mathcal{A}_m)^\kappa, i^-(\mathcal{A}_m))$. Moreover, everything is natural and in particular compatible with the automorphisms $\Phi[t,t^{-1}]^\kappa\colon \mathcal{A}_0[t,t^{-1}]^\kappa \to \mathcal{A}_0[t,t^{-1}]^\kappa$ and $\Phi[t^\pm]^\kappa\colon \mathcal{A}_0[t^\pm]^\kappa \to \mathcal{A}_0[t^\pm]^\kappa$. Hence these data define analogously to the construction of Subsection 14.E restriction functors

$$i^0\colon \mathcal{C}[t,t^{-1}]^\kappa \to \mathcal{C}^\kappa; \tag{14.6}$$

$$i^\pm\colon \mathcal{C}[t^{\pm 1}]^\kappa \to \mathcal{C}^\kappa, \tag{14.7}$$

such that we have adjunctions $((i_0)^\kappa, i^0)$, $((i_+)^\kappa, i^+)$, and $((i_-)^\kappa, i^-)$. Everything is natural in Φ.

14.H. **Truncation functors.** Put

$$\overline{\mathbb{Z}} := \mathbb{Z} \amalg \{-\infty, \infty\}.$$

Notation 14.8 (Truncation for objects)**.** Let $\underline{A} = (A_m)_{m\in\mathbb{N}}$ and $\underline{A'} = (A'_m)_{m\in\mathbb{N}}$ be objects in $\mathcal{S}(\mathcal{A}_*)$. Consider elements $\underline{a} = (a_m)_{m\in\mathbb{N}}$ and $\underline{b} = (b_m)_{m\in\mathbb{N}}$ in $\prod_{m\in\mathbb{N}} \overline{\mathbb{Z}}$. Define an object in $\mathcal{C}^\kappa$ by

$$\underline{A}[\underline{a},\underline{b}] = \left\{ \bigoplus_{k_i=a_i}^{b_i} A_m \;\middle|\; m \in \mathbb{N} \right\},$$

where $\bigoplus_{k_i=a_i}^{b_i} A_m$ is to be defined to be zero if $a_i > b_i$ or if we have $a_i = b_i$ and $b_i \in \{\pm\infty\}$. Note that the direct sum $\bigoplus_{k_i=a_i}^{b_i} A_m$ has as entry for k_i always the same object, namely A_m. Since we are working in $\mathcal{C}^\kappa$, this definition makes sense also in the case where $a_i = -\infty$ or $b_i = \infty$.

Given an element $c \in \overline{\mathbb{Z}}$, denote by $\underline{c}$ the element in $\prod_{m\in\mathbb{N}} \overline{\mathbb{Z}}$, whose value at every $m \in \mathbb{N}$ is c. Then we get for any object $\underline{A}$ in $\mathcal{C}[t,t^{-1}]$, which is the same as an object in $\mathcal{S}(\mathcal{A}_*)$,

$$i^0\underline{A} = \underline{A}[\underline{-\infty},\underline{\infty}].$$

Given a morphism $\underline{f}\colon \underline{A} \to \underline{A'}$ in $\mathcal{C}[t,t^{-1}]$ and $\underline{a}, \underline{b}, \underline{a'}$, and $\underline{b'}$ in $\prod_{m\in\mathbb{N}} \overline{\mathbb{Z}}$, define the $\mathcal{C}^\kappa$-morphism $f[\,]\colon \underline{A}[\underline{a},\underline{b}] \to \underline{A'}[\underline{a'},\underline{b'}]$ to be the composite

$$\underline{f}[\,]\colon \underline{A}[\underline{a},\underline{b}] \xrightarrow{\underline{i}} \underline{A}[\underline{-\infty},\underline{\infty}] = i^0\underline{A} \xrightarrow{i^0 \underline{f}} \underline{A'}[\underline{-\infty},\underline{\infty}] = i^0\underline{A'} \xrightarrow{\underline{p}} \underline{A'}[\underline{a'},\underline{b'}],$$

where $\underline{i}$ is the obvious inclusion and $\underline{p}$ is the obvious projection in $\mathcal{C}^\kappa$.

The morphism $f[\,]\colon i^0\underline{A} = A[\underline{-\infty},\underline{\infty}] \to i^0\underline{A'} = A'[\underline{-\infty},\underline{\infty}]$ agrees with $i^0 f$ for a morphism $\underline{f}\colon \underline{A} \to \underline{A'}$ in $\mathcal{C}[t,t^{-1}]$. If f belongs to $\mathcal{C}[t^\pm]$, we abbreviate $(i_\pm f)[\,]$ by $f[\,]$ again.

Note that $(\underline{g} \circ \underline{f})[\,]$ is in general *not* equal to $\underline{g}[\,] \circ \underline{f}[\,]$ and $\mathrm{id}[\,]$ is in general *not* the identity. As a typical example, let $\underline{f^+}\colon \underline{A} \to \underline{A}$ be the morphism $\mathrm{id}_{\underline{A}} \cdot t$ and $\underline{f^-}\colon \underline{A} \to \underline{A}$ be the morphism $\mathrm{id}_{\underline{A}} \cdot t^{-1}$.Then

$$(f^+ \circ f^-)[\,]\colon A[0,\underline{\infty}] \to A[0,\underline{\infty}]$$

is the identity. The map

$$f^-[\,]\colon \underline{A}[\underline{0},\underline{\infty}] \to \underline{A}[\underline{0},\underline{\infty}]$$

is given at each $m \in \mathbb{N}$ by the map $\bigoplus_{k=0}^{\infty} A_m \to \bigoplus_{k=0}^{\infty} A_m$ sending $\{u_k \mid k = 0, 1, 2, \ldots\}$ to $\{u_{k+1} \mid k = 0, 1, 2, \ldots\}$. Since $f^-[\,]$ is not injective, we do *not* have $f^+[\,] \circ f^-[\,] = (f^+ \circ f^-)[\,]$.

As another example,

$$\mathrm{id}_{\underline{A}}[\,] \colon \underline{A}[\underline{-\infty}, \underline{\infty}] \to A[\underline{0}, \underline{0}]$$

is given at every $m \in \mathbb{N}$ by the projection onto the 0th summand $\bigoplus_{k=-\infty}^{\infty} A_m \to A_m$.

Notation 14.9 (Truncation for chain complexes)**.** If C^+ is a $\mathcal{C}[t]$-chain complex and $\underline{a}, \underline{b} \in \prod_{m \in \mathbb{N}} \overline{\mathbb{Z}}$, then we obtain a $\mathcal{C}^\kappa$-chain complex $C^+[\underline{a}, \underline{b}]$ by defining the nth chain object to be $C_n^+[\underline{a}, \underline{b}]$ and the n-th differential to be $c_n[\,] \colon C_n^+[\underline{a}, \underline{b}] \to C_{n-1}^+[\underline{a}, \underline{b}]$, if c_n is the differential of C^+. (One has to check that $c_n[\,] \circ c_{n+1}[\,] = 0$.) A chain map $\underline{f} \colon C^+ \to D^+$ of $\mathcal{C}[t]$-chain complexes induces a $\mathcal{C}^\kappa$-chain map denoted by $f[\,] \colon C^+[\underline{a}, \underline{b}] \to D^+[\underline{a'}, \underline{b'}]$, provided that $\underline{a'} \leq \underline{a}$, i.e., $a'_m \leq a_m$ for all $m \in \mathbb{N}$, and $\underline{b'} \leq \underline{b}$ hold.

If C^- is an $\mathcal{C}[t^{-1}]$-chain complex and $\underline{a}, \underline{b} \in \prod_{n \in \mathbb{N}} \overline{\mathbb{Z}}$, define the $\mathcal{C}^\kappa$-chain complex $C^-[\underline{a}, \underline{b}]$ analogously. A chain map $f \colon C^- \to D^-$ of $\mathcal{A}[t^{-1}]$-chain complexes induces a $\mathcal{C}^\kappa$-chain map denoted by $f[\,] \colon C^-[\underline{a}, \underline{b}] \to D^-[\underline{a'}, \underline{b'}]$, provided that $\underline{a'} \geq \underline{a}$, i.e., $a'_m \geq a_m$ for all $m \in \mathbb{N}$, and $\underline{b'} \geq \underline{b}$ hold.

Note that Notation 14.9 (in contrast to Notation 14.8) does in this generality *not* make sense for chain complexes in $\mathcal{C}[t, t^{-1}]$, e.g., $c_n[\,] \circ c_{n+1}[\,] = 0$ does not hold anymore.

14.I. **Some basic tools for non-connective K-theory.** Recall that we have defined the negative K-theory of an additive category using the delooping construction based on the Bass-Heller-Swan decomposition of [13]. In this section we present another definition based on the non-connective K-theory spectrum associated to appropriate Waldhausen categories due to Bunke-Kasprowski-Winges [2].

The next definition is taken from [2, Definition 2.1].

Definition 14.10.

(i) The Waldhausen category $\mathcal{W}$ admits *factorizations*, if every morphism in $\mathcal{W}$ can be factorized into a cofibration followed by a weak equivalence; no functoriality of this factorization is assumed;

(ii) The Waldhausen category $\mathcal{W}$ is *homotopical*, if it admits factorizations and the weak equivalences satisfy the *two-out-of-six property*, i.e., if for composable morphisms $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3$ in $\mathcal{W}$ both $f_2 \circ f_1$ and $f_3 \circ f_2$ are weak equivalences, then also f_1, f_2, f_3, and $f_3 \circ f_2 \circ f_1$ are weak equivalences.

Let $\mathsf{Wald}^{\mathrm{ho}}$ be the category of homotopical Waldhausen categories. In the sequel we denote by

$$\mathbf{K}^{\infty, \mathrm{W}} \colon \mathsf{Wald}^{\mathrm{ho}} \to \mathsf{Spectra} \tag{14.11}$$

the non-connective K-theory functor constructed in [2, Definition 2.37].

Remark 14.12. Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ becomes a Waldhausen category, if we define the weak equivalences to be the isomorphisms and the cofibration to be the morphisms $f \colon A \to B$, for which there exists an object $A^\perp$ and an isomorphism $u \colon A \oplus A^\perp \xrightarrow{\cong} B$ such that the composite of u with the canonical inclusion $A \to A \oplus A^\perp$ is f. Note that this Waldhausen category is *not* homotopical, as it does not satisfy factorization. So we cannot apply (14.11) to the Waldhausen category $\mathcal{A}$.

Let $\mathsf{Ch}(\mathcal{A})$ be the Waldhausen category of bounded chain complexes over $\mathcal{A}$, where a cofibration $f_*\colon C_* \to D_*$ is a chain map such that $f_n\colon C_n \to D_n$ is a cofibration in $\mathcal{A}$ and the weak equivalences are the chain homotopy equivalences. Then $\mathsf{Ch}(\mathcal{A})$ is homotopical thanks to the mapping cylinder construction. Hence we can apply (14.11) to the Waldhausen category $\mathsf{Ch}(\mathcal{A})$ and can consider its non-connective K-theory spectrum $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{A}))$.

More generally, if $\mathcal{A}$ is an exact category, then the Waldhausen category $\mathsf{Ch}(\mathcal{A})$ can be defined analogously and is homotopical.

Suppose that $\mathcal{W}$ is a category with cofibrations and that $\mathcal{W}$ is equipped with two categories of weak equivalences, one finer than the other, $v\mathcal{W} \subseteq w\mathcal{W}$. Thus $\mathcal{W}$ becomes a Waldhausen category in two ways. Suppose that in both cases $\mathcal{W}$ is a homotopical Walhausen category. Let $\mathcal{W}^w$ denote the full subcategory of $\mathcal{W}$ given by the objects C in $\mathcal{W}$ having the property that the map $C \to \mathrm{pt}$ belongs to $w\mathcal{W}$. Then $\mathcal{W}^w$ inherits two Waldhausen structures, if we put $v\mathcal{W}^w = \mathcal{W}^w \cap v\mathcal{W}$ and $w\mathcal{W}^w = \mathcal{W}^w \cap w\mathcal{W}$. Both yield homotopical Waldhausen categories.

Theorem 14.13 (Fibration Theorem)**.** *Under the assumptions above we get a weak homotopy fibration of spectra*

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}^w, v\mathcal{W}^w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}, v\mathcal{W}) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}, w\mathcal{W}).$$

Proof. This follows from [2, Theorem 2.35]. □

Theorem 14.14 (Cisinski's Approximation Theorem)**.** *Let $F\colon \mathcal{W}_0 \to \mathcal{W}_1$ be an exact functor of homotopical Waldhausen categories. Assume:*

(i) An arrow in $\mathcal{W}_0$ is a weak equivalence in $\mathcal{W}_0$, if and only if its image in $\mathcal{W}_1$ is a weak equivalence in $\mathcal{W}_1$;

(ii) Given any object C_0 in $\mathcal{W}_0$ and any map $f\colon F(C_0) \to C_1$ in $\mathcal{W}_1$, there exists a commutative diagram in $\mathcal{W}_1$

$$\begin{array}{ccc} F(C_0) & \xrightarrow{f} & C_1 \\ {\scriptstyle F(u)}\downarrow & & \downarrow{\scriptstyle \simeq\, v} \\ F(D_0) & \xrightarrow[w]{\simeq} & D_1 \end{array}$$

for a morphism $u\colon C_0 \to D_0$ in $\mathcal{W}_0$ and weak equivalences $v\colon C_1 \to D_1$ and $w\colon F(D_0) \to D_1$ in $\mathcal{W}_1$.

Then the map of spectra $\mathbf{K}^{\infty,\mathrm{W}}(F)\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}_0) \xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}_1)$ is a weak homotopy equivalence.

Proof. This follows from [2, Theorem 2.16]. □

Theorem 14.15 (Cofinality Theorem)**.** *Let $I\colon \mathcal{W}_0 \to \mathcal{W}_1$ be the inclusion of a full homotopical Waldhausen subcategory $\mathcal{W}_0$ into a homotopical Waldhausen category $\mathcal{W}_1$. Assume:*

(i) The functor F admits a mapping cylinder argument, i.e., for every morphism $f\colon C_0 \to C_1$ in $\mathcal{W}_1$ such that C_0 belongs to $\mathcal{W}_0$ and C_1 is the target of a weak equivalence with some object in $\mathcal{W}_0$ as source, there is a factorization in $\mathcal{W}_1$

$$C_0 \xrightarrow{f'} C' \xrightarrow{f''} C_1$$

such that C' belongs to $\mathcal{W}_0$ and f'' is a weak equivalence;

(ii) The category $\mathcal{W}_1$ is dominated by $\mathcal{W}_0$, i.e., for any object C_1 in $\mathcal{W}_1$ there exists an object C_0 in $\mathcal{W}_0$ and an object C_1' in $\mathcal{W}_1$ and morphisms $r\colon C_0 \to C_1$ and $i\colon C_1' \to C_0$ such that $r \circ i$ is a weak equivalence.

Then $\mathbf{K}^{\infty,\mathrm{W}}(I)\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}_0) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{W}_1)$ *is a weak homotopy equivalence.*

Proof. This follows from [2, Theorem 2.30] and the fact that on the level of stable ∞-categories non-connective K-theory is inverting the passage to the idempotent completion. □

For the proof of the following result we refer to [11, Theorem 18.17].

Theorem 14.16 (The weak homotopy fibration sequence of a stable Karoubi filtration for K-theory in the setting of Waldhausen categories)**.** *Let $\mathcal{A}$ be a additive category and $i\colon \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory. If the additive category $\mathcal{A}$ is stably $\mathcal{U}$-filtered, then*

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{U})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{A}/\mathcal{U}))$$

is a weak homotopy fibration of non-connective spectra, where $\mathbf{K}^{\infty,\mathrm{W}}$ has been defined in (14.11)*,*

The proof of the next result can be found in [11, Theorem 18.33].

Theorem 14.17 (Gillet-Waldhausen zigzag for non-connective K-theory)**.** *There is a zigzag of weak homotopy equivalences, natural in $\mathcal{A}$, from the non-connective K-theory spectrum $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{A}))$ in the sense of Bunke-Kasprowski-Winges* [2] *to the non-connective K-theory spectrum $\mathbf{K}^{\infty}(\mathcal{A})$ in the sense of Lück-Steimle* [13].

14.J. The projective line.

Definition 14.18 (Projective line)**.** We define *the projective line $\mathcal{X}$* to be the following additive category. Objects are triples (C^+, f, C^-) consisting of objects C^+ in $\mathcal{C}[t^+]$ and C^- in $\mathcal{C}[t^-]$ and an isomorphism $f\colon j_+(C^+) \to j_-(C^-)$ in $\mathcal{C}[t,t^{-1}]$. A morphism $(\varphi^+,\varphi^-)\colon (C^+,f,C^-) \to (D^+,g,D^-)$ in $\mathcal{X}$ consists of morphisms $\varphi^+\colon C^+ \to D^+$ in $\mathcal{C}[t]$ and $\varphi^-\colon C^- \to D^-$ in $\mathcal{C}[t^{-1}]$ such that the following diagram commutes in $\mathcal{C}[t,t^{-1}]$

$$\begin{array}{ccc} j_+(C^+) & \xrightarrow{f} & j_-(C^-) \\ \downarrow{\scriptstyle j_+(\varphi^+)} & & \downarrow{\scriptstyle j_-(\varphi^-)} \\ j_+(D^+) & \xrightarrow[g]{} & j_-(D^-). \end{array}$$

Let

$$k^{\pm}\colon \mathcal{X} \quad \to \quad \mathcal{C}[t^{\pm 1}] \tag{14.19}$$

be the functor sending (C^+,f,C^-) to $C^{\pm}$.

The category $\mathcal{X}$ is naturally an exact category by declaring a sequence to be exact, if and only if becomes (split) exact both after applying k^+ and k^-. With this exact structure we obtain the structure of a homotopical Waldhausen category on $\mathsf{Ch}(\mathcal{X})$. Hence $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X}))$ is defined, in contrast to $\mathbf{K}^{\infty}(\mathcal{X})$ which does not make sense, since $\mathcal{X}$ is neither an additive category nor a homotopical Waldhausen category.

The next theorem will be a main ingredient in the proof of Theorem 14.31.

Theorem 14.20 (The algebraic K-theory of the projective line)**.** *Consider the following (not necessarily commutative) diagram of spectra*

$$\begin{array}{ccc} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) & \xrightarrow{\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])) \\ \downarrow{\scriptstyle \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^+))} & & \downarrow{\scriptstyle \mathbf{K}^{\infty,\mathrm{W}}\mathsf{Ch}((j_-))} \\ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) & \xrightarrow[\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j_+))]{} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}])). \end{array}$$

There is a natural equivalence of functors $T\colon j_+ \circ k^+ \xrightarrow{\cong} j_- \circ k^-$, *which is given on an object* (A^+, f, A^-) *by* f. *It induces a preferred homotopy* $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j_+)) \circ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^+)) \simeq \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j_-)) \circ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))$.

Then the diagram above is a weak homotopy pullback, i.e., the canonical map from $\mathbf{K}^{\infty,\mathrm{W}}(\mathcal{X})$ *to the homotopy pullback of*

$$\begin{array}{ccc} & & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])) \\ & & \downarrow{\scriptstyle \mathbf{K}^{\infty,\mathrm{W}}\mathsf{Ch}((j_-))} \\ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) & \xrightarrow[\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j_+))]{} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}])). \end{array}$$

is a weak homotopy equivalence.

For the proof of Theorem 14.20, we can use the ideas of [14, Section 5] taking into account that meanwhile the basic tools appearing in [14, Section 4] have been proved also for non-connective K-theory.

In the first step of the proof of Theorem 14.20 we replace the additive category $\mathcal{C}[t]$ by a larger exact category $\mathcal{Y}$ with equivalent K-theory. It is defined as follows: An object of $\mathcal{Y}$ is a triple (A^+, f, A) consisting of objects A^+ in $\mathcal{C}[t]$ and A^- of $\mathcal{C}[t,t^{-1}]$ and an isomorphism $f\colon j_+A^+ \to A$ in $\mathcal{C}[t,t^{-1}]$. A morphism from (A^+, f, A) to (B^+, g, B) is a morphism $\varphi^+\colon A^+ \to B^+$ in $\mathcal{C}[t]$ and a morphism $\varphi\colon A \to B$ in $\mathcal{C}[t,t^{-1}]$ such that diagram in $\mathcal{C}[t,t^{-1}]$

$$\begin{array}{ccc} j_+(A^+) & \xrightarrow{f} & A \\ {\scriptstyle j_+(\varphi^+)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ j_+(B^+) & \xrightarrow[g]{} & B \end{array}$$

commutes. Note that φ is determined already by φ^+. The category $\mathcal{Y}$ is exact in the same way as $\mathcal{X}$ is. Note that $\mathcal{X}$ and $\mathcal{Y}$ have the same set of objects but $\mathcal{Y}$ contains more morphisms, since in contrast to $\mathcal{X}$ we do not require that φ belongs to $\mathcal{C}[t^{-1}]$.

Lemma 14.21. *The functors*

$$a\colon \mathcal{C}[t] \to \mathcal{Y}, \quad A \mapsto (A, \mathrm{id}, j_-A)$$

and

$$b\colon \mathcal{Y} \to \mathcal{C}[t], \quad (A^+, f, A^-) \mapsto A^+$$

are exact. The composite $b \circ a$ *is the identity and the composite* $a \circ b$ *is naturally isomorphic to the identity functor. In particular, they induce homotopy equivalences on non-connective* K*-theory, homotopy inverse to each other,*

$$\begin{aligned} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(a))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) &\xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y})); \\ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(b))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y})) &\xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])). \end{aligned}$$

Proof. It is clear that the functors are exact. Obviously $b \circ a$ is the identity. The composite $a \circ b$ is naturally isomorphic to the identity functor: the isomorphism in $\mathcal{Y}$ at the object (A^+, f, A) is given by $(\mathrm{id}, f)\colon (A^+, \mathrm{id}, j_+A^+) \xrightarrow{\cong} (A^+, f, A)$. This implies $\mathbf{K}(a) \circ \mathbf{K}(b) \simeq \mathrm{id}$. ☐

Denote by

$$k'\colon \mathcal{X} \to \mathcal{Y}$$

the inclusion functor, and define

$$j'\colon \mathsf{Ch}(\mathcal{Y}) \to \mathsf{Ch}(\mathcal{C}[t,t^{-1}]), \quad (A^+, f, A) \mapsto A.$$

Then the square

$$\begin{array}{ccc} \mathsf{Ch}(\mathcal{X}) & \xrightarrow{\mathsf{Ch}(k^-)} & \mathsf{Ch}(\mathcal{C}[t^{-1}]) \\ {\scriptstyle \mathsf{Ch}(k')}\downarrow & & \downarrow{\scriptstyle \mathsf{Ch}(j^-)} \\ \mathsf{Ch}(\mathcal{Y}) & \xrightarrow[\mathsf{Ch}(j')]{} & \mathsf{Ch}(\mathcal{C}[t,t^{-1}]) \end{array}$$

is strictly commutative, and we are going to show that it induces a weak homotopy pullback after applying $\mathbf{K}^{\infty,\mathrm{W}}$. To show that the square is a weak homotopy pullback on non-connective K-theory, we are going to show that the horizontal homotopy fibers of $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))$ and $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j'))$ agree up to weak homotopy equivalence.

Let $w\mathsf{Ch}(\mathcal{X})$ be the subcategory of $\mathsf{Ch}(\mathcal{X})$ consisting of all chain maps, which become after applying $\mathsf{Ch}(k^-)$ weak equivalences in $\mathsf{Ch}(\mathcal{C}[t^{-1}])$. Let $\mathsf{Ch}(\mathcal{X})^w$ be the full subcategory of $\mathsf{Ch}(\mathcal{X})$ of all objects, which are w-acyclic. In other words, an object (C^+, f, C^-) belongs to $\mathsf{Ch}(\mathcal{X})^w$, if and only if C^- is contractible as an $\mathcal{C}[t^{-1}]$-chain complex. Similarly, denote by $w\mathsf{Ch}(\mathcal{Y})$ the subcategory of all morphisms f such that $\mathsf{Ch}(j')(f)$ is a chain homotopy equivalence in $\mathsf{Ch}(\mathcal{C}[t,t^{-1}])$, and adopt the notation $\mathsf{Ch}(\mathcal{Y})^w$ for the w-acyclic objects.

Lemma 14.22. *The maps*

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X}), w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}]));$$
$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j'))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y}), w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}])),$$

are homotopy equivalences.

Proof. We want to apply the Cisinki's Approximation Theorem 14.14. We give the details only for $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))$, the analogous proof for $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(j'))$ is left to the reader. We have to verify the conditions (i) and (ii) appearing in Cisinki's Approximation Theorem 14.14.

A morphism f in $\mathsf{Ch}(\mathcal{X})$ is by definition in $w\mathsf{Ch}(\mathcal{X})$, if and only if $\mathsf{Ch}(k^-)(f)$ is a chain homotopy equivalence in $\mathsf{Ch}(\mathcal{C}[t^{-1}])$. This takes care of condition (i) for $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))$.

Next we deal with condition (ii). Consider an object (C^+, f, C^-) in $\mathsf{Ch}(\mathcal{X})$ and a morphism $\varphi^- : C^- \to D^-$ in $\mathsf{Ch}(\mathcal{C}[t^{-1}])$. We will extend φ^- to a morphism

$$\varphi = (\varphi^+, \varphi^-)\colon (C^+, f, C^-) \to (D^+, g, D^-)$$

in $\mathsf{Ch}(\mathcal{X})$. If we have achieved this, we are done by the following argument. Note that $\varphi = (\varphi^+, \varphi^-)$ is a morphism in $\mathsf{Ch}(\mathcal{X})$ projecting to φ^- under $\mathsf{Ch}(k^-)$. Then, factorizing $\varphi = \mu \circ \psi$ into a cofibration ψ followed by a weak equivalence μ (using the mapping cylinder), we can write $\varphi^- = \mu^- \circ \mathsf{Ch}(k^-)(\psi)$, where ψ is a cofibration and μ^- is a weak equivalence, as required in condition (ii).

The construction of D^+ and φ^+ require the following preparation.

Consider an object $\underline{A} \in \mathcal{S}(\mathcal{A}_*)$ and an element $\underline{k} = \{k_m \mid m \in \mathbb{N}\} \in \prod_{m\in\mathbb{N}} \mathbb{Z}$. Define a morphism $t^{\underline{k}}\colon \underline{A} \to \underline{A}$ in $\mathcal{S}(\mathcal{A}_*[t,t^{-1}])$ by $(t^{k_m})_{m\in\mathbb{N}}$, where $t^{k_m} : A_m \to A_m$ is the obvious morphism in $\mathcal{A}_m[t,t^{-1}]$ determined by t^{k_m}. We can and will regard $t^{\underline{k}}\colon \underline{A} \to \underline{A}$ also as a morphism in $\mathcal{C}[t,t^{-1}]$. One easily checks for any morphism $f\colon A \to B$ in $\mathcal{C}[t,t^{-1}]$ that $t^{\underline{k}} \circ f = f \circ t^{\underline{k}}$ holds. Moreover we have $t^{\underline{k}+\underline{k'}} = t^{\underline{k}} \circ t^{\underline{k'}}$ for $\underline{k}+\underline{k'}$ defined by the componentwise addition in $\prod_{m\in\mathbb{N}} \mathbb{Z}$ and $t^{\underline{0}} = \mathrm{id}$ for $\underline{0} \in \prod_{m\in\mathbb{N}} \mathbb{Z}$ given by the element which has zero as entry for the component for $m \in \mathbb{N}$. The following property is important for us. Given a morphism $f\colon A \to B$ in $\mathcal{C}[t,t^{-1}]$, there exists $\underline{k}_f \in \prod_{m\in\mathbb{N}} \mathbb{Z}$ such that for every $\underline{k} \in \prod_{m\in\mathbb{N}} \mathbb{Z}$ with $\underline{k} \geq \underline{k}_f$ we have $t^{\underline{k}} \circ f$ in $\mathcal{C}[t]$, where $\underline{k} \geq \underline{k}_f$ is to be understood componentwise.

Choose a natural number N such that $C_m^+ = D_m^- = 0$ for $|m| > N$. Consider $\underline{k} \in \prod_{m\in\mathbb{N}} \mathbb{Z}$. For any integer n define $n \cdot \underline{k}$ to be the element, whose i-th entry is $n \cdot k_m$. Then we obtain the following commutative diagram in $\mathcal{C}[t, t^{-1}]$

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \xrightarrow{\mathrm{id}} & 0 & \xrightarrow{\mathrm{id}} & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
C_N^+ & \xrightarrow{t^{\underline{k}}\circ\varphi_N^-\circ f_N} & D_N^- & \xrightarrow{t^{-\underline{k}}} & D_N^- \\
\big\downarrow{\scriptstyle c_N^+} & & \big\downarrow{\scriptstyle t^{\underline{k}}\circ d_N^-} & & \big\downarrow{\scriptstyle d_N^-} \\
C_{N-1}^+ & \xrightarrow{t^{2\cdot\underline{k}}\circ\varphi_{N-1}^-\circ f_{N-1}} & D_{N-1}^- & \xrightarrow{t^{-2\cdot\underline{k}}} & D_{N-1}^- \\
\big\downarrow{\scriptstyle c_{N-1}^+} & & \big\downarrow{\scriptstyle t^{\underline{k}}\circ d_{N-1}^-} & & \big\downarrow{\scriptstyle d_{N-1}^-} \\
\vdots & & \vdots & & \vdots \\
\big\downarrow{\scriptstyle c_{-N+1}^+} & & \big\downarrow{\scriptstyle t^{\underline{k}}\circ d_{-N+1}^-} & & \big\downarrow{\scriptstyle d_{-N+1}^-} \\
C_{-N+1}^+ & \xrightarrow{t^{2N\cdot\underline{k}}\circ\varphi_{-N+1}^-\circ f_{-N+1}} & D_{-N+1}^- & \xrightarrow{t^{-2N\cdot\underline{k}}} & D_{-N+1}^- \\
\big\downarrow{\scriptstyle c_{-N+1}^+} & & \big\downarrow{\scriptstyle t^{\underline{k}}\circ d_{-N+1}^-} & & \big\downarrow{\scriptstyle d_{-N+1}^-} \\
C_{-N}^+ & \xrightarrow{t^{(2N+1)\cdot\underline{k}}\circ\varphi_{-N}^-\circ f_{-N}} & D_{-N}^- & \xrightarrow{t^{-(2N+1)\cdot\underline{k}}} & D_{-N}^- \\
\big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} \\
0 & \xrightarrow{\mathrm{id}} & 0 & \xrightarrow{\mathrm{id}} & 0 \\
\big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} \\
\vdots & & \vdots & & \vdots
\end{array}
$$

where the left column is the chain complex C^+, the right column is the chain complex D^- and the middle column is obtained from D^- by composing each differential with $t^{\underline{k}}$. Since there are only finitely many objects and morphisms in the diagram above, which are different from zero, we can choose $\underline{k}$ such that each horizontal arrow from the first column to the second column and each vertical arrow in the middle column belong to $\mathcal{C}[t]$. Hence the middle column defines a chain complex D^+ over $\mathcal{C}[t]$, the horizontal arrows from the left column to the middle column define a chain map $\phi^+ \colon C^+ \to D^+$ over $\mathcal{C}[t]$ and the collection of the morphisms from the middle column to the right column define a chain homotopy equivalence $g \colon D^+ \to D^-$ over $\mathcal{C}[t, t^{-1}]$. By construction we have $j_-(\varphi^-) \circ f = g \circ j_+(\varphi^+)$. This finishes the construction of the desired morphism ϕ and hence the proof of Lemma 14.22. □

Theorem 14.23. *There are weak homotopy fibration sequences*

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})^w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{C}[t^{-1}]);$$
$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y})^w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathcal{C}[t, t^{-1}]).$$

Proof. We give the details only for the first sequence, the analogous proof for the second one is left to the reader.

We apply the Fibration Theorem 14.13 in the case $\mathcal{W} = \mathsf{Ch}(\mathcal{X})$, w as described above and v the structure of weak equivalences coming from chain homotopy equivalences. The necessary conditions appearing in the Fibration Theorem 14.13 are satisfied by [14, Lemma 3.1 and Lemma 4.12]. Thus we obtain a weak homotopy fibration of spectra

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})^w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X}), w).$$

Because of Lemma 14.22 we obtain a weak homotopy fibration

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})^w) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) \to \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])).$$

□

Lemma 14.24. *The functor k' induces a weak homotopy equivalence*

$$\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})^w) \xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{Y})^w).$$

Proof. Again we will use the Cisinki's Approximation Theorem 14.14. We have to verify the conditions (i) and (ii) appearing in Cisinki's Approximation Theorem 14.14.

Let

$$(14.25) \qquad \begin{array}{ccc} j_+C^+ & \xrightarrow{f} & j_-C^- \\ {\scriptstyle j_+\varphi^+}\downarrow & & \downarrow{\scriptstyle j_-\varphi^-} \\ j_+D^+ & \xrightarrow{g} & j_-D^- \end{array}$$

represent a morphism in $\mathsf{Ch}(\mathcal{X})^w$, which maps to a weak equivalence in $\mathsf{Ch}(\mathcal{Y})^w$. Then φ^+ is a chain homotopy equivalence in $\mathcal{C}[t]$ and φ^- is a chain homotopy equivalence in $\mathcal{C}[t, t^{-1}]$. By assumption, C^- and D^- are contractible in $\mathcal{C}[t^{-1}]$, so φ^- has to be an equivalence in $\mathcal{C}[t^{-1}]$. It follows that the morphism given by (14.25) is a weak equivalence in $\mathsf{Ch}(\mathcal{X})^w$ already. This takes care of condition (i).

It remains to check condition (ii). Suppose now that

$$(14.26) \qquad \begin{array}{ccc} j_+C^+ & \xrightarrow{f} & C^- \\ {\scriptstyle j_+\varphi^+}\downarrow & & \downarrow{\scriptstyle \varphi^-} \\ j_+D^+ & \xrightarrow{g} & D^- \end{array}$$

represents a morphism in $\mathsf{Ch}(\mathcal{Y})^w$ satisfying (C^+, f, C^-) in $\mathsf{Ch}(\mathcal{X})^w$. We have to factor this morphism through a map in $\mathsf{Ch}(\mathcal{X})^w$ (which we may then replace by a cofibration using the mapping cylinder) and a weak equivalence in $\mathsf{Ch}(\mathcal{Y})^w$.

Note that the morphism φ^- is a chain homotopy equivalence in $\mathcal{C}[t, t^{-1}]$, as both C^- and D^- are contractible in that category by assumption. We conclude from [14, Lemma 3.1 (ix)] that there is a chain isomorphism of the shape

$$\begin{pmatrix} \varphi^- & y \\ x & z \end{pmatrix} : C^- \oplus E \xrightarrow{\cong} D^- \oplus E',$$

where E and E' are elementary chain complexes in $\mathcal{C}[t, t^{-1}]$, or even in $\mathcal{C}$, since both categories have the same objects.

For appropriate $\underline{k} \in \prod_{m\in\mathbb{N}} \mathbb{Z}$, the commutative diagram

$$\begin{array}{ccc} j_+C^+ & \xrightarrow{\quad f \quad} & C^- \\ {\scriptstyle \begin{pmatrix} j_+\varphi^+ \\ t^{\underline{k}} \circ x \circ f \end{pmatrix}} \downarrow & \begin{pmatrix} g^{-1}\circ\varphi^- & g^{-1}\circ y \\ t^{\underline{k}}\circ x & t^{\underline{k}}\circ z \end{pmatrix}^{-1} & \downarrow {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \\ j_+D^+ \oplus i_0E' & \longrightarrow & C^- \oplus i_0E \\ {\scriptstyle (1 \quad 0)} \downarrow & & \downarrow {\scriptstyle (\varphi^- \quad y)} \\ j_+D^+ & \xrightarrow{\quad g \quad} & D^- \end{array}$$

provides the desired factorization of (14.26). ☐

Proof of Theorem 14.20. Theorem 14.20 follows from Lemma 14.21, Lemma 14.23, and Lemma 14.24. ☐

14.K. **The global section functor.** Recall that we have defined truncation functors in Subsection 14.H.

Definition 14.27 (Global section functor)**.** The *global section functor*

$$\Gamma\colon \mathsf{Ch}(\mathcal{X}) \to \mathsf{Ch}(\mathcal{C}^\kappa)$$

sends an object (C^+, f, C^-) to the $\mathcal{C}^\kappa$-chain complex

$$\Sigma^{-1}\operatorname{cone}\bigl(f[\,]\colon C^+[0,\infty] \to C^-[1,\infty]\bigr).$$

A morphism $(\varphi^+,\varphi^-)\colon (C^+,f,C^-) \to (D^+,g,D^-)$ of $\mathsf{Ch}(\mathcal{X})$ is sent to the morphism in $\mathsf{Ch}(\mathcal{A}^\kappa)$ obtained from the commutative diagram (using the trivial homotopy)

$$\begin{array}{ccc} C^+[0,\infty] & \xrightarrow{\quad f[\,] \quad} & C^-[-1,\infty] \\ {\scriptstyle \varphi^+[\,]}\downarrow & & \downarrow{\scriptstyle \varphi^-[\,]} \\ D^+[0,\infty] & \xrightarrow[\quad g[\,] \quad]{} & D^-[-1,\infty]. \end{array}$$

Note that $f[\,]$ is indeed a chain map, as it is the composite of the three chain maps $C^+[0,\infty] \xrightarrow{\mathrm{id}[\,]} C^+[-\infty,\infty] \xrightarrow{i^0 f = f[\,]} C^-[-\infty,\infty] \xrightarrow{\mathrm{id}[\,]} C^-[-1,\infty]$ Let $\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}) \subset \mathsf{Ch}(\mathrm{Idem}(\mathcal{C}^\kappa))$ be the full subcategory of homotopy finite chain complexes over $\mathrm{Idem}(\mathcal{C}^\kappa)$, i.e., chain complexes over $\mathrm{Idem}(\mathcal{C}^\kappa)$, which are homotopy equivalent to a bounded chain complex over $\mathrm{Idem}(\mathcal{C})$.

It follows from [14, Lemma 3.5] that this category is closed under pushouts along a cofibration, so it is a Waldhausen subcategory of $\mathsf{Ch}(\mathrm{Idem}(\mathcal{C}^\kappa))$. The Approximation Theorem 14.14 of Cisinski shows that the inclusion $I(\mathcal{C})\colon \mathsf{Ch}(\mathrm{Idem}(\mathcal{C})) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$ induces an equivalence

$$\mathbf{K}^{\infty,\mathrm{W}}(I(\mathcal{C}))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathrm{Idem}(\mathcal{C}))) \xrightarrow{\sim} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})) \tag{14.28}$$

on non-connective K-theory.

Lemma 14.29. *(i) The functor Γ is Waldhausen exact (for the canonical Waldhausen structures);*

(ii) For any object (C^+, f, C^-) of $\mathsf{Ch}(\mathcal{X})$, the chain complex given by the projective line $\Gamma(C^+, f, C^-) \in \mathsf{Ch}(\mathcal{C}^\kappa) \subseteq \mathsf{Ch}(\mathrm{Idem}(\mathcal{C}^\kappa))$ is chain homotopy equivalent to an object in $\mathsf{Ch}(\mathrm{Idem}(\mathcal{C}))$.

Thus, Γ defines a Waldhausen exact functor

$$\Gamma(\mathcal{C})\colon \mathsf{Ch}(\mathcal{X}) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}).$$

Proof. (i) It is not hard to check using [14, Section 4.1] that the two functors $k^{\pm}\colon \mathsf{Ch}(\mathcal{X}) \to \mathsf{Ch}(\mathcal{C}[t^{\pm 1}])$ are Waldhausen exact. The restriction functors from $\mathsf{Ch}(\mathcal{C}[t])$, $\mathsf{Ch}(\mathcal{C}[t^{-1}])$ and $\mathsf{Ch}(\mathcal{C}[t,t^{-1}])$ to $\mathsf{Ch}(\mathcal{C}^{\kappa})$ are defined on the level of additive categories and hence are Waldhausen exact. Taking cones and suspensions is also Waldhausen exact.

(ii) Write the morphism $f_n^{-1}\colon C_n^+ \to C_n^-$ in $\mathcal{C}[t,t^{-1}]$ as a sum $\sum_{l_n=a_n}^{b_n} f_n[l_n]\cdot s^{l_n}$ for appropriate morphisms $f_n[l_n]\colon \mathcal{S}(\Phi[t,t^{-1}])^{l_n}(C_n^-) \to C_n^+$ in $\mathcal{S}(\mathcal{A}_*[t,t^{-1}])$. Note that $f_n[l_n] = (f_n[l_n]_m)_{m\in\mathbb{N}}$ for appropriate morphisms $f_n[l_n]_m\colon \Phi^{l_n}((C_n^-)_m) \to (C_n^+)_m$ in $\mathcal{A}_0[t,t^{-1}]$. Now choose for $n\in\mathbb{Z}$, $l_n\in\mathbb{Z}$, and $m\in\mathbb{N}$ a natural number $k_n[l]_m$ such that we can write $f_n[l_n]_m = \sum_{j_{n,l_n,m}=-k_n[l_n]_m}^{k_n[l_n]_m} f_n[l_n]_m[j_{n,l_n,m}]\cdot t^{j_{n,l_n,m}}$ for appropriate morphisms $f_n[l_n]_m[j_{n,l_n,m}]\colon \Phi^{l_n}((C_n^-)_m) \to (C_n^+)_m$ in $\mathcal{A}_0$. Since C^+ and C^- are bounded, there exists a natural number N such that $C_n^- = 0$ and $C_n^+ = 0$ holds for $|n| > N$. Hence we get for every $n\in\mathbb{Z}$ with $|n|>N$ and every $m\in\mathbb{N}$ that $(C_n^-)_m = 0$ and $(C_n^+)_m = 0$ hold. Now define for $m\in\mathbb{N}$ a natural number k_i by

$$k_m = 1 + \max\{k_n[l_n]_m \mid -N \le n \le N, a_n \le l_n \le b_n\}.$$

This definition makes sense since the set $\bigcup_{n=-N}^{N}\{l_n \in \mathbb{Z} \mid a_n \le l_n \le b_n\}$ is finite. Note that k_m depends only on m, in contrast to $k_n[l_n]_m$. We conclude that $f_n[l_n]_m = \sum_{j_{n,l_n,m}=-k_m+1}^{k_m-1} f_n[l_n]_m[j_{n,l_n,m}]\cdot t^{j_{n,l_n,m}}$ holds for every $m\in\mathbb{N}$.

Define $\underline{k} \in \prod_{m\in\mathbb{N}}\mathbb{Z}$ by the collection $\{k_m \mid m\in\mathbb{N}\}$. Then $f^{-1}[\,]\colon C^-[\underline{-\infty},\underline{\infty}] \to C^+[\underline{k},\underline{\infty}]$ factorizes as

$$\begin{array}{ccc} C^-[\underline{-\infty},\underline{\infty}] & \xrightarrow{f^{-1}[\,]} & C^+[\underline{k},\underline{\infty}] \\ \big\downarrow{\scriptstyle \mathrm{id}[\,]} & \nearrow{\scriptstyle \overline{f^{-1}}} & \\ C^-[\underline{1},\underline{\infty}] & & \end{array}$$

and the composite

$$C^+[\underline{k},\underline{\infty}] \xrightarrow{f[\,]} C^-[\underline{1},\underline{\infty}] \xrightarrow{\overline{f^{-1}}} C^+[\underline{k},\infty]$$

is the identity map.

Hence over $\mathrm{Idem}(\mathcal{C}^{\kappa})$, the chain complex $C^-[1,\infty]$ splits as

$$C^-[1,\infty] \cong C^+[\underline{k},\underline{\infty}] \oplus R,$$

where R_n is given by $(C_n^-[\underline{1},\underline{\infty}], \mathrm{id}-p_n)$ for the projection

$$p_n\colon C^-[\underline{1},\underline{\infty}] \xrightarrow{\overline{f^{-1}}} C^+[\underline{k},\infty] \xrightarrow{f[\,]} C^-[\underline{1},\underline{\infty}].$$

Since the composite

$$C^-[2\cdot\underline{k},\underline{\infty}] \xrightarrow{f^{-1}[\,]} C^+[\underline{k},\underline{\infty}] \xrightarrow{f[\,]} C^-[2\cdot\underline{k},\underline{\infty}]$$

is the identity, $\mathrm{id}-p_n$ restricted to $C^-[2\cdot\underline{k},\underline{\infty}]$ is trivial. Hence we can find a projection $q_n\colon C_n^-[\underline{1}, 2\cdot\underline{k}-\underline{1}] \to C_n^-[\underline{1}, 2\cdot\underline{k}-\underline{1}]$ such that R_n and $(C_n^-[\underline{1}, 2\cdot\underline{k}-\underline{1}], q_n)$ are isomorphic. Since $C_n^-[\underline{1}, 2\cdot\underline{k}-\underline{1}]$ belongs to $\mathcal{C}$, R_n is isomorphic to an object in $\mathrm{Idem}(\mathcal{C})$ for every $n\in\mathbb{Z}$. This implies that the bounded $\mathrm{Idem}(\mathcal{C}^{\kappa})$-chain complex R is isomorphic to a bounded $\mathrm{Idem}(\mathcal{C})$-chain complex. We obtain an exact sequence of $\mathrm{Idem}(\mathcal{C}^{\kappa})$-chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^+[\underline{k},\infty] & \longrightarrow & C^+[0,\infty] & \longrightarrow & C^+[0,\underline{k}-\underline{1}] & \longrightarrow & 0 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow -f[\,] & & \big\downarrow g & & \\ 0 & \longrightarrow & C^+[\underline{k},\infty] & \xrightarrow{-f[\,]} & C^-[1,\infty] & \xrightarrow{r} & R & \longrightarrow & 0 \end{array}$$

where $r_n\colon C_n^-[1,\infty] \to R_n$ is the canonical projection given by $\mathrm{id} - p_n$ and g is the induced map on the quotients. We conclude that $\Sigma^{-1}\operatorname{cone}(-f[\,]) \simeq \Sigma^{-1}\operatorname{cone}(g)$, which is isomorphic to a chain complex in $\mathrm{Idem}(\mathcal{C})$. Hence $\Gamma(C^+, f, C^-)$ belongs to $\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$. □

14.L. **Embedding lower-K-theory in higher K-theory.** For $k \in \mathbb{Z}$ and a spectrum $\mathbf{E}$, let $\Sigma^k\mathbf{E}$ be the spectrum obtained by shifting, i.e., $(\Sigma^k\mathbf{E})_n = \mathbf{E}_{n-k}$. Let

$$\mathbf{d}(\mathbf{E})\colon \Sigma^{-1}S^1_+ \wedge \mathbf{E} \to \mathbf{E} \tag{14.30}$$

be the weak homotopy equivalence, natural in $\mathbf{E}$, which is given in dimension n by the n-th structure map of $\mathbf{E}$. Let $i\colon S^1 \to S^1 \vee S^1$ be the pinching map. For a spectrum $\mathbf{E}$, define a map of spectra

$$\nabla\colon \Sigma^{-1}S^1_+ \wedge \mathbf{E} \to (\Sigma^{-1}S^1_+ \wedge \mathbf{E}) \vee (\Sigma^{-1}S^1_+ \wedge \mathbf{E}),$$

natural in $\mathbf{E}$, by

$$\begin{aligned}(\Sigma^{-1}S^1_+ \wedge \mathbf{E})_n = S^1_+ \wedge E_{n+1} &\xrightarrow{i_+ \wedge E_{n+1}} (S^1 \vee S^1)_+ \wedge E_{n+1} \\ &\xrightarrow{\cong} (S^1_+ \wedge E_{n+1}) \vee (S^1_+ \wedge E_{n+1}) = (\Sigma^{-1}S^1_+ \wedge \mathbf{E})_n \vee (\Sigma^{-1}S^1_+ \wedge \mathbf{E})_n.\end{aligned}$$

The next theorem essentially says that $S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))$ is up to weak homotopy equivalence a retract of $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}]))$.

Theorem 14.31. *There is a diagram of spectra*

$$\begin{array}{ccc} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})) & & \\ \big\uparrow \simeq\ \mathrm{id}_{S^1_+} \wedge \mathbf{d}(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) & & \\ S^1_+ \wedge (\Sigma^{-1}S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) & \xrightarrow[\simeq]{\mathbf{i}} & \\ \big\downarrow \mathbf{s} & & \\ \mathbf{H}(\mathcal{C}) & \xrightarrow{\mathbf{r}} & S^1_+ \wedge \big(\Sigma^{-1}S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}))\big), \\ \big\downarrow \simeq\ \mathbf{f} & & \\ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}])) & & \end{array}$$

such that the triangle commutes up to a preferred homotopy $\mathbf{h}$ *and the maps marked with* $\simeq$ *are weak homotopy equivalences. Moreover, everything including the preferred homotopy* $\mathbf{h}$ *is natural in* Φ.

For its proof we need the next Lemma 14.33,

We define for $j = 0, 1$ a functor of additive categories

$$L_j\colon \mathcal{C} \to \mathcal{X} \tag{14.32}$$

as follows. The functor $L_j(\mathcal{C})$ sends an object C in $\mathcal{C}$ to the object $(C, t^{\underline{j}}, C)$ in $\mathcal{X}$, where $\underline{j} \in \prod_{m\in\mathbb{N}}\mathbb{Z}$ is given by the constant function $I \to \mathbb{Z}$ with value j. A morphism $u\colon C \to C'$ in $\mathcal{C}$ is sent to the morphism from $(C, t^{\underline{j}}, C)$ to $(C', t^{\underline{j}}, C')$ in $\mathcal{X}$ given by the morphism $j_+(u)\colon C \to C$ in $\mathcal{C}[t]$ and the morphism $j_-(u)\colon C \to C$ in

$\mathcal{C}[t^{-1}]$. Let $J(C)\colon \mathcal{C} \to \operatorname{Idem}(\mathcal{C})$ and $I(\mathcal{C})\colon \mathsf{Ch}(\operatorname{Idem}(\mathcal{C})) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$ be the obvious inclusions and $0\colon \mathsf{Ch}(\mathcal{C}) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$ be the constant functor with value the chain complex, all of whose chain objects are 0.

Lemma 14.33.

(i) There is a natural equivalence of functors $\mathsf{Ch}(\mathcal{C}) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$

$$T_0\colon I(\mathcal{C}) \circ \mathsf{Ch}(J(C)) \xrightarrow{\simeq} \Gamma \circ \mathsf{Ch}(L_0),$$

which is natural in Φ;

(ii) There is a natural equivalence of functors $\mathsf{Ch}(\mathcal{C}) \to \mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$

$$T_1\colon 0 \xrightarrow{\simeq} \Gamma \circ \mathsf{Ch}(L_1),$$

which is natural in Φ;

Proof. (i) Fix an object C in $\mathsf{Ch}(\mathcal{C})$. Then $\Gamma(L_0(C))$ is by definition

$$\Sigma^{-1} \operatorname{cone}\bigl(\mathrm{id}[\,]\colon C[0,\infty] \to C[1,\infty]\bigr).$$

We have the obvious split exact sequence in $\mathsf{Ch}(\mathcal{C}^{\kappa})$

$$0 \to C[\underline{0},\underline{0}] \xrightarrow{\mathrm{id}[\,]} C[\underline{0},\underline{\infty}] \xrightarrow{\mathrm{id}[\,]} C[\underline{1},\underline{\infty}] \to 0.$$

It induces a chain homotopy equivalence in $\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$

$$T_0(C)\colon C \xrightarrow{\simeq} \Gamma(L_0(C)).$$

(ii) Fix an object C in $\mathsf{Ch}(\mathcal{C})$. Then $\Gamma(L_1(C))$ is by definition

$$\Sigma^{-1} \operatorname{cone}\bigl(t^{\underline{1}}[\,]\colon C[0,\infty] \to C[1,\infty]\bigr).$$

The chain map $t^{\underline{1}}[\,]\colon C[0,\infty] \to C[1,\infty]$ is a chain isomorphism, its inverse is $t^{\underline{-1}}[\,]\colon C[\underline{1},\underline{\infty}] \to C[\underline{0},\underline{\infty}]$. Hence $\Gamma(L_1(C))$ is contractible and we obtain a chain homotopy equivalence in $\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})$

$$T_1(C)\colon 0 \xrightarrow{\simeq} \Gamma(L_0(C)).$$

□

Proof of Theorem 14.31. Denote by $*$ the trivial spectrum. Consider the following diagram of non-connective spectra

$$\begin{array}{ccccc}
* & \longleftarrow & S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})) & \longrightarrow & * \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \nabla} & & \Big\downarrow \\
 & & (S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) \vee (S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) & & \\
 & & \Big\downarrow{\scriptstyle \mathbf{a}} & & \\
S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) & \xleftarrow{\mathbf{b}^+} & S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) & \xrightarrow{\mathbf{b}^-} & S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle S\wedge\mathbf{K}^{\infty,\mathrm{W}}(\Gamma)} & & \Big\downarrow \\
* & \longleftarrow & S \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C})) & \longrightarrow & *
\end{array}$$

where we abbreviate $S = \Sigma^{-1} S^1_+$, $u\colon S^1 \to S^1$ sends z to z^{-1}, and the maps $\mathbf{a}$ and $\mathbf{b}^{\pm}$ are given by

$$\begin{aligned}
\mathbf{a} &:= (\Sigma^{-1}\mathrm{id}_{S^1_+} \wedge \mathbf{K}(\mathsf{Ch}(L_0))) \vee (\Sigma^{-1} u \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(L_1)));\\
\mathbf{b}^{\pm} &:= \Sigma^{-1}\mathrm{id}_{S^1_+} \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^{\pm})).
\end{aligned}$$

Note that u is a base point preserving selfmap of S^1 of degree -1. Fix a pointed nullhomotopy for the composite $S^1 \xrightarrow{i} S^1 \vee S^1 \xrightarrow{\mathrm{id}_{S^1} \vee u} S^1$. Since $k^{\pm} \circ L_0$ and $k^{\pm} \circ L_1$ agree, both are equal to with $i^{\pm}$, the upper left square and the upper right square commute up to a preferred homotopy, natural in Φ. The lower left and the lower right square commute. The homotopy pushout of the upper row is $S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})))$ and of the lower row is $S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}\mathcal{C}))$. Let $\mathbf{HP}(\mathcal{C})$ be the homotopy pushout of the middle row. Then we get from the data above maps of spectra

$$\begin{aligned} \mathbf{s}\colon S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) &\to \mathbf{HP}(\mathcal{C}); \\ \mathbf{r}\colon \mathbf{HP}(\mathcal{C}) &\to S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}))\bigr). \end{aligned}$$

The canonical inclusion $J(C)\colon \mathcal{C} \to \mathrm{Idem}(\mathcal{C})$ induces a weak homotopy equivalence $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(J(\mathcal{C})))\colon \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})) \xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathrm{Idem}(\mathcal{C})))$. This follows from Theorem 14.15, where the elementary proof that the conditions appearing in Theorem 14.15 are satisfied is left to the reader. Define the weak homotopy equivalence

$$\begin{aligned} \mathbf{i} =\colon S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))\bigr) &\xrightarrow{S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(J(C))))} \\ S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathrm{Idem}(\mathcal{C})))\bigr) & \\ &\xrightarrow{\mathrm{id}_{S^1_+} \wedge \Sigma^{-1} \mathrm{id}_{S^1_+} \wedge \mathbf{K}^{\infty,\mathrm{W}}(I(\mathcal{C}))} S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}))\bigr), \end{aligned}$$

where $\mathbf{K}^{\infty,\mathrm{W}}(I(\mathcal{C}))$ is the weak homotopy equivalence of (14.28).

The composite

$$\mathbf{r} \circ \mathbf{s}\colon S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))) \to S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{C}))\bigr)$$

is homotopic to $\mathbf{i}$ by Lemma 14.33. From the commutative diagram

$$\begin{array}{ccc} \Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) & \xrightarrow{\mathbf{d}} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t])) \\ \uparrow{\scriptstyle \Sigma^{-1} \mathrm{id}_{S^1_+} \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^+))} & & \uparrow{\scriptstyle \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^+))} \\ \Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) & \xrightarrow{\mathbf{d}} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{X})) \\ \downarrow{\scriptstyle \Sigma^{-1} \mathrm{id}_{S^1_+} \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))} & & \downarrow{\scriptstyle \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(k^-))} \\ \Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])) & \xrightarrow{\mathbf{d}} & \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t^{-1}])) \end{array}$$

and Theorem 14.20, we obtain a weak homotopy equivalence of spectra, natural in Φ,

$$\mathbf{f}\colon \mathbf{HP}(\mathcal{C}) \xrightarrow{\simeq} \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}[t,t^{-1}])).$$

We get from the weak homotopy equivalence $\mathbf{d}(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})))$ of (14.30) a weak homotopy equivalence

$$\mathrm{id}_{S^1_+} \wedge \mathbf{d}(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})))\colon S^1_+ \wedge \bigl(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C}))\bigr) \xrightarrow{\simeq} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{C})).$$

This finishes the proof of Theorem 14.31. □

Next we repeat the same construction in the easy case, where we do not take the automorphisms Φ into account. So, given $\mathcal{A}_*$, we define additive categories

$$\begin{aligned} \widehat{\mathcal{C}} &= \mathcal{S}(\mathcal{A}_*); \\ \widehat{\mathcal{C}}[t^{\pm}] &= \mathcal{S}(\mathcal{A}_*[t^{\pm}]); \\ \widehat{\mathcal{C}}[t,t^{-1}] &= \mathcal{S}(\mathcal{A}_*[t,t^{-1}]), \end{aligned}$$

induction functors fitting in a commutative dagram

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{\widehat{i}_+} & \widehat{\mathcal{C}}[t] \\ {\scriptstyle \widehat{i}_-}\downarrow & \searrow^{\widehat{i}_0} & \downarrow{\scriptstyle \widehat{j}_+} \\ \widehat{\mathcal{C}}[t^{-1}] & \xrightarrow[\widehat{j}_-]{} & \widehat{\mathcal{C}}[t,t^{-1}] \end{array}$$

additive categories

$$\begin{aligned} \widehat{\mathcal{C}}^{\kappa} &= \mathcal{S}(\mathcal{A}_*^{\kappa}); \\ \widehat{\mathcal{C}}[t^{\pm}]^{\kappa} &= \mathcal{S}(\mathcal{A}_*[t^{\pm}]^{\kappa}); \\ \widehat{\mathcal{C}}[t,t^{-1}]^{\kappa} &= \mathcal{S}(\mathcal{A}_*[t,t^{-1}]^{\kappa}), \end{aligned}$$

and restriction functors

$$\begin{aligned} \widehat{i}^0 \colon \widehat{\mathcal{C}}[t,t^{-1}]^{\kappa} &\to \widehat{\mathcal{C}}^{\kappa}; \\ \widehat{i}^{\pm} \colon \widehat{\mathcal{C}}[t^{\pm 1}]^{\kappa} &\to \widehat{\mathcal{C}}^{\kappa}, \end{aligned}$$

such that we have adjunctions $((\widehat{i}_0)^{\kappa}, \widehat{i}^0)$, $((\widehat{i}_+)^{\kappa}, \widehat{i}^+)$, and $((\widehat{i}_-)^{\kappa}, \widehat{i}^-)$. There is an obvious definition of the projective line $\widehat{\mathcal{X}}$ and of the global section functor $\widehat{\Gamma} \colon \widehat{\mathcal{X}} \to \mathsf{Ch}^{\mathrm{hf}}(\widehat{\mathcal{C}}^{\kappa})$.

Now analogously to the proof of Theorem 14.31 one can show

Theorem 14.34. *There is a diagram of spectra*

$$\begin{array}{ccc} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\widehat{\mathcal{C}})) & & \\ \simeq \Big\uparrow {\scriptstyle \mathrm{id}_{S^1_+} \wedge \mathbf{d}(\mathbf{K}^{\infty,\mathrm{W}}(\widehat{\mathcal{C}}))} & & \\ S^1_+ \wedge (\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\widehat{\mathcal{C}}))) & & \\ \Big\downarrow {\scriptstyle \widehat{\mathbf{s}}} & \searrow^{\widehat{\mathbf{i}}}_{\simeq} & \\ \mathbf{HP}(\widehat{\mathcal{C}}) & \xrightarrow{\widehat{\mathbf{r}}} & S^1_+ \wedge \big(\Sigma^{-1} S^1_+ \wedge \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}^{\mathrm{hf}}(\widehat{\mathcal{C}}))\big), \\ \simeq \Big\downarrow {\scriptstyle \widehat{\mathbf{f}}} & & \\ \mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\widehat{\mathcal{C}}[t,t^{-1}])) & & \end{array}$$

such that the triangle commutes up to a preferred homotopy $\widehat{\mathbf{h}}$ and the maps marked with $\simeq$ are weak homotopy equivalence. Moreover, everything including the preferred homotopy $\widehat{\mathbf{h}}$ is natural in Φ.

There are obvious inclusions $l \colon \widehat{\mathcal{C}} \to \mathcal{C}$, $l[t^{\pm}] \colon \widehat{\mathcal{C}}[t^{\pm}] \to \mathcal{C}[t^{\pm}]$, and $l[t,t^{-1}] \colon \widehat{\mathcal{C}}[t,t^{-1}] \to \mathcal{C}[t,t^{-1}]$. They induce a map denoted by $\mathbf{l}$ from the diagram appearing in Theorem 14.34 to the one appearing in Theorem 14.31.

The pro-automorphism $\Phi \colon \mathcal{A}_* \to \mathcal{A}_*$ induces in the obvious way an automorphism of the diagram appearing in Theorem 14.34, denoted by $\mathbf{t}$. Taking the mapping torus in each entry of the diagram appearing in Theorem 14.34 yields a

diagram of the form
(14.35)

$$\begin{array}{ccc}
S^1_+ \wedge T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\Phi)))} & & \\
\simeq \Big\uparrow \mathrm{id}_{S^1_+} \wedge \mathbf{d}(T_{\mathbf{K}^\infty,\mathrm{W}(\mathcal{S}(\Phi))}) & & \\
S^1_+ \wedge \big(\Sigma^{-1} S^1_+ \wedge T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\Phi)))}\big) & \xrightarrow[\simeq]{T_{\widehat{\mathbf{i}}}} & \\
\Big\downarrow \widehat{\mathbf{s}} & & \\
T_{\mathbf{HP}(\mathcal{S}(\Phi))} & \xrightarrow{T_{\widehat{\mathbf{r}}}} & S^1_+ \wedge \big(\Sigma^{-1} S^1_+ \wedge T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}^{\mathrm{hf}}(\mathcal{S}(\Phi)))}\big). \\
\simeq \Big\downarrow T_{\widehat{\mathbf{f}}} & & \\
T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\Phi[t,t^{-1}])))} & &
\end{array}$$

It is not true that $\mathbf{l} \circ \mathbf{t}$ agrees with $\mathbf{l}$ again. However, this is true up to preferred homotopy, and therefore we get a map from the diagram (14.35) to the diagram appearing in Theorem 14.31. These homotopies are all induced by obvious natural transformations of functors. For instance, for the automorphism $\widehat{\Phi} = \mathcal{S}(\Phi)\colon \widehat{\mathcal{C}} = \mathcal{S}(\mathcal{A}_*) \xrightarrow{\cong} \widehat{\mathcal{C}} = \mathcal{S}(\mathcal{A}_*)$ and the inclusion $l\colon \widehat{\mathcal{C}} = \mathcal{S}(\mathcal{A}_*) \to \mathcal{C} = \mathcal{S}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}]$ we get a natural transformation $l \to l \circ \widehat{\Phi}$, if we assign an object $\underline{A}$ in $\widehat{\mathcal{C}} = \mathcal{S}(\mathcal{A}_*)$ the morphism $\underline{A} \to \underline{A}$ in $\mathcal{C} = \mathcal{S}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}]$ given by $\mathrm{id}_{\widehat{\Phi}(\underline{A})} \cdot t$. From these data we get a map from the diagram (14.35) to the diagram appearing in Theorem 14.31.

Now we apply the functor π_n to the diagram (14.35), the diagram appearing in Theorem 14.31 and the map between them constructed above. Taking into account that some of the arrows appearing in the diagram (14.35) and the diagram Theorem 14.31 are weak equivalences, we get for every $n \in \mathbb{Z}$ a commutative diagram
(14.36)

$$\begin{array}{ccc}
\pi_{n-1}\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\phi)))}\big) & \longrightarrow & \pi_{n-1}\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{S}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}]))\big) \\
\Big\downarrow & & \Big\downarrow \\
\pi_n\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\Phi[t,t^{-1}])))}\big) & \longrightarrow & \pi_n\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{S}(\mathcal{A}_*[t,t^{-1}]))_{\mathcal{S}(\Phi[t,t^{-1}])}[\mathbb{Z}])\big) \\
\Big\downarrow & & \Big\downarrow \\
\pi_{n-1}\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{S}(\phi)))}\big) & \longrightarrow & \pi_{n-1}\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{S}(\mathcal{A}_*)_{\mathcal{S}(\Phi)}[\mathbb{Z}]))\big)
\end{array}$$

such that the composite of the left two vertical arrows and the composite of the right two vertical arrows are isomorphisms.

Next we explain, how we get the corresponding diagram for $\mathcal{L}$ instead of $\mathcal{S}$
(14.37)

$$\begin{array}{ccc}
\pi_{n-1}\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{L}(\phi)))}\big) & \longrightarrow & \pi_{n-1}\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*)_{\mathcal{L}(\Phi)}[\mathbb{Z}]))\big) \\
\Big\downarrow & & \Big\downarrow \\
\pi_n\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{L}(\Phi[t,t^{-1}])))}\big) & \longrightarrow & \pi_n\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}])_{\mathcal{L}(\Phi[t,t^{-1}])}[\mathbb{Z}]))\big) \\
\Big\downarrow & & \Big\downarrow \\
\pi_{n-1}\big(T_{\mathbf{K}^\infty,\mathrm{W}(\mathsf{Ch}(\mathcal{L}(\phi)))}\big) & \longrightarrow & \pi_{n-1}\big(\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*)_{\mathcal{L}(\Phi)}[\mathbb{Z}]))\big)
\end{array}$$

such that the composite of the left two vertical arrows and the composite of the right two vertical arrows are isomorphisms. Note that the upper horizontal arrow is an isomorphism, if the middle arrow is an isomorphism and that the upper arrow deals with the homotopy groups in a dimension, which is the dimension of the middle arrow minus 1.

The point is that the construction of map from the diagram (14.35) to the diagram Theorem 14.31 including the statement that some maps are weak homotopy equivalences, carries over word by word, if we replace $\mathcal{T}$ by $\mathcal{S}$ everywhere. These two constructions are compatible with the various inclusions and therefore yields also a version of a map from the diagram (14.35) to the diagram Theorem 14.31 including the statement that some maps are weak homotopy equivalences, where we replace $\mathcal{S}$ by $\mathcal{L}$ everywhere and use Lemma 13.4. Now diagram (14.37) is derived from this map analogously to (14.36).

Recall that $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*)))$ is a spectrum with a $\mathbb{Z}$-action, which comes from $\mathbf{K}^{\infty,\mathrm{W}}(\mathsf{Ch}(\mathcal{L}(\Phi)))$. We obtain a covariant functor, see for instance [1, Section 9],

$$\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*))}\colon \mathsf{Or}(\mathbb{Z}) \to \mathsf{Spectra}. \tag{14.38}$$

It determines a $\mathbb{Z}$-homology theory $H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*))})$ with the property that for every subgroup $H \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$ we have the natural isomorphism

$$H_n^{\mathbb{Z}}(\mathbb{Z}/H,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*))}) \xrightarrow{\cong} K_n(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*)) \rtimes_{\mathsf{Ch}(\mathcal{L}(\Phi))|_H} H)$$

as explained for instance in [1, Section 9]. Analogously we get $\mathbb{Z}$-homology theories $H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}]))})$, $H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)})$, and $H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}[t,t^{-1}]_*)})$ with the property that for every subgroup $H \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$ we have the natural isomorphisms

$$\begin{aligned}
H_n^{\mathbb{Z}^r}(\mathbb{Z}/H,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}]))}) &\xrightarrow{\cong} K_n(\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}])) \rtimes_{\mathsf{Ch}(\mathcal{L}(\Phi[t,t^{-1}]))|_H} H);\\
H_n^{\mathbb{Z}}(\mathbb{Z}/H,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) &\xrightarrow{\cong} K_n(\mathcal{L}(\mathcal{A}_*) \rtimes_{\mathcal{L}(\Phi)|_H} H);\\
H_n^{\mathbb{Z}}(\mathbb{Z}/H,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[t,t^{-1}])}) &\xrightarrow{\cong} K_n(\mathcal{L}(\mathcal{A}_*[t,t^{-1}]) \rtimes_{\mathcal{L}(\Phi[t,t^{-1}])|_H} H).
\end{aligned}$$

Lemma 14.39. *The are natural equivalences of $\mathbb{Z}$-homology theories*

$$\begin{aligned}
H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*))}) &\xrightarrow{\cong} H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)});\\
H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty,\mathrm{W}}_{\mathsf{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}]))}) &\xrightarrow{\cong} H_n^{\mathbb{Z}}(-,\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[t,t^{-1}])}).
\end{aligned}$$

Proof. They are induced by the zigzag of natural weak homotopy equivalences appearing in Theorem 14.17 using [5, Lemma 4.6] □

Lemma 14.40. *For every $n \in \mathbb{Z}$ there exists a commutative diagram*

$$\begin{array}{ccc}
H_{n-1}^{\mathbb{Z}}(E\mathbb{Z},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) & \longrightarrow & H_{n-1}^{\mathbb{Z}}(\mathrm{pt},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)})\\
\downarrow & & \downarrow\\
H_{n}^{\mathbb{Z}}(E\mathbb{Z},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[t,t^{-1}])}) & \longrightarrow & H_{n}^{\mathbb{Z}}(\mathrm{pt},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[t,t^{-1}])})\\
\downarrow & & \downarrow\\
H_{n-1}^{\mathbb{Z}}(E\mathbb{Z},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) & \longrightarrow & H_{n-1}^{\mathbb{Z}}(\mathrm{pt},\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)})
\end{array}$$

such that the composite of the two vertical arrows appearing in the left and in the right column are isomorphisms and the horizontal arrows are induced by the projection $E\mathbb{Z} \to \mathrm{pt}$.

In particular the upper horizontal arrow is bijective, if the middle horizontal arrow is bijective.

Proof. Because of Lemma 14.39 it suffices to proof Lemma 14.40 in the case, where we replace $\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}$ by $\mathbf{K}^{\infty,\mathrm{W}}_{\mathtt{Ch}(\mathcal{L}(\mathcal{A}_*))}$ and $\mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[t,t^{-1}])}$ by $\mathbf{K}^{\infty}_{\mathtt{Ch}(\mathcal{L}(\mathcal{A}_*[t,t^{-1}]))}$.

Now one easily checks unravelling the definitions that the rows of the diagram above after this replacement agree with the rows of the diagram (14.37). Now the claim follows from the diagram (14.37). □

14.M. Reduction to the connective case.

Theorem 14.41 (Reduction to the connective case)**.** *Fix a natural number n_0. Suppose that for every natural number d and every natural number n satisfying $n \geq n_0$ the map*

$$H_n^{\mathbb{Z}}(E\mathbb{Z}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])}) \to H_n^{\mathbb{Z}}(\mathrm{pt}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])}) = K_n(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]) \times_{\mathcal{L}(\Phi[\mathbb{Z}^d])} \mathbb{Z})$$

is an isomorphism.

Then for every $n \in \mathbb{Z}$ the map

$$H_n^{\mathbb{Z}}(E\mathbb{Z}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) \to H_n^{\mathbb{Z}}(\mathrm{pt}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*)}) = K_n(\mathcal{L}(\mathcal{A}_*) \times_{\mathcal{L}(\Phi)} \mathbb{Z})$$

is an isomorphism.

Proof. One can iterate the passage from $\mathcal{A}_*$ to $\mathcal{A}_*[t,t^{-1}] = \mathcal{A}_*[\mathbb{Z}]$ and obtain a passage from $\mathcal{A}_*$ to $\mathcal{A}_*[\mathbb{Z}^d] = (\mathcal{A}_*[\mathbb{Z}^{d-1}])[t,t^{-1}]$ for every natural number d. Now Theorem 14.41 follows from Lemma 14.40. □

14.N. **Proof of Theorem 14.1.** Now we are ready to finalize the proof of Theorem 14.1.

Thanks to Lemma 14.2, we can assume without loss of generality that $r = 1$. Because of Theorem 14.41 it suffices to show that for every natural number d and every natural number n satisfying $n \geq 2$ the map

$$H_n^{\mathbb{Z}}(E\mathbb{Z}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])}) \to H_n^{\mathbb{Z}}(\mathrm{pt}, \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])}) = K_n(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]) \times_{\mathcal{L}(\Phi[\mathbb{Z}^d])} \mathbb{Z})$$

is an isomorphism.

Let $\mathbf{K}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}$ be the connective version of $\mathbf{K}^{\infty}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}$. Since we have $\dim(E\mathbb{Z}) \leq 1$, we conclude from Lemma 3.3 that the vertical arrows appearing in the commutative diagram

$$\begin{array}{ccc}
H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) & \longrightarrow & H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) \\
\cong\downarrow & & \downarrow\cong \\
H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}^{\infty}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) & \longrightarrow & H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}^{\infty}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) \\
\cong\uparrow & & \uparrow\cong \\
H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])}) & \longrightarrow & H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}^{\infty}_{\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])})
\end{array}$$

are bijective for $n \geq 2$. Hence it suffices show that for every natural number d the map

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) \to H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}_{\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))}) \tag{14.42}$$

is bijective for $n \in \mathbb{Z}, n \geq 2$.

By assumption the category $\mathcal{A}_m[\mathbb{Z}^d]$ is uniformly $l(d)$-regular coherent and the inclusion $\mathcal{A}_m[\mathbb{Z}^d] \to \mathcal{A}_{m+1}[\mathbb{Z}^d]$ is flat for every $m \geq 0$. We conclude from Lemma 13.10 that $\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d])$ is uniformly $l(d)$-regular coherent. We conclude from Lemma 6.4 (vi) that $\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))$ is uniformly $l(d)$-regular coherent. Hence $\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))$ is idempotent complete and regular coherent. Now the bijectivity of (14.42) for $n \in \mathbb{Z}, n \geq 2$ follows from Theorem 7.5 applied to $\mathrm{Idem}(\mathcal{L}(\mathcal{A}_*[\mathbb{Z}^d]))$, since for the map $\mathbf{a}$ appearing there the homomorphism $\pi_n(\mathbf{a})$ can be identified with the map (14.42) for $n \geq 2$. This finishes the proof of Theorem 14.1.

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