

LEHMER'S PROBLEM FOR ARBITRARY GROUPS

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ABSTRACT. We consider the problem whether for a group G there exists a constant $\Lambda(G) > 1$ such that for any (r, s) -matrix A over the integral group ring $\mathbb{Z}G$ the Fuglede-Kadison determinant of the G -equivariant bounded operator $L^2(G)^r \rightarrow L^2(G)^s$ given by right multiplication with A is either one or greater or equal to $\Lambda(G)$. If G is the infinite cyclic group and we consider only $r = s = 1$, this is precisely Lehmer's problem.

0. INTRODUCTION

Lehmer's problem is the question whether the Mahler measure of a polynomial with integer coefficients is either one or bounded from below by a fixed constant $\Lambda > 1$. If one views the polynomial as an element in the integral group ring $\mathbb{Z}[\mathbb{Z}]$ ring of \mathbb{Z} , then its Mahler measure agrees with the Fuglede-Kadison determinant of the \mathbb{Z} -equivariant bounded operator $r_p^{(2)}: L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})$ given by right multiplication with p . This suggests to consider for any group G the following generalization.

Let A be an (r, s) -matrix over the integral group ring $\mathbb{Z}G$. We propose to study the problem whether there is a constant $\Lambda(G) > 1$ such that the Fuglede-Kadison determinant of $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ is either one or larger or equal to $\Lambda(G)$. If we only allow $r = s = 1$, we denote such a constant by $\Lambda_1(G)$. If we consider only the case $r = s$ or the case $r = s = 1$ and additionally require that $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^r$ is a weak isomorphism, or, equivalently, is injective, we denote such a constant by $\Lambda^w(G)$ or $\Lambda_1^w(G)$. Lehmer's problem is equivalent to the question whether $\Lambda_1(\mathbb{Z}) > 1$ holds.

For obvious reasons we have $\Lambda(G) \leq \Lambda^w(G)$, $\Lambda_1(G) \leq \Lambda_1^w(G)$, $\Lambda(G) \leq \Lambda_1(G)$, and $\Lambda^w(G) \leq \Lambda_1^w(G)$. Since for a group G which contains \mathbb{Z} as a subgroup we have $\Lambda_1^w(G) \leq \Lambda_1^w(\mathbb{Z}) = \Lambda_1(\mathbb{Z})$ by Lemma 5.1 (1) and Theorem 10.1, we see that a counterexample to Lehmer's problem would imply $\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = 1$ for any group which contains \mathbb{Z} as subgroup. Hence all the discussions in this paper are more or less void if a counterexample to Lehmer's problem exists which is not known and fortunately not expected to be true.

If there is no upper bound on the order of finite subgroups of G , then $\Lambda_1^w(G) = 1$ by Remark 9.1. Indeed, there is a finitely presented group G with $\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = 1$, see Example 9.2. Therefore we will concentrate on torsionfree groups.

The most optimistic scenario would be that for any torsionfree group G all the constants $\Lambda(G)$, $\Lambda^w(G)$, $\Lambda_1(G)$ and $\Lambda_1^w(G)$ are conjectured to be the Mahler measure $M(L)$ of Lehmer's polynomial $L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$. But this is not the case in general, there is a hyperbolic closed 3-manifold, the Week's manifold W satisfying $\Lambda^w(\pi_1(W)) < M(L)$, see Example 13.2.

We will not make any new contributions to Lehmer's problem in this article. However, we think that it is interesting to put Lehmer's problem, which itself is

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already very interesting and has many intriguing connections to number theory, topology and geometry, in a more general context. Moreover, we will give some evidence for the hope that $\Lambda(G) > 1$ or even $\Lambda(G) \geq \Lambda_1(\mathbb{Z})$ holds for some torsionfree groups G . Namely, we will show in Theorem 10.1 that $\Lambda^w(\mathbb{Z}^d) = \Lambda_1(\mathbb{Z}^d) = \Lambda_1^w(\mathbb{Z}^d)$ holds for all natural numbers $d \geq 1$ and that this value is actually independent of $d \geq 1$. We can also prove $\Lambda(\mathbb{Z}) = \Lambda(\mathbb{Z}^d)$ for $d \geq 1$, but have not been able to relate $\Lambda(\mathbb{Z})$ to $\Lambda^w(\mathbb{Z})$ except for the obvious inequality $\Lambda(\mathbb{Z}) \leq \Lambda^w(\mathbb{Z})$. In particular we do not know whether $\Lambda^w(\mathbb{Z}) > 1 \implies \Lambda(\mathbb{Z}) > 1$. Conjecturally one may hope for $\Lambda(\mathbb{Z}) = \sqrt{\Lambda^w(\mathbb{Z})}$.

Moreover, we will explain in Section 7 how to use approximation techniques to potentially extend the class of groups for which $\Lambda(G) > 1$ holds.

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1. LEHMER’S PROBLEM

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-trivial element. Its *Mahler measure* is defined by

$$(1.1) \quad M(p) := \exp \left(\int_{S^1} \ln(|p(z)|) d\mu \right).$$

By Jensen's equality we have

$$(1.2) \quad \exp \left(\int_{S^1} \ln(|p(z)|) d\mu \right) = |c| \cdot \prod_{\substack{i=1,2,\dots,r \\ |a_i|>1}} |a_i|,$$

if we write $p(z)$ as a product $p(z) = c \cdot z^k \cdot \prod_{i=1}^r (z - a_i)$ for an integer $r \geq 0$, non-zero complex numbers c, a_1, \dots, a_r and an integer k . This implies $M(p) \geq 1$ if p has integer coefficients, i.e., belongs to $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$.

The following problem goes back to a question of Lehmer [22].

Problem 1.3 (Lehmer's Problem). *Does there exist a constant $\Lambda > 1$ such that for all non-trivial elements $p(z) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ with $M(p) \neq 1$ we have*

$$M(p) \geq \Lambda.$$

Remark 1.4 (Lehmer's polynomial). There is even a candidate for which the minimal Mahler measure is attained, namely, *Lehmer's polynomial*

$$L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

It is conceivable that for any non-trivial element $p \in \mathbb{Z}[\mathbb{Z}]$ with $M(p) > 1$

$$M(p) \geq M(L) = 1.17628\dots$$

holds.

Actually, $L(z)$ is $-z^5 \cdot \Delta(z)$, where $\Delta(z)$ is the Alexander polynomial of the pretzel knot given by $(-2, 3, 7)$.

For a survey on Lehmer's problem we refer for instance to [3, 4, 5, 33].

2. THE MAHLER MEASURE AS FUGLEDE-KADISON DETERMINANT

The following result is proved in [26, (3.23) on page 136]. We will recall the Fuglede-Kadison determinant and its basic properties in the Appendix, see Section 14.

Theorem 2.1 (Mahler measure and Fuglede-Kadison determinants over \mathbb{Z}). *Consider an element $p = p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$. It defines a bounded \mathbb{Z} -equivariant operator $r_p^{(2)}: L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})$ by multiplication with p . Suppose that p is not zero.*

Then the Fuglede-Kadison determinant $\det_{\mathcal{N}(\mathbb{Z})}(r_p^{(2)})$ of $r_p^{(2)}$ agrees with the Mahler measure, i.e.,

$$\det_{\mathcal{N}(\mathbb{Z})}(r_p^{(2)}) = M(p).$$

Note that the identification of the Fuglede-Kadison determinant with the Mahler measure holds also for non-trivial elements p in $\mathbb{C}[\mathbb{Z}^d] = \mathbb{C}[z_1^{\pm 1}, z_d^{\pm 1}, \dots, z_d^{\pm 1}]$, where d is any natural number, see [26, Example 3.13 on page 128].

3. LEHMER'S PROBLEM FOR ARBITRARY GROUPS

Given a group G , we consider $L^2(G)$ as a Hilbert space with the obvious isometric linear G -action from the left and write an element in $L^2(G)^r := \bigoplus_{i=1}^r L^2(G)$ as a row (x_1, x_2, \dots, x_r) for $x_i \in L^2(G)$, in other words as a $(1, r)$ -matrix. Given a A in $M_{r,s}(\mathbb{Z}G)$ or $M_{r,s}(\mathbb{C}G)$, we obtain by right multiplication with A a bounded G -equivariant operator

$$(3.1) \quad r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s, \quad (x_i)_{i=1,2,\dots,r} \mapsto \left(\sum_{k=1}^r x_k \cdot a_{k,j} \right)_{j=1,2,\dots,s}.$$

Note that with these conventions we have $r_{AB}^{(2)} = r_B^{(2)} \circ r_A^{(2)}$ for $A \in M_{r,s}(\mathbb{C}G)$ and $B \in M_{s,t}(\mathbb{C}G)$.

Definition 3.2 (Lehmer's constant of a group). Define *Lehmer's constant* of a group G

$$\Lambda(G) \in [1, \infty)$$

to be the infimum of the set of Fuglede-Kadison determinants

$$\det_{\mathcal{N}(G)}(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s),$$

where A runs through all (r, s) -matrices $A \in M_{r,s}(\mathbb{Z}G)$ for all $r, s \in \mathbb{Z}$ with $r, s \geq 1$ for which $\det_{\mathcal{N}(G)}(r_A^{(2)}) > 1$ holds.

If we only allow $(1, 1)$ -matrices A with $\det_{\mathcal{N}(G)}(r_A) > 1$, we denote the corresponding infimum by

$$\Lambda_1(G) \in [1, \infty).$$

If we only allow (r, r) -matrices A for any natural number r such that $r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^r$ is a weak isomorphism, or, equivalently, is injective, and $\det_{\mathcal{N}(G)}(r_A) > 1$, we denote the corresponding infimum by

$$\Lambda^w(G) \in [1, \infty).$$

If we only allow $(1, 1)$ -matrices A such that $r_A^{(2)} : L^2(G) \rightarrow L^2(G)$ is weak isomorphism, or, equivalently, is injective, and $\det_{\mathcal{N}(G)}(r_A) > 1$, then we denote the corresponding infimum by

$$\Lambda_1^w(G) \in [1, \infty).$$

Obviously we have

$$\begin{aligned} \Lambda(G) &\leq \Lambda^w(G) \leq \Lambda_1^w(G); \\ \Lambda(G) &\leq \Lambda_1(G) \leq \Lambda_1^w(G). \end{aligned}$$

A priori there is no obvious relation between $\Lambda^w(G)$ and $\Lambda_1(G)$.

Problem 3.3 (Lehmer's problem for arbitrary groups). *For which groups G is $\Lambda(G) > 1$, $\Lambda_1(G) > 1$, $\Lambda^w(G) > 1$ or $\Lambda_1^w(G) > 1$ true?*

For amenable groups this problem is already considered in [6, Question 4.7]. See also [8, 9].

Remark 3.4 ($\Lambda_1(\mathbb{Z})$ and Lehmer's problem). In view of Theorem 2.1 we see that Lehmer's Problem 1.3 is equivalent to the question whether $\Lambda_1(\mathbb{Z}) > 1$. In view of Remark 1.4 one would expect that $\Lambda_1(\mathbb{Z})$ is the Mahler measure $M(L)$ of Lehmer's polynomial. We conclude $\Lambda_1(\mathbb{Z}) = \Lambda_1^w(\mathbb{Z}) = \Lambda^w(\mathbb{Z})$ from Theorem 10.1. We do not know how $\Lambda(\mathbb{Z})$ and $\Lambda^w(\mathbb{Z})$ are related except for the obvious inequality $\Lambda(\mathbb{Z}) \leq \Lambda^w(\mathbb{Z})$.

Remark 3.5 (Why matrices and why $\Lambda(G)$?). We are also interested besides $\Lambda_1(G)$ in the numbers $\Lambda^w(G)$, $\Lambda_1^w(G)$ and $\Lambda(G)$ for the following reasons. There is the notion of L^2 -torsion, see for instance [26, Chapter 3], which is essentially defined in terms of the Fuglede-Kadison determinants of the differentials of the L^2 -chain complex of the universal covering of a finite CW-complex or closed manifold. These differentials are given by (r, s) -matrices over $\mathbb{Z}G$, where r and s can be any natural numbers. Therefore it is important to consider matrices and not only elements in $\mathbb{Z}G$. Moreover, these differentials are not injective in general.

Another reason to consider matrices is the possibility to consider restriction to a subgroup of finite index since this passage turns a $(1, 1)$ -matrix into a matrix of the size $([G : H], [G : H])$.

One advantage of $\Lambda(G)$ or $\Lambda_1(G)$ in comparison with $\Lambda^w(G)$ or $\Lambda_1^w(G)$ is the better behavior under approximation, see Sections 7 and 8. The problem is that for a square matrix A over G such that $r_A^{(2)}$ is a weak isomorphism, the operator $r_{p(A)}^{(2)}$ is not necessarily again a weak isomorphism, if we have a not necessarily injective

group homomorphism $p: G \rightarrow Q$ and $p(A)$ is the reduction of A to a matrix over Q .

Remark 3.6 (Dobrowolski's estimate). Dobrowolski [12] shows for a monic polynomial $p(z)$ with $p(0) \neq 0$ which is not a product of cyclotomic polynomials

$$M(p) \geq 1 + \frac{1}{a \exp(bk^k)},$$

where k is the number of non-zero coefficients of p and a and b are given constants.

This triggers the question, whether for a given number k and group G there exists a constant $\Lambda_1(k, G) > 1$ such that for every element $x = \sum_{g \in G} n_g \cdot g$ in $\mathbb{Z}G$ for which at most k of the coefficients n_g are not zero and $\det_{\mathcal{N}(G)}(r_x^{(2)}) \neq 1$ holds, we have $\det_{\mathcal{N}(G)}(r_x^{(2)}) \geq \Lambda_1(k, G)$.

For $G = \mathbb{Z}$ the existence of $\Lambda_1(k, G) > 1$ follows from Dobrowolski's result, and extends to $G = \mathbb{Z}^d$ by the iterated limit appearing in Remark 8.1.

4. THE DETERMINANT CONJECTURE

Recall that the Mahler measure satisfies $M(p) \geq 1$ for any non-trivial polynomial p with integer coefficients. This is expected to be true for the Fuglede-Kadison determinant for all groups, namely, there is the

Conjecture 4.1 (Determinant Conjecture). *Let G be a group. Then for any $A \in M_{r,s}(\mathbb{Z}G)$ the Fuglede-Kadison determinant of the morphism $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ given by right multiplication with A satisfies*

$$\det_{\mathcal{N}(G)}(r_A^{(2)}) \geq 1.$$

Remark 4.2 (Status of the Determinant Conjecture). The following is known for the class \mathcal{D} of groups for which the Determinant Conjecture 4.1 is true, see [13, Theorem 5], [26, Section 13.2 on pages 459 ff], [31, Theorem 1.21].

- (1) Amenable quotient
Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{D}$ and the quotient G/H is amenable. Then $G \in \mathcal{D}$;
- (2) Colimits
If $G = \text{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I (with not necessarily injective structure maps) and each G_i belongs to \mathcal{D} , then G belongs to \mathcal{D} ;
- (3) Inverse limits
If $G = \text{lim}_{i \in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to \mathcal{D} , then G belongs to \mathcal{D} ;
- (4) Subgroups
If H is isomorphic to a subgroup of a group G with $G \in \mathcal{D}$, then $H \in \mathcal{D}$;
- (5) Quotients with finite kernel
Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of groups. If K is finite and G belongs to \mathcal{D} , then Q belongs to \mathcal{D} ;
- (6) Sofic groups belong to \mathcal{D} .

The class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups. One expects that there exists non-sofic groups but no example is known. More information about sofic groups can be found for instance in [14] and [30].

Remark 4.3 (Invertible matrices and the Determinant Conjecture). Let G be a group. Consider a matrix $A \in GL_r(\mathbb{Z}G)$. Then we get from Theorem 14.18 (1)

$$\det_{\mathcal{N}(G)}(r_A^{(2)}) \cdot \det_{\mathcal{N}(G)}(r_{A^{-1}}^{(2)}) = 1.$$

If G satisfies the Determinant Conjecture 4.1, we get

$$(4.4) \quad \det_{\mathcal{N}(G)}(r_A) = 1 \quad \text{for } A \in GL_r(\mathbb{Z}G).$$

The argument appearing in the proof of [28, Theorem 6.7 (2)] shows that the K -theoretic Farrell-Jones Conjecture for $\mathbb{Z}G$ also implies (4.4).

5. BASIC PROPERTIES OF LEHMER'S CONSTANT FOR ARBITRARY GROUPS

Lemma 5.1. (1) *If H is a subgroup of G , then*

$$\begin{aligned} \Lambda(G) &\leq \Lambda(H); \\ \Lambda_1(G) &\leq \Lambda_1(H); \\ \Lambda^w(G) &\leq \Lambda^w(H); \\ \Lambda_1^w(G) &\leq \Lambda_1^w(H); \end{aligned}$$

(2) *If $H \subseteq G$ has finite index, then*

$$\begin{aligned} \Lambda(H)^{[G:H]^{-1}} &\leq \Lambda(G); \\ \Lambda^w(H)^{[G:H]^{-1}} &\leq \Lambda^w(G); \end{aligned}$$

(3) *We have*

$$\begin{aligned} \Lambda_1(\{1\}) = \Lambda^w(\{1\}) = \Lambda_1^w(\{1\}) &= 2, \\ \Lambda(\{1\}) &= \sqrt{2}, \end{aligned}$$

and

$$\begin{aligned} \Lambda^w(\mathbb{Z}/2) = \Lambda_1^w(\mathbb{Z}/2) &= \sqrt{3}; \\ \Lambda_1(\mathbb{Z}/2) &= \sqrt{2}; \\ 2^{1/4} &\leq \Lambda(\mathbb{Z}/2) \leq \sqrt{2}; \end{aligned}$$

(4) *If G is finite and $|G| \geq 3$, we get*

$$\begin{aligned} 2^{|G|^{-1}} &\leq \Lambda^w(G) \leq \Lambda_1^w(G) \leq (|G| - 1)^{|G|^{-1}}; \\ 2^{(2|G|)^{-1}} &\leq \Lambda(G) \leq \Lambda_1(G) \leq (|G| - 1)^{|G|^{-1}}; \end{aligned}$$

(5) *Let G be a group. Then*

$$\begin{aligned} \Lambda(G) &= \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1(G) &= \inf\{\Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda^w(G) &= \inf\{\Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1^w(G) &= \inf\{\Lambda_1^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\}. \end{aligned}$$

Proof. (1) Consider $A \in M_{r,s}(\mathbb{Z}H)$. Let $i: H \rightarrow G$ be the inclusion. By applying the ring homomorphism $\mathbb{Z}H \rightarrow \mathbb{Z}G$ induced by i to the entries of A , we obtain a matrix $i_*A \in M_{r,s}(\mathbb{Z}G)$. Then we get

$$\det_{\mathcal{N}(G)}(r_{i_*A}^{(2)}) = \det_{\mathcal{N}(H)}(r_A^{(2)})$$

from Theorem 14.18 (5) and hence $\Lambda(H) \leq \Lambda(G)$ and $\Lambda_1(H) \leq \Lambda_1(G)$.

If $r = s$ and $r_A^{(2)}$ is injective, then $r_{i_*A}^{(2)}$ is injective because of [26, Lemma 1.24 (3) on page 30]. This implies $\Lambda^w(G) \leq \Lambda^w(H)$ and $\Lambda_1^w(G) \leq \Lambda_1^w(H)$.

(2) Consider a matrix $A \in M_{r,s}(\mathbb{Z}G)$. We have introduced the bounded G -equivariant operator $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ in (3.1). Let $i^*r_A^{(2)}: i^*L^2(G)^r \rightarrow i^*L^2(G)^s$ be the bounded H -equivariant operator obtained by restricting the G -action to an H -action. Since $[G:H]$ is finite, there is an H -equivariant isometric isomorphism

of Hilbert spaces from $L^2(H)^{[G:H]}$ to $i^*L^2(G)$. Hence for an appropriate matrix $B \in M_{r \cdot [G:H], s \cdot [G:H]}(\mathbb{Z}H)$ the bounded H -equivariant operator $i^*r_A^{(2)}: i^*L^2(G)^r \rightarrow i^*L^2(G)^s$ can be identified with $r_B^{(2)}: L^2(H)^{r \cdot [G:H]} \rightarrow L^2(H)^{s \cdot [G:H]}$. We conclude from Theorem 14.18 (4)

$$\det_{\mathcal{N}(H)}(r_B^{(2)}) = \det_{\mathcal{N}(H)}(i^*r_A^{(2)}) = \det_{\mathcal{N}(G)}(r_A^{(2)})^{[G:H]}.$$

This implies $\Lambda(G)^{[G:H]} \geq \Lambda(H)$.

If $r = s$ and $r_A^{(2)}$ is injective, then $r_{i^*A}^{(2)}$ is injective. This implies $\Lambda^w(G)^{[G:H]} \geq \Lambda^w(H)$.

(3) Consider $A \in M_{r,s}(\mathbb{Z}) = M_{r,s}(\mathbb{Z}[\{1\}])$. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the eigenvalues of AA^* (listed with multiplicity), which are different from zero. We get from Example 14.15

$$\det_{\mathcal{N}(\{1\})}(r_A^{(2)}) = \sqrt{\prod_{i=1}^r \lambda_i}.$$

Let $p(t) = \det_{\mathbb{C}}(t - AA^*)$ be the characteristic polynomial of AA^* . It can be written as $p(t) = t^a \cdot q(t)$ for some polynomial $q(t)$ with integer coefficients and $q(0) \neq 0$. One easily checks

$$|q(0)| = \prod_{i=1}^r \lambda_i.$$

Since q has integer coefficients, we conclude $\det(r_A^{(2)}) = \sqrt{n}$ for some integer $n \geq 1$.

A direct calculation shows $\det(r_A^{(2)}) = \sqrt{2}$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Hence we get

$$\Lambda(\{1\}) = \sqrt{2}.$$

Consider the square matrix $A \in M_{r,r}(\mathbb{Z}) = M_{r,r}(\mathbb{Z}[\{1\}])$ such that $r_A^{(2)}$ is a weak isomorphism, or, equivalently, A is invertible as a matrix over \mathbb{C} . Then we conclude from Example 14.15

$$\det_{\mathcal{N}(\{1\})}(r_A^{(2)}) = |\det_{\mathbb{C}}(A)| = |\det_{\mathbb{Z}}(A)| \in \{n \in \mathbb{Z} \mid n \geq 1\}.$$

This implies

$$\Lambda_1(\{1\}) = \Lambda^w(\{1\}) = \Lambda_1^w(\{1\}) = 2.$$

Consider $A \in M_{r,r}(\mathbb{Z}[\mathbb{Z}/2])$. It induces a $\mathbb{Z}[\mathbb{Z}/2]$ -homomorphism $r_A: \mathbb{Z}[\mathbb{Z}/2]^r \rightarrow \mathbb{Z}[\mathbb{Z}/2]^r$. There exists an obvious short exact sequence of $\mathbb{Z}[\mathbb{Z}/2]$ -modules $0 \rightarrow \mathbb{Z}^- \rightarrow \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}^+ \rightarrow 0$, where \mathbb{Z} is the underlying abelian group of \mathbb{Z}^\pm and the generator of $\mathbb{Z}/2$ acts by $\pm \text{id}$ on \mathbb{Z}^\pm . We obtain a commutative diagram of endomorphisms of finitely generated free \mathbb{Z} -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^- & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^+ \longrightarrow 0 \\ & & \downarrow r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^-} & & \downarrow r_A & & \downarrow r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^+} \\ 0 & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^- & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2]^r \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^+ \longrightarrow 0 \end{array}$$

This implies

$$\det_{\mathbb{Z}}(r_A) = \det_{\mathbb{Z}}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^-}) \cdot \det_{\mathbb{Z}}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^+}).$$

Since $\mathbb{Z}^+ \otimes_{\mathbb{Z}} \mathbb{F}_2$ and $\mathbb{Z}^- \otimes_{\mathbb{Z}} \mathbb{F}_2$ are isomorphic as $\mathbb{F}_2[\mathbb{Z}/2]$ -modules, we get

$$\det_{\mathbb{F}_2}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^-} \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}_2}) = \det_{\mathbb{F}_2}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^+} \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}_2}).$$

Since the reduction to \mathbb{F}_2 of $\det_{\mathbb{Z}}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^\pm})$ is $\det_{\mathbb{F}_2}(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^\pm} \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}_2})$, we conclude

$$\det(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^-}) = \det(r_A \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \text{id}_{\mathbb{Z}^+}) \pmod{2}.$$

This implies that $\det_{\mathbb{Z}}(r_A)$ is odd or divisible by four. In particular $|\det_{\mathbb{Z}}(r_A)|$ is different from 2. This implies that for any matrix $A \in M_{r,r}(\mathbb{Z}[\mathbb{Z}/2])$ for which $r_A: \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}[\mathbb{Z}/2]$ is injective, we have $|\det_{\mathbb{Z}}(r_A)| = 1$ or $|\det_{\mathbb{Z}}(r_A)| \geq 3$.

One easily checks that $|\det_{\mathbb{Z}}(r_{t+2}: \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}[\mathbb{Z}/2])| = 3$. Since for any matrix $A \in M_{r,r}(\mathbb{Z}[\mathbb{Z}/2])$ with injective $r_A^{(2)}: L^2(\mathbb{Z}/2)^r \rightarrow L^2(\mathbb{Z}/2)^r$ we have $\det_{\mathcal{N}(\mathbb{Z}/2)}(r_A^{(2)}) = \sqrt{|\det_{\mathbb{Z}}(r_A)|}$ by Example 14.15, we conclude $\Lambda^w(\mathbb{Z}/2) = \Lambda_1^w(\mathbb{Z}/2) = \sqrt{3}$.

Since $\det_{\mathcal{N}(\mathbb{Z}/2)}(r_{t+1}: L^2(\mathbb{Z}/2) \rightarrow L^2(\mathbb{Z}/2)) = \sqrt{2}$ holds by Example 14.16 and Theorem 14.18 (4), we get $\Lambda_1(\mathbb{Z}/2) \leq \sqrt{2}$. We conclude $\sqrt{2} \leq \Lambda_1(\mathbb{Z}/2)$ from $\Lambda_1(\{1\}) = 2$ and assertion (2). This implies $\Lambda_1(\mathbb{Z}/2) = \sqrt{2}$.

We conclude $2^{1/4} \leq \Lambda(\mathbb{Z}/2)$ from $\Lambda_1(\{1\}) = \sqrt{2}$ and assertion (2).

(4) We conclude from assertions (2) and (3) for the finite group G

$$\begin{aligned} \Lambda(G) &\geq 2^{(2|G|)^{-1}}; \\ \Lambda^w(G) &\geq 2^{|G|^{-1}}. \end{aligned}$$

Consider the norm element $N_G := \sum_{g \in G} g$. Let $e \in G$ be the unit element. Put $x = N_G - e \in \mathbb{Z}G$. We have a canonical $\mathbb{C}G$ -decomposition $\mathbb{C}G = \mathbb{C} \oplus V$, where \mathbb{C} is the trivial G -representation and V is a direct sum of irreducible G -representations with $V^G = 0$. Then $r_{N_G}: \mathbb{C}G \rightarrow \mathbb{C}G$ is the direct sum of $|G| \cdot \text{id}: \mathbb{C} \rightarrow \mathbb{C}$ and $0: V \rightarrow V$. Hence $r_x: \mathbb{C}G \rightarrow \mathbb{C}G$ is the direct sum of $(|G| - 1) \cdot \text{id}_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C}$ and of $-\text{id}_V: V \rightarrow V$. This implies that $r_x: \mathbb{C}G \rightarrow \mathbb{C}G$ is a \mathbb{C} -isomorphism and

$$\det_{\mathbb{C}}(r_x: \mathbb{C}G \rightarrow \mathbb{C}G) = |G| - 1.$$

We conclude from by Example 14.15

$$\det_{\mathcal{N}(G)}(r_x^{(2)}) = (|G| - 1)^{|G|^{-1}}.$$

Since $|G| \geq 3$ and hence $(|G| - 1)^{|G|^{-1}}$ is different from 1, we get

$$\Lambda_1(G) \leq \Lambda_1^w(G) \leq (|G| - 1)^{|G|^{-1}}.$$

(5) We obtain

$$\begin{aligned} \Lambda(G) &\leq \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1(G) &\leq \inf\{\Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda^w(G) &\leq \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1^w(G) &\leq \inf\{\Lambda_1^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\}, \end{aligned}$$

from assertion (1).

Consider any matrix $A \in M_{r,s}(\mathbb{Z}G)$. Let H be the subgroup of G which is generated by the finite set consisting of those elements $g \in G$ for which for at least one entry in A the coefficient of g is non-trivial. Then $H \subseteq G$ is finitely generated and $A = i_* B$ for some matrix $B \in M_{r,s}(\mathbb{Z}H)$ for the inclusion $i: H \rightarrow G$. We get

$$\det_{\mathcal{N}(G)}(r_B^{(2)}) = \det_{\mathcal{N}(H)}(r_A^{(2)})$$

from Theorem 14.18 (5). This implies

$$\begin{aligned} \Lambda(G) &\geq \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1(G) &\geq \inf\{\Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\}. \end{aligned}$$

If $r = s$ and $r_A^{(2)}$ is injective, then also $r_B^{(2)}$ is injective. Hence we get

$$\begin{aligned}\Lambda^w(G) &\geq \inf\{\Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\}; \\ \Lambda_1^w(G) &\geq \inf\{\Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\}.\end{aligned}$$

This finishes the proof of Lemma 5.1. \square

Example 5.2 (Finite cyclic group of odd order). Let n be an odd natural number. Then we get for the finite cyclic group \mathbb{Z}/n the equality

$$(5.3) \quad \Lambda^w(\mathbb{Z}/n) = \Lambda_1^w(\mathbb{Z}/n) = 2^{n-1}.$$

Namely, let $t \in \mathbb{Z}/n$ be a generator. Consider the element $t + 1$. Then the $\mathbb{Z}[\mathbb{Z}/n]$ -homomorphism $r_{t+1}: \mathbb{Z}[\mathbb{Z}/n] \rightarrow \mathbb{Z}[\mathbb{Z}/n]$ defines after forgetting the \mathbb{Z}/n -action an \mathbb{Z} -automorphism of \mathbb{Z}^n given by the matrix

$$B[n] = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

Since n is odd, we compute by developing after the first row $\det(B[n]) = 2$. Hence $r_{t+1}^{(2)}$ is injective and we get from Example 14.15.

$$\det_{\mathcal{N}(\mathbb{Z}/n)}(r_{t+1}^{(2)}) = 2^{n-1}.$$

This together with Lemma 5.1 (4) implies (5.3).

Moreover, we get

$$(5.4) \quad 2^{(2n)-1} \leq \Lambda(\mathbb{Z}/n) \leq \Lambda_1(\mathbb{Z}/n) \leq 2^{n-1}.$$

Remark 5.5 (Computations for finite abelian groups). Lind [23, Definition 1.1] has introduced a Lehmer constant for compact abelian groups. If G is a finite abelian group, then his constant agrees with $\ln(\Lambda_1^w(G))$ for the number $\Lambda_1^w(G)$ introduced in Definition 3.2. Lind gives some precise values and some estimates for $\ln(\Lambda_1^w(G))$ for finite abelian groups which were considerably improved by Kaiblinger [18] for finite cyclic groups.

Next we show that $\Lambda_1^w(G) = \Lambda^w(G)$ holds for finite abelian G . The classical determinant $\det_{\mathbb{C}G}$ induces an isomorphism $K_1(\mathbb{C}G) \xrightarrow{\cong} \mathbb{C}G^\times$. The Fuglede-Kadison determinant $\det_{\mathcal{N}(G)}$ induces a homomorphism $K_1(\mathbb{C}G) \rightarrow \{r \in \mathbb{R} \mid r > 0\}$. We have $\det_{\mathbb{Z}G}(A) = \det_{\mathbb{C}G}(A)$ for $A \in M_{r,r}(\mathbb{Z}G)$. For $A \in M_{r,r}(\mathbb{Z}G)$ the map $r_A^{(2)}$ is a weak isomorphism if and only if it is an isomorphism, or, equivalently, $\det_{\mathbb{Z}G}(A) = \det_{\mathbb{C}G}(A)$ is a unit in $\mathbb{C}G$. This implies for $A \in M_{r,r}(\mathbb{Z}G)$ for which $r_A^{(2)}$ is a weak isomorphism, that for $d := \det_{\mathbb{Z}G}(A)$ the map $r_d^{(2)}$ is a weak isomorphism satisfying $\det_{\mathcal{N}(G)}(r_A^{(2)}) = \det_{\mathcal{N}(G)}(r_d^{(2)})$. Hence we get $\Lambda_1^w(G) \leq \Lambda^w(G)$ and therefore $\Lambda_1^w(G) = \Lambda^w(G)$.

In general we have $\Lambda_1(\mathbb{Z}/n) \neq \Lambda_1^w(\mathbb{Z}/n)$ and $\Lambda(\mathbb{Z}/n) \neq \Lambda^w(\mathbb{Z}/n)$, see Lemma 5.1 (3).

Computations for finite dihedral groups can be found in [2].

6. TORSIONFREE ELEMENTARY AMENABLE GROUPS

Throughout this section let G be an amenable group for which $\mathbb{Q}G$ has no non-trivial zero-divisor. Examples for G are torsionfree elementary amenable groups, see [20, Theorem 1.2], [24, Theorem 2.3]. Then $\mathbb{Q}G$ has a skewfield of fractions $S^{-1}\mathbb{Q}G$ given by the Ore localization with respect to the multiplicative closed subset S of non-trivial elements in $\mathbb{Q}G$, see [26, Example 8.16 on page 324].

Next want to define a homomorphism

$$(6.1) \quad \Delta: K_1(S^{-1}\mathbb{Q}G) \rightarrow \mathbb{R}^{>0}$$

as follows. Consider any natural number r and a matrix $A \in GL_r(S^{-1}\mathbb{Q}G)$. We can choose $a \in \mathbb{Q}G$ with $a \neq 0$ such that $A[a] := (a \cdot I_r) \cdot A$ belongs to $M_{r,r}(\mathbb{Q}G)$, where $(a \cdot I_r)$ is the diagonal (r, r) -matrix whose entries on the diagonal are all equal to a . Since G satisfies the Determinant Conjecture 4.1 by Remark 4.2, the Fuglede-Kadison determinants of both $r_{A[a]}: L^2(G)^r \rightarrow L^2(G)^r$ and $r_a: L^2(G)^r \rightarrow L^2(GF)^r$ are well-defined real numbers. If $[A]$ denotes the class represented by A in $K_1(S^{-1}\mathbb{Q}G)$, we want to define

$$\Delta([A]) := \frac{\det_{\mathcal{N}(G)}(r_{A[a]}^{(2)})}{\det_{\mathcal{N}(G)}(r_{a \cdot I_r}^{(2)})}.$$

Note for the sequel that $r_{A[a]}: L^2(G)^r \rightarrow L^2(G)^r$ and $r_{a \cdot I_r}: L^2(G)^r \rightarrow L^2(G)^r$ are weak isomorphisms by Lemma 6.4 (1). The proof that that this is a well-defined homomorphism of abelian groups can be found in in [28] on the pages following (7.14), take $F = \mathbb{Q}$ and V to be the trivial 1-dimensional representation there.

There is a Dieudonne determinant for invertible matrices over a skewfield K which takes values in the abelianization of the group of units of the skewfield $K^\times/[K^\times, K^\times]$ and induces an isomorphism, see [32, Corollary 4.3 in page 133]

$$(6.2) \quad \det_D: K_1(K) \xrightarrow{\cong} K^\times/[K^\times, K^\times].$$

The inverse

$$(6.3) \quad \iota: K^\times/[K^\times, K^\times] \xrightarrow{\cong} K_1(K)$$

sends the class of a unit to the class of the corresponding $(1, 1)$ -matrix. In the sequel K is chosen to be $S^{-1}\mathbb{Q}G$.

The next result is a special case of [28, Lemma 7.23], take $F = \mathbb{Q}$ and V to be the trivial 1-dimensional representation.

Lemma 6.4. *Consider any matrix $A \in M_{r,r}(\mathbb{Q}G)$. Then*

(1) *The following statements are equivalent:*

- (a) $r_A: \mathbb{Q}G^r \rightarrow \mathbb{Q}G^r$ is injective;
- (b) $r_A: S^{-1}\mathbb{Q}G^r \rightarrow S^{-1}\mathbb{Q}G^r$ is injective;
- (c) $r_A: S^{-1}\mathbb{Q}G^r \rightarrow S^{-1}\mathbb{Q}G^r$ is bijective, or, equivalently A becomes invertible over $S^{-1}\mathbb{Q}G$;
- (d) $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^r$ is injective;
- (e) $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^r$ is a weak isomorphism;

(2) *If one of the equivalent conditions above is satisfied, then $r_A^{(2)}$ is a weak isomorphism of determinant class and we get the equation*

$$\det_{\mathcal{N}(G)}(r_A^{(2)}) = \Delta \circ \iota(\det_D(A)),$$

where the homomorphisms Δ and ι have been defined in (6.1) and (6.3).

In particular $\det_{\mathcal{N}(G)}(r_A^{(2)})$ agrees with the quotient $\frac{\det_{\mathcal{N}(G)}(r_x^{(2)})}{\det_{\mathcal{N}(G)}(r_y^{(2)})}$ for two appropriate elements $x, y \in \mathbb{Q}G$ with $x, y \neq 0$.

On the first glance Lemma 6.4 (2) seems to be enough to show $\Lambda^w(G) = \Lambda_1^w(G)$ but this is not the case since we would need to replace $\frac{\det_{\mathcal{N}(G)}(r_x^{(2)})}{\det_{\mathcal{N}(G)}(r_y^{(2)})}$ by $\det_{\mathcal{N}(G)}(r_x^{(2)})$. The following example illustrates why we do not know whether this is true in general.

Example 6.5. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $(2, 2)$ -matrix over a skewfield K , its Dieudonne determinant in $K^\times/[K^\times, K^\times]$ is defined to be the class of $-cb$ if $a = 0$ and to be the class of $ad - aca^{-1}b$ otherwise. It can happen that the matrix A lives over $\mathbb{Q}G$, but the obvious representative of the Dieudonne determinant does not. The following example is due to Peter Linnell. Let G be the metabelian group

$$\mathbb{Z} \wr \mathbb{Z} = \langle x_i, y \mid x_i x_j = x_j x_i, y^{-1} x_i y = x_{i+1} \text{ for all } i, j \in \mathbb{Z} \rangle.$$

Then we have $\mathbb{Q}G \subset L^1(G) \subset \mathcal{U}(G)$, and division ring of quotients for $\mathbb{Q}G$ is contained in $\mathcal{U}(G)$, where $\mathcal{U}(G)$ is the algebra of affiliated operators. Consider the element $2 - x_0 \in \mathbb{Q}G$. Then $(2 - x_0)y(2 - x_0)^{-1}$ is not contained in $\mathbb{Q}G$ by the following observation. This element is the same as $y(1 - x_1/2)(1 - x_0/2)^{-1}$ and now we work inside $L^1(G)$, so we get $y(1 - x_1/2)(1 + x_0/2 + x_0^2/4 + \dots)$. So the Dieudonne determinant of the matrix $A = \begin{pmatrix} 2 - x_0 & 1 \\ y & 0 \end{pmatrix}$ is represented by the element $(2 - x_0)y(2 - x_0)^{-1}$ which is not contained in $\mathbb{Q}G$ although all entries of A belong to $\mathbb{Q}G$.

Remark 6.6. If in the situation of Lemma 6.4 the group G happens to be abelian, then the Dieudonne determinant reduces to the standard determinant $\det_{\mathbb{Q}G}$ for the commutative ring $\mathbb{Q}G$ and it has the property that for a square matrix A over $\mathbb{Q}G$ its value $\det_{\mathbb{Q}G}(A)$ is an element in $\mathbb{Q}G$. Moreover, we can replace in Lemma 6.4 (2) the fraction $\frac{\det_{\mathcal{N}(G)}(r_x^{(2)})}{\det_{\mathcal{N}(G)}(r_y^{(2)})}$ by $\det_{\mathcal{N}(G)}(r_x^{(2)})$ for some $x \in \mathbb{Q}G$ with $x \neq 0$.

We conclude from Lemma 6.4 (2) and Remark 6.6

Lemma 6.7. *We have $\Lambda^w(\mathbb{Z}^d) = \Lambda_1^w(\mathbb{Z}^d)$.*

Remark 6.8. Example 6.5 does not rule out the possibility that for every square matrix A over $\mathbb{Q}G$, which is invertible over $S^{-1}\mathbb{Q}G$, there exists a non-trivial element $u \in \mathbb{Q}G$ such that the Dieudonne determinant of A regarded as invertible matrix over the skewfield $S^{-1}\mathbb{Q}G$ in the abelian group $S^{-1}\mathbb{Q}G^\times/[S^{-1}\mathbb{Q}G^\times, S^{-1}\mathbb{Q}G^\times]$ is represented by u . We neither have a proof for this claim nor a counterexample. This question is also interesting in connection with the L^2 -polytope homomorphism appearing in [15, Section 3.2]. Moreover, a positive answer implies [19, Theorem 5.14].

7. GENERAL APPROXIMATION RESULTS

In this section we explain how approximation techniques may help in the future to extend the class of groups for which one can give a positive answer to Lehmer's problem.

7.1. Approximation Conjecture for Fuglede-Kadison determinants. We have the following conjecture which was formulated as a question in [26, Question 13.52 on page 478], see also [27, Section 15].

Conjecture 7.1 (Approximation Conjecture for Fuglede-Kadison determinants). *Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i \in I} G_i = \{1\}$. Put*

$Q_i := G/G_i$. Consider any matrix $A \in M_{r,s}(\mathbb{Z}G)$. Denote by $A_i \in M_{r,s}(\mathbb{Z}Q_i)$ the reduction of A to $\mathbb{Z}Q_i$ coming from the projection $G \rightarrow Q_i$.

Then we get for the Fuglede-Kadison determinants

$$\begin{aligned} \det_{\mathcal{N}(G)}(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s) \\ = \lim_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)} : L^2(Q_i)^r \rightarrow L^2(Q_i)^s). \end{aligned}$$

Unfortunately, the status of Conjecture 7.1 is very poor, it does follow for virtually cyclic groups G from the special case proved in [26, Lemma 13.53 on page 478], but we are not aware of a proof for a group which is not virtually cyclic. Nevertheless there is hope that Conjecture 7.1 is true for torsionfree groups.

Remark 7.2 (Integer coefficients are necessary). There are counterexamples to Conjecture 7.1 if one replaces the coefficients in \mathbb{Z} by coefficients in \mathbb{C} , see [26, Example 13.69 on page 481]. This is in contrast to Theorem 8.4.

Conjecture 7.1 has the following interesting consequence.

Theorem 7.3 (Consequence of the Approximation Theorem Conjecture for Fuglede-Kadison determinants). *Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i \in I} G_i = \{1\}$. Put $Q_i := G/G_i$. Assume that each group Q_i satisfies the Determinant Conjecture 4.1. Moreover, suppose that G satisfies the Approximation Conjecture for Fuglede-Kadison determinants 7.1. Then*

(1) *We have*

$$\begin{aligned} \Lambda(G) &\geq \limsup_{i \in I} \Lambda(Q_i); \\ \Lambda_1(G) &\geq \limsup_{i \in I} \Lambda_1(Q_i); \end{aligned}$$

(2) *Suppose that for any element $A \in M_{r,s}(\mathbb{Z}G)$ there exists a constant $\beta(A) > 0$ and an index $i_0(A) \in I$ such that the implication $\dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) > 0 \implies \dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) \geq \beta(A)$ holds for all $i \in I$ with $i \geq i_0(A)$. (We will recall the notion of the von Neumann dimension $\dim_{\mathcal{N}(G)}$ in Appendix 14.)*

Then we have

$$\Lambda^w(G) \geq \limsup_{i \in I} \Lambda^w(Q_i);$$

(3) *Suppose that for any element $A \in M_{1,1}(\mathbb{Z}G)$ there exists a constant $\beta_1(A) > 0$ and an index $i_0(A) \in I$ such that the implication $\dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) > 0 \implies \dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) \geq \beta_1(A)$ holds for all $i \in I$ with $i \geq i_0(A)$.*

Then we have

$$\Lambda_1^w(G) \geq \limsup_{i \in I} \Lambda_1^w(Q_i);$$

Proof. (1) This is obvious.

(2) and (3) Consider a matrix $A \in M_{r,r}(\mathbb{Z}G)$ such that $r_A^{(2)}$ is injective, or, equivalently, $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = 0$. We conclude from [26, Theorem 13.19 (2) on page 461]

$$0 = \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = \lim_{i \in I} \dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})).$$

Hence there exists i_0 such that $\dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) = 0$ holds for $i \geq i_0$ and hence that $r_{A_i}^{(2)}$ is injective for $i \geq i_0$. \square

Remark 7.4 (Atiyah Conjecture). A version of the *Atiyah Conjecture* says for a group G for which there exists a natural number D such that the order of any finite subgroup of G divides D that for any element $A \in M_{r,s}(\mathbb{Z}G)$ we get

$$D \cdot \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s)) \in \mathbb{Z}.$$

If G happens to be torsionfree, we can choose $D = 1$ and get the implication $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s)) \in \mathbb{Z}$.

Suppose that there exists a natural number D such that for every $i \in I$ and every finite subgroup $H \subseteq Q_i$ the order $|H|$ divides D . Then the implication $\dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) > 0 \implies \dim_{\mathcal{N}(Q_i)}(\ker(r_{A_i}^{(2)})) \geq \beta(A)$ appearing in assertions (2) and (3) of Theorem 7.3 is automatically satisfied if each group Q_i satisfies the Atiyah Conjecture above, just take $\beta(A) := \frac{1}{D}$.

A survey on the Atiyah Conjecture and the groups for which it is known to be true can be found in [26, Chapter 10]. We mention that the Atiyah Conjecture holds for G if G is elementary amenable and there is a upper bound on the orders of the finite subgroups of G .

We conclude from Theorem 7.3 and Remark 7.4

Theorem 7.5 (Residually torsionfree elementary amenable groups). *Let G be a residually torsionfree elementary amenable group in the sense that we can find a sequence of in G normal subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ such that G/G_n is torsionfree elementary amenable for $n = 0, 1, 2, \dots$ and $\bigcap_{n>0} G_n = \{1\}$. Suppose that G satisfies the Approximation Conjecture 7.1 for Fuglede-Kadison determinants.*

Then

$$\begin{aligned} \Lambda^w(G) &\geq \limsup_{i \in I} \Lambda^w(Q_i); \\ \Lambda_1^w(G) &\geq \limsup_{i \in I} \Lambda_1^w(Q_i). \end{aligned}$$

Example 7.6 (Examples of (virtually) residually torsionfree nilpotent). Free groups are examples of residually torsionfree nilpotent groups, see [29, § 2].

Let M be a compact orientable irreducible 3-manifold whose boundary is empty or is a disjoint union of incompressible tori and whose fundamental group π is infinite and not solvable. Then its fundamental group π is virtually residually torsionfree nilpotent, see for instance [1, page 84].

This implies that $\Lambda(\pi) > 1$ if π satisfies the Approximation Conjecture 7.1 and $\Lambda^w(H) = \Lambda(\mathbb{Z})$ holds for any torsionfree nilpotent group. Note that we cannot conclude $\Lambda(\pi) = \Lambda(\mathbb{Z})$ since we only know that π is virtually residually torsionfree nilpotent and not that π is residually torsionfree nilpotent.

Remark 7.7. In order to apply Theorem 7.5 one needs to know Conjecture 7.1 which we have already discussed above and also have some information for $\Lambda^w(H)$, for nilpotent groups. Not much is known for these groups. Not even the three-dimensional Heisenberg group is fully understood. See for instance [10, Section 5].

7.2. Sub-Approximation Theorem. At least we can prove an inequality in a situation which is more general than the case of a normal chain considered in Subsection 7.1.

Given a matrix $A \in M_{r,s}(\mathbb{C}G)$, we will in the sequel denote by A^* the element in $M_{s,r}(\mathbb{C}G)$, whose (i, j) -th entry is $a_{j,i}^*$, where for an element $x = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we denote by x^* the element $\sum_{g \in G} \overline{\lambda_g} \cdot g^{-1} \in \mathbb{C}G$. With this convention the adjoint $(r_A^{(2)})^*$ of the bounded operator $r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s$ is $r_{A^*}^{(2)} : L^2(G)^s \rightarrow L^2(G)^r$.

Theorem 7.8 (Sub-Approximation Theorem). *Let G be a group. Suppose that I is a directed set and we have a collection of groups $\{Q_i \mid i \in I\}$ together with group homomorphisms $q_i: G \rightarrow Q_i$ for each $i \in I$. Given $A \in M_{r,s}(\mathbb{Z}G)$, denote by $A_i \in M_{r,s}(\mathbb{Z}Q_i)$ the reduction of A to $\mathbb{Z}Q_i$ coming from the projection $G \rightarrow Q_i$. Suppose:*

- *For any finite subset $F \subseteq G$ there exists an index $i_0(F) \in I$ such that for all $i \geq i_0(F)$ and $f \in F$ the implication $q_i(f) \neq e \implies f \neq e$ holds, where e denotes the unit element in G and Q_i ; (This is automatically satisfied if there is an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i \in I} G_i = \{1\}$, $Q_i = G/G_i$ and $q_i: G \rightarrow Q_i$ is the projection.)*
- *Each group Q_i satisfies the Determinant Conjecture 4.1.*

Then for any element $A \in M_{r,s}(\mathbb{Z}G)$ we have

$$\det_{\mathcal{N}(G)}(r_A^{(2)}) \geq \limsup_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}).$$

Proof. We conclude from Theorem 14.18 (3)

$$\begin{aligned} \det_{\mathcal{N}(G)}(r_A) &= \sqrt{\det_{\mathcal{N}(G)}(r_{AA^*}^{(2)})}; \\ \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) &= \sqrt{\det_{\mathcal{N}(Q_i)}(r_{A_i A_i^*}^{(2)})}. \end{aligned}$$

Hence we can assume without loss of generality that $r = s$ and that $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^r$ and $r_{A_i}^{(2)}: L^2(Q_i)^r \rightarrow L^2(Q_i)^r$ are positive operators, otherwise replace A by AA^* .

We want to apply [26, Theorem 13.19 on page 461] in the case, where G_i in the notation of [26, Theorem 13.19 on page 461] is Q_i , $A_i \in M_{r,r}(\mathbb{Z}Q_i)$ is the matrix above and $\text{tr}_i = \text{tr}_{\mathcal{N}(G_i)}$. Then the claim does not follow directly from the assertions in [26, Theorem 13.19 on page 461] but from the inequality

$$\ln \left(\det_{\mathcal{N}(G)}(r_A^{(2)}) \right) \geq \limsup_{i \in I} \ln \left(\det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) \right)$$

appearing at the very end of the proof of [26, Theorem 13.19 on page 465]. It remains to check all the assumptions appearing in [26, Theorem 13.19 on page 461].

Choose a real number K satisfying

$$K \geq \sqrt{(2r-1) \cdot r} \cdot \max \{ \|A_{j,k}\|_1 \mid 1 \leq j \leq r, 1 \leq k \leq s \}.$$

Then we also have

$$K \geq \sqrt{(2r-1) \cdot r} \cdot \max \{ \|(A_i)_{j,k}\|_1 \mid 1 \leq j \leq r, 1 \leq k \leq s \}.$$

This implies the inequalities $\|r_A^{(2)}\| \leq K$ and $\|r_{A_i}^{(2)}\| \leq K$ for the operator norms of $r_A^{(2)}$ and $r_{A_i}^{(2)}$.

Consider a polynomial p with real coefficients. Let F be the finite set of elements g in G for which there exists a natural number j with $1 \leq j \leq r$ such that the coefficient of g of the (j, j) -th entry of $p(A)$ is different from zero. Choose $i_0 \in I$ such for all $i \geq i_0$ and $f \in F$ the implication $q_i(f) \neq e \implies f \neq e$ holds. This implies

$$\text{tr}_{\mathcal{N}(G)}(p(A)) = \text{tr}_{\mathcal{N}(Q_i)}(p(A_i)) \quad \text{for } i \geq i_0.$$

In particular we get

$$\text{tr}_{\mathcal{N}(G)}(p(A)) = \lim_{i \in I} \text{tr}_{\mathcal{N}(Q_i)}(p(A_i)).$$

This finishes the proof of Theorem 7.8. \square

Remark 7.9 (Sub-Approximation and Lehmer's problem). Theorem 7.8 looks more promising than Theorem 7.3 if one is interested in the conclusion (1) of Theorem 7.3 only. Theorem 7.3 applies to more general systems of group homomorphisms $g \rightarrow Q_i$ than in Theorem 7.3 and it does not need in contrast to Theorem 7.3 the condition that G satisfies the Approximation Conjecture for Fuglede-Kadison determinants 7.1, but only requires only a milder version to imply the same conclusion as in Theorem 7.3 (1).

Namely, we have additionally to assume that $\det_{\mathcal{N}(G)}(r_A^{(2)}) > 1$ implies the existence of an index i_0 such that $\det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) > 1$ holds for $i \geq i_0$, because then $\det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) > \Lambda(Q_i)$ holds for $i \in I$ with $i \geq i_0$ and hence the inequality $\limsup_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) \geq \limsup_{i \in I} \Lambda(Q_i)$ is true.

Or we have additionally to assume that $\limsup_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) = 1$ implies $\det_{\mathcal{N}(G)}(r_A^{(2)}) = 1$ and that there exists a number $\Lambda > 1$ and an index i_0 such that $\Lambda(Q_i) \geq \Lambda$ holds for $i \geq i_0$, because then either $\limsup_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) = 1$ or the inequality $\limsup_{i \in I} \det_{\mathcal{N}(Q_i)}(r_{A_i}^{(2)}) \geq \Lambda$ holds.

8. APPROXIMATION RESULTS OVER \mathbb{Z}^d

For $G = \mathbb{Z}^d$ we get much better approximation results. Essentially we will generalize the approximation results of Boyd and Lawton to arbitrary matrices over $\mathbb{Z}[\mathbb{Z}^d]$. This will be important for the proof of Theorem 10.1.

Remark 8.1 (Approximating Mahler measures for polynomials in several variables by polynomials in one variable). There is a case, where the inequality in Theorem 7.8 becomes an equality with \limsup replaced by \lim . Namely, let $p(z_1, z_2)$ be a non-trivial polynomial with complex coefficients in two variables z_1 and z_2 . For a natural number k let $p(z, z^k)$ be the polynomial with complex coefficients in one variable z obtained from $p(z_1, z_2)$ by replacing $z_1 = z$ and $z_2 = z^k$ in $p(z_1, z_2)$. This corresponds to the homomorphism $q_k: \mathbb{Z}^2 \rightarrow \mathbb{Z}$, $(n_1, n_2) \mapsto n_1 + k \cdot n_2$. If k is large enough, then $p(z, z^k)$ is again non-trivial. We have the formula

$$\det_{\mathcal{N}(\mathbb{Z}^2)}(r_{p(z_1, z_2)}^{(2)}: L^2(\mathbb{Z}^2) \rightarrow L^2(\mathbb{Z}^2)) = \lim_{k \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{p(z, z^k)}^{(2)}: L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})).$$

Its proof can be found in [4, Appendix 3]. The corresponding formula for a non-trivial polynomial $p(z_1, z_2, \dots, z_d)$ with complex coefficients in d -variables z_1, z_2, \dots, z_d

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{p(z_1, z_2, \dots, z_d)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{p(z, z^{k_2}, \dots, z^{k_d})}^{(2)}: L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})) \end{aligned}$$

is proved in [4, Appendix 4] and [21, Theorem 2]. Note that for given p and natural numbers k_2, k_3, \dots, k_d , we can find natural numbers $N_2, N_3(k_2), \dots, N_d(k_2, k_3, \dots, k_{d-1})$ such that we have $p(z, z^{k_2}, \dots, z^{k_d}) \neq 0$, provided that $k_2 \geq N_2, k_3 \geq N_3(k_2), \dots, k_d \geq N_d(k_2, k_3, \dots, k_d)$ hold.

Consider a natural number $d \geq 2$. For natural numbers k_2, k_3, \dots, k_d define a group homomorphism

$$q(k_2, k_3, \dots, k_d): \mathbb{Z}^d \rightarrow \mathbb{Z}, \quad (a_1, a_2, \dots, a_d) \mapsto a_1 + \sum_{i=2}^d k_i \cdot a_i.$$

It induces a ring homomorphism $\hat{q}(k_2, k_3, \dots, k_d): \mathbb{C}[\mathbb{Z}^d] \rightarrow \mathbb{C}[\mathbb{Z}]$. Given a matrix A , let

$$A[k_2, \dots, k_d] \in M_{r,s}(\mathbb{C}[\mathbb{Z}])$$

be the matrix obtained from A by applying the ring homomorphism $\widehat{q}(k_2, k_3, \dots, k_d)$ to each entry. If $p = p(z_1, z_2, \dots, z_d)$ is polynomial and we regard it as element in $M_{1,1}(\mathbb{C}[\mathbb{Z}^d])$, then $p[k_2, \dots, k_d]$ is the $(1, 1)$ -matrix over $\mathbb{C}[\mathbb{Z}]$ given by the polynomial $p(z, z^{k_2}, \dots, z^{k_d})$. Note that for any element $p \in \mathbb{C}[\mathbb{Z}^d] = \mathbb{C}[z^{\pm 1}, \dots, z_d^{\pm 1}]$ we can find a monomial $z_1^{n_1} \cdot z_2^{n_2} \cdot \dots \cdot z_d^{n_d}$ such that $p_0 := z_1^{n_1} \cdot z_2^{n_2} \cdot \dots \cdot z_d^{n_d} \cdot p$ is a polynomial in variables z_1, \dots, z_d and we have $\det_{\mathcal{N}(\mathbb{Z}^d)}(r_p^{(2)}) = \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{p_0}^{(2)})$. Hence we conclude from the iterated limit appearing in Remark 8.1 that for a $(1, 1)$ -matrix A over $\mathbb{C}[\mathbb{Z}^d]$ we have

$$(8.2) \quad \det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \dots, k_d]}^{(2)}: L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})).$$

We want to extend this to matrices of arbitrary finite size. For this purpose the following formula is useful, which reduces the computation of the Fuglede-Kadison determinant $\det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)})$ for $A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d])$ to the special case, where $r = s$ and $r_A^{(2)}$ is injective, and finally to the case $r = s = 1$.

Lemma 8.3. *Consider a matrix $A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d])$. Then there exists an integer $q \geq 0$ and a matrix $B \in M_{q,r}(\mathbb{C}[\mathbb{Z}^d])$ such that the sequence $0 \rightarrow L^2(\mathbb{Z}^d)^q \xrightarrow{r_B^{(2)}} L^2(\mathbb{Z}^d)^r \xrightarrow{r_A^{(2)}} L^2(\mathbb{Z}^d)^s$ is weakly exact, i.e., $r_B^{(2)}$ is injective and the closure of the image of $r_B^{(2)}$ is the kernel of $r_A^{(2)}$. For any such choice we get for the matrices $D_1 = B^*B + AA^*$ in $M_{r,r}(\mathbb{C}[\mathbb{Z}^d])$ and $D_2 = BB^*$ in $M_{q,q}(\mathbb{C}[\mathbb{Z}^d])$:*

- (1) *The operators $r_{D_1}^{(2)}: L^2(\mathbb{Z}^d)^r \rightarrow L^2(\mathbb{Z}^d)^r$ and $r_{D_2}^{(2)}: L^2(\mathbb{Z}^d)^q \rightarrow L^2(\mathbb{Z}^d)^q$ are injective;*
- (2) *We have*

$$\det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)}) = \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)})}{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})}};$$

- (3) *The elements $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)$ and $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)$ in $\mathbb{C}[\mathbb{Z}^d]$ are different from zero and we get*

$$\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)}) = \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)); \\ \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)}) = \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)).$$

Proof. (1) Let $\mathbb{C}[\mathbb{Z}^d]_{(0)}$ be the quotient field of $\mathbb{C}[\mathbb{Z}^d]$. Let q be the $\mathbb{C}[\mathbb{Z}^d]_{(0)}$ -dimension of the kernel of $r_A: \mathbb{C}[\mathbb{Z}^d]_{(0)}^r \rightarrow \mathbb{C}[\mathbb{Z}^d]_{(0)}^s$. We can choose a matrix $B \in M_{q,r}(\mathbb{C}[\mathbb{Z}^d]_{(0)})$ such that the sequence of $\mathbb{C}[\mathbb{Z}^d]_{(0)}$ -modules $0 \rightarrow \mathbb{C}[\mathbb{Z}^d]_{(0)}^q \xrightarrow{r_B} \mathbb{C}[\mathbb{Z}^d]_{(0)}^r \xrightarrow{r_A} \mathbb{C}[\mathbb{Z}^d]_{(0)}^s$ is exact. By possibly multiplying each entry of B with the same non-trivial element in $\mathbb{C}[\mathbb{Z}^d]$, we can additionally arrange that B lies in $M_{q,r}(\mathbb{C}[\mathbb{Z}^d])$. We conclude from [26, Lemma 1.34 (1) on page 35] that the following sequence is weakly exact

$$0 \rightarrow L^2(\mathbb{Z}^d)^q \xrightarrow{r_B^{(2)}} L^2(\mathbb{Z}^d)^r \xrightarrow{r_A^{(2)}} L^2(\mathbb{Z}^d)^s.$$

We consider it as a chain complex of Hilbert $\mathcal{N}(\mathbb{Z}^d)$ -modules concentrated in dimensions 2, 1 and 0. Then its first and second L^2 -homology vanishes. The Laplace operators of it in dimensions 1 and 2 is given by $r_{D_1}^{(2)}$ and $r_{D_2}^{(2)}$. We conclude from [26, Lemma 3.39 on page 145] applied in the case where we replace $L^2(\mathbb{Z}^d)^s$ by the closure of the image of $r_A^{(2)}$ that $r_{D_1}^{(2)}$ and $r_{D_2}^{(2)}$ are weak isomorphisms and in particular injective.

(2) We conclude from Theorem 14.18 (3) and from [26, Lemma 3.30 on page 140] applied to the chain complex of Hilbert modules above

$$\begin{aligned}
& \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)})) \\
&= \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_B^{(2)})) - \frac{1}{2} \cdot \left(2 \cdot \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})) - \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)})) \right) \\
&= \frac{1}{2} \cdot \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})) - \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})) + \frac{1}{2} \cdot \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)})) \\
&= -\frac{1}{2} \cdot \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})) + \frac{1}{2} \cdot \ln(\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)})) \\
&= \ln \left(\sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)})}{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)})}} \right).
\end{aligned}$$

(3) This follows from assertion (1) and Lemma 6.4 and Remark 6.6. This finishes the proof of Lemma 8.3. \square

Theorem 8.4 (Approximating Mahler measures for matrices over $\mathbb{C}[\mathbb{Z}^d]$ by matrices over $\mathbb{C}[\mathbb{Z}]$). *Consider a matrix $A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d])$. Then we get*

$$\begin{aligned}
& \det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)}): L^2(\mathbb{Z}^d)^r \rightarrow L^2(\mathbb{Z}^d)^s \\
&= \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \dots, k_d]}^{(2)}): L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^s.
\end{aligned}$$

Proof. Because of Lemma 8.3, we can choose for A an integer $q \geq 0$ and a matrix $B \in M_{q,r}(\mathbb{C}[\mathbb{Z}^d])$ such that the sequence $0 \rightarrow L^2(\mathbb{Z})^q \xrightarrow{r_B^{(2)}} L^2(\mathbb{Z}^d)^r \xrightarrow{r_A^{(2)}} L^2(\mathbb{Z}^d)^s$ is weakly exact. Put $D_1 = B^*B + AA^*$ in $M_{r,r}(\mathbb{C}[\mathbb{Z}^d])$ and $D_2 = BB^*$ in $M_{q,q}(\mathbb{C}[\mathbb{Z}^d])$ as in Lemma 8.3. We conclude from Lemma 8.3 that $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)$ and $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)$ in $\mathbb{C}[\mathbb{Z}^d]$ are different from zero and we get

$$(8.5) \quad \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_1}^{(2)}) = \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d));$$

$$(8.6) \quad \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{D_2}^{(2)}) = \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d));$$

$$(8.7) \quad \det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)}) = \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d))}{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)}^{(2)}: L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d))}}.$$

Fix $i = 1, \dots, d-1$. Let b_i be the maximum over the norms of those integers n_i for which there exists a monomial of the form $z_1^{n_1} z_2^{n_2} \dots z_d^{n_d}$ such that its coefficients in the description of $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)$ and $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)$ as a sum of such monomials is non-trivial. Define $c_i = 2 \cdot \sum_{j=1}^i b_j$ for $i = 1, 2, \dots, (d-1)$.

Consider any sequence of natural numbers k_1, k_2, \dots, k_d satisfying $k_2 > c_1$, $k_3 > c_2 \cdot k_2$, $k_4 > c_3 \cdot k_3$, \dots , $k_d > c_{d-1} \cdot k_{d-1}$. Let $l = 1, 2$. Then we get for any two d -tuples (m_1, m_2, \dots, m_d) and (n_1, n_2, \dots, n_d) for which there exists a monomial of the form $z_1^{m_1} z_2^{m_2} \dots z_d^{m_d}$ and $z_1^{n_1} z_2^{n_2} \dots z_d^{n_d}$ such that its coefficients in the description of $\det_{\mathbb{C}[\mathbb{Z}^d]}(D_l)$ as a sum of such monomials is non-trivial

$$\begin{aligned}
q(k_1, k_3, \dots, k_d)((m_1, m_2, \dots, m_d)) &= q(k_1, k_3, \dots, k_d)((n_1, n_2, \dots, n_d)) \\
&\implies (m_1, m_2, \dots, m_d) = (n_1, n_2, \dots, n_d).
\end{aligned}$$

This implies that $(\det_{\mathbb{C}[\mathbb{Z}^d]}(D_l))[k_2, \dots, k_d]$ is not zero. One easily checks

$$\begin{aligned} \det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)[k_2, \dots, k_d] &= \det_{\mathbb{C}[\mathbb{Z}^d]}(D_1[k_2, \dots, k_d]); \\ \det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)[k_2, \dots, k_d] &= \det_{\mathbb{C}[\mathbb{Z}^d]}(D_2[k_2, \dots, k_d]); \\ D_1[k_2, \dots, k_d] &= B[k_2, \dots, k_d]^* B[k_2, \dots, k_d] + A[k_2, \dots, k_d] A[k_2, \dots, k_d]^*; \\ D_2[k_2, \dots, k_d] &= B[k_2, \dots, k_d] B[k_2, \dots, k_d]^*. \end{aligned}$$

We conclude that $\det_{\mathbb{C}[\mathbb{Z}]}(D_1[k_2, \dots, k_d])$ and $\det_{\mathbb{C}[\mathbb{Z}]}(D_2[k_2, \dots, k_d])$ are non-trivial.

Lemma 6.4 (1) implies that $r_{D_1[k_2, \dots, k_d]}^{(2)} : L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^r$ and $r_{D_2[k_2, \dots, k_d]}^{(2)} : L^2(\mathbb{Z})^q \rightarrow$

$L^2(\mathbb{Z})^q$ are weak isomorphism. This implies that the sequence $0 \rightarrow L^2(\mathbb{Z})^q \xrightarrow{r_{B[k_2, \dots, k_d]}^{(2)}}$

$L^2(\mathbb{Z})^r \xrightarrow{r_{A[k_2, \dots, k_d]}^{(2)}} L^2(\mathbb{Z})^s$ is weakly exact since we can view it as a chain complex of Hilbert $\mathcal{N}(\mathbb{Z})$ -modules, its Laplace operator in dimension 1 and 2 is given by the weak isomorphisms $r_{D_1[k_2, \dots, k_d]}^{(2)}$ and $r_{D_2[k_2, \dots, k_d]}^{(2)}$ and we can apply [26, Lemma 3.39 on page 145] applied in the case where we replace $L^2(\mathbb{Z})^s$ by the closure of the image of $r_{A[k_2, \dots, k_d]}^{(2)}$. We conclude from Lemma 8.3 applied to $A[k_2, \dots, k_d]$ and $B[k_2, \dots, k_d]$

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \dots, k_d]}^{(2)} : L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^s) \\ = \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_1[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_2[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}}, \end{aligned}$$

provided that $k_2 > c_1$, $k_3 > c_2 \cdot k_2$, $k_4 > c_3 \cdot k_3$, \dots , $k_d > c_{d-1} \cdot k_{d-1}$. This implies

$$\begin{aligned} (8.8) \quad \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \dots, k_d]}^{(2)} : L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^s) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_1[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_2[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}}. \end{aligned}$$

We get from (8.2), (8.5) and (8.6)

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_1)}^{(2)} : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_1[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})), \end{aligned}$$

and

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}^d]}(D_2)}^{(2)} : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_2[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z})). \end{aligned}$$

This implies

$$\begin{aligned} (8.9) \quad \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_1)}^{(2)} : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d))}{\det_{\mathcal{N}(\mathbb{Z}^d)}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_2)}^{(2)} : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d))}} \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \sqrt{\frac{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_1[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}{\det_{\mathcal{N}(\mathbb{Z})}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(D_2[k_2, \dots, k_d])}^{(2)} : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}))}}. \end{aligned}$$

Putting (8.7), (8.8) and (8.9) together yields the desired equality

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z}^d)}(r_A^{(2)} : L^2(\mathbb{Z}^d)^r \rightarrow L^2(\mathbb{Z}^d)^s) \\ = \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \dots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \dots, k_d]}^{(2)} : L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^s). \end{aligned}$$

This finishes the proof of Theorem 8.4. \square

9. LEHMER'S PROBLEM FOR GROUPS WITH TORSION

Remark 9.1 (Bound on the order of finite subgroups). Let G be a group with $\Lambda_1^w(G) > 1$. Let $H \subseteq G$ be any finite subgroup. We conclude from Lemma 5.1 (1) and (4) that $\Lambda_1^w(G) \leq \Lambda_1^w(H) \leq (|H| - 1)^{|H|^{-1}}$ holds if $|H| \geq 3$. Since we have $\lim_{m \rightarrow \infty} (m - 1)^{m^{-1}} = 1$, there is a constant $C > 1$ depending only on $\Lambda_1^w(G)$ such that $|H| \leq C$ holds.

Example 9.2 (Lamplighter group). Let $L = \mathbb{Z}/2\wr\mathbb{Z}$ be the lamplighter group. It is finitely generated and contains finite subgroups of arbitrary large order. Hence we get $\Lambda(L) = \Lambda^w(L) = \Lambda_1(L) = \Lambda_1^w(L) = 1$ from Remark 9.1. The lamplighter group is a subgroup of a finitely presented group, see for instance [26, Remark 10.24 on page 380]. Hence there exists a finitely presented group G with $\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = 1$ by Lemma 5.1 (1).

The considerations above explain why we will concentrate on torsionfree groups in the sequel.

10. FINITELY GENERATED FREE ABELIAN GROUPS

Theorem 10.1 (Finitely generated free abelian groups). *Let $d \geq 1$ be an integer. Then we have:*

$$\Lambda_1(\mathbb{Z}^d) = \Lambda_1^w(\mathbb{Z}^d) = \Lambda^w(\mathbb{Z}^d) = \Lambda_1(\mathbb{Z}) = \Lambda_1^w(\mathbb{Z}) = \Lambda^w(\mathbb{Z}),$$

and

$$\Lambda(\mathbb{Z}^d) = \Lambda(\mathbb{Z}).$$

Proof. Theorem 10.1 follows after we have shown the following equalities

$$(10.2) \quad \Lambda(\mathbb{Z}^d) = \Lambda(\mathbb{Z});$$

$$(10.3) \quad \Lambda_1^w(\mathbb{Z}^d) = \Lambda_1^w(\mathbb{Z});$$

$$(10.4) \quad \Lambda^w(\mathbb{Z}^d) = \Lambda^w(\mathbb{Z});$$

$$(10.5) \quad \Lambda_1(\mathbb{Z}^d) = \Lambda_1(\mathbb{Z}).$$

Equality (10.2) follows from Theorem 8.4.

Equation (10.3) follows from the conclusion of Lemma 6.4 (1) that for $a \in \mathbb{Z}[\mathbb{Z}^d]$ with $a \neq 0$ the map $r_a^{(2)} : L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d)$ is injective.

Equation (10.4) has already been proved in Lemma 6.7.

Equation (10.5) follows from Lemma 5.1 (1) and Remark 8.1, see in particular (8.2). \square

11. NON ABELIAN FREE GROUPS

We cannot prove anything for non-abelian free groups so far, but we want at least to list some good properties and explain the main problem. Throughout this section F denotes a non-abelian free group.

A ring R is called *semifir* if every finitely generated submodule of a free R -module is free and for a free module any two basis have the same cardinality. For any field K the group ring $K[F]$ is a semifir [7, Corollary on page 68], [11]. Moreover, $\mathbb{C}G$

embeds into a skewfield $\mathcal{D}(F)$ and the inclusion is $\Sigma(\mathbb{C}G \subseteq \mathcal{U}(G))$ -inverting as explained in [26, Section 10.3.6 and Lemma 10.82 on page 408].

A problem is that for a matrix $A \in M_{r,s}(\mathbb{C}F)$ it can happen that $r_A: \mathbb{C}G^r \rightarrow \mathbb{C}G^s$ is injective but $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ is not injective. An example comes for the free group F_r in $r \geq 2$ generators s_1, s_2, \dots, s_r from the $\mathbb{C}[F_r]$ -chain complex of the universal covering of $\bigvee_{i=1}^r S^1$ whose first differential $r_A: \mathbb{C}[F_r]^r \rightarrow \mathbb{C}[F_r]$ is given by the matrix A which is the transpose of $(s_1 - 1, s_2 - 1, \dots, s_r - 1)$. The map r_A is injective since the universal covering is 1-connected. The induced F -equivariant bounded operator $r_A^{(2)}: L^2(F_r)^r \rightarrow L^2(F_r)$ is not injective since Lemma 14.8 implies

$$\dim_{\mathcal{N}(F_r)}(\ker(r_A^{(2)})) \geq \dim_{\mathcal{N}(F_r)}(L^2(F_r)^r) - \dim_{\mathcal{N}(F)}(L^2(F_r)) = r - 1 > 0.$$

Actually we have $\dim_{\mathcal{N}(F_r)}(\ker(r_A^{(2)})) = r - 1$.

12. LEHMER'S PROBLEM FOR TORSIONFREE GROUPS

Theorem 10.1 leads to state the following version of Lehmer's problem for torsionfree groups.

Problem 12.1 (Lehmer's problem for torsionfree groups.). *For which torsionfree group G does*

$$\Lambda^w(G) > 1$$

hold?

13. 3-MANIFOLDS

Let M be a closed hyperbolic 3-manifold with fundamental group π . Then the L^2 -torsion of its universal covering satisfies

$$-\rho^2(\widetilde{M}) = \frac{1}{6\pi} \cdot \text{vol}(M)$$

where $\text{vol}(M)$ is the volume, see for instance [26, Theorem 3.152 on page 187]. There is a natural number n and a (n, n) -matrix A with entries in $\mathbb{Z}\pi$ such that $r_A^{(2)}: L^2(\pi)^n \rightarrow L^2(\pi)^n$ is a weak isomorphisms and its Fulgede-Kadison determinant satisfies

$$-\rho^2(\widetilde{M}) = \ln(\det^{(2)}(r_A^{(2)})).$$

This follows from the argument in the proof of [25, Theorem 2.4]. We conclude

$$\det_{\mathcal{N}(\pi)}^{(2)}(r_A^{(2)}) = \exp\left(\frac{1}{6\pi} \cdot \text{vol}(M)\right).$$

Hence we get

$$(13.1) \quad \Lambda^w(\pi) \leq \exp\left(\frac{1}{6\pi} \cdot \text{vol}(M)\right).$$

Example 13.2 (Weeks manifold). There is a closed hyperbolic 3-manifold W , the so called Weeks manifold, which is the unique closed hyperbolic 3-manifold with smallest volume, see [16, Corollary 1.3]. Its volume is between 0,942 and 0,943. Hence we get from (13.1) the estimate

$$\Lambda^w(\pi) \leq \exp\left(\frac{1}{6\pi} \cdot 0,943\right) \leq 1,06$$

This implies $\Lambda^w(\pi) < M(L)$.

14. APPENDIX: L^2 -INVARIANTS

In this appendix we give some basic definitions and properties about L^2 -invariants.

14.1. Group von Neumann algebras. Denote by $L^2(G)$ the Hilbert space $L^2(G)$ consisting of formal sums $\sum_{g \in G} \lambda_g \cdot g$ for complex numbers λ_g such that $\sum_{g \in G} |\lambda_g|^2 < \infty$. This is the same as the Hilbert space completion of the complex group ring $\mathbb{C}G$ with respect to the pre-Hilbert space structure for which G is an orthonormal basis. Notice that left multiplication with elements in G induces an isometric G -action on $L^2(G)$. Given a Hilbert space H , denote by $\mathcal{B}(H)$ the C^* -algebra of bounded (linear) operators from H to itself, where the norm is the operator norm and the involution is given by taking adjoints.

Definition 14.1 (Group von Neumann algebra). The *group von Neumann algebra* $\mathcal{N}(G)$ of the group G is defined as the algebra of G -equivariant bounded operators from $L^2(G)$ to $L^2(G)$

$$\mathcal{N}(G) := \mathcal{B}(L^2(G))^G.$$

In the sequel we will view the complex group ring $\mathbb{C}G$ as a subring of $\mathcal{N}(G)$ by the embedding of \mathbb{C} -algebras $\rho_r: \mathbb{C}G \rightarrow \mathcal{N}(G)$ which sends $g \in G$ to the G -equivariant operator $r_{g^{-1}}: L^2(G) \rightarrow L^2(G)$ given by right multiplication with g^{-1} .

Remark 14.2 (The general definition of von Neumann algebras). In general a *von Neumann algebra* \mathcal{A} is a sub- $*$ -algebra of $\mathcal{B}(H)$ for some Hilbert space H , which is closed in the weak topology and contains $\text{id}: H \rightarrow H$. Often in the literature the group von Neumann algebra $\mathcal{N}(G)$ is defined as the closure in the weak topology of the complex group ring $\mathbb{C}G$ considered as $*$ -subalgebra of $\mathcal{B}(L^2(G))$. This definition and Definition 14.1 agree, see [17, Theorem 7.2 on page 434].

Example 14.3 (The von Neumann algebra of a finite group). If G is finite, then nothing happens, namely $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$.

Example 14.4 (The von Neumann algebra of \mathbb{Z}^d). In general there is no concrete model for $\mathcal{N}(G)$. However, for $G = \mathbb{Z}^d$, there is the following illuminating model for the group von Neumann algebra $\mathcal{N}(\mathbb{Z}^d)$. Let $L^2(T^d)$ be the Hilbert space of equivalence classes of L^2 -integrable complex-valued functions on the n -dimensional torus T^d , where two such functions are called equivalent if they differ only on a subset of measure zero. Define the ring $L^\infty(T^d)$ by equivalence classes of essentially bounded measurable functions $f: T^d \rightarrow \mathbb{C}$, where essentially bounded means that there is a constant $C > 0$ such that the set $\{x \in T^d \mid |f(x)| \geq C\}$ has measure zero. An element (k_1, \dots, k_n) in \mathbb{Z}^d acts isometrically on $L^2(T^d)$ by pointwise multiplication with the function $T^d \rightarrow \mathbb{C}$, which maps (z_1, z_2, \dots, z_n) to $z_1^{k_1} \dots z_n^{k_n}$. The Fourier transform yields an isometric \mathbb{Z}^d -equivariant isomorphism $L^2(\mathbb{Z}^d) \xrightarrow{\cong} L^2(T^d)$. Hence $\mathcal{N}(\mathbb{Z}^d) = \mathcal{B}(L^2(T^d))^{\mathbb{Z}^d}$. We obtain an isomorphism (of C^* -algebras)

$$L^\infty(T^d) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^d)$$

by sending $f \in L^\infty(T^d)$ to the \mathbb{Z}^d -equivariant operator $M_f: L^2(T^d) \rightarrow L^2(T^d)$, $g \mapsto g \cdot f$, where $(g \cdot f)(x)$ is defined by $g(x) \cdot f(x)$.

14.2. The von Neumann dimension. An important feature of the group von Neumann algebra is its trace.

Definition 14.5 (Von Neumann trace). The *von Neumann trace* on $\mathcal{N}(G)$ is defined by

$$\text{tr}_{\mathcal{N}(G)}: \mathcal{N}(G) \rightarrow \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)},$$

where $e \in G \subseteq L^2(G)$ is the unit element.

Definition 14.6 (Finitely generated Hilbert module). A finitely generated *Hilbert $\mathcal{N}(G)$ -module* V is a Hilbert space V together with a linear isometric G -action such that there exists an isometric linear G -embedding of V into $L^2(G)^r$ for some natural

number r . A map of Hilbert $\mathcal{N}(G)$ -modules $f: V \rightarrow W$ is a bounded G -equivariant operator.

Definition 14.7 (Von Neumann dimension). Let V be a finitely generated Hilbert $\mathcal{N}(G)$ -module. Choose a matrix $A = (a_{i,j}) \in M_{r,r}(\mathcal{N}(G))$ with $A^2 = A$ such that the image of the G -equivariant bounded operator $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^r$ given by A is isometrically G -isomorphic to V . Define the *von Neumann dimension* of V by

$$\dim_{\mathcal{N}(G)}(V) := \sum_{i=1}^r \operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}) \in [0, \infty).$$

The von Neumann dimension $\dim_{\mathcal{N}(G)}(V)$ depends only on the isomorphism class of the Hilbert $\mathcal{N}(G)$ -module V but not on the choice of r and the matrix A . The von Neumann dimension $\dim_{\mathcal{N}(G)}$ is *faithful*, i.e. $\dim_{\mathcal{N}(G)}(V) = 0 \Leftrightarrow V = 0$ holds for any finitely generated Hilbert $\mathcal{N}(G)$ -module V . It is weakly exact in the following sense, see [26, Theorem 1.12 on page 21].

Lemma 14.8. Let $0 \rightarrow V_0 \xrightarrow{i} V_1 \xrightarrow{p} V_2 \rightarrow 0$ be a sequence of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Suppose that it is weakly exact, i.e., i is injective, the closure of i is the kernel of p and the image of p is dense. Then

$$\dim_{\mathcal{N}(G)}(V_1) = \dim_{\mathcal{N}(G)}(V_0) + \dim_{\mathcal{N}(G)}(V_2).$$

Example 14.9 (Von Neumann dimension for finite groups). If G is finite, then $\dim_{\mathcal{N}(G)}(V)$ is $\frac{1}{|G|}$ -times the complex dimension of the underlying complex vector space V .

Example 14.10 (Von Neumann dimension for \mathbb{Z}^d). Let $X \subset T^d$ be any measurable set and $\chi_X \in L^\infty(T^d)$ be its characteristic function. Denote by $M_{\chi_X}: L^2(T^d) \rightarrow L^2(T^d)$ the \mathbb{Z}^d -equivariant unitary projection given by multiplication with χ_X . Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^d)$ -module with $\dim_{\mathcal{N}(\mathbb{Z}^d)}(V) = \operatorname{vol}(X)$.

14.3. Weak isomorphisms. A bounded G -equivariant operator $f: L^2(G)^r \rightarrow L^2(G)^s$ is called a *weak isomorphism* if and only if it is injective and its image is dense. If there exists a weak isomorphism $L^2(G)^r \rightarrow L^2(G)^s$, then we must have $r = s$ by Lemma 14.8. The following statements are equivalent for a bounded G -equivariant operator $f: L^2(G)^r \rightarrow L^2(G)^r$, see [26, Lemma 1.13 on page 23]:

- (1) f is a weak isomorphism;
- (2) Its adjoint f^* is a weak isomorphism;
- (3) f is injective;
- (4) f has dense image;
- (5) The von Neumann dimension of the closure of the image of f is r .

Consider a matrix $A \in M_{r,s}(\mathbb{C}G)$. If $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ is injective, then the $\mathbb{C}G$ -homomorphism $r_A: \mathbb{C}G^r \rightarrow \mathbb{C}G^s$ is injective. The converse is not true in general but in the special case that G is amenable, this follows from [26, Theorem 6.24 on page 249 and Theorem 6.37 on page 259].

14.4. The Fuglede-Kadison determinant.

Definition 14.11 (Spectral density function). Let $f: V \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Denote by $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator f^*f . Define the *spectral density function* of f by

$$F_f: \mathbb{R} \rightarrow [0, \infty) \quad \lambda \mapsto \dim_{\mathcal{N}(G)}(\operatorname{im}(E_{\lambda^2}^{f^*f})).$$

The spectral density function is monotone non-decreasing and right-continuous.

Example 14.12 (Spectral density function for finite groups). Let G be finite and $f: U \rightarrow V$ be a map of finitely generated Hilbert $\mathcal{N}(G)$ -modules, i.e., of finite-dimensional unitary G -representations. Then $F(f)$ is the right-continuous step function whose value at λ is the sum of the complex dimensions of the eigenspaces of f^*f for eigenvalues $\mu \leq \lambda^2$ divided by the order of G , or, equivalently, the sum of the complex dimensions of the eigenspaces of $|f|$ for eigenvalues $\mu \leq \lambda$ divided by the order of G .

Example 14.13 (Spectral density function for \mathbb{Z}^d). Let $G = \mathbb{Z}^d$. In the sequel we use the notation and the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of Example 14.4. For $f \in L^\infty(T^d)$ the spectral density function $F(M_f)$ of $M_f: L^2(T^d) \rightarrow L^2(T^d)$ sends λ to the volume of the set $\{z \in T^d \mid |f(z)| \leq \lambda\}$.

Definition 14.14 (Fuglede-Kadison determinant). Let $f: V \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Let $F_f(\lambda)$ be the spectral density function of Definition 14.11 which is a monotone non-decreasing right-continuous function. Let dF be the unique measure on the Borel σ -algebra on \mathbb{R} which satisfies $dF([a, b]) = F(b) - F(a)$ for $a < b$. Then define the *Fuglede-Kadison determinant*

$$\det_{\mathcal{N}(G)}(f) \in [0, \infty)$$

to be the positive real number

$$\det_{\mathcal{N}(G)}(f) = \exp \left(\int_{0+}^{\infty} \ln(\lambda) dF \right)$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda) dF$ converges to a real number and by 0 otherwise.

Notice that in the definition above we do not require that the source and domain of f agree or that f is injective or that f is surjective. Our conventions imply that the Fuglede-Kadison operator of the zero operator $0: V \rightarrow V$ is 1.

Example 14.15 (Fuglede-Kadison determinant for finite groups). To illustrate this definition, we look at the example where G is finite. We essentially get the classical determinant $\det_{\mathbb{C}}$. Namely, let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the non-zero eigenvalues of f^*f with multiplicity μ_i . Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f ,

$$\det_{\mathcal{N}(G)}(f) = \exp \left(\sum_{i=1}^r \frac{\mu_i}{|G|} \cdot \ln(\sqrt{\lambda_i}) \right) = \prod_{i=1}^r \lambda_i^{\frac{\mu_i}{2 \cdot |G|}} = \det_{\mathbb{C}}(\overline{f^*f})^{\frac{1}{2 \cdot |G|}}.$$

where $\det_{\mathbb{C}}(\overline{f^*f})$ is put to be 1 if f is the zero operator and hence $\overline{f^*f}$ is $\text{id}: \{0\} \rightarrow \{0\}$. If $f: \mathbb{C}G^m \rightarrow \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{N}(G)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}.$$

Example 14.16 (Fuglede-Kadison determinant for $(2, 2)$ -matrices over the trivial group). Consider $A \in M_{2,2}(\mathbb{C})$. Let A^* be the conjugate transpose of A and denote by $\text{tr}_{\mathbb{Z}}(AA^*)$ the trace of AA^* . We conclude from Example 14.15 for the trivial group $\{1\}$

$$\det_{\mathcal{N}(\{1\})}(r_A: \mathbb{C}^2 \rightarrow \mathbb{C}^2) = \begin{cases} |\det_{\mathbb{C}}(A)| & \text{if } \det_{\mathbb{C}}(A) \neq 0; \\ \sqrt{\text{tr}_{\mathbb{Z}}(AA^*)} & \text{if } \det_{\mathbb{C}}(A) = 0 \text{ and } A \neq 0; \\ 1 & \text{if } A = 0. \end{cases}$$

Example 14.17 (Fuglede-Kadison determinant for $\mathcal{N}(\mathbb{Z}^d)$). Let $G = \mathbb{Z}^d$. We use the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of Example 14.4. For $f \in L^\infty(T^d)$ we conclude from Example 14.13

$$\det_{\mathcal{N}(\mathbb{Z}^d)}(M_f: L^2(T^d) \rightarrow L^2(T^d)) = \exp \left(\int_{T^d} \ln(|f(z)|) \cdot \chi_{\{u \in S^1 \mid f(u) \neq 0\}} d\text{vol}_z \right)$$

using the convention $\exp(-\infty) = 0$.

Let $i: H \rightarrow G$ be an injective group homomorphism. Let V be a finitely generated Hilbert $\mathcal{N}(H)$ -module. There is an obvious pre-Hilbert structure on $\mathbb{C}G \otimes_{\mathbb{C}H} V$ for which G acts by isometries since $\mathbb{C}G \otimes_{\mathbb{C}H} V$ as a complex vector space can be identified with $\bigoplus_{G/H} V$. Its Hilbert space completion is a finitely generated Hilbert $\mathcal{N}(G)$ -module and denoted by i_*M . A morphism of finitely generated Hilbert $\mathcal{N}(H)$ -modules $f: V \rightarrow W$ induces a map of finitely generated Hilbert $\mathcal{N}(G)$ -modules $i_*f: i_*V \rightarrow i_*W$.

The following theorem can be found with proof in [26, Theorem 3.14 on page 128 and Lemma 3.15 (4) on page 129].

Theorem 14.18 (Fuglede-Kadison determinant).

- (1) Let $f: U \rightarrow V$ and $g: V \rightarrow W$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$ -modules such that f has dense image and g is injective. Then

$$\det_{\mathcal{N}(G)}(g \circ f) = \det_{\mathcal{N}(G)}(f) \cdot \det_{\mathcal{N}(G)}(g);$$

- (2) Let $f_1: U_1 \rightarrow V_1$, $f_2: U_2 \rightarrow V_2$ and $f_3: U_2 \rightarrow V_1$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$ -modules such that f_1 has dense image and f_2 is injective. Then

$$\det_{\mathcal{N}(G)} \begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} = \det_{\mathcal{N}(G)}(f_1) \cdot \det_{\mathcal{N}(G)}(f_2);$$

- (3) Let $f: U \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Then

$$\det_{\mathcal{N}(G)}(f) = \det_{\mathcal{N}(G)}(f^*) = \sqrt{\det_{\mathcal{N}(G)}(f^*f)} = \sqrt{\det_{\mathcal{N}(G)}(ff^*)};$$

- (4) Let $i: H \rightarrow G$ be the inclusion of a subgroup of finite index $[G : H]$. Let $i^*f: i^*U \rightarrow i^*V$ be the morphism of finitely generated Hilbert $\mathcal{N}(H)$ -modules obtained from f by restriction. Then

$$\det_{\mathcal{N}(H)}(i^*f) = \det_{\mathcal{N}(G)}(f)^{[G:H]};$$

- (5) Let $i: H \rightarrow G$ be an injective group homomorphism and let $f: U \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(H)$ -modules. Then

$$\det_{\mathcal{N}(G)}(i_*f) = \det_{\mathcal{N}(H)}(f).$$

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