UNIVERSAL L^2-TORSION, POLYTOPES AND APPLICATIONS TO 3-MANIFOLDS

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Abstract. Given an L^2-acyclic connected finite CW-complex, we define its universal L^2-torsion in terms of the chain complex of its universal covering. It takes values in the weak Whitehead group $\mathrm{Wh}^w(G)$. We study its main properties such as homotopy invariance, sum formula, product formula and Poincaré duality. Under certain assumptions, we can specify certain homomorphisms from the weak Whitehead group $\mathrm{Wh}^w(G)$ to abelian groups such as the real numbers or the Grothendieck group of integral polytopes, and the image of the universal L^2-torsion can be identified with many invariants such as the L^2-torsion, the L^2-torsion function, twisted L^2-Euler characteristics and, in the case of a 3-manifold, the dual Thurston norm polytope.

0. Introduction

We assign to a group G the weak K_1-groups $K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$ and the weak Whitehead group $\mathrm{Wh}^w(G)$, see Definition 1.1. These groups are, as the names suggest, variations on the corresponding classical groups. More precisely, the name 'weak' comes from the fact that in the definitions we study matrices over the group ring $\mathbb{Z}G$, but we no longer require that they are invertible over the group ring, but we only require that they are weakly invertible in the sense of L^2-invariants. These groups are in general much larger than their classical analogues. For instance, for $G = \mathbb{Z}$ we have $\mathrm{Wh}(\mathbb{Z}) = 0$ but $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^\times / \{\pm z^n \mid n \in \mathbb{Z}\}$.

Furthermore, to an L^2-acyclic finite based free $\mathbb{Z}G$-chain complex C_* we assign its *universal L^2-torsion*, see Definition 1.7

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(G). \tag{0.1}$$

It is characterized by the universal properties that

$$\rho_u^{(2)}\big(0 \to \mathbb{Z}G \xrightarrow{\pm\,\mathrm{id}} \mathbb{Z}G \to 0\big) = 0$$

and that for any short based exact sequence $0 \to C_* \to D_* \to E_* \to 0$ we get $\rho_u^{(2)}(D_*) = \rho_u^{(2)}(C_*) + \rho_u^{(2)}(E_*)$, as explained in Remark 1.16. If X is an L^2-acyclic finite free G-CW-complex, it defines an element

$$\rho_u^{(2)}(X; \mathcal{N}(G)) \in \mathrm{Wh}^w(G) := \widetilde{K}_1^w(G)/ \pm G, \tag{0.2}$$

determined by $\rho_u^{(2)}(C_*(X))$, where $C_*(X)$ is the cellular $\mathbb{Z}G$-chain complex of X, see Definition 2.1. As an example, if X is the free abelian cover of a knot exterior $S^3 \setminus \nu K$, then under the aforementioned isomorphism $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^\times / \{\pm z^n \mid n \in \mathbb{Z}\}$ the universal torsion $\rho_u^{(2)}(X; \mathcal{N}(\mathbb{Z}))$ agrees with the Turaev-Milnor torsion [29, 37, 36] of the knot.

If X is a finite connected CW-complex with fundamental group π, and universal covering $\widetilde{X}$, we call X L^2-acyclic if $\widetilde{X}$ is L^2-acyclic as finite free π-CW-complex

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and abbreviate

$$\rho_u^{(2)}(\widetilde{X}) := \rho_u^{(2)}(\widetilde{X}; \mathcal{N}(\pi)) \in \mathrm{Wh}^w(\pi). \tag{0.3}$$

The basic properties of these invariants including homotopy invariance, sum formula, product formula, and Poincaré duality are collected in Theorem 2.5 and Theorem 2.11. One can show for a finitely presented group G, for which there exists at least one L^2-acyclic finite connected CW-complex X with $\pi_1(X) = G$, that every element in $\mathrm{Wh}^w(G)$ can be realized as $\rho_u^{(2)}(\widetilde{Y})$ for some L^2-acyclic finite connected CW-complex Y with $G = \pi_1(Y)$, see Lemma 2.8.

The point of this new invariant is that it encompasses many other well-known invariants. We illustrate this by considering some examples. Although these invariants do make sense in all dimensions, we often restrict ourselves to the case of admissible 3-manifolds M:

Definition 0.4 (Admissible 3-manifold)**.** A 3-manifolds is called *admissible* if it is compact, connected, orientable, and irreducible, its boundary is empty or a disjoint union of tori, and its fundamental group is infinite.

0.1. **L^2-torsion and the L^2-Alexander torsion.** In Section 2.4 we will see that the universal L^2-torsion of a 3-manifold M determines the L^2-torsion and more generally the L^2-torsion function (also called L^2-Alexander polynomial and L^2-Alexander torsion) that recently was intensively studied, see e.g., [1, 6, 7, 16, 18, 19]. The former invariant is determined by the volumes of the hyperbolic pieces in the Jaco-Shalen-Johannson decomposition of M, see [26, Theorem 0.7]. The latter invariant is a function $\rho^{(2)}(\widetilde{M}, \phi) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ associated to M and a class $\phi \in H^1(M; \mathbb{Z})$. This function captures a lot of interesting topological information, in particular it was shown by the authors [8] and independently by Liu [20] that it determines the Thurston norm of ϕ.

0.2. **Twisted L^2-Euler characteristic.** In Section 3.2 we recall the statement of the Atiyah Conjecture. For a short discussion of the Atiyah Conjecture for a torsionfree group G and its status which is adapted to the needs of this paper, we refer to [9, Section 3], a more general introduction is given in [23, Chapter 10]. In Remark 3.30, given a torsionfree group G satisfying the Atiyah Conjecture, we will introduce a pairing

$$\mathrm{Wh}^w(\pi) \times H^1(G) \to \mathbb{Z}, \tag{0.5}$$

which has the following property: given an L^2-acyclic finite free $\mathbb{Z}G$-chain complex C_* and an element $\phi \in H^1(G) = \mathrm{Hom}(G, \mathbb{Z})$ the image of $\rho_u^{(2)}(C_*; \mathcal{N}(G)) \otimes \phi$ under the pairing above equals the L^2-Euler characteristic we introduced in [9].

It is now easy to see that for an admissible 3-manifold M and $\phi \colon \pi_1(M) \to \mathbb{Z}$ the L^2-Euler characteristic of [9] is determined by the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$. Similarly, using [9, Section 8] one can show that the degrees of the higher-order Alexander polynomials of Cochran [3] and Harvey [15, Section 3] of knots and 3-manifolds are determined by universal L^2-torsions of appropriate covers of M.

0.3. **Polytopes.** For a finitely generated free abelian group H we denote by $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ the abelian group given by the Grothendieck construction applied to the abelian monoid of integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum, modulo translation with elements in H, see (3.2). Given a group G that satisfies the Atiyah Conjecture we will construct the *polytope homomorphism*

$$\mathbb{P} \colon \mathrm{Wh}^w(\mathbb{Z}G) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f)$$

in (3.17) and define the *L^2-torsion polytope* $P(\widetilde{M})$ to be the image of the universal L^2-torsion under the polytope homomorphism.

Of particular interest is the composition of the universal torsion with the polytope homomorphism. For example let M be an admissible 3-manifold that is not a closed graph manifold. Then we obtain a well-defined element

$$P(\widetilde{M}) := \mathbb{P}(\rho_u(\widetilde{M})) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f).$$

We now explain the information contained in this invariant. Following Thurston [34] we can use the minimal complexity of surfaces in a 3-manifold M to assign to M an integral polytope $T(M) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi)_f$, called the *dual Thurston polytope*, see (3.36) for details. One of our main results is that we show in Theorem 3.37 that

$$[T(M)^*] \quad = \quad 2 \cdot P(\widetilde{M}) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\pi)_f).$$

We can also use this approach to assign formal differences of polytopes to many other groups, e.g. free-by-cyclic groups and two-generator one-relator groups. We will discuss these examples in more details in a joint paper [10] with Stephan Tillmann.

Finally let G be any group that admits a finite model for BG and that satisfies the Atiyah Conjecture and let $f\colon G \to G$ be a monomorphism. Then we can associate to this monomorphism the polytope invariant of the corresponding ascending HNN-extension. If $G = F_2$ is the free group on two generators this polytope invariant has been studied by Funke–Kielak [14]. We hope that this invariant of monomorphisms of groups will have other interesting applications.

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Contents

1. Universal L^2-torsion for chain complexes

In this section we define the universal L^2-torsion and the algebraic K-group where it takes values in.

1.1. **From finite based free $\mathbb{Z}G$-chain complexes to finite Hilbert $\mathcal{N}(G)$-chain complexes.** Let G be a group. We will always work with left modules. A *$\mathbb{Z}G$-basis* for a finitely generated free $\mathbb{Z}G$-module M is a finite subset $B \subseteq M$ such that the canonical map

$$\bigoplus_{b\in B} \mathbb{Z}G \to M, \qquad (r_b)_{b\in B} \mapsto \sum_{b\in B} r_b \cdot b$$

is a $\mathbb{Z}G$-isomorphism. Notice that we do not require that B is ordered. A *finitely generated based free $\mathbb{Z}G$-module* is a pair (M, B) consisting of a finitely generated free $\mathbb{Z}G$-module M together with a basis B. (Sometimes we omit B from the notation.) In the sequel we will equip $\mathbb{Z}G^n = \bigoplus_{i=1}^n \mathbb{Z}G$ with the standard basis. Throughout this paper a basis is understood to be unordered.

A *finite Hilbert $\mathcal{N}(G)$-module* V is a Hilbert space V together with a linear isometric left G-action such that there exists an isometric linear embedding into $L^2(G)^n$ for some natural number n. (Here $\mathcal{N}(G)$ stands for the group von Neumann algebra which is the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$.) Obviously $L^2(G)^n$ is a finite Hilbert $\mathcal{N}(G)$-module. Morphisms of finite Hilbert $\mathcal{N}(G)$-modules are bounded G-equivariant operators (which are not necessarily isometric). For a basic introduction to Hilbert $\mathcal{N}(G)$-chain complexes and L^2-Betti numbers we refer for instance to [23, Chapter 1]. For a finitely generated based free $\mathbb{Z}G$-module (M, B) define a finite Hilbert $\mathcal{N}(G)$-module $\Lambda^G(M)$ by $L^2(G) \otimes_{\mathbb{Z}G} M$ equipped with the Hilbert space structure, for which the induced map

$$\bigoplus_{b\in B} L^2(G) \to L^2(G) \otimes_{\mathbb{Z}G} M, \quad (x_b)_{b\in B} \mapsto \sum_{b\in B} x_b \otimes_{\mathbb{Z}G} b$$

is a G-equivariant isometric invertible operator of Hilbert spaces. (If G is clear from the context, we often abbreviate Λ^G by Λ.) Obviously $\Lambda(\mathbb{Z}G^n)$ can be identified with $L^2(G)^n$. One easily checks that a $\mathbb{Z}G$-map of finitely generated based free $\mathbb{Z}G$-modules $f\colon M \to N$ induces a morphism $\Lambda(f)$ of Hilbert $\mathcal{N}(G)$-modules. This construction is functorial. Moreover we have $r_1 \cdot \Lambda(f_1) + r_2 \cdot \Lambda(f_2) = \Lambda(r_1 \cdot f_0 + r_2 \cdot f_2)$ for integers r_1, r_2 and $\mathbb{Z}G$-maps $f_0, f_1\colon M \to N$ of finitely generated based free $\mathbb{Z}G$-modules. Obviously Λ is compatible with direct sums, i.e., for two finitely generated based free $\mathbb{Z}G$-modules M_0 and M_1 the canonical map $\Lambda(M_0) \oplus \Lambda(M_1) \to \Lambda(M_0 \oplus M_1)$ is an isometric G-equivariant invertible operator. Let $f\colon M \to N$ be a $\mathbb{Z}G$-homomorphism of finitely generated based free $\mathbb{Z}G$-modules. If B and C are the bases of M and N, let $A(f)$ be the B-C-matrix defined by f, namely, if for

$b \in B$ we write $f(b) = \sum_{c \in C} a_{b,c} \cdot c$ for $a_{b,c} \in \mathbb{Z}G$, then $A = (a_{b,c})$. Let A^* be the C-B-matrix, whose entry at $(c,b) \in C \times B$ is $\overline{a_{b,c}}$, where $\overline{\sum_{g \in G} r_g \cdot g} := \sum_{g \in G} r_g \cdot g^{-1}$ for $\sum_{g \in G} r_g \cdot g \in \mathbb{Z}$. Define $f^* \colon N \to M$ to be the $\mathbb{Z}G$-homomorphism associated to A^*. One easily checks that $\Lambda(f^*) = \Lambda(f)^*$, where $\Lambda(f)^*$ is the adjoint operator associated to $\Lambda(f)$. All in all one may summarize by saying that Λ defines a functor of additive categories with involution from the category of finitely generated based free $\mathbb{Z}G$-modules to the category of finite Hilbert $\mathcal{N}(G)$-modules.

Given an (m,n)-matrix $A = (a_{i,j})$, where $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$, right multiplication with A defines a $\mathbb{Z}G$-homomorphism

$$r_A \colon \mathbb{Z}G^m \to \mathbb{Z}G^n, \quad (x_1,x_2,\ldots,x_m) \mapsto (x_1,x_2,\ldots,x_m) \cdot A = \Big(\sum_{i=1}^m x_i \cdot a_{i,j}\Big)_{j=1,2,\ldots,n}$$

Notice that A is the matrix $A(r_A)$ associated to r_A.

A finite based free $\mathbb{Z}G$-chain complex C_* is a $\mathbb{Z}G$-chain complex C_* such that there exists a natural number N with $C_n = 0$ for $|n| > N$ and such that each chain module C_n is a finitely generated based free $\mathbb{Z}G$-module. Denote by $\Lambda(C_*)$ the finite Hilbert $\mathcal{N}(G)$-chain complex which is obtained by applying the functor Λ. Here by a finite Hilbert $\mathcal{N}(G)$-chain complex D_* we mean a chain complex in the category of finite Hilbert $\mathcal{N}(G)$-chain complexes such that there exists a natural number N with $D_n = 0$ for $n > N$.

1.2. **The weak K_1-group $K_1^w(\mathbb{Z}G)$.** The universal L^2-torsion, that we will define in Section 1.3, will take values in the following K_1-group. Recall that a morphism $f \colon V \to W$ of finite Hilbert $\mathcal{N}(G)$-modules is called a *weak isomorphism* if it is injective and has dense image.

Definition 1.1 ($K_1^w(\mathbb{Z}G)$)**.** Define the *weak K_1-group*

$$K_1^w(\mathbb{Z}G)$$

to be the abelian group defined in terms of generators and relations as follows. Generators $[f]$ are given by of $\mathbb{Z}G$-endomorphisms $f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ for $n \in \mathbb{Z}, n \geq 0$ such that $\Lambda(f)$ is a weak isomorphism of finite Hilbert $\mathcal{N}(G)$-modules. If $f_1, f_2 \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ are $\mathbb{Z}G$-endomorphisms such that $\Lambda(f_1)$ and $\Lambda(f_2)$ are weak isomorphisms, then we require the relation

$$[f_2 \circ f_1] = [f_1] + [f_2].$$

If $f_0 \colon \mathbb{Z}G^m \to \mathbb{Z}G^m$, $f_2 \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ and $f_1 \colon \mathbb{Z}G^n \to \mathbb{Z}G^m$ are $\mathbb{Z}G$-maps such that $\Lambda(f_0)$ and $\Lambda(f_2)$ are weak isomorphisms, then we require for the $\mathbb{Z}G$-map

$$f = \begin{pmatrix} f_0 & f_1 \\ 0 & f_2 \end{pmatrix} \colon \mathbb{Z}G^{m+n} = \mathbb{Z}G^m \oplus \mathbb{Z}G^n \to \mathbb{Z}G^m \oplus \mathbb{Z}G^n$$

the relation

$$[f] = [f_0] + [f_2].$$

Let

$$\widetilde{K}_1^w(\mathbb{Z}G)$$

be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup generated by the element $[-\operatorname{id} \colon \mathbb{Z}G \to \mathbb{Z}G]$. This is the same as the cokernel of the obvious composite $K_1(\mathbb{Z}) \to K_1^w(\mathbb{Z}G)$. Define the *weak Whitehead group* of G

$$\operatorname{Wh}^w(G)$$

to be the cokernel of the homomorphism

$$\{\sigma \cdot g \mid \sigma \in \{\pm 1\}, g \in G\} \to K_1^w(\mathbb{Z}G), \quad \sigma \cdot g \mapsto [r_{\sigma \cdot g} \colon \mathbb{Z}G \to \mathbb{Z}G].$$

Definition 1.1 makes sense since the morphisms appearing above $\Lambda(f_2 \circ f_2)$ and $\Lambda(f)$ are again weak isomorphism by [23, Lemma 3.37 on page 144].

Remark 1.2. We obtain the classical notions of $K_1(\mathbb{Z}G)$ and $\widetilde{K}_1(\mathbb{Z}G)$ if we replace in Definition 1.1 above everywhere the condition that $\Lambda(f)$ is a weak isomorphism of finite Hilbert $\mathcal{N}(G)$-modules by the stronger condition that f is a $\mathbb{Z}G$-isomorphism.

1.3. **The universal L^2-torsion for chain complexes.** Recall that a $\mathbb{Z}G$-chain complex C_* is *contractible*, if C_* possesses a *chain contraction* γ_*, i.e., a sequence of $\mathbb{Z}G$-maps $\gamma_n\colon C_n \to C_{n+1}$ such that $c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n = \mathrm{id}_{C_n}$ holds for all $n \in \mathbb{Z}$. Moreover, for a finite based free contractible $\mathbb{Z}G$-chain complex C_* its *(classical) torsion*

$$\rho(C_*) \in \widetilde{K}_1(\mathbb{Z}G) \tag{1.3}$$

is defined for any choice of chain contraction γ_* by the class $[(c+\gamma)_{\mathrm{odd}}]$ of the $\mathbb{Z}G$-isomorphism of finitely generated based free $\mathbb{Z}G$-modules $(c+\gamma)_{\mathrm{odd}}\colon C_{\mathrm{odd}} \xrightarrow{\cong} C_{\mathrm{ev}}$, where here and in the sequel we write

$$C_{\mathrm{odd}} \;=\; \bigoplus_{n \in \mathbb{Z}} C_{2n+1} \quad \text{ and } \quad C_{\mathrm{ev}} \;=\; \bigoplus_{n \in \mathbb{Z}} C_{2n}.$$

As we mentioned above, a basis is understood to be unordered. This implies that the class $[(c+\gamma)_{\mathrm{odd}}]$ is not well-defined in $K_1(\mathbb{Z}G)$, but it is indeed well-defined in $\widetilde{K}_1(\mathbb{Z}G) = K_1(\mathbb{Z}G)/[-\mathrm{id}]$.

We want to carry over these two notions to finite based free $\mathbb{Z}G$-chain complexes C_* for which $\Lambda(C_*)$ is L^2-acyclic. Recall that a finite Hilbert $\mathcal{N}(G)$-chain complex D_* is *L^2-acyclic* if for every $n \in \mathbb{Z}$ its *n-th L^2-homology group* $H_n^{(2)}(D_*)$, which is defined to be the finite Hilbert $\mathcal{N}(G)$-module given by the quotient of the kernel of the n-th differential by the closure of the image of the $(n+1)$-st differential, vanishes, or, equivalently, if for each $n \in \mathbb{Z}$ the *n-th L^2-Betti number* $b_n^{(2)}(D_*)$, which is defined to be the von Neumann dimension of $H_n^{(2)}(D_*)$, vanishes.

Definition 1.4 (Weak chain contraction)**.** Consider a $\mathbb{Z}G$-chain complex C_*. A *weak chain contraction* (γ_*, u_*) for C_* consists of a $\mathbb{Z}G$-chain map $u_*\colon C_* \to C_*$ and a $\mathbb{Z}G$-chain homotopy $\gamma_*\colon u_* \simeq 0_*$ such that $\Lambda(u_n)$ is a weak isomorphism for all $n \in \mathbb{Z}$ and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ holds for all $n \in \mathbb{Z}$.

Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Denote by $\Delta_n\colon C_n \to C_n$ its *n-th combinatorial Laplace operator* which is the $\mathbb{Z}G$-map $\Delta_n = c_{n+1} \circ c_{n+1}^* + c_n^* \circ c_n\colon C_n \to C_n$. It has the property that $\Lambda(\Delta_n)$ is the Laplace operator of the finite Hilbert $\mathcal{N}(G)$-chain complex $\Lambda(C_*)$, where the n-th Laplace operator of a finite Hilbert $\mathcal{N}(G)$-chain complex D_* is the morphisms of finite Hilbert $\mathcal{N}(G)$-modules $d_{n+1} \circ d_{n+1}^* + d_n^* \circ d_n\colon D_n \to D_n$.

Lemma 1.5. *Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Then the following assertions are equivalent:*

1. *$\Lambda(\Delta_n)$ is a weak isomorphism for all $n \in \mathbb{Z}$;*
2. *There exists a weak chain contraction (γ_*, u_*) with $\gamma_n \circ \gamma_{n-1} = 0$ for all $n \in \mathbb{Z}$;*
3. *There exists a weak chain contraction (γ_*, u_*);*
4. *$\Lambda(C_*)$ is L^2-acyclic.*

Proof. (1) $\implies$ (2) Define $\gamma_n\colon C_n \to C_{n+1}$ by c_{n+1}^*. Then $\gamma_n \circ \gamma_{n-1} = (c_{n+1})^* \circ c_n^* = (c_n \circ c_{n+1})^* = 0$. Put $u_n = \Delta_n\colon C_n \to C_n$. Then the collection of the u_n's defines a $\mathbb{Z}G$-chain map $u_*\colon C_* \to C_*$ such that γ_* is a $\mathbb{Z}G$-chain homotopy $u_* \simeq 0_*$, we have $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ for $n \in \mathbb{Z}$, and $\Lambda(u_n)$ is a weak isomorphism for $n \in \mathbb{Z}$.

(2) $\implies$ (3) is obvious.

(3) $\implies$ (4) Since $\Lambda(u_n)$ is a weak isomorphism for all $n \in \mathbb{Z}$, the induced map $H_n^{(2)}(\Lambda(u_*))\colon H_n^{(2)}(\Lambda(C_*)) \to H_n^{(2)}(\Lambda(C_*))$ is a weak isomorphism for all $n \in \mathbb{Z}$

by [23, Lemma 3.44 on page 149]. Since $\Lambda(u_*)$ is nullhomotopic, $H_n^{(2)}(\Lambda(u_*))$ is the zero map for all $n \in \mathbb{Z}$. This implies that $H_n^{(2)}(\Lambda(C_*))$ vanishes for all $n \in \mathbb{Z}$. Hence $\Lambda(C_*)$ is L^2-acyclic.

(4) $\implies$ (1) This follows from [23, Lemma 1.13 on page 23 and Lemma 1.18 on page 24] since $\Lambda(\Delta_n)$ is the n-th Laplace operator of $\Lambda(C_*)$. □

Remark 1.6 (Homomorphisms of finitely generated based free $\mathbb{Z}G$-modules and $\widetilde{K}_1^w(\mathbb{Z}G)$)**.** Let $f\colon M \to N$ be a $\mathbb{Z}G$-homomorphism of finitely generated based free $\mathbb{Z}G$-modules such that $\Lambda(f)$ is a weak isomorphism. Then we conclude from [23, Lemma 1.13 on page 23]

$$\mathrm{rk}_{\mathbb{Z}G}(M) = \dim_{\mathcal{N}(G)}(\Lambda(M)) = \dim_{\mathcal{N}(G)}(\Lambda(N)) = \mathrm{rk}_{\mathbb{Z}G}(N).$$

Hence we can choose a bijection between the two bases for M and N. It induces a $\mathbb{Z}G$-isomorphism $b\colon N \to M$. Now define an element

$$[f] := [b \circ f] \in \widetilde{K}_1^w(\mathbb{Z}G).$$

Since we are working in $\widetilde{K}_1^w(\mathbb{Z}G)$, the choice of the bijection of the bases does not matter.

Definition 1.7 (Universal L^2-torsion for L^2-acyclic finite based free $\mathbb{Z}G$-chain complexes)**.** Let C_* be a finite based free $\mathbb{Z}G$-chain complex such that $\Lambda(C_*)$ is L^2-acyclic. Define its *universal L^2-torsion*

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(\mathbb{Z}G)$$

by

$$\rho_u^{(2)}(C_*) = [(uc+\gamma)_{\mathrm{odd}}] - [u_{\mathrm{odd}}],$$

where (γ_*, u_*) is any weak chain contraction of C_*.

We have to explain that this is well-defined. The existence of a weak chain contraction follows from Lemma 1.5. We have to show that $\Lambda((uc+\gamma)_{\mathrm{odd}})$ and $\Lambda(u_{\mathrm{odd}})$ are weak isomorphisms and that the definition is independent of the choice of weak chain contraction. Let (δ_*, v_*) be another weak chain contraction. Define $\Theta_1\colon C_{\mathrm{ev}} \to C_{\mathrm{ev}}$ by the lower triangle matrix

$$\Theta_1 := (v \circ u + \delta \circ \gamma)_{\mathrm{ev}} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \ddots & vu & 0 & 0 & \dots \\ \dots & \delta\gamma & vu & 0 & \dots \\ \dots & 0 & \delta\gamma & vu & \dots \\ \ddots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Then the composite

$$\Theta_2\colon C_{\mathrm{odd}} \xrightarrow{(uc+\gamma)_{\mathrm{odd}}} C_{\mathrm{ev}} \xrightarrow{\Theta_1} C_{\mathrm{ev}} \xrightarrow{(vc+\delta)_{\mathrm{ev}}} C_{\mathrm{odd}}$$

is given by the lower triangle matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \dots & (v^2u^2)_{2n-1} & 0 & 0 & \dots \\ \dots & * & (v^2u^2)_{2n+1} & 0 & \dots \\ \dots & * & * & (v^2u^2)_{2n+3} & \dots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We conclude from [23, Lemma 3.37 on page 144] that $\Lambda(\Theta_1)$ and $\Lambda(\Theta_2)$ are weak isomorphisms and we have

$$\Theta_2 = (vc+\delta)_{\mathrm{ev}} \circ \Theta_1 \circ (uc+\gamma)_{\mathrm{odd}}.$$

Analogously we find $\mathbb{Z}G$-homomorphisms Θ_3 and Θ_4 such that $\Lambda(\Theta_3)$ and $\Lambda(\Theta_4)$ are weak isomorphism and we have

$$\Theta_4 = (vc+\delta)_{\mathrm{odd}} \circ \Theta_3 \circ (uc+\gamma)_{\mathrm{ev}}.$$

Since we can interchange the roles of (γ_*, u_*) and (δ_*, v_*), we can find $\mathbb{Z}G$-homomorphisms Θ_i for $i = 5, 6, 7, 8$ such that $\Lambda(\Theta_i)$ is a weak isomorphism and we have

$$\begin{array}{lcl} \Theta_6 & = & (uc+\gamma)_{\mathrm{ev}} \circ \Theta_5 \circ (vc+\delta)_{\mathrm{odd}}; \\ \Theta_8 & = & (uc+\gamma)_{\mathrm{odd}} \circ \Theta_7 \circ (vc+\delta)_{\mathrm{ev}}. \end{array}$$

We conclude that $\Lambda((uc+\gamma)_{\mathrm{odd}})$, $\Lambda(uc+\gamma)_{\mathrm{ev}})$, $\Lambda((vc+\delta)_{\mathrm{odd}})$, and $\Lambda((vc+\delta)_{\mathrm{ev}})$ are weak isomorphisms. So we get well-defined elements $[(uc+\gamma)_{\mathrm{odd}}]$, $[u_{\mathrm{odd}}]$, $[(vc+\delta)_{\mathrm{ev}}]$, and $[v_{\mathrm{ev}}]$ in $\widetilde{K}_1^w\mathbb{Z}G)$. We have $u_{\mathrm{ev}} \circ (uc+\gamma)_{\mathrm{odd}} = (uc+\gamma)_{\mathrm{odd}} \circ u_{\mathrm{odd}}$ and $v_{\mathrm{ev}} \circ (vc+\delta)_{\mathrm{odd}} = (vc+\delta)_{\mathrm{odd}} \circ v_{\mathrm{odd}}$. This implies

$$[u_{\mathrm{odd}}] \;=\; [u_{\mathrm{ev}}] \quad \text{and} \quad [v_{\mathrm{odd}}] \;=\; [v_{\mathrm{ev}}].$$

Since

$$\begin{array}{lcl} [\Theta_1] & = & [v_{\mathrm{ev}}] + [u_{\mathrm{ev}}]; \\ [\Theta_2] & = & 2 \cdot [v_{\mathrm{odd}}] + 2 \cdot [u_{\mathrm{odd}}]; \\ [\Theta_2] & = & [(vc+\delta)_{\mathrm{ev}}] + [\Theta_1] + [(uc+\gamma)_{\mathrm{odd}}], \end{array}$$

hold in $\widetilde{K}_1^w(\mathbb{Z}G)$, we get in $\widetilde{K}_1^w(\mathbb{Z}G)$

$$[(uc+\gamma)_{\mathrm{odd}}] - [u_{\mathrm{odd}}] \;=\; -[(vc+\delta)_{\mathrm{ev}}] + [v_{\mathrm{ev}}]. \tag{1.8}$$

Since the right hand side of the equation (1.8) above is independent of (γ_*, u_*), we conclude that Definition 1.7 makes sense.

We call an exact short sequence $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ of finitely generated based free $\mathbb{Z}G$-modules *based exact* if $i(B_0) \subseteq B_1$ holds and p maps $B_1 \setminus i(B_0)$ bijectively onto B_2 for the given $\mathbb{Z}G$-basis $B_i \subseteq M_i$ for $i = 0, 1, 2$. This extends in the obvious way to $\mathbb{Z}G$-chain complexes.

Lemma 1.9. *Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ be a based exact short sequence of finite based free $\mathbb{Z}G$-chain complexes. Suppose that two of the finite Hilbert $\mathcal{N}(G)$-chain complexes $\Lambda(C_*)$, $\Lambda(D_*)$ and $\Lambda(E_*)$ are L^2-acyclic. Then all three are L^2-acyclic and we get in $\widetilde{K}_1^w(\mathbb{Z}G)$*

$$\rho_u^{(2)}(D_*) = \rho_u^{(2)}(C_*) + \rho_u^{(2)}(E_*).$$

Proof. All three finite Hilbert $\mathcal{N}(G)$-chain complexes $\Lambda(C_*)$, $\Lambda(D_*)$ and $\Lambda(E_*)$ are L^2-acyclic by [23, Theorem 1.21 on page 27]. Since E_n is free and $0 \to C_n \to D_n \to E_n$ is exact, we can choose $\mathbb{Z}G$-homomorphisms $r_n \colon D_n \to C_n$ and $s_n \colon E_n \to D_n$ satisfying $r_n \circ s_n = 0$, $r_n \circ i_n = \mathrm{id}_{C_n}$ and $p_n \circ s_n = \mathrm{id}_{E_n}$ for each $n \in \mathbb{Z}$. Because of Lemma 1.5 we can choose weak chain contractions (γ_*, u_*) for C_* and (ϵ_*, w_*) for E_* such that $\gamma_{n+1} \circ \gamma_n = 0$ and $\epsilon_{n+1} \circ \epsilon_n = 0$ holds for all $n \in \mathbb{Z}$. Define

$$\delta_n := i_{n+1} \circ \gamma_n \circ r_n + s_{n+1} \circ \epsilon_n \circ p_n \colon D_n \to D_{n+1}.$$

We compute

$$\begin{array}{lcl} \delta \circ \delta & = & (i \circ \gamma \circ r + s \circ \epsilon \circ p) \circ (i \circ \gamma \circ r + s \circ \epsilon \circ p) \\ & = & i \circ \gamma \circ r \circ i \circ \gamma \circ r + i \circ \gamma \circ r \circ s \circ \epsilon \circ p \\ & & \qquad +s \circ \epsilon \circ p \circ i \circ \gamma \circ r + s \circ \epsilon \circ p \circ s \circ \epsilon \circ p \\ & = & i \circ \gamma \circ \mathrm{id} \circ \gamma \circ r + i \circ \gamma \circ 0 \circ \epsilon \circ p + s \circ \epsilon \circ 0 \circ \gamma \circ r + s \circ \epsilon \circ \mathrm{id} \circ \epsilon \circ p \\ & = & i \circ \gamma^2 \circ r + s \circ \epsilon^2 \circ p \;=\; i \circ 0 \circ r + s \circ 0 \circ p \;=\; 0. \end{array}$$

Put $v_n := d_{n+1} \circ \delta_n + \delta_{n-1} \circ d_n$. One easily checks $v_n \circ \delta_{n-1} = \delta_{n-1} \circ v_{n-1}$ for every $n \in \mathbb{Z}$. The following diagrams commute

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r, "i_n"] \arrow[d, "\gamma_n"] & D_n \arrow[r, "p_n"] \arrow[d, "\delta_n"] & E_n \arrow[r] \arrow[d, "\epsilon_n"] & 0 \\ 0 \arrow[r] & C_{n+1} \arrow[r, "i_{n+1}"] & D_{n+1} \arrow[r, "p_{n+1}"] & E_{n+1} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r, "i_n"] \arrow[d, "u_n"] & D_n \arrow[r, "p_n"] \arrow[d, "v_n"] & E_n \arrow[r] \arrow[d, "w_n"] & 0 \\ 0 \arrow[r] & C_n \arrow[r, "i_n"] & D_n \arrow[r, "p_n"] & E_n \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & C_{\mathrm{odd}} \arrow[r, "i_{\mathrm{odd}}"] \arrow[d, "(uc+\gamma)_{\mathrm{odd}}"] & D_{\mathrm{odd}} \arrow[r, "p_{\mathrm{odd}}"] \arrow[d, "(vd+\delta)_{\mathrm{odd}}"] & E_{\mathrm{odd}} \arrow[r] \arrow[d, "(we+\epsilon)_{\mathrm{odd}}"] & 0 \\ 0 \arrow[r] & C_{\mathrm{odd}} \arrow[r, "i_{\mathrm{odd}}"] & D_{\mathrm{odd}} \arrow[r, "p_{\mathrm{odd}}"] & E_{\mathrm{odd}} \arrow[r] & 0. \end{tikzcd}$$

Since $\Lambda(u_n)$ and $\Lambda(w_n)$ are weak isomorphisms, the same is true for $\Lambda(v_n)$ by [23, Lemma 3.37 on page 144]. Hence (δ_*, v_*) is a weak chain contraction for D_*. Since each of the short exact sequences $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{p_n} E_n \to 0$ is based exact by assumption, we get in $\widetilde{K}_1^w(\mathbb{Z}G)$

$$\begin{aligned} [(vd+\delta)_{\mathrm{odd}}] &= [(uc+\gamma)_{\mathrm{odd}}] + [(we+\epsilon)_{\mathrm{odd}}]; \\ [v_{\mathrm{odd}}] &= [u_{\mathrm{odd}}] + [w_{\mathrm{odd}}]; \\ \rho_u^{(2)}(D_*) &= \rho_u^{(2)}(C_*) + \rho_u^{(2)}(E_*). \end{aligned}$$

This finishes the proof of Lemma 1.9. □

Let $f\colon (C_*, c_*) \to (D_*, d_*)$ be a $\mathbb{Z}G$-chain map of $\mathbb{Z}G$-chain complexes. We denote by $\operatorname{cone}(f_*)$ the *cone of* f, this is the chain complex which is given by the modules $C_{n+1} \oplus D_n$ and where the differential from $C_n \oplus D_{n+1} \to C_{n-1} \oplus D_n$ is given by

$$\begin{pmatrix} -c_n & 0 \\ f_n & d_{n+1} \end{pmatrix}.$$

Furthermore the *supension* of a chain complex (C_*, c_*) is defined as the chain complex where the n-th boundary map is given by $C_{n-1} \xrightarrow{-c_{n-1}} C_{n-2}$.

Lemma 1.10. *Let $f\colon C_* \to D_*$ be a $\mathbb{Z}G$-chain homotopy equivalence of finite based free $\mathbb{Z}G$-chain complexes. Denote by $\rho(\operatorname{cone}(f_*)) \in \widetilde{K}_1(\mathbb{Z}G)$ the classical torsion of the finite based free contractible $\mathbb{Z}G$-chain complex $\operatorname{cone}(f_*)$. Suppose that $\Lambda(C_*)$ or $\Lambda(D_*)$ is L^2-acyclic. Then both $\Lambda(C_*)$ and $\Lambda(D_*)$ are L^2-acyclic, and we get in $\widetilde{K}_1^w(\mathbb{Z}G)$*

$$\rho_u^{(2)}(D_*) - \rho_u^{(2)}(C_*) = \zeta(\rho(\operatorname{cone}(f_*)))$$

for the canonical homomorphism $\zeta\colon \widetilde{K}_1(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G)$.

Proof. Since f_* is a $\mathbb{Z}G$-chain homotopy equivalence, $\operatorname{cone}(f_*)$ is a finite based free contractible $\mathbb{Z}G$-chain complex. In particular $\Lambda(\operatorname{cone}(f_*))$ is L^2-acyclic. One easily checks that $\zeta\colon \widetilde{K}_1(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G)$ maps $\rho(\operatorname{cone}(f_*))$ to $\rho_u^{(2)}(\operatorname{cone}(f_*))$ since a chain contraction γ_* for $\operatorname{cone}(f_*)$ defines a weak chain contraction $(\gamma_*, \mathrm{id}_{\operatorname{cone}(f_*)})$ for $\operatorname{cone}(f_*)$. Now apply Lemma 1.9 to the obvious based exact short sequences of finite based free $\mathbb{Z}G$-chain complexes $0 \to D_* \to \operatorname{cone}(f_*) \to \Sigma C_* \to 0$ and use the obvious fact that $\Lambda(\Sigma C_*)$ is L^2-acyclic, if and only if $\Lambda(C_*)$ is L^2-acyclic, and in this case $\rho_u^{(2)}(\Sigma C_*) = -\rho_u^{(2)}(C_*)$ holds. □

1.4. The K-group $K_1^{w,\mathrm{ch}}(\mathbb{Z}G)$.

Definition 1.11 ($\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ and $\widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G)$)**.** Let $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ be the abelian group defined in terms of generators and relations as follows. Generators $[C_*]$ are given by (basis preserving isomorphism classes of) finite based free $\mathbb{Z}G$-chain complexes C_* such that $\Lambda(C_*)$ is L^2-acyclic. Whenever we have a short based exact sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ such that two (and hence all) of the finite Hilbert $\mathcal{N}(G)$-chain complexes $\Lambda(C_*)$, $\Lambda(D_*)$, and $\Lambda(E_*)$ are L^2-acyclic, we require the relation

$$[D_*] = [C_*] + [E_*].$$

We also require that for any n we have

$$\big[0 \to \mathbb{Z}G \xrightarrow{\mathrm{id}_{\mathbb{Z}G}} \mathbb{Z}G \to 0\big] = \big[0 \to \mathbb{Z}G \xrightarrow{-\mathrm{id}_{\mathbb{Z}G}} \mathbb{Z}G \to 0\big] = 0,$$

where the two non-zero terms lie in dimension n and $n+1$. Let $\widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G)$ be the abelian group defined analogously, where we replace everywhere the condition that $\Lambda(C_*)$ is L^2-acyclic by the stronger condition that C_* is contractible as $\mathbb{Z}G$-chain complex.

Theorem 1.12 (Chain complexes versus automorphisms)**.** *The classical torsion and the universal L^2-torsion induce isomorphisms such that the following diagram commutes, where the horizontal maps are the obvious forgetful homomorphisms.*

$$\begin{tikzcd}
\widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G) \arrow[r, "\zeta"] \arrow[d, "\rho"', "\cong"] & \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G) \arrow[d, "\cong"', "\rho_u^{(2)}"] \\
\widetilde{K}_1(\mathbb{Z}G) \arrow[r, "\zeta"'] & \widetilde{K}_1^{w}(\mathbb{Z}G).
\end{tikzcd}$$

The inverses of ρ and $\rho_u^{(2)}$ are given by the obvious maps regarding a homomorphism of finitely generated based free $\mathbb{Z}G$-modules as a 1-dimensional finite based free $\mathbb{Z}G$-chain complex.

Proof. The map $\rho_u^{(2)} \colon \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G) \to \widetilde{K}_1^{w}(\mathbb{Z}G)$ is well-defined by Lemma 1.9. One easily checks that the diagram appearing in Theorem 1.12 commutes. It remains to show that the two vertical maps are isomorphisms. We start out with the left vertical map.

Next we define a homomorphism

$$\mathrm{el} \colon \widetilde{K}_1^{w}(\mathbb{Z}G) \xrightarrow{\cong} \widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G). \tag{1.13}$$

Consider a $\mathbb{Z}G$-endomorphism $f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ such that $\Lambda(f)$ is a weak isomorphism. We want to define

$$\mathrm{el}([f]) := [\mathrm{el}(f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n)]. \tag{1.14}$$

where $\mathrm{el}(f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n)$ is the 1-dimensional finite based free $\mathbb{Z}G$-chain complex whose first differential is f. Since $\Lambda(f)$ is a weak isomorphism, $\Lambda\big(\mathrm{el}(f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n)\big)$ is L^2-acyclic and defines an element in $[\mathrm{el}(f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n)]$ in $K_1^{w,\mathrm{ch}}(\mathbb{Z}G)$. We have to check that the relations in $K_1^{w}(\mathbb{Z}G)$ are satisfied. This follows for the additivity relation directly from Lemma 1.9. It remains to show for $\mathbb{Z}G$-endomorphisms $f_1, f_2 \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ such that $\Lambda(f_1)$ and $\Lambda(f_2)$ are weak isomorphisms that we get in $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$

$$[\mathrm{el}(f_2 \circ f_1)] = [\mathrm{el}(f_1)] + [\mathrm{el}(f_2)].$$

Consider the chain map

$$h_* \colon \Sigma^{-1}\,\mathrm{el}(f_2) \to \mathrm{el}(f_1)$$

given by $h_1 = \mathrm{id}_{\mathbb{Z}G^n}$ and $h_i = 0$ for $i \neq 1$. We conclude from the short based exact sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to \mathrm{el}(f_1) \to \mathrm{cone}(h_*) \to \mathrm{el}(f_2) \to 0$ that we get in $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$

$$[\mathrm{cone}(h_*)] = [\mathrm{el}(f_1)] + [\mathrm{el}(f_2)].$$

There is also a based exact short sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to \mathrm{el}(f_2 \circ f_1) \xrightarrow{j} \mathrm{cone}(h_*) \to \mathrm{el}(\mathrm{id}_{\mathbb{Z}G^n}) \to 0$, given by

$$\begin{tikzcd}[column sep=huge]
0 \arrow[r] & \mathbb{Z}G^n \arrow[r, "{\begin{pmatrix} f_1 \\ -\mathrm{id}_{\mathbb{Z}G^n}\end{pmatrix}}"] \arrow[d, "f_2 \circ f_1"] & \mathbb{Z}G^n \oplus \mathbb{Z}G^n \arrow[r, "{\begin{pmatrix} \mathrm{id}_{\mathbb{Z}G^n} & f_1\end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} f_2 & 0 \\ \mathrm{id}_{\mathbb{Z}G^n} & f_1\end{pmatrix}}"] & \mathbb{Z}G^n \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{Z}G^n}"] & 0 \\
0 \arrow[r] & \mathbb{Z}G^n \arrow[r, "{\begin{pmatrix} \mathrm{id}_{\mathbb{Z}G^n} \\ 0\end{pmatrix}}"'] & \mathbb{Z}G^n \oplus \mathbb{Z}G^n \arrow[r, "{\begin{pmatrix} 0 & \mathrm{id}_{\mathbb{Z}G^n}\end{pmatrix}}"'] & \mathbb{Z}G^n \arrow[r] & 0.
\end{tikzcd}$$

It yields the equation in $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$

$$[\mathrm{cone}(h_*)] = [\mathrm{el}(f_2 \circ f_1)] + [\mathrm{el}(\mathrm{id}_{\mathbb{Z}G^n})].$$

This implies

$$\begin{aligned}
(1.15)\quad [\mathrm{el}(f_2 \circ f_1)] &= [\mathrm{cone}(h_*)] - [\mathrm{el}(\mathrm{id}_{\mathbb{Z}G^n})] \\
&= [\mathrm{el}(f_1] + [\mathrm{el}(f_2)] - [\mathrm{el}(\mathrm{id}_{\mathbb{Z}G^n})] \\
&= [\mathrm{el}(f_1)] + [\mathrm{el}(f_2)] - n \cdot [\mathrm{el}(\mathrm{id}_{\mathbb{Z}G})] \;=\; [\mathrm{el}(f_1)] + [\mathrm{el}(f_2)].
\end{aligned}$$

This finishes the proof that the homomorphism el announced in (1.14) is well-defined.

One easily checks that $\rho_u^{(2)} \circ \mathrm{el} \colon K_1^w(\mathbb{Z}G) \to K_1^w(\mathbb{Z}G)$ is the identity. In order to show that the homomorphisms $\rho_u^{(2)}$ and el are bijective and inverse to one another, it remains to show that $\mathrm{el} \colon K_1^w(\mathbb{Z}G) \to K_1^{w,\mathrm{ch}}(\mathbb{Z}G)$ is surjective.

We have to show for any finite based free $\mathbb{Z}G$-chain complex C_* with the property that $\Lambda(C_*)$ is L^2-acyclic that $[C_*]$ lies in the image of el. By possibly suspending C_*, we can assume without loss of generality that $C_n = 0$ for $n \leq -1$. Now we do induction over the dimension d of C_*. The induction beginning $d \leq 1$ is obvious, the induction step from $d - 1 \geq 1$ to d is done as follows.

Choose a weak chain contraction γ_* for C_*. Define a $\mathbb{Z}G$-chain map

$$i_* \colon\ \mathrm{el}(c_1 \circ \gamma_0 \colon C_0 \to C_0) \to C_*$$

by $i_1 = \gamma_0$, $i_0 = \mathrm{id}_{C_0}$ and $i_k = 0$ for $k \neq 0, 1$. We conclude from Lemma 1.9 applied to the short based exact sequence $0 \to C_* \to \mathrm{cone}(i_*) \to \Sigma\,\mathrm{el}(c_1 \circ \gamma_0 \colon C_0 \to C_0) \to 0$ that $\mathrm{cone}(i_*)$ is a finite based free $\mathbb{Z}G$-chain complex such that $\Lambda(\mathrm{cone}(f_*))$ is L^2-acyclic and we get in $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$

$$[\mathrm{cone}(i_*)] = [(C_*)] - \big[\mathrm{el}(c_1 \circ \gamma_0 \colon C_0 \to C_0)\big].$$

Define a $\mathbb{Z}G$-chain map

$$j_* \colon\ \mathrm{el}(\mathrm{id}_{C_0}) \to \mathrm{cone}(i_*)$$

by $j_0 = \mathrm{id}_{C_0} \colon C_0 \to C_0$, $j_1 = \begin{pmatrix} \mathrm{id}_{C_0} \\ 0 \end{pmatrix} \colon C_0 \to C_0 \oplus C_1$. Then we can equip $\mathrm{coker}(j_*)$ with an obvious $\mathbb{Z}G$-basis such that we obtain a based exact short sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to \mathrm{el}(\mathrm{id}_{C_0}) \xrightarrow{j_*} \mathrm{cone}(i_*) \to \mathrm{coker}(j_*) \to 0$. We conclude from Lemma 1.9 that $\mathrm{coker}(j_*)$ is L^2-acyclic and we get in $\widetilde{K}_1^{w,\mathrm{ch}}(\mathbb{Z}G)$

$$[\mathrm{cone}(i_*)] = [\mathrm{el}(\mathrm{id}_{C_0})] + [\mathrm{coker}(j_*)].$$

Since $[\mathrm{el}(\mathrm{id}_{C_0})]$ and $\big[\mathrm{el}(c_1 \circ \gamma_0 \colon C_0 \to C_0)\big]$ lie in the image of el, it suffices to show that $[\mathrm{coker}(j_*)]$ lies in the image of el. Since $\mathrm{coker}(j_*)$ has dimension $\le d$ and its zeroth-chain module is trivial, this follows from the induction hypothesis. This finishes the proof that the homomorphisms $\rho_u^{(2)}$ and el are bijective and inverse to one another.

The proofs for $\rho\colon \widetilde{K}_1^{\mathrm{ch}}(\mathbb{Z}G) \to \widetilde{K}_1(\mathbb{Z}G)$ are analogous. □

Remark 1.16 (Universal property of the universal L^2-torsion)**.** An *additive L^2-torsion invariant* (A, a) consists of an abelian group A and an assignment which associates to a finite based free $\mathbb{Z}G$-chain complex C_* such that $\Lambda(C_*)$ is L^2-acyclic, an element $a(C_*) \in A$ such that for any based exact short sequence of such $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ we get

$$a(D_*) = a(C_*) + a(E_*),$$

and we have

$$a(\mathrm{el}(\pm\,\mathrm{id}_{\mathbb{Z}G})) = 0.$$

We call an additive L^2-torsion invariant (U, u) *universal* if for every additive L^2-torsion invariant (A, a) there is precisely one group homomorphism $f\colon U \to A$ such that for every finite based free $\mathbb{Z}G$-chain complex C_* for which $\Lambda(C_*)$ is L^2-acyclic, we have $f(u(C_*)) = a(C_*)$.

Theorem 1.12 implies that $(\widetilde{K}_1^w(\mathbb{Z}G), \rho_u^{(2)})$ is the universal additive L^2-torsion invariant.

We obtain an involution $*\colon \widetilde{K}_1^w(\mathbb{Z}G) \to \widetilde{K}_1^w(\mathbb{Z}G)$ by sending $[r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n]$ to $[r_{A^*}\colon \mathbb{Z}G^n \to \mathbb{Z}G^n]$.

Lemma 1.17. *Let C_* be a finite based free $\mathbb{Z}G$-chain complex such that $\Lambda(C_*)$ is L^2-acyclic. Then for $n \in \mathbb{Z}$ the combinatorial Laplace operator is a $\mathbb{Z}G$-map $\Delta_n\colon C_n \to C_n$ such that $\Lambda(\Delta_n)$ is a weak isomorphism, and we get in $\widetilde{K}_1^w(\mathbb{Z}G)$*

$$\rho_u^{(2)}(C_*) + *(\rho_u^{(2)}(C_*)) = -\sum_{n \ge 0} (-1)^n \cdot n \cdot [\Delta_n].$$

Proof. Lemma 1.5 shows that $\Lambda(\Delta_n)$ is a weak isomorphism for all $n \in \mathbb{Z}$. A finite based free $\mathbb{Z}G$-chain complex has length $\le l$ if there exist natural numbers n_- and n_+ with $n_- \le n_+$ such that $C_n \ne 0$ implies $n_- \le n \le n_+$ and $l = n_+ - n_-$. We prove Lemma 1.17 by induction over l. The induction beginning $l = 1$ is done as follows. Choose a natural number n_+ such that C_* is concentrated in dimensions n_+ and $n_+ - 1$. Since $\Delta_{n_+-1} \circ c_{n_+} = c_{n_+} \circ \Delta_{n_+}$ holds, we get in $\widetilde{K}_1^w(\mathbb{Z}G)$

$$[\Delta_{n_+-1}] = [\Delta_{n_+-1} \circ c_{n_+}] - [c_{n_+}] = [c_{n_+} \circ \Delta_{n_+}] - [c_{n_+}] = [\Delta_{n_+}].$$

Now we compute

$$\begin{aligned}
\rho_u^{(2)}(C_*) + *(\rho_u^{(2)}(C_*)) &= (-1)^{n_++1} \cdot [c_{n_+}] + *\big((-1)^{n_++1} \cdot [c_{n_+}]\big) \\
&= (-1)^{n_++1} \cdot [c_{n_+}] + (-1)^{n_++1} \cdot [c_{n_+}^*] \\
&= -(-1)^{n_+} \cdot [c_{n_+}^* \circ c_{n_+}] \\
&= -(-1)^{n_+} \cdot [\Delta_{n_+}] \\
&= -\big((-1)^{n_+} \cdot n_+ \cdot [\Delta_{n_+}] + (-1)^{n_+-1} \cdot (n_+ - 1) \cdot [\Delta_{n_+}]\big) \\
&= -\big((-1)^{n_+} \cdot n_+ \cdot [\Delta_{n_+}] + (-1)^{n_+-1} \cdot (n_+ - 1) \cdot [\Delta_{n_+-1}]\big) \\
&= -\sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n].
\end{aligned}$$

The induction beginning from $l - 1 \ge 1$ to l is done as follows. Choose integers n_- and n_+ with $n_- \le n_+$ such that $C_n \ne 0$ implies $n_- \le n \le n_+$ and $n_+ - n_- \le l$. Let

$\mathrm{el}_{n_-}(\Delta_{n_-})_*$ be the finite based free $\mathbb{Z}G$-chain complex concentrated in dimensions n_-+1 and n_- whose (n_-+1)-st differential is $\Delta_{n_-}: C_{n_-} \to C_{n_-}$. Define a $\mathbb{Z}G$-chain map $f_*\colon \mathrm{el}_{n_-}(\Delta_{n_-})_* \to C_*$ by putting $f_{n_-} = \mathrm{id}_{C_{n_-}}$ and $f_{n_-+1} = c^*_{n_-+1}$. Let $\mathrm{cone}(f_*)$ be its mapping cone. Then we obtain a based exact short sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to C_* \to \mathrm{cone}(f_*) \to \Sigma\,\mathrm{el}_{n_-}(\Delta_{n_-})_* \to 0$. Since $\Lambda(\mathrm{el}_{n_-}(\Delta_{n_-})_*)$ and $\Lambda(C_*)$ are L^2-acyclic, Lemma 1.9 implies that also $\mathrm{cone}(f_*)$ is L^2-acyclic and we get in $\widetilde{K}_1^w(\mathbb{Z}G)$

$$\rho_u^{(2)}(C_*) \quad = \quad \rho_u^{(2)}(\mathrm{cone}(f_*)) - \rho_u^{(2)}\bigl(\Sigma\,\mathrm{el}_{n_-}(\Delta_{n_-})_*\bigr). \tag{1.18}$$

Let $\mathrm{el}_{n_-}(\mathrm{id}_{C_{n_-}})_*$ be the finite based free $\mathbb{Z}G$-chain complex concentrated in dimensions n_-+1 and n_- whose (n_-+1)-st differential is $\mathrm{id}\colon C_{n_-} \to C_{n_-}$. Let $i_*\colon \mathrm{el}_{n_-}(\mathrm{id}_{C_{n_-}})_* \to \mathrm{cone}(f_*)_*$ be the $\mathbb{Z}G$-chain map which is given by the identity $\mathrm{id}_{C_{n_-}}$ in degree n_- and by the obvious inclusion $C_{n_-} \to C_{n_-} \oplus C_{n_-+1}$ in degree n_-+1. The cokernel of i_* is the finite based free $\mathbb{Z}G$-chain complex D_* which is concentrated in dimensions n for $n_-+1 \le n \le n_+$ and given by

$$\cdots \to 0 \to C_{n_+} \xrightarrow{c_{n_+}} C_{n_+-1} \xrightarrow{c_{n_+-1}} \cdots \xrightarrow{c_{n_-+4}} C_{n_-+3} \xrightarrow{\begin{pmatrix} 0 & c_{n_-+3}\end{pmatrix}} C_{n_-} \oplus C_{n_-+2}$$
$$\xrightarrow{\begin{pmatrix} -c^*_{n_-+1} & c_{n_-+2}\end{pmatrix}} C_{n_-+1} \to 0 \to \cdots$$

We obtain a based exact short sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to \mathrm{el}_{n_-}(\mathrm{id}_{C_{n_-}})_* \xrightarrow{i_*} \mathrm{cone}(f_*) \to D_* \to 0$. Lemma 1.9 implies that $\Lambda(D_*)$ is L^2-acyclic and we get in $\widetilde{K}_1^w(\mathbb{Z}G)$

$$\rho_u^{(2)}(\mathrm{cone}(f_*)) = \rho_u^{(2)}\bigl(\mathrm{el}_{n_-}(\mathrm{id}_{C_{n_-}})_*\bigr) + \rho_u^{(2)}(D_*) = \rho_u^{(2)}(D_*). \tag{1.19}$$

Combining (1.18) and (1.19) yields

$$\rho_u^{(2)}(C_*) \quad = \quad \rho_u^{(2)}(D_*) - \rho_u^{(2)}\bigl(\Sigma\,\mathrm{el}_{n_-}(\Delta_{n_-})_*\bigr). \tag{1.20}$$

The induction hypothesis applies to D_* since its length is $\le l-1$. Hence we get, if Δ'_n is the combinatorial Laplace operator of D_*, that

$$\begin{aligned}
& \rho_u^{(2)}(D_*) + *\bigl(\rho_u^{(2)}(D_*)\bigr) \\
= \;& -\textstyle\sum_{n\in\mathbb{Z}}(-1)^n \cdot n \cdot [\Delta'_n] \\
= \;& -(-1)^{n_-+1} \cdot (n_-+1) \cdot [\Delta_{n_-+1}] - (-1)^{n_-+2} \cdot (n_-+2) \cdot [\Delta_{n_-} \oplus \Delta_{n_-+2}] \\
& \qquad\qquad\qquad\qquad\qquad\qquad - \sum_{n=n_-+3}^{n_+} (-1)^n \cdot n \cdot [\Delta_n] \\
= \;& -\sum_{n=n_-+1}^{n_+} (-1)^n \cdot n \cdot [\Delta_n] - (-1)^{n_-+2} \cdot (n_-+2) \cdot [\Delta_{n_-}] \\
= \;& -\sum_{n=n_-}^{n_+} (-1)^n \cdot n \cdot [\Delta_n] - (1)^{n_-+2} \cdot 2 \cdot [\Delta_{n_-}] \\
= \;& -\sum_{n\in\mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n] + (1)^{n_-+1} \cdot 2 \cdot [\Delta_{n_-}].
\end{aligned}$$

We compute

$$\begin{aligned}
& \rho_u^{(2)}\bigl(\Sigma\,\mathrm{el}_{n_-}(\Delta_{n_-})_*\bigr) + *\Bigl(\rho_u^{(2)}\bigl(\Sigma\,\mathrm{el}_{n_-}(\Delta_{n_-})_*\bigr)\Bigr) \\
& = \;(-1)^{n_-+1} \cdot [-\Delta_{n_-}] + *\bigl((-1)^{n_-+1}[-\Delta_{n_-}]\bigr) \;=\; (-1)^{n_-+1} \cdot 2 \cdot [\Delta_{n_-}].
\end{aligned}$$

Now Lemma 1.17 follows from (1.20) together with the last two equalities. $\square$

1.5. **Review of division and rational closure.** Let R be a subring of the ring S. (Here and throughout the paper a ring is understood to be an associative ring with 1, which is not necessarily commutative.) The *division closure* $\mathcal{D}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is division closed, i.e., any element $x \in \mathcal{D}(R \subseteq S)$ which is invertible in S is already invertible in $\mathcal{D}(R \subseteq S)$. The *rational closure* $\mathcal{R}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is rationally closed, i.e., for any natural number n and matrix $A \in M_{n,n}(\mathcal{R}(R \subseteq S))$ which is invertible in S is already invertible over $\mathcal{R}(R \subseteq S)$. The division closure and the rational closure always exist. Obviously $R \subseteq \mathcal{D}(R \subseteq S) \subseteq \mathcal{R}(R \subseteq S) \subseteq S$.

Consider a group G. Let $\mathcal{N}(G)$ be the group von Neumann algebra which can be identified with the algebra $\mathcal{B}(L^2(G), L^2(G))^G$ of bounded G-equivariant operators $L^2(G) \to L^2(G)$. Denote by $\mathcal{U}(G)$ the algebra of operators which are affiliated to the group von Neumann algebra, see [23, Section 8] for details. This is the same as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors in $\mathcal{N}(G)$, see [23, Theorem 8.22 (1)]. By the right regular representation we can embed $\mathbb{C}G$ and hence also $\mathbb{Z}G$ as subring in $\mathcal{N}(G)$. We will denote by $\mathcal{R}(G)$ and $\mathcal{D}(G)$ the division and the rational closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$. So we get a commutative diagram of inclusions of rings

$$\begin{tikzcd}
\mathbb{Z}G \arrow[r] \arrow[d] & \mathcal{N}(G) \arrow[dd] \\
\mathcal{D}(G) \arrow[d] & \\
\mathcal{R}(G) \arrow[r] & \mathcal{U}(G)
\end{tikzcd}$$

Lemma 1.21. *Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Then the following assertions are equivalent:*

(1) *$\Lambda(C_*)$ is L^2-acyclic;*
(2) *The operator $\Lambda(\Delta_n)\colon \Lambda(C_n) \to \Lambda(C_n)$ is a weak isomorphism for all $n \in \mathbb{Z}$;*
(3) *The $\mathcal{U}(G)$-homomorphism $\mathrm{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Z}G} \Delta_n \colon \mathcal{U}(G) \otimes_{\mathbb{Z}G} C_n \to \mathcal{U}(G) \otimes_{\mathbb{Z}G} C_n$ is an isomorphism for all $n \in \mathbb{Z}$;*
(4) *The $\mathcal{R}(G)$-homomorphism $\mathrm{id}_{\mathcal{R}(G)} \otimes_{\mathbb{Z}G} \Delta_n \colon \mathcal{R}(G) \otimes_{\mathbb{Z}G} C_n \to \mathcal{R}(G) \otimes_{\mathbb{Z}G} C_n$ is an isomorphism for all $n \in \mathbb{Z}$;*
(5) *The $\mathcal{U}(G)$-chain complex $\mathcal{U}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible;*
(6) *The $\mathcal{R}(G)$-chain complex $\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible.*

Proof. (1) $\iff$ (2) This has already been proved in Lemma 1.5.

(2) $\iff$ (3) This follows from [23, Theorem 6.24 on page 249 and Theorem 8.22 (5) on page 327].

(3) $\iff$ (4) This follows from the definition of the rational closure.

(4) $\implies$ (6) The collection of the $\mathbb{Z}G$-maps $c_{n+1}^*\colon C_n \to C_{n+1}$ defines $\mathbb{Z}G$-chain homotopy $\Delta_* \simeq 0_*$, where $\Delta_*\colon C_* \to C_*$ is the $\mathbb{Z}G$-chain map given by Δ_n in degree n. Therefore we get a $\mathcal{R}(G)$-chain isomorphism $\mathrm{id}_{\mathcal{R}(G)} \otimes_{\mathbb{Z}G} \Delta_*\colon \mathcal{R}(G) \otimes_{\mathbb{Z}G} C_* \to \mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*$ which is $\mathcal{R}(G)$-nullhomotopic. Hence $\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*$ is contractible.

(6) $\implies$ (5) This is obvious.

(5) $\implies$ (1) We conclude from [23, Theorem 6.24 on page 249 and Theorem 8.29 (5) on page 330] that $b_n^{(2)}(\Lambda(C_*)) = 0$ for all $n \in \mathbb{Z}$. This finishes the proof of Lemma 1.21. □

With the notation we just introduced we can now formulate the following proposition of Linnell–Lück, see [25, Theorem 0.1]. For many torsionfree groups it gives an alternative description of $K_1^w(\mathbb{Z}[G])$.

Proposition 1.22. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let G be a torsionfree group which belongs to $\mathcal{C}$. Then the following hold:*

1. $K_1^w(\mathbb{Z}[G])$ *is isomorphic to* $K_1(\mathcal{D}(G))$.
2. *The ring* $\mathcal{D}(G)$ *is a skew field.*
3. *The group* $K_1(\mathcal{D}(G))$ *is the abelianization of the multiplicative group of units in* $\mathcal{D}(G)$.

2. Universal L^2-torsion for CW-complexes and manifolds

We will define the universal L^2-torsion $\rho_u^{(2)}(X;\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$ for an L^2-acyclic finite free G-CW-complex X by applying the notion of the universal L^2-torsion of Section 1 to the cellular chain complex. We will present the basic properties of this invariant in Theorem 2.5.

If G is a group such that there exists a connected L^2-acyclic finite free G-CW-complex, then we will see that every element in $\mathrm{Wh}^w(G)$ occurs as $\rho_u^{(2)}(X;\mathcal{N}(G))$, see Lemma 2.8.

2.1. The universal L^2-torsion for G-CW-chain complexes.

Notice that the cellular G-CW-structure on a finite free G-CW-complex defines only an equivalence class of $\mathbb{Z}G$-bases on $C_*(X)$, where we call two $\mathbb{Z}G$-basis B and B' equivalent if there exists a bijection $\sigma\colon B \to B'$ such that for every $b \in B$ there exists $g \in G$ and $\epsilon \in \{\pm 1\}$ with $\epsilon \cdot g \cdot \sigma(b) = b$. The Hilbert $\mathcal{N}(G)$-chain complex $C_*^{(2)}(X)$ is independent of the choice of a $\mathbb{Z}G$-basis within the equivalence class of cellular $\mathbb{Z}G$-bases. So we can define the n-th L^2-Betti number $b_n^{(2)}(X;\mathcal{N}(G))$ to be $b_n^{(2)}(C_*^{(2)}(X);\mathcal{N}(G))$.

Definition 2.1 (Universal L^2-torsion for G-CW-complexes)**.** Let X be a finite free G-CW-complex which is L^2-acyclic, i.e., its n-th L^2-Betti number $b_n^{(2)}(X;\mathcal{N}(G))$ vanishes for all $n \geq 0$. Then we define its *universal L^2-torsion*

$$\rho_u^{(2)}(X;\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$$

to be the image of $\rho_u^{(2)}(C_*(X);\mathcal{N}(G))$ under the projection $\widetilde{K}_1^w(\mathbb{Z}G) \to \mathrm{Wh}^w(G)$ after any choice of $\mathbb{Z}G$-basis for $C_*(X)$ which represents the equivalence class of cellular $\mathbb{Z}G$-basis.

This definition extends to pairs (X,Y) of finite free G-CW-complexes in the obvious way, consider the cellular $\mathbb{Z}G$-CW-complex $C_*(X,Y)$ and require that $b_n^{(2)}(X,Y;\mathcal{N}(G))$ vanishes for all $n \geq 0$.

Remark 2.2 (Universal L^2-torsion for manifolds)**.** Every compact topological manifold has a preferred simple homotopy type, see [17, IV]. We can therefore extend the definition of the universal L^2-torsion from CW-complexes to manifolds in the usual way.

Since we consider the universal L^2-torsion as an element in $\mathrm{Wh}^w(G)$, the choice of the $\mathbb{Z}G$-basis representing the equivalence class of cellular $\mathbb{Z}G$-basis does not matter and $\rho_u^{(2)}(X;\mathcal{N}(G))$ depends only on the finite free G-CW-structure on X.

Example 2.3 ($G = \mathbb{Z}$)**.** The quotient field $\mathbb{Q}(z,z^{-1})$ of the integral domain $\mathbb{Z}[z,z^{-1}] = \mathbb{Z}[\mathbb{Z}]$ consists of rational functions with coefficients in $\mathbb{Q}$ in one variable z. Given a matrix $A \in M_{n,n}(\mathbb{Z}[\mathbb{Z}])$, the operator $\Lambda\big(r_A\colon \mathbb{Z}[\mathbb{Z}]^n \to \mathbb{Z}[\mathbb{Z}]^n\big)$ is a weak isomorphism if and only if $\det_{\mathbb{Z}[\mathbb{Z}]}(A) \in \mathbb{Z}[\mathbb{Z}]$ is non-zero. This follows from [23, Lemma 1.34 on page 35]. Hence we obtain a well-defined homomorphism

$$\det_{\mathbb{Z}[\mathbb{Z}]}\colon K_1^w(\mathbb{Z}[\mathbb{Z}]) \to \mathbb{Q}(z,z^{-1})^\times, \quad \big[r_A\colon \mathbb{Z}[\mathbb{Z}]^n \to \mathbb{Z}[\mathbb{Z}]^n\big] \mapsto \det_{\mathbb{Z}[\mathbb{Z}]}(A).$$

Define

$$i\colon \mathbb{Q}(z,z^{-1})^\times \to K_1^w(\mathbb{Z}[\mathbb{Z}])$$

by sending an element $x \in \mathbb{Q}(z, z^{-1})^\times$ to $[r_p \colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]] - [r_q \colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]]$ for any two elements $p, q \in \mathbb{Z}[\mathbb{Z}]$ with $p \neq 0, q \neq 0$ satisfying $x = p \cdot q^{-1}$ in $\mathbb{Q}(z, z^{-1})^\times$. One easily checks that i is well-defined and $\det_{\mathbb{Z}[\mathbb{Z}]} \circ i = \mathrm{id}_{\mathbb{Q}(z,z^{-1})^\times}$. It is not hard to prove using the standard Euclidean algorithm on the polynomial ring $\mathbb{Q}[z]$ that i is surjective. Hence $\det_{\mathbb{Z}[\mathbb{Z}]}$ and i are inverses of each other. We obtain an induced isomorphism

$$\mathrm{Wh}^w(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}(z, z^{-1})^\times / \{\sigma \cdot z^n \mid \sigma \in \{\pm 1\}, n \in \mathbb{Z}\}. \tag{2.4}$$

Consider $\mathbb{R}$ with the standard $\mathbb{Z}$-action given by translation. This is the universal covering of S^1 with the standard action of $\pi_1(S^1) = \mathbb{Z}$. The cellular $\mathbb{Z}[\mathbb{Z}]$-chain complex of $\mathbb{R}$ is 1-dimensional and has as first differential $r_{z-1} \colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]$. Hence the isomorphism (2.4) sends $\rho_u^{(2)}(\mathbb{R}; \mathcal{N}(\mathbb{Z})) \in \mathrm{Wh}^w(\mathbb{Z})$ to the class of the element $(z-1)$.

More generally, let X be any finite free $\mathbb{Z}$-CW-complex. Then X is L^2-acyclic if and only if all of the $\mathbb{Q}[\mathbb{Z}]$-modules $H_n(X; \mathbb{Q})$ are torsion $\mathbb{Q}[\mathbb{Z}]$-modules, see [23, Lemma 1.34 on page 35]. It is now straightforward to see that under the isomorphism 2.4 the universal L^2-torsion of X corresponds to the Milnor-Turaev torsion [28, 36] of X.

Theorem 2.5 (Main properties of the universal L^2-torsion)**.**

(1) (G-homotopy invariance) *Let $f \colon X \to Y$ be a G-homotopy equivalence of finite free G-CW-complexes. Suppose that X or Y is L^2-acyclic. Then both X and Y are L^2-acyclic and we get*

$$\rho_u^{(2)}(Y) - \rho_u^{(2)}(X) = \zeta(\tau(f)),$$

where $\tau(f) \in \mathrm{Wh}(G)$ is the Whitehead torsion of f and $\zeta \colon \mathrm{Wh}(G) \to \mathrm{Wh}^w(G)$ is the obvious homomorphism;

(2) (Sum formula) *Consider a G-pushout of finite free G-CW-complexes*

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of G-CW-complexes and X has the obvious G-CW-structure coming from the ones on X_0, X_1 and X_2. Suppose that X_0, X_1 and X_2 are L^2-acyclic. Then X is L^2-acyclic and we get

$$\rho_u^{(2)}(X; \mathcal{N}(G)) = \rho_u^{(2)}(X_1; \mathcal{N}(G)) + \rho_u^{(2)}(X_2; \mathcal{N}(G)) - \rho_u^{(2)}(X_0; \mathcal{N}(G));$$

(3) (Induction) *Let $j \colon H \to G$ be an inclusion of groups. Denote by $j_* \colon \mathrm{Wh}^w(H) \to \mathrm{Wh}^w(G)$ the induced homomorphism. Let X be an L^2-acyclic finite free H-CW-complex. Then $j_*X = G \times_H X$ is an L^2-acyclic finite free G-CW-complex and we get*

$$\rho_u^{(2)}(j_*X; \mathcal{N}(G)) = j_*\big(\rho_u^{(2)}(X; \mathcal{N}(H))\big);$$

(4) (Restriction) *Let $j \colon H \to G$ be an inclusion of groups such that the index $[G : H]$ is finite. Denote by $j^* \colon \mathrm{Wh}^w(G) \to \mathrm{Wh}^w(H)$ the homomorphism given by restriction. Let X be a finite free G-CW-complex. Let j^*X be the restriction to H by j. Then j^*X is a finite free H-CW-complex, X is L^2-acyclic if and only if j^*X is L^2-acyclic, and, if this is the case, we get*

$$\rho_u^{(2)}(j^*X; \mathcal{N}(H)) = j^*\big(\rho_u^{(2)}(X; \mathcal{N}(G))\big);$$

(5) (Product formula) *Let G_0 and G_1 be groups and denote by $j\colon G_0 \to G_0 \times G_1$ the obvious inclusion. Denote by $j_*\colon \operatorname{Wh}^w(G_0) \to \operatorname{Wh}^w(G_0 \times G_1)$ the induced homomorphism. Let X_i be a finite free G_i-CW-complex for $i = 0, 1$. Suppose that X_0 is L^2-acyclic. Then $X_0 \times X_1$ is L^2-acyclic and we get*
$$\rho_u^{(2)}(X_0 \times X_1; \mathcal{N}(G_0 \times G_1)) = \chi(X_1/G_1) \cdot j_*\big(\rho_u^{(2)}(X_0; \mathcal{N}(G_0))\big);$$

(6) (Fibrations) *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose that F and B are finite CW-complexes. Let $q\colon \overline{E} \to E$ be a G-covering. Let $\overline{F} \to F$ be the G-covering obtained from q by the pullback construction with i. Suppose that $\overline{F}$ is L^2-acyclic. Assume that* $\operatorname{Wh}(G)$ *vanishes. Then $\overline{E}$ up to G-homotopy equivalence and $\overline{F}$ are L^2-acyclic finite free G-CW-complexes and we get in* $\operatorname{Wh}^w(G)$
$$\rho_u^{(2)}(\overline{E}; \mathcal{N}(G)) = \chi(B) \cdot \rho_u^{(2)}(\overline{F}; \mathcal{N}(G));$$

(7) (S^1-actions) *Let X be a connected finite S^1-CW-complex. Fix a base point $x \in X$. Let $\mu\colon \pi_1(X, x) \to G$ be a group homomorphism. Suppose that the composite*
$$\pi_1(S^1, 1) \xrightarrow{\pi_1(\operatorname{ev}_x, 1)} \pi_1(X, x) \xrightarrow{\mu} G$$
is injective, where $\operatorname{ev}_x\colon S^1 \to X$ sends z to $z \cdot x$. Let I_n be the set of open n-dimensional S^1-cells of X. For each $e \in I_n$ choose a point $x(e)$ in its interior and a path $w(e)$ from $x(e)$ to x. Denote by $S^1_{x(e)} \subseteq S^1$ the isotropy group of $x(e)$ which must be finite because of the assumptions above. Let $\overline{\operatorname{ev}}_e\colon S^1/S^1_{x(e)} \to X$ be the injective map which sends $z \cdot S^1_{x(e)}$ to $z \cdot x(e)$. Choose a homeomorphism $f_e\colon S^1 \xrightarrow{\cong} S^1/S^1_{x(e)}$ with $f_e(1) = 1 \cdot S^1_{x(e)}$. Identify $\mathbb{Z} = \pi_1(S^1, 1)$. Let $j(e)\colon \mathbb{Z} \to G$ be the injective homomorphism given by the composite
$$j(e)\colon \mathbb{Z} = \pi_1(S^1, 1) \xrightarrow{\pi_1(f_e, 1)} \pi_1(S^1/S^1_{x(e)}, 1 \cdot S^1_{x(e)}) \xrightarrow{\pi_1(\overline{\operatorname{ev}}_e, 1 \cdot S^1_{x(e)})} \pi_1(X, x(e)) \xrightarrow{c_{w(e)}} \pi_1(X, x) \xrightarrow{\mu} G,$$
where $c_{w(e)}$ is given by conjugation with the path $w(e)$. Denote by
$$j(e)_*\colon \operatorname{Wh}^w(\mathbb{Z}) \to \operatorname{Wh}^w(G)$$
the homomorphism induced by $j(e)$. Equip $\mathbb{R}$ with the $\mathbb{Z}$-action given by translation. Then $\overline{X}$ is L^2-acyclic and we get
$$\rho_u^{(2)}(\overline{X}; \mathcal{N}(G)) = \sum_{n \geq 0} (-1)^n \cdot \sum_{e \in I_n} j(e)_*\big(\rho_u^{(2)}(\mathbb{R}, \mathcal{N}(\mathbb{Z}))\big);$$

(8) (Poincaré duality) *Let M be an orientable n-dimensional manifold with free proper G-action and boundary ∂M. Let $w\colon G \to \{\pm 1\}$ be the orientation homomorphism sending $g \in G$ to 1, if multiplication with g respects the orientation and to -1 otherwise. Equip $\operatorname{Wh}^w(G)$ with the involution $*$ coming from the w-twisted involution on $\mathbb{Z}G$ sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g \cdot w(g) \cdot g^{-1}$. Then*
$$\rho_u^w(M, \partial M; \mathcal{N}(G)) = (-1)^{n+1} \cdot *\big(\rho_u^w(M; \mathcal{N}(G))\big).$$

Proof. (1) This follows from Lemma 1.10.

(2) This follows from Lemma 1.9.

(3) This follows directly from the definitions and [23, Lemma 1.24 (4) on page 30] using the canonical isomorphism $\mathbb{Z}G \otimes_{\mathbb{Z}H} C_*(X) \cong C_*(j_*X)$.

(4) Obviously j^*X is a finite free H-CW-complex. We conclude
$$b_n^{(2)}(j^*X; \mathcal{N}(H)) = [G : H] \cdot b_n^{(2)}(X; \mathcal{N}(G))$$

from [23, Theorem 1.35 (9) on page 38]. Hence X is L^2-acyclic if and only if j^*X is L^2-acyclic. Since the index $[G : H]$ is finite, there is an obvious homomorphism $j^*\colon \operatorname{Wh}^w(G) \to \operatorname{Wh}^w(H)$ given by restriction with j. Now the formula $\rho_u^{(2)}(j^*X;\mathcal{N}(H)) = j^*(\rho_u^{(2)}(X;\mathcal{N}(G)))$ follows from the definitions since the restriction of the $\mathbb{Z}G$-chain complex $C_*(X)$ to $\mathbb{Z}H$ with j agrees with $C_*(j^*X)$.

(5) This follows from assertions (1), (2) and (3) by induction over the equivariant cells in Y.

(6) This follows from assertions (1) and (2) by induction over the equivariant cells in B.

(7) This follows from assertions (1), (2) and (3) by induction over the S^1-cells in X using the fact that the finite free $\mathbb{Z}$-CW-complex $\mathbb{R} = \widetilde{S^1}$ is L^2-acyclic.

(8) There is a simple $\mathbb{Z}G$-chain homotopy equivalence $C^{n-*}(M) \to C_*(M, \partial M)$, where $C^{n-*}(M)$ is the dual $\mathbb{Z}G$-chain complex with respect to the w-twisted involution. Lemma 1.10 implies

$$\rho_u^w(C^{n-*}(M)) = \rho_u^w(C_*(M, \partial M)).$$

We conclude directly from the definitions.

$$\rho_u^w(C^{n-*}(M)) = (-1)^{n+1} \cdot *\bigl(\rho_u^w(C_*(M)).$$

This finishes the proof of Theorem 2.5. □

Remark 2.6 (Universal L^2-torsion in terms of the combinatorial Laplace operator)**.** Let M be an orientable n-dimensional manifold with free proper G-action and empty boundary such that the G action is orientation preserving. Suppose that the dimension of M is odd. Let $\Delta_n\colon C_*(M) \to C_*(M)$ be its combinatorial Laplace operator. Then we conclude from Lemma 1.17 and Theorem 2.5 (8) that we get in $\operatorname{Wh}^w(G)$

$$2 \cdot \rho_u^{(2)}(M;\mathcal{N}(G)) = -\sum_{n\in\mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n].$$

The advantage of the formula above is that one can derive the right hand side directly from the differentials without having to find an explicit weak chain contraction. Notice that in the case, where the dimension of M is even, we do not get interesting information about the universal L^2-torsion, namely, we just get

$$0 = -\sum_{n\in\mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n].$$

Example 2.7 (Torus T^n)**.** Let T^n be the n-dimensional torus for $n \geq 2$. Let G be a torsion-free group. Let $\mu\colon \pi_1(T^n) \to G$ be a non-trivial group homomorphism. Let $\overline{T^n} \to T^n$ be the G-covering associated to μ. Then $\overline{T^n}$ is L^2-acyclic and we get

$$\rho_u^{(2)}(\overline{T^n};\mathcal{N}(G)) = 0$$

by the following argument.

We can choose an integer k with $k \geq 1$, an isomorphism $\nu\colon \pi_1(T^n) \xrightarrow{\cong} \mathbb{Z}^k \times \mathbb{Z}^{n-k}$ and an injection $i\colon \mathbb{Z}^k \to G$ such that $\mu = i \circ \operatorname{pr} \circ \nu$ for $\operatorname{pr}\colon \mathbb{Z}^k \times \mathbb{Z}^{n-k} \to \mathbb{Z}^k$ the projection. We can find a homeomorphism $f\colon T^n \xrightarrow{\cong} T^k \times T^{n-k}$ such that $\pi_1(f) = \nu$. Hence we obtain a G-homeomorphism

$$i_*\bigl(\widetilde{T^k} \times T^{n-k}\bigr) \xrightarrow{\cong} \overline{T^n},$$

where the $\mathbb{Z}^k$-action on $\widetilde{T^k} \times T^{n-k}$ is given by the standard $\mathbb{Z}^k$-action on $\widetilde{T^k}$ and the trivial $\mathbb{Z}^k$-action on T^{n-k}. We conclude from Theorem 2.5 (3) and (5) that $\overline{T^n}$ is L^2-acyclic and

$$\rho_u^{(2)}(\overline{T^n};\mathcal{N}(G)) \;=\; \rho_u^{(2)}(\widetilde{T^k} \times T^{n-k};\mathcal{N}(\mathbb{Z}^k)) \;=\; \chi(T^{n-k}) \cdot \rho_u^{(2)}(\widetilde{T^k};\mathcal{N}(\mathbb{Z}^k)).$$

If $k \neq n$, then the claim follows from $\chi(T^{n-k}) = 0$. Suppose that $k = n$. Then we have to show $\rho_u^{(2)}(\widetilde{T^k}; \mathcal{N}(\mathbb{Z}^k)) = 0$ for $k \geq 2$. This follows from Theorem 2.5 (5) applied to $\widetilde{S^1 \times T^{k-1}}$ using $\chi(T^{k-1}) = 0$.

Lemma 2.8 (Realizability of the universal L^2-torsion for G-CW-complexes)**.** *Let G be a group such that there exists a connected L^2-acyclic finite free G-CW-complex X. Consider any element $\omega \in \mathrm{Wh}^w(G)$. Then there exists a connected L^2-acyclic finite free G-CW-complex Y obtained from X by attaching trivially equivariant 2-cells and attaching equivariant 3-cells with $\rho_u^{(2)}(Y; \mathcal{N}(G)) = \omega$.*

Proof. Choose a matrix $A \in M_{n,n}(\mathbb{Z}G)$ such that $\Lambda(r_A)$ is a weak isomorphism and $[r_A] = \omega - \rho_u^{(2)}(X; \mathcal{N}(G))$ holds in $\mathrm{Wh}^w(G)$. Choose base points $x \in X$ and $s \in S^2$. Let $k\colon G \to X$ be the G-map sending g to $g \cdot x$ and let $l\colon G \to G \times S^2$ be the G-map sending g to (g, s). Let X' be the finite free G-CW-complex X' given by the G-pushout

$$\begin{array}{ccc} \coprod_{j=1}^n G & \xrightarrow{\coprod_{i=1}^n k} & X \\ {\scriptstyle \coprod_{j=1}^n l}\big\downarrow & & \big\downarrow \\ \coprod_{j=1} G \times S^2 & \longrightarrow & X'. \end{array}$$

For $g \in G$ choose a path v_g in X from x to gx. Let $t_g\colon \pi_2(X', gx) \to \pi_2(X', x)$ be the standard isomorphism of abelian groups given by v_g. Fix elements $i, j \in \{1, 2, \dots, n\}$. Let $a[i,j] = \sum_{g \in G} a[i,j]_g \cdot g \in \mathbb{Z}G$ be the entry of A at (i,j). Choose for $g \in G$ a pointed map $q[i,j]_g\colon (S^2, s) \to (X', x)$ such that its class $[q[i,j]_g]$ in $\pi_2(X', x)$ is $a[i,j]_g$-times the image under $t_g\colon \pi_2(X', gx) \to \pi_2(X', x)$ of the element in $\pi_2(X', gx)$ given by the composite $S^2 \to G \times S^2, y \mapsto (g, y)$ with the inclusion of the j-th summand of $\coprod_{j=1} G \times S^n$ into X'. Let $q_i\colon (S^2, s) \to (X', x)$ be a pointed map representing in $\pi_2(X', x)$ the element $\sum_{j=1}^n \sum_{g \in G} [q[i,j]_g]$. Let $\widehat{q_i}\colon G \times S^2 \to X$ be the G-map sending (g, z) to $g \cdot q_i(z)$. Define a finite free G-CW-complex Y by the G-pushout

$$\begin{array}{ccc} \coprod_{i=1}^n G \times S^2 & \xrightarrow{\coprod_{i=1}^n \widehat{q_i}} & X' \\ \big\downarrow & & \big\downarrow \\ \coprod_{i=1}^n G \times D^3 & \longrightarrow & Y. \end{array}$$

By construction there is a based exact sequence of finite based free $\mathbb{Z}G$-chain complexes $0 \to C_*(X) \to C_*(Y) \to D_* \to 0$, where D_* is concentrated in dimensions 2 and 3 and has as third differential $r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$. Since $\Lambda(C_*(X))$ and $\Lambda(D_*)$ are L^2-acyclic, Lemma 1.9 implies that also $\Lambda(C_*(Y))$ is L^2-acyclic and we get in $\mathrm{Wh}^w(G)$

$$\begin{aligned} \rho_u^{(2)}(Y) &= \rho_u^{(2)}(C_*(Y; \mathcal{N}(G)) \\ &= \rho_u^{(2)}(C_*(X; \mathcal{N}(G)) + \rho_u^{(2)}(D_*; \mathcal{N}(G)) \\ &= \rho_u^{(2)}(C_*(X; \mathcal{N}(G)) + [r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n] \\ &= \rho_u^{(2)}(C_*(X; \mathcal{N}(G)) + \omega - \rho_u^{(2)}(C_*(X; \mathcal{N}(G)) \;=\; \omega. \quad \square \end{aligned}$$

Lemma 2.9 (Realizability of the universal L^2-torsion for manifolds without boundary and cocompact free proper G-action)**.** *Let G be a group such that there exists a connected L^2-acyclic finite free G-CW-complex X. Consider any element $\omega \in \mathrm{Wh}^w(G)$ and any integer $d \geq 2 \cdot \max\{\dim(X), 3\} + 1$. Then there exists a connected smooth d-dimensional manifold M without boundary and cocompact free proper smooth G-action such that M is L^2-acyclic and we have*

$$\rho_u^{(2)}(M; \mathcal{N}(G)) = (-1)^{d+1} \cdot *(\omega) + \omega.$$

Furthermore, if X is simply-connected, then M can also be chosen to be simply-connected.

Proof. By Lemma 2.8 we can find a connected L^2-acyclic finite free G-CW-complex Y of dimension $\max\{3, \dim(X)\}$ with $\rho^2(Y; \mathcal{N}(G)) = \omega$. We can embed Y/G into $\mathbb{R}^d$ and choose a regular neighborhood N. This is a compact manifold N with boundary ∂N such that the inclusion $i\colon Y \to N$ is a simple homotopy equivalence. Let $\overline{N} \to N$ be the G-covering obtained from the G-covering $Y \to Y/G$ by the pullback construction applied to any homotopy inverse of $i\colon Y \to N$. Let $\overline{\partial N} \to \partial N$ be the restriction of $\overline{N} \to N$ to ∂N. Since Y is L^2-acyclic, $\overline{N}$ is L^2-acyclic. We conclude from Poincaré duality and the long weak exact L^2-homology sequence, see [23, Theorem 1.21 on page 27 and Theorem 1.35 on page 37] that $\overline{\partial N}$ and $(\overline{N}, \overline{\partial N})$ are L^2-acyclic. Let M be $\overline{N} \cup_{\overline{\partial N}} \overline{N}$. This is a smooth manifold without boundary with proper free smooth G-action. One easily checks using Lemma 1.9 and Theorem 2.5 that N is L^2-acyclic and we get in $\mathrm{Wh}^w(G)$

$$\begin{aligned}
\rho_u^{(2)}(M; \mathcal{N}(G)) &= \rho_u^{(2)}(\overline{N}; \mathcal{N}(G)) - \rho_u^{(2)}(\overline{\partial N}; \mathcal{N}(G)) + \rho_u^{(2)}(\overline{N}; \mathcal{N}(G)) \\
&= \rho_u^{(2)}(\overline{N}; \mathcal{N}(G)) + \rho_u^{(2)}(\overline{N}, \overline{\partial N}; \mathcal{N}(G)) \\
&= \rho_u^{(2)}(\overline{N}; \mathcal{N}(G)) + (-1)^{d+1} \cdot *\bigl(\rho_u^{(2)}(\overline{N}; \mathcal{N}(G))\bigr) \\
&= \omega + (-1)^{d+1} \cdot *(\omega).
\end{aligned}$$

Notice that for an element $\omega \in \mathrm{Wh}^w(G)$ the equality $\omega = (-1)^{d+1} \cdot *(\omega)$ is a necessary condition for ω to be realized as $\omega = \rho_u^{(2)}(M; \mathcal{N}(G))$ for a smooth orientable manifold M without boundary and proper free orientation preserving G-action such that M is L^2-acyclic, see Theorem 2.5 (8). The construction above shows that for an element $\omega \in \mathrm{Wh}^w(G)$ satisfying $\omega = (-1)^{d+1} \cdot *(\omega)$ we can find a smooth orientable manifold M without boundary and proper free orientation preserving G-action such that M is L^2-acyclic and $\rho_u^{(2)}(M; \mathcal{N}(G)) = 2 \cdot \omega$.

Finally, if X is simply-connected, then $\overline{N}$ is simply-connected and M is also simply-connected. $\square$

2.2. **The universal L^2-torsion for universal coverings.** The most natural and interesting case is the one of a universal covering. For the reader's convenience we record the basic properties of the universal L^2-torsion in this setting.

Definition 2.10 (Universal L^2-torsion for universal coverings)**.** Let X be a finite connected CW-complex. We call it *L^2-acyclic* if its universal covering $\widetilde{X}$ is L^2-acyclic, i.e., the n-th L^2-Betti number $b_n^{(2)}(\widetilde{X}) := b_n^{(2)}(\widetilde{X}; \mathcal{N}(\pi_1(X)))$ vanishes for all $n \geq 0$. Then we define its *universal L^2-torsion*

$$\rho_u^{(2)}(\widetilde{X}) \in \mathrm{Wh}^w(\pi_1(X))$$

by $\rho_u^{(2)}(\widetilde{X}; \mathcal{N}(\pi_1(X)))$ as introduced in Definition 2.1.

If X is a finite CW-complex, we call it *L^2-acyclic* if each path component $C \in \pi_0(C)$ is L^2-acyclic in the sense above and we define

$$\begin{aligned}
\mathrm{Wh}^w(\Pi(X)) &:= \bigoplus_{C \in \pi_0(X)} \mathrm{Wh}^w(\pi_1(C)); \\
\rho_u^{(2)}(\widetilde{X}) &:= \bigl(\rho_u^{(2)}(\widetilde{C})\bigr)_{C \in \pi_0(X)} \quad \in \mathrm{Wh}^w(\Pi(X)).
\end{aligned}$$

This definition extends to CW-pairs (X, A) by

$$\rho_u^{(2)}(\widetilde{X}, \widetilde{A}) := \bigl(\rho_u^{(2)}(\widetilde{C}), \widetilde{A \cap C}\bigr)_{C \in \pi_0(X)} \quad \in \mathrm{Wh}^w(\Pi(X)),$$

where we denote for a path component C of X by $\widetilde{A \cap C} \to A \cap C$ the restriction of the universal covering $\widetilde{C} \to C$ to $A \cap C$.

Given a map $f\colon X \to Y$ of finite C-complexes such that $\pi_1(f,x)\colon \pi_1(X,x) \to \pi_1(Y,f(x))$ is injective for all $x \in X$, we get a homomorphism $f_*\colon \mathrm{Wh}^w(\Pi(X)) \to \mathrm{Wh}^w(\Pi(Y)$ by the collection of maps $(f|_C)_*\colon \mathrm{Wh}^w(\pi_1(C)) \to \mathrm{Wh}^w(\pi_1(D))$ for $C \in \pi_0(X)$ and $D \in \pi_1(Y)$ with $f(C) \subseteq D$.

It is evident that Theorem 2.5 gives rise to statements about the universal L^2-torsions for universal coverings. Most statements of Theorem 2.5 specialize in an obvious way. Therefore in the next theorem we spell out only three properties.

Theorem 2.11 (Main properties of the universal L^2-torsion for universal coverings)**.**

(2) (Sum formula) *Consider a pushout of finite CW-complexes*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of CW-complexes and X has the obvious CW-structure coming from the ones on X_0, X_1 and X_2. Suppose that X_0, X_1 and X_2 are L^2-acyclic and that for $i = 0,1,2$ and any point $x_i \in X$ the homomorphism $\pi_1(j_i,x_i)\colon \pi_1(X_i,x_i) \to \pi_1(X,j_i(x_i))$ is injective. Then X is L^2-acyclic and we get

$$\rho_u^{(2)}(\widetilde{X}) = (j_1)_*\bigl(\rho_u^{(2)}(\widetilde{X_1})\bigr) + (j_2)_*\bigl(\rho_u^{(2)}(\widetilde{X_2})\bigr) - (j_0)_*\bigl(\rho_u^{(2)}(\widetilde{X_0})\bigr).$$

(5) (Fibrations) *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose that $F$and B are connected finite CW-complexes. Assume that $\pi_1(i)\colon \pi_1(F) \to \pi_1(E)$ is injective and that F is L^2-acyclic. Suppose that* $\mathrm{Wh}(\pi_1(E))$ *vanishes. Then E is up to homotopy an L^2-acyclic connected finite CW-complex and we get*

$$\rho_u^{(2)}(\widetilde{E}) = \chi(B)\cdot i_*\bigl(\rho_u^{(2)}(\widetilde{F})\bigr).$$

(6) (S^1-actions) *Let X be a connected finite S^1-CW-complex. If we use the notation and make the assumptions of Theorem 2.5* (7) *in the special case $G = \pi_1(X)$ and $\mu = \mathrm{id}_{pi_1(X)}$, then X is L^2-acyclic and we get*

$$\rho_u^{(2)}(\widetilde{X}) = \sum_{n\geq 0}(-1)^n\cdot\sum_{e\in I_n} j(e)_*\bigl(\rho_u^{(2)}(\widetilde{S^1})\bigr).$$

Remark 2.12 (Realizability of the universal L^2-torsion for universal coverings.)**.** Let π be group such that there exists a connected finite CW-complex X with $\pi = \pi_1(X)$ which is L^2-acyclic. Consider any element $\omega \in \mathrm{Wh}^w(\pi)$. As a special case of Lemma 2.8 we get that there is a connected finite CW-complex Y with $\pi = \pi_1(Y)$ such that Y is L^2-acyclic and $\rho_u^{(2)}(\widetilde{Y}) = \omega$.

Also Lemma 2.9 has an obvious analogue for $\rho_u^{(2)}(\widetilde{M})$ for a closed manifold M.

Example 2.13 ($\rho_u^{(2)}(\widetilde{T^n})$)**.** If $n \geq 2$, we conclude $\rho_u^{(2)}(\widetilde{T^n}) = 0$ from Example 2.7. We have computed $\rho_u^{(2)}(\widetilde{S^1})$ in Example 2.3.

Theorem 2.14 (Jaco-Shalen-Johannson decomposition)**.** *Let M be an admissible 3-manifold and let M_1, M_2, ..., M_r be its pieces in the Jaco-Shalen-Johannson decomposition. Let $j_i\colon \pi_1(M_i) \to \pi_1(M)$ be the injection induced by the inclusion $M_i \to M$. Then each M_i and M are L^2-acyclic and we have*

$$\rho_u^{(2)}(\widetilde{M}) \;=\; \sum_{i=1}^{r}(j_i)_*\bigl(\rho_u^{(2)}(\widetilde{M_i})\bigr).$$

Proof. Each piece M_i is L^2-acyclic by [21, Theorem 0,1]. Now the claim follows from Example 2.7 and Theorem 2.11 (2). $\square$

Remark 2.15 (Graph manifolds)**.** We recall that a *Seifert manifold* is loosely speaking a singular S^1-bundle over a surface. A *graph manifold* is a 3-manifold that is obtained by gluing Seifert manifolds along boundary tori. We refer to [2] for precise definitions. There is an obvious analogue of Theorem 2.11 (6) for Seifert manifolds M with infinite fundamental groups where the role of S^1 is played by the regular fiber whose inclusion to M always induces an injection on the fundamental groups, and the cells corresponds to tubular neighborhoods of fibers. So we get again a formula for $\rho_u^{(2)}(\widetilde{M})$ which essentially reduces the computation to the one of $\rho_u^{(2)}(\widetilde{S^1})$. In view of Theorem 2.14 this extends to graph manifolds. The hyperbolic pieces in the Jaco-Shalen-Johannson decomposition are much harder to deal with.

2.3. **Mapping tori.** Let $f\colon X \to X$ be a self-map of a connected finite CW-complex. Denote by T_f the mapping torus. For $p\colon T_f \to S^1$ the obvious projection, consider any factorization $\pi_1(p)\colon \pi_1(T_f) \xrightarrow{\mu} G \xrightarrow{\phi} \pi_1(S^1) = \mathbb{Z}$. Let $\overline{T_f} \to T_f$ be the G-covering associated to $\mu\colon \pi_1(T_f) \to G$. Then $\overline{T_f}$ is L^2-acyclic by [22, Theorem 2.1] and we can consider

$$\rho_u^{(2)}(\overline{T_f};\mathcal{N}(G)) \in K_1^w(\mathbb{Z}G), \tag{2.16}$$

and especially

$$\rho_u^{(2)}(\widetilde{T_f}) \in K_1^w(\mathbb{Z}[\pi_1(T_f)]. \tag{2.17}$$

In particular the latter is an interesting invariant of f. The following makes it possible to reduce the complexity of calculating the invariant for mapping tori.

Lemma 2.18. (1) *Consider a pushout of finite connected CW-complexes*

$$\begin{tikzcd} X_0 \arrow[r, "l_1"] \arrow[d, "l_2"] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of CW-complexes and X has the obvious CW-structure coming from the ones on X_0, X_1 and X_2. Suppose that the homomorphism $\pi_1(j_i)\colon \pi_1(X_i) \to \pi_1(X)$ is injective for $i = 0, 1, 2$. Consider self-homotopy equivalences $f_i\colon X_i \to X_i$ satisfying $f_i \circ l_i = l_i \circ f_0$ for $i = 1, 2$. Let $f\colon X \to X$ be the self homotopy equivalence determined by the pushout property.

Then we obtain a pushout of connected finite CW-complexes

$$\begin{tikzcd} T_{f_0} \arrow[r] \arrow[d] \arrow[dr, "k_0"] & T_{f_1} \arrow[d, "k_1"] \\ T_{f_2} \arrow[r, "k_2"] & T_f \end{tikzcd}$$

such that $\pi_1(k_i)$ is injective for $i = 0, 1, 2$, and we get

$$\rho_u^{(2)}(\widetilde{T_f}) = (k_1)_*\big(\rho_u^{(2)}(\widetilde{T_{f_1}})\big) + (k_2)_*\big(\rho_u^{(2)}(\widetilde{T_{f_2}})\big) - (k_0)_*\big(\rho_u^{(2)}(\widetilde{T_{f_0}})\big);$$

(2) *If $f\colon X \to X$ is a self-homotopy equivalence of a connected finite CW-complex X such that X is L^2-acyclic, then*

$$\rho_u^{(2)}(\widetilde{T_f}) = 0.$$

Proof. (1) This follows from Theorem 2.11 (2).

(2) This follows from Theorem 2.11 (2) applied to the pushout

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, "\mathrm{id}_X \amalg f"] \arrow[d] & X \arrow[d] \\ X \times [0,1] \arrow[r] & T_f. \end{tikzcd}$$

□

2.4. **L^2-torsion and the L^2-Alexander torsion.** Throughout this section let M be an admissible 3-manifold. We write $\pi = \pi_1(M)$. Taking the Fuglede-Kadison determinant yields a homomorphism

$$\det_{\mathcal{N}(\pi)} \colon \mathrm{Wh}^w(\pi) \to \mathbb{R}.$$

The image of $\rho_u^{(2)}(\widetilde{M})$ under this homomorphism is easily seen to be the L^2-torsion $\rho^{(2)}(\widetilde{M})$ which can be computed as $-1/6\pi$-times the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen-Johannson decomposition of M, see [26, Theorem 0.7].

We can generalize this discussion. Let $\mathrm{pr}\colon \pi \to H_1(\pi)_f := H_1(\pi)/\operatorname{tors}(H_1(\pi))$ be the projection. Denote by $\mathrm{Rep}_{\mathbb{C}}(H_1(\pi)_f)$ the representation ring of finite-dimensional complex $H_1(\pi)_f$-representations. There is a pairing

$$\mathrm{Wh}^w(\pi) \otimes \mathrm{Rep}_{\mathbb{C}}(H_1(\pi)_f) \to \mathbb{R}$$

given by the Fuglede-Kadison determinant twisted with $\mathrm{pr}^* V$ for some finite-dimensional $H_1(\pi)_f$-representation V. It sends $\big(\rho_u^{(2)}(\widetilde{M}), [V]\big)$ to the $\mathrm{pr}^* V$-twisted L^2-torsion $\rho^{(2)}(\widetilde{M}; \mathrm{pr}^* V)$, see [24].

Given $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(\pi)_f, \mathbb{Z})$, we get for every $t \in (0,\infty)$ an element in $\mathrm{Rep}_{\mathbb{C}}(H_1(\pi)_f)$ by the 1-dimensional representation $\mathbb{C}_{t,\phi}$, which is given by the $H_1(\pi)_f$-action on $\mathbb{C}$ determined by $g \cdot \lambda := t^{\phi(g)} \cdot \lambda$ for $g \in H_1(\pi)_f$ and $\lambda \in \mathbb{C}$. Thus we obtain an L^2-torsion function

$$\rho^{(2)}(\widetilde{M}) \colon (0,\infty) \to \mathbb{R}, \quad t \mapsto \rho^{(2)}(\widetilde{M}; \mathrm{pr}^* \mathbb{C}_{t,\phi}).$$

A standard argument shows that this function is a well-defined invariant of the pair (M,ϕ) up to the addition of a function of the form $t \mapsto k\ln(t)$ for some $k \in \mathbb{Z}$. The L^2-torsion function $\rho^{(2)}(\widetilde{M})$ which is determined by $\rho_u^{(2)}(\widetilde{M}) \in \mathrm{Wh}^w(\pi)$ and whose value at $t = 1$ is the L^2-torsion $\rho^{(2)}(\widetilde{M})$. This L^2-torsion function is studied for instance in [6, 8, 16, 18, 19, 20]. Moreover, $\limsup_{t\to\infty} \frac{\rho^{(2)}(\widetilde{M})(t)}{\ln(t)}$ and $\liminf_{t\to 0} \frac{\rho^{(2)}(\widetilde{M})(t)}{\ln(t)}$ exist as real numbers and their difference is called the degree of the L^2-torsion function. The negative of the degree turns out to be the Thurston seminorm $x_M(\phi)$ of ϕ, see [8, 20].

3. The Grothendieck group of integral polytopes and $K_1^w(\mathbb{Z}G)$

In this section we want to detect elements in $\mathrm{Wh}^w(G)$, in particular $\rho_u^{(2)}(X;\mathcal{N}(G))$, in terms of integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f$ and establish for admissible 3-manifolds different from $S^1 \times D^2$ a relation to the Thurston seminorm.

3.1. **The Grothendieck group of integral polytopes.** Next we recall the definition of and give some information about the polytope group from Friedl-Lück [9, Section 6.2].

A *polytope* in a finite-dimensional real vector space V is a subset which is the convex hull of a finite subset of V. An element p in a polytope is called *extreme* if the implication $p = \frac{q_1}{2} + \frac{q_2}{2} \implies q_1 = q_2 = p$ holds for all elements q_1 and q_2 in the polytope. Denote by $\operatorname{Ext}(P)$ the set of extreme points of P. If P is the convex hull of the finite set S, then $\operatorname{Ext}(P) \subseteq S$ and P is the convex hull of $\operatorname{Ext}(P)$. The *Minkowski sum* of two polytopes P_1 and P_2 is defined to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p \in P_2\}.$$

It is the convex hull of the set $\{p_1 + p_2 \mid p_1 \in \operatorname{Ext}(P_1), p_2 \in \operatorname{Ext}(P_2)\}$.

Let H be a finitely generated free abelian group. We obtain a finite-dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} H$. An integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H$ is a polytope such that $\operatorname{Ext}(P)$ is contained in H, where we consider H as a lattice in $\mathbb{R} \otimes_{\mathbb{Z}} H$ by the standard embedding $H \to \mathbb{R} \otimes_{\mathbb{Z}} H,\ h \mapsto 1 \otimes h$. The Minkowski sum of two integral polytopes is again an integral polytope. Hence the integral polytopes form an abelian monoid under the Minkowski sum with the integral polytope $\{0\}$ as neutral element.

Definition 3.1 (Grothendieck group of integral polytopes)**.** Let $\mathcal{P}_{\mathbb{Z}}(H)$ be the abelian group given by the Grothendieck construction applied to the abelian monoid of integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum.

Notice that for polytopes P_0, P_1 and Q in a finite-dimensional real vector space we have the implication $P_0 + Q = P_1 + Q \Longrightarrow P_0 = P_1$, see [30, Lemma 2]. Hence elements in $\mathcal{P}_{\mathbb{Z}}(H)$ are given by formal differences $[P] - [Q]$ for integral polytopes P and Q in $\mathbb{R} \otimes_{\mathbb{Z}} H$ and we have $[P_0] - [Q_0] = [P_1] - [Q_1] \Longleftrightarrow P_0 + Q_1 = P_1 + Q_0$.

There is an obvious homomorphism of abelian groups $i\colon H \to \mathcal{P}_{\mathbb{Z}}(H)$ which sends $h \in H$ to the class of the polytope $\{h\}$. Denote its cokernel by

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \quad = \quad \operatorname{coker}\bigl(i\colon H \to \mathcal{P}_{\mathbb{Z}}(H)\bigr). \tag{3.2}$$

Put differently, in $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ two polytopes are identified if they are obtained by translation with some element in the lattice H from one another.

Example 3.3. An integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}$ is given by an interval $[m, n]$ for integers m, n with $m \le n$. The Minkowski sum becomes $[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$. One easily checks that one obtains isomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}) \quad \xrightarrow{\cong} \quad \mathbb{Z}^2 \quad [[m,n]] \mapsto (n-m, m); \tag{3.4}$$

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(\mathbb{Z}) \quad \xrightarrow{\cong} \quad \mathbb{Z}, \quad [[m,n]] \mapsto n-m. \tag{3.5}$$

Given a homomorphism of finitely generated abelian groups $f\colon H \to H'$, we can assign to an integral polytope $P \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H$ an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H'$ by the image of P under $\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f\colon \mathbb{R} \otimes_{\mathbb{Z}} H \to \mathbb{R} \otimes_{\mathbb{Z}} H'$ and thus we obtain homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f)\colon \mathcal{P}_{\mathbb{Z}}(H) \quad \to \quad \mathcal{P}_{\mathbb{Z}}(H'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)]; \tag{3.6}$$

$$\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(f)\colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \quad \to \quad \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H'). \tag{3.7}$$

Lemma 3.8. *Let H be a finitely generated free abelian group. Then:*

(1) *The homomorphism*

$$\xi\colon \mathcal{P}_{\mathbb{Z}}(H) \to \prod_{\phi \in \operatorname{Hom}_{\mathbb{Z}}(H,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad [P] - [Q] \mapsto \bigl(\mathcal{P}_{\mathbb{Z}}(\phi)([P] - [Q])\bigr)_{\phi}$$

is injective;

(2) *The canonical short sequence of abelian groups*

$$0 \to H \xrightarrow{i} \mathcal{P}_{\mathbb{Z}}(H) \xrightarrow{\mathrm{pr}} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \to 0$$

is split exact;

(3) *The abelian groups $\mathcal{P}_{\mathbb{Z}}(H)$ and $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ are free.*

Proof. (1) Consider $[P] - [Q]$ in $\mathcal{P}_{\mathbb{Z}}(H)$. Suppose that $[P] - [Q]$ is not the trivial element. Then the polytopes P and Q are different. Hence we can assume without loss of generality that there exists $q \in Q$ with $q \notin P$ (if not, consider $[Q] - [P]$). By the Separating Hyperplane Theorem, see [31, Theorem V.4 on page 130], there exists an $\mathbb{R}$-linear map $\psi\colon \mathbb{R} \otimes_{\mathbb{Z}} H \to \mathbb{R}$ and $r \in \mathbb{R}$ such that $\psi(x) < r$ holds for all $x \in P$ and $\psi(q) > r$. By continuity we can find a $\mathbb{Z}$-linear map $\phi'\colon H \to \mathbb{Q}$ such that the same holds for the $\mathbb{R}$-linear map $\mathbb{R} \otimes H \to \mathbb{R}$ induced by ϕ. Choose a $\mathbb{Z}$-map $\phi\colon H \to \mathbb{Z}$ such that for some natural number n we have $n \cdot \phi' = \phi$. Then we have $\phi(P) < n \cdot r$ and $\phi(q) > n \cdot r$. This implies $\phi(P) \neq \phi(Q)$. Hence $\mathcal{P}_{\mathbb{Z}}(\phi)([P] - [Q]) \neq 0$ and therefore $\xi([P] - [Q]) \neq 0$.

(2) We pick an identification of H with $\mathbb{Z}^n$. We endow $\mathbb{Z}^n$ with the lexicographical order. It is straightforward to verify that there exists a unique homomorphism

$$\mathcal{P}_{\mathbb{Z}}(H) = \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n) \to H = \mathbb{Z}^n$$

with the property that a polytope gets sent to the extreme point of P of lowest order. This is clearly a splitting of the map $\mathbb{Z}^n = H \to \mathcal{P}_{\mathbb{Z}}(H) = \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$.

(3) It follows from Example 3.3 and assertion (1) that $\mathcal{P}_{\mathbb{Z}}(H)$ embeds into a countable free abelian group, hence it is free abelian by [33]. It follows from assertion (2) that $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H)$ is also free-abelian. Explicit bases of the free abelian groups are given by Funke [13]. □

3.2. **The polytope homomorphism.** In the following we will always mostly work with groups that satisfy the Atiyah Conjecture. For the reader's convenience we recall the statement.

Definition 3.9 (Atiyah Conjecture)**.** We say that a torsion-free group G satisfies the *Atiyah Conjecture* if for any matrix $A \in M_{m,n}(\mathbb{Q}G)$ the von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A is an integer.

The precise statement of the Atiyah Conjecture is not relevant to us, what is important is that we have the following proposition which is [23, Lemma 10.39].

Proposition 3.10. *Let G be a group that is torsion-free and that satisfies the Atiyah Conjecture. Then the rational closure $\mathcal{R}(G)$ agrees with the division closure $\mathcal{D}(G)$ of $\mathbb{Z}G \subseteq \mathcal{U}(G)$ and $\mathcal{D}(G)$ is a skew-field.*

From now on we suppose that G is torsion-free, satisfies the Atiyah Conjecture and $H_1(G)_f$ is finitely generated. In Friedl-Lück [9, Section 6.2] the main ingredients of the so called *polytope homomorphism*

$$\mathbb{P}\colon K_1^w(\mathbb{Z}G) \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f) \tag{3.11}$$

have been established. We briefly recall its definition for the reader's convenience.

There is a homomorphism, well-defined by Lemma 1.21,

$$\theta\colon K_1^w(\mathbb{Z}G) \to K_1(\mathcal{R}(G)), \quad [f] \mapsto [\mathrm{id}_{\mathcal{R}(G)} \otimes_{\mathbb{Z}G} f]. \tag{3.12}$$

Then by Proposition 3.10 the rational closure $\mathcal{R}(G)$ agrees with the division closure $\mathcal{D}(G)$ of $\mathbb{Z}G \subseteq \mathcal{U}(G)$ and $\mathcal{D}(G)$ is a skew-field. There is a Dieudonné determinant for invertible matrices over a skew field which takes values in the abelianization of the group of units, see [32, Corollary 4.3 in page 133]. Hence we obtain an isomorphism

$$\det_D\colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^{\times}_{\mathrm{abel}} := \mathcal{D}(G)^{\times}/[\mathcal{D}(G)^{\times}, \mathcal{D}(G)^{\times}]. \tag{3.13}$$

The inverse

$$J_{\mathcal{D}(G)}\colon \mathcal{D}(G)^{\times}_{\mathrm{abel}} \xrightarrow{\cong} K_1(\mathcal{D}(G)) \tag{3.14}$$

sends the class of a unit in D to the class of the corresponding $(1,1)$-matrix.

Next we want to define a homomorphism

$$\mathbb{P}' \colon \mathcal{D}(G)^{\times}_{\mathrm{abel}} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f). \tag{3.15}$$

We denote by K the kernel of $\mathrm{pr} \colon G \to H_1(G)_f$. Choose a map of sets $s \colon H_1(G)_f \to G$ with $\mathrm{pr} \circ s = \mathrm{id}_{H_1(G)_f}$. Then there is an isomorphism

$$\widehat{j_s} \colon T^{-1}(\mathcal{D}(K) *_s H_1(G)_f) \xrightarrow{\cong} \mathcal{D}(G),$$

where $T^{-1}(\mathcal{D}(K) *_s H_1(G)_f)$ is the Ore localization of the integral domain given by the crossed product $\mathcal{D}(K) *_s H_1(G)_f$ with respect to the multiplicative set T of non-trivial elements. Consider an element $u \in \mathcal{D}(K) *_s H_1(G)_f$ with $u \neq 0$. We can write $u = \sum_{h \in H} u_h \cdot h \in \mathcal{D}(K) *_s H_1(G)_f$ for appropriate elements $u_h \in \mathcal{D}(K)$. Define an integral polytope $P(u) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f$ by the convex hull of the subset $\{h \in H_1(G)_f \mid u_h \neq 0\}$ of $H_1(G)_f$. (This may be viewed as a non-commutative version of the Newton polytope of a polynomial in several variables.) This construction is compatible with the multiplication in $\mathcal{D}(K) *_s H_1(G)_f$ and the Minkowski sum, namely, for $u, v \in \mathcal{D}(K) *_s H_1(G)_f$ with $u, v \neq 0$ we get $P(uv) = P(u) + P(v)$. Thus we obtain a homomorphism of abelian groups

$$\big(T^{-1}(\mathcal{D}(K) *_s H_1(G)_f)\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f), \quad uv^{-1} \mapsto [P(u)] - [P(v)].$$

If we compose it with the isomorphism coming from $\widehat{j_s}$ and take into account that $\mathcal{P}_{\mathbb{Z}}(H_1(G)_f)$ is abelian, we get the desired well-defined homomorphism $\mathbb{P}'$ announced in (3.15).

The polytope homomorphism (3.11) is defined to be the composite

$$\mathbb{P} \colon K_1^w(\mathbb{Z}G) \xrightarrow{\theta} K_1(\mathcal{D}(G)) \xrightarrow{\det_D} \mathcal{D}(G)^{\times}_{\mathrm{abel}} \xrightarrow{\mathbb{P}'} \mathcal{P}_{\mathbb{Z}}(H_1(G)_f)$$

of the homomorphisms defined in (3.12), (3.13), and (3.15).

One easily checks that the polytope homomorphism (3.11) induces homomorphisms denoted by the same symbol $\mathbb{P}$

$$\mathbb{P} \colon \widetilde{K}_1^w(\mathbb{Z}G) \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f); \tag{3.16}$$

$$\mathbb{P} \colon \mathrm{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f). \tag{3.17}$$

Sending the class of an integral polytope P to the class of the integral polytope $-P = \{-p \mid p \in P\}$, yields involutions

$$\begin{aligned} * \colon \mathcal{P}_{\mathbb{Z}}(H_1(G)_f) &\xrightarrow{\cong} \mathcal{P}_{\mathbb{Z}}(H_1(G)_f), \\ * \colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) &\xrightarrow{\cong} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f). \end{aligned}$$

Consider any group homomorphism $w \colon G \to \{\pm 1\}$. Equip $\mathbb{Z}G$ with the involution of rings sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g \cdot w(g) \cdot g^{-1}$ and $K_1^w(\mathbb{Z}G)$, $\widetilde{K}_1^w(\mathbb{Z}G)$, and $\mathrm{Wh}^w(G)$ with the induced involutions. One easily checks

Lemma 3.18. *The polytope homomorphisms introduced in* (3.11), (3.16), *and* (3.18) *are compatible with the involutions defined above.*

If $f \colon G \to K$ is an injective group homomorphism, then the following diagram

$$\begin{tikzcd} \mathrm{Wh}^w(G) \arrow[r, "\mathrm{Wh}^w(f)"] \arrow[d, "\mathbb{P}^G"'] & \mathrm{Wh}^w(K) \arrow[d, "\mathbb{P}^K"] \\ \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) \arrow[r, "\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(f)_f)"'] & \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(K)_f) \end{tikzcd} \tag{3.19}$$

commutes and is compatible with the involutions.

Remark 3.20 (Computational complexity)**.** The main problem when one wants to compute the image of an element in $\mathrm{Wh}^w(G)$ under the polytope homomorphism $\mathbf{P}^G\colon \mathrm{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f)$ is that for an (n,n)-matrix over $\mathbb{Z}G$ it is usually very hard to figure out the corresponding unit in $\mathcal{D}(G)^\times$ since the Dieudonné determinant is not at all easy to compute. See also [24, Remark 6.24]. The situation is easy if $n = 1$, as exploited in Subsection 4.2.

3.3. **The L^2-torsion polytope.**

Definition 3.21 (The L^2-torsion polytope)**.** Let G be a torsion-free group satisfying the Atiyah Conjecture. Suppose that $H_1(G)_f$ is finitely generated. Let X be a free finite G-CW-complex which is L^2-acyclic. Then we define its *L^2-torsion polytope*

$$P(X;G) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f)$$

to be the image of the *negative* of the universal L^2-torsion $\rho_u^{(2)}(X;\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$ introduced in Definition 2.1 under the polytope homomorphism introduced in (3.17), i.e,

$$P(X;G) = \mathbb{P}^G\bigl(-\rho_u^{(2)}(X;\mathcal{N}(G))\bigr).$$

We take a minus sign in the definition above in order to get nicer formulas when relating $P(X;G)$ to the Thurston norm and the dual Thurston polytope, see Theorem 3.29 and Theorem 3.37.

If X is an L^2-acyclic connected finite CW-complex, we abbreviate

$$P(\widetilde{X}) := P(\widetilde{X};\mathcal{N}(\pi_1(X)) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(X)_f).$$

Now there are obvious analogues of the Theorems 2.5 and 2.11 which are sometimes simpler to state. As an illustration we go through a few examples for $P(\widetilde{X})$ for L^2-acyclic connected finite CW-complexes X and Y. Recall that we assume that the fundamental group is torsion-free and satisfies the Atiyah Conjecture.

(1) (Homotopy invariance) If X and Y are simple homotopy equivalent or if X and Y are homotopy equivalent and $\pi_1(X)$ satisfies the K-theoretic Farrell-Jones Conjecture, then the image of $P(\widetilde{X})$ under the isomorphism $\mathbb{R} \otimes_{\mathbb{Z}} H_1(X)_f \xrightarrow{\cong} \mathbb{R} \otimes_{\mathbb{Z}} H_1(Y)_f$ induced by f is $P(\widetilde{Y})$;

(2) (S^1-actions) Let X be a connected finite S^1-CW-complex. Suppose that for one and hence all $x \in X$ the map $\pi_1(S^1,1) \xrightarrow{\pi_1(\mathrm{ev}_x,1)} \pi_1(X,x)$ is injective, where $\mathrm{ev}_x\colon S^1 \to X$ sends z to $z \cdot x$. Define the S^1-orbifold Euler characteristic of X by

$$\chi_{\mathrm{orb}}^{S^1}(X) \;=\; \sum_{n \geq 0} (-1)^n \cdot \sum_{e \in I_n} \frac{1}{|S_e^1|},$$

where I_n is the set of open n-dimensional S^1-cells of X and for such an S^1-cell $e \in I_n$ we denote by S_e^1 the isotropy group of any point in e.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$P(\widetilde{X}) = \chi_{\mathrm{orb}}^{S^1}(X) \cdot \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(\mathrm{ev}_x)([J]),$$

where the integral polytope $J \subseteq \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}$ is the polytope given by $[0,1] \subseteq \mathbb{R} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}$.

(3) (Seifert and graph manifolds) An analogous formula holds for Seifert manifolds with infinite fundamental group, compare Remark 2.15.

3.4. **The L^2-torsion and the Thurston seminorm.** Let H be a finitely generated torsion-free abelian group. Let $P \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H$ be a polytope. It defines a seminorm on $\operatorname{Hom}_{\mathbb{Z}}(H, \mathbb{R}) = \operatorname{Hom}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} H, \mathbb{R})$ by

$$\|\phi\|_P \quad := \quad \tfrac{1}{2} \sup\{\phi(p_0) - \phi(p_1) \mid p_0, p_1 \in P\}. \tag{3.22}$$

It is compatible with the Minkowski sum, namely, for two integral polytopes $P, Q \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H$ we have

$$\|\phi\|_{P+Q} \quad = \quad \|\phi\|_P + \|\phi\|_Q. \tag{3.23}$$

Put

$$\begin{aligned}\mathcal{SN}(H) := \{f \colon \operatorname{Hom}_{\mathbb{Z}}(H; \mathbb{R}) \to \mathbb{R} \mid & \text{ there exist integral polytopes} \\ & P \text{ and } Q \text{ in } \mathbb{R} \otimes_{\mathbb{Z}} H \text{ with } f = \|\ \|_P - \|\ \|_Q\}.\end{aligned} \tag{3.24}$$

This becomes an abelian group by $(f - g)(\phi) = f(\phi) - g(\phi)$ because of (3.23). Again because of (3.23) we obtain an epimorphism of abelian groups

$$\operatorname{sn} \colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H) \to \mathcal{SN}(H) \tag{3.25}$$

by sending $[P] - [Q]$ for two polytopes $P, Q \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H$ to the function

$$\operatorname{Hom}_{\mathbb{Z}}(H, \mathbb{R}) \to \mathbb{R}, \quad \phi \mapsto \|\phi\|_P - \|\phi\|_Q.$$

Theorem 3.26 (The L^2-torsion and the Thurston seminorm)**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$ and is not a closed graph manifold. We write $\pi = \pi_1(M)$. Then there is a virtually finitely generated free abelian group Γ, and a factorization $\operatorname{pr}_M \colon \pi \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f$ of the canonical projection $\operatorname{pr}_M \colon \pi \to H_1(\pi)_f$ into epimorphisms, such that the following holds:*

Consider a torsion-free group G which satisfies the Atiyah Conjecture and for which $H_1(G)_f$ is finitely generated, and any factorization of $\alpha \colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$. Let $\overline{M} \to M$ be the G-covering associated to μ. Let $\phi \colon H_1(G)_f \to \mathbb{Z}$ be any group homomorphism. Then $\overline{M}$ is L^2-acyclic and its universal L^2-torsion $\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))$ is sent under the composite

$$\mathbb{PN} \colon \operatorname{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) \xrightarrow{\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\beta \circ \nu)_f)} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f) \xrightarrow{\operatorname{sn}} \mathcal{SN}(H_1(M)_f)$$

of the homomorphisms $\mathbb{P}$, $\mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(\beta \circ \nu)_f)$, and sn *defined in* (3.17), (3.7), *and* (3.25) *to the element given by the negative of half the Thurston seminorm*

$$\operatorname{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{R}) \to \mathbb{R}, \quad \phi \mapsto -\tfrac{1}{2} x_M(\phi).$$

Proof. Before we start with the proof we need to introduce some notation. We write $\pi = \pi_1(M)$. Given a homomorphism $\mu \colon \pi \to G$ and a homomorphism $\psi \colon G \to \mathbb{Z}$ we denote by $\overline{M}$ the cover of M corresponding to μ, we write $C_* = C_*(\overline{M})$ and we define

$$\chi^{(2)}(M; \mu, \psi) \ = \ \chi^{(2)}\big((\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^* C_*; \mathcal{N}(K)\big),$$

where K is the kernel of ψ and $i \colon K \to G$ is the inclusion.

Now can go ahead with the proof. By [9, Theorem 0.4] there exists a virtually finitely generated free abelian group Γ, and a factorization $\operatorname{pr}_M \colon \pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f$ of the canonical projection $\operatorname{pr}_M \colon \pi \to H_1(\pi)_f$ into epimorphisms, such that the following holds for any $\phi \in H^1(M; \mathbb{Z}) = \operatorname{Hom}(H_1(M)_f; \mathbb{Z})$ and any torsion-free group G satisfying the Atiyah Conjecture and any factorization of $\alpha \colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$:

$$\chi^{(2)}(M; \mu, \phi \circ \beta \circ \nu) \quad = \quad -x_M(\phi), \tag{3.27}$$

Now let $\mu\colon \pi \to G$ be as above. Let K be the kernel of $\phi \circ \beta \circ \nu$. We use the notation introduced in the beginning of the proof. We start out with proving the claim that

$$\mathbb{PN}\big(\rho_u^{(2)}(C_*)\big)(\phi) \quad = \quad \tfrac{1}{2}\chi^{(2)}\big((\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*; \mathcal{N}(K)\big). \tag{3.28}$$

provided that $\phi \circ \beta \circ \nu$ is surjective.

In order to prove the claim we first consider an (n,n)-matrix A over $\mathbb{Z}G$ which becomes invertible over $\mathcal{D}(G)$. It defines a class $[A] \in \mathrm{Wh}^w(G)$ by Lemma 1.21 since our hypothesis that G satisfies the Atiyah Conjecture implies by Proposition 3.10 that $\mathcal{D}(G) = \mathcal{R}(G)$. We conclude from [9, Lemma 6.12 and Lemma 6.16] that we get in the notation of [9]

$$\mathbb{PN}([A])(\phi) \quad = \quad -\tfrac{1}{2}\dim_{\mathcal{D}(K)}\big(\mathrm{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^n \to \mathcal{D}(K)_t[u^{\pm 1}]^n\big)\big).$$

If $\mathrm{el}(r_A)$ denotes the $\mathbb{Z}G$-chain complex concentrated in dimension 0 and 1 with first differential $r_A\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ and $i\colon K \to G$ is the inclusion, then we conclude from [9, Theorem 3.6 (4)]

$$\begin{aligned}\chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*\,\mathrm{el}(r_A); \mathcal{N}(K)\big)& \\ &= \dim_{\mathcal{D}(K)}\big(\mathrm{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^n \to \mathcal{D}(K_t[u^{\pm 1}]^n\big)\big).\end{aligned}$$

This implies

$$\mathbb{PN}([A])(\phi) \;=\; (\mathrm{sn} \circ \mathbb{P})\big(\rho_u^{(2)}(\mathrm{el}(r_A)\big)(\phi) \;=\; \tfrac{1}{2}\chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*\,\mathrm{el}(r_A); \mathcal{N}(K)\big).$$

We conclude from Remark 1.16 that for any L^2-acyclic finite based free $\mathbb{Z}G$-chain complex C_* we get

$$\mathbb{PN}(\rho_u^{(2)}(C_*))(\phi) \quad = \quad \tfrac{1}{2}\chi^{(2)}\big((\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*; \mathcal{N}(K)\big),$$

since $C_* \mapsto \chi^{(2)}\big((\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*; \mathcal{N}(K)\big)$ defines an additive L^2-torsion invariant with values in $\mathbb{R}$. This concludes the proof of (3.28).

We now turn to the actual proof of the theorem. If we apply the above claim to $C_* = C_*(\overline{M})$ and if we combine the resulting equality with (3.27) we see that

$$\mathbb{PN}\big(\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))\big)(\phi) \quad = \quad -\tfrac{1}{2}x_M(\phi).$$

provided that ϕ is surjective, since the surjectivity of ϕ implies the surjectivity of $\phi \circ \beta \circ \nu$.

Both maps

$$\mathbb{PN}\big(\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))\big) \text{ and } x_M\colon \mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{R}) \to \mathbb{R}$$

are continuous since seminorms are continuous maps, and satisfy

$$\begin{aligned}\mathbb{PN}\big(\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))\big)(r \cdot \phi) \quad &= \quad |r| \cdot \mathbb{PN}\big(\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))\big)(\phi);\\ -x_M(r \cdot \phi) \quad &= \quad |r| \cdot (-x_M(\phi)),\end{aligned}$$

for $r \in \mathbb{R}$ and $\phi \in \mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{R})$. Hence we get for every $\phi \in \mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{R})$

$$\mathbb{PN}\big(\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))\big)(\phi) \quad = \quad -\tfrac{1}{2}x_M(\phi),$$

since $\mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{Q})$ is dense in $\mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{R})$ and for any non-trivial $\psi \in \mathrm{Hom}_{\mathbb{Z}}(H_1(M)_f, \mathbb{Q})$ there exists an epimorphism $\phi\colon H_1(M)_f \to \mathbb{Z}$ and a rational number r with $r \cdot \phi = \psi$. This finishes the proof of Theorem 3.26. □

Theorem 3.29 (The L^2-torsion for universal coverings and the Thurston seminorm)**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Suppose $\pi_1(M)$ satisfies the Atiyah Conjecture.*

Then the L^2-torsion polytope $P(\widetilde{M})$ of Definition 3.21 is sent under the homomorphism $\mathrm{sn}\colon \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(M)_f) \to \mathcal{SN}(H_1(M)_f)$ to half of the Thurston seminorm x_M.

We point out that for "almost all" 3-manifolds M the fundamental group $\pi_1(M)$ satisfies the Atiyah Conjecture. More precisely, $\pi_1(M)$ satisfies the Atiyah Conjecture if M is not a closed graph manifold or if M is a closed graph manifold which admits a Riemannian metric of non-positive sectional curvature.) We refer to [9, Theorem 3.2] for details.

Proof. If M is not a graph manifold, the claim follows from Theorem 3.26. (Note that here we use the sign convention in the definition of $P(\widetilde{M})$ that we introduced in the beginning of Section 3.3.) The case of a graph manifold is handled analogously using [9, Theorem 2.14]. □

Remark 3.30. The pairing (0.5) is given by the homomorphism

$$\mathrm{Wh}^w(G) \xrightarrow{\mathbb{P}} \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) \xrightarrow{\mathrm{sn}} \mathcal{SN}(H_1(G)_f)$$

namely, an element $x \otimes \phi \in \mathrm{Wh}^w(G) \otimes H^1(G)$ is sent to the evaluation at $\phi\colon G \to \mathbb{Z}$ of the element in $\mathcal{SN}(H_1(G)_f)$ given by the image of $\rho_u^{(2)}(C_*)$ under this homomorphism. The same argument as appearing in the proof of (3.28) together with the formula $\chi^{(2)}(C_*;\mathcal{N}(G), k \cdot \phi) = k \cdot \chi^{(2)}(C_*;\mathcal{N}(G),\phi)$ for $k \in \mathbb{Z}$ show the claim in Subsection 0.2 that for an L^2-acyclic finite free $\mathbb{Z}G$-chain complex C_* and an element $\phi \in H^1(G)$ the image of $\rho_u^{(2)}(C_*;\mathcal{N}(G)) \otimes \phi$ under the pairing above is $\chi^{(2)}(C_*;\mathcal{N}(G),\phi)$.

In Theorem 3.29 we just saw that the polytope $P(\widetilde{M})$ determines the Thurston norm of M. In the coming sections we will prove a more precise statement, namely that the polytope $P(\widetilde{M})$ agrees, up to translation, with the dual of the Thurston norm ball.

3.5. **Seminorms and compact convex subsets.** Let V be a finite-dimensional real vector space. We write $V^* = \mathrm{Hom}(V,\mathbb{R})$. Given any subset X of V, we define its dual X^* to be

$$X^* \quad = \quad \{\phi \in V^* \,|\, \phi(v) \leq 1 \text{ for all } v \in X\}. \tag{3.31}$$

Given a compact convex subset $X \subseteq V$ we use the definition of (3.22) to define a seminorm

$$\| \ \|_X \colon V^* \to [0,\infty), \quad \phi \mapsto \|\phi\|_X := \tfrac{1}{2}\sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in X\}.$$

We define $(-X)$ to be the compact convex subset $\{-x \mid x \in X\}$. The Minkowski sum $X + (-X)$ is again a compact convex subset and we get $\| \ \|_X = \| \ \|_{-X}$ and $2 \cdot \| \ \|_X = \| \ \|_{X+(-X)}$.

Given a seminorm s on V, we assign to it its unit ball

$$B_s := \{v \in V \mid s(v) \leq 1\}$$

and denote by B_s^* the associated dual. A straightforward argument shows that we have the equality

$$B_s^* = \{\phi \in V^* \mid \phi(v) \leq s(v) \text{ for all } v \in V\}. \tag{3.32}$$

In the sequel we will identify $V = V^{**}$ by the canonical isomorphism.

Lemma 3.33. *If X is a closed convex subset containing 0, then under the identification $V = V^{**}$ we have $X = X^{**}$.*

Proof. It is straightforward to see that $X \subset X^{**}$. We now show the reverse inclusion. Consider $y \in V$ with $y \notin X$. By the Separating Hyperplane Theorem, see [31, Theorem V.4 on page 130], we can find $\psi \in V^*$ and $r \in \mathbb{R}$ such that $\psi(x) < r$ holds for all $x \in X$ and $\psi(y) > r$. Since 0 is contained in X and since $\psi(0) = 0$ we deduce that $r > 0$. Define $\phi := r^{-1} \cdot \psi \in V^*$. Then $\phi(x) \le 1$ for $x \in X$ and $\phi(y) > 1$. This implies $y \notin X^{**}$. $\square$

Lemma 3.34. *Let s be a seminorm on V and $X \subseteq V$ be a compact convex subset. Then*

(1) *The convex set B_s is compact if and only if s is a norm;*
(2) *B_s^* is convex and compact;*
(3) *For any $v \in V$ we have*

$$s(v) \;=\; \frac{1}{\sup\{r \in [0,\infty) \mid rv \in B_s\}};$$

(4) *We have $B_s = (B_s^*)^*$;*
(5) *We have $\| \; \|_{B_s^*} = s$;*
(6) *We have $X + (-X) = (B_{\| \; \|_X})^*$.*

Proof. (1) This is obvious.

(2) It is straightforward to see that B_s^* is convex and it follows from the description (3.32) that B_s^* is compact.

(3) Consider $r \in (0,\infty)$ and $v \in V$ with $s(v) \neq 0$. Then we get $rv \in B_s \iff r \le s(v)^{-1}$ and hence $\sup\{r \in [0,\infty) \mid rv \in B_s\} \le s(v)^{-1}$. This implies $s(v) \le \frac{1}{\sup\{r\in[0,\infty)\mid rv\in B_s\}}$. Since for $v \in V$ with $s(v) \neq 0$ we have $s(s(v)^{-1} \cdot v) \in B_s$, the claim follows.

(4) This follows from Lemma 3.33.

(5) We compute for $v \in V$

$$\begin{aligned}
\|v\|_{B_s^*} &= \tfrac{1}{2}\sup\{\phi_0(v) - \phi_1(v) \mid \phi_0,\phi_1 \in B_s^*\} \\
&= \tfrac{1}{2}\sup\{\phi_0(v) - \phi_1(v) \mid \phi_i \in V^*, \phi_i(w) \le s(w) \text{ for all } w \in V \text{ and } i = 0,1\} \\
&= \tfrac{1}{2}\sup\{\phi(v) \mid \phi \in V^*, \phi(w) \le s(w) \text{ for all } w \in V\} \\
&\qquad\qquad + \tfrac{1}{2}\sup\{\phi(-v) \mid \phi \in V^*, \phi(w) \le s(w) \text{ for all } w \in V)\}.
\end{aligned}$$

The Hahn-Banach Theorem, see [31, Theorem III.5 on page 75], implies for all $v \in V$

$$\sup\{\phi(v) \mid \phi \in V^*, \phi(w) \le s(w) \text{ for all } w \in V\} = s(v).$$

Since $s(v) = s(-v)$, assertion (5) follows.

(6) Since 0 is contained in $X + (-X)$, Lemma 3.33 implies

$$X + (-X) = (X + (-X))^{**}.$$

We get directly from the definitions

$$\begin{aligned}
(X + (-X))^{**} &= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in X + (-X)^*\} \\
&= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in V^* \\
&\qquad \text{satisfying } \phi(w) \le 1 \text{ for all } w \in X + (-X)\} \\
&= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in V^* \\
&\qquad \text{satisfying } \operatorname{supp}\{\phi(w_0) - \phi(w_1) \mid w_0, w_1 \in X + (-X)\} \le 2\big\} \\
&= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in V^* \text{ satisfying } \|\phi\|_{X+(-X)} \le 2\big\} \\
&= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in V^* \text{ satisfying } \|\phi\|_X \le 1\big\} \\
&= \big\{v \in V \mid \phi(v) \le 1 \text{ for all } \phi \in B_{\| \; \|_X}\big\} \\
&= (B_{\| \; \|_X})^*.
\end{aligned}$$

This finishes the proof of Lemma 3.34. $\square$

3.6. **The dual Thurston polytope.** Now let M be a compact oriented 3-manifold. In the sequel we will identify $\mathbb{R}\otimes_{\mathbb{Z}} H_1(M)_f = H_1(M;\mathbb{R})$ and $H^1(M;\mathbb{R}) = H_1(M;\mathbb{R})^*$ and $V = V^{**}$ by the obvious isomorphisms. We refer to

$$B_{x_M} \;:=\; \{\phi \in H^1(M;\mathbb{R}) \,|\, x_M(\phi) \leq 1\} \tag{3.35}$$

as the *Thurston norm ball* and we refer to

$$T(M)^* := B^*_{x_M} \subset (H^1(M;\mathbb{R}))^* = H_1(M;\mathbb{R}) \tag{3.36}$$

as the *dual Thurston polytope.* Explicitly by (3.32) we have

$$T(M)^* = \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

Thurston [35, Theorem 2 on page 106 and first paragraph on page 107] has shown that $T(M)^*$ is an integral polytope.

Theorem 3.37 (The dual Thurston polytope and the L^2-torsion polytope)**.** *Let M be an admissible 3-manifold which is not homeomorphic to $S^1 \times D^2$. Suppose that $\pi_1(M)$ satisfies the Atiyah Conjecture. Then*

$$[T(M)^*] \;=\; 2 \cdot P(\widetilde{M}) \;\in\; \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f).$$

Here we recall that we had pointed out after the statement of Theorem 3.29 that the fundamental group $\pi_1(M)$ of "almost all" 3-manifolds satisfies the Atiyah Conjecture.

The proof of Theorem 3.37 will require the remainder of this section.

3.7. **The dual Thurston polytope and the L^2-torsion polytope.** Recall that we have defined an involution $*\colon \mathcal{P}_{\mathbb{Z}}(H) \to \mathcal{P}_{\mathbb{Z}}(H)$ by sending $[P]-[Q]$ to $[-P]-[-Q]$. It induces an involution $*\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H) \to \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)$.

Next we define a homomorphism of abelian groups

$$\mathrm{poly}\colon \mathcal{SN}(H) \to \mathrm{im}\big(\mathrm{id} + *\colon \mathcal{P}_{\mathbb{Z}}(H) \to \mathcal{P}_{\mathbb{Z}}(H)\big) \tag{3.38}$$

by sending an element in $\mathcal{SN}(H)$ represented by $\|\;\|_P - \|\;\|_Q$ for integral polytopes P and Q in $\mathbb{R}\otimes_{\mathbb{Z}} H$ to the element $[P]+[-P]-[Q]-[-Q]$. This is well-defined because of (3.23) and Lemma 3.34 (6) which implies $\|\;\|_P = \|\;\|_Q \iff P+(-P) = Q+(-Q)$ for integral polytopes P and Q in $\mathbb{R}\otimes_{\mathbb{Z}} H$.

Lemma 3.39.

(1) *The map* $\mathrm{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H) \to \mathcal{SN}(H)$ *defined in* (3.25) *satisfies* $\mathrm{sn}\circ * = \mathrm{sn}$*;*

(2) *We have the following commutative diagram*

$$\begin{tikzcd}
\mathcal{P}_{\mathbb{Z}}(H) \arrow[d, "\mathrm{pr}"] \arrow[drr, "\mathrm{id}+*"] & & \\
\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H) \arrow[r, "\mathrm{sn}"] & \mathcal{SN}(H) \arrow[r, "\mathrm{poly}"] & \mathrm{im}\big(\mathrm{id}+*\colon \mathcal{P}_{\mathbb{Z}}(H) \to \mathcal{P}_{\mathbb{Z}}(H)\big)
\end{tikzcd}$$

where pr *is the canonical projection;*

(3) *The map* sn *induces an injective map*

$$\mathrm{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2} \to \mathcal{SN}(H)$$

whose cokernel is annihilated by multiplication with 2*, and a surjective map*

$$\overline{\mathrm{sn}}\colon \mathbb{Z}\otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H) \to \mathcal{SN}(H)$$

whose kernel is annihilated by multiplication with 2*.*

Proof. (1) This follows from $\| \ \|_P = \| \ \|_{-P}$.

(2) This follows from the definitions.

(3) By definition of $\mathcal{SN}(H)$ the map $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H) \to \mathcal{SN}(H)$ is surjective. Hence also the map $\overline{\operatorname{sn}}$ is surjective. Consider $x \in \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2}$ with $\operatorname{sn}(x) = 0$. Choose $y \in \mathcal{P}_{\mathbb{Z}}(H)$ with $\operatorname{pr}(y) = x$. We get

$$y + *(y) = \operatorname{poly} \circ \operatorname{sn} \circ \operatorname{pr}(y) = \operatorname{poly} \circ \operatorname{sn}(x) = 0.$$

We get in $\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)$

$$0 = \operatorname{pr}(y + *(y)) = x + *(x) = 2 \cdot x.$$

This implies that the kernel of $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2} \to \mathcal{SN}(H)$ is annihilated by multiplication with 2. Since $\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)$ is torsionfree, see Lemma 3.8 (3), the map $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2} \to \mathcal{SN}(H)$ is injective. Since the kernel and the cokernel of the map

$$j\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2} \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H), \quad z \mapsto 1 \otimes z$$

are annihilated by multiplication with 2 and $\overline{\operatorname{sn}} \circ j = \operatorname{sn}|_{\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H)^{\mathbb{Z}/2}}$, Lemma 3.39 follows. □

Now we can give the proof of Theorem 3.37.

Proof of Theorem 3.37. We conclude from Theorem 3.29 that the L^2-polytope $P(\widetilde{M}) \in \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)$ is sent under $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f) \to \mathcal{SN}(H_1(M)_f)$ to $-\frac{1}{2}x_M$. By Lemma 3.34 (5) we have

$$\operatorname{sn}([T(M)^*]) \ = \ \| - \|_{T(M)^*} \ = \ \| - \|_{B^*_{x_M}} \ = \ x_M.$$

This shows that $(-2)\cdot P(\widetilde{M})$ and $[T(M)^*]$ are sent by the homomorphism $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f) \to \mathcal{SN}(H_1(M)_f)$ to x_M.

By construction $T(M)^* = -T(M)^*$. Hence $[T(M)^*]$ lies in $\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)^{\mathbb{Z}/2}$.

We conclude

$$*\big(\rho^{(2)}_u(\widetilde{M})\big) = \rho^{(2)}_u(\widetilde{M}, \widetilde{M}|_{\partial M})$$

from the version of Poincaré duality for compact manifolds with boundary, see Theorem 2.5 (8). From the additivity of the universal L^2-torsion, see Lemma 1.9, we get

$$\rho^{(2)}_u(\widetilde{M}, \widetilde{M}|_{\partial M}) = \rho^{(2)}_u(\widetilde{M}) - \rho^{(2)}_u(\widetilde{M}|_{\partial M}).$$

Since ∂M is a union of incompressible tori, we conclude $\rho^{(2)}_u(\widetilde{M}|_{\partial M}) = 0$ from Example 2.7. This implies

$$*\big(\rho^{(2)}_u(\widetilde{M}, \widetilde{M}|_{\partial M})\big) = \rho^{(2)}_u(\widetilde{M}).$$

Recall that the L^2-polytope $P(\widetilde{M})$ is defined be the image of $-\rho^{(2)}_u(\widetilde{M})$ under the polytope homomorphism $\mathbb{P}\colon \mathrm{Wh}^w(\pi_1(M)) \to \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)$. We conclude from Lemma 3.18 that $P(\widetilde{M}) \in \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)^{\mathbb{Z}/2}$.

Since both $[T(M)^*]$ and $(-2)\cdot P(\widetilde{M})$ lie in $\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)^{\mathbb{Z}/2}$ and have the same image under $\operatorname{sn}\colon \mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f) \to \mathcal{SN}(H_1(M)_f)$, namely $-x_M$, we conclude from Lemma 3.39 (3) that $[T(M)^*] = (-2)\cdot P(\widetilde{M})$ holds in $\mathcal{P}^{\mathrm{Wh}}_{\mathbb{Z}}(H_1(M)_f)$. This finishes the proof of Theorem 3.37. □

4. Examples

4.1. **L^2-acyclic groups.** Let G be a group with a finite model for BG which is L^2-acyclic, i.e., all its Betti numbers $b_n^{(2)}(G;\mathcal{N}(G))$ vanish. Furthermore suppose that its Whitehead group $\mathrm{Wh}(G)$ is trivial. (The Farrell-Jones Conjecture, which is known for a large class of groups containing for instance hyperbolic groups, CAT(0)-groups, lattices in almost connected Lie groups and solvable groups, implies the vanishing of $\mathrm{Wh}(G)$ for torsionfree G.) Then we get by Theorem 2.5 (1) a well-defined invariant

$$\rho_u^{(2)}(G) \quad := \quad \rho_u^{(2)}(EG;\mathcal{N}(G)) \in \mathrm{Wh}^w(G). \tag{4.1}$$

If G satisfies the Atiyah Conjecture, we can apply Definition 3.21 to EG and obtain an element

$$P(G) \quad := \quad P(EG;G) \in \mathcal{P}_{\mathbb{Z}}^{\mathrm{Wh}}(H_1(G)_f) \tag{4.2}$$

which is the image of $-\rho_u^{(2)}(G)$ under the polytope homomorphism in (3.17). There is an obvious sum formula for amalgamated products coming from Theorem 2.11 (2). Namely, if we have injective group homomorphisms $G_0 \to G_i$ for $i = 1, 2$ and G_i has a finite model for BG_i and is L^2-acyclic for $i = 0, 1, 2$, then $G = G_1 *_{G_0} G_2$ has a finite model for BG, is L^2-acyclic and we get in $\mathrm{Wh}^w(G)$

$$\rho_u^{(2)}(G) \quad = \quad (j_1)_*\rho_u^{(2)}(G_1) + (j_2)_*\rho_u^{(2)}(G_2) - (j_0)_*\rho_u^{(2)}(G_0), \tag{4.3}$$

where $j_i \colon G_i \to G$ is the inclusion. There are also obvious analogues of the finite covering formula, the product formula and the statement about fibrations of Theorem 2.5.

4.2. **Torsion-free one-relator groups with two generators.** Let G be a torsion-free one-relator group with two generators which is not the free group. Choose any presentation $\langle x, y \mid R\rangle$ with two generators and one relation R. Let X be the associated presentation complex which has one zero-cell, two 1-cells, one for each generator x and y, and one 2-cell which is attached to the 1-skeleton which is the wedge of two copies of S^1 according to the word R. Then $\pi_1(X)$ is isomorphic to G and X is a model for BG, see [27, Chapter III §§9–11]. The finite based free $\mathbb{Z}G$-chain complex of the universal covering $\widetilde{X}$ is given in terms of Fox derivatives by

$$\mathbb{Z}G \xrightarrow{\left(\frac{\partial R}{\partial x}\ \frac{\partial R}{\partial y}\right)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} x-1 \\ y-1\end{pmatrix}} \mathbb{Z}G$$

It is known that $b_n^{(2)}(EG;\mathcal{N}(G)) = 0$ holds for all $n \geq 0$, see [5, Theorem 4.2]. There is an obvious short based exact sequence of finite based free $\mathbb{Z}G$-chain complexes

$$0 \to \Sigma\,\mathrm{el}(r_{\frac{\partial R}{\partial x}} \colon \mathbb{Z}G \to \mathbb{Z}G) \to C_*(\widetilde{X}) \to \mathrm{el}(r_{y-1} \colon \mathbb{Z}G \to \mathbb{Z}G) \to 0$$

Since $C_*(\widetilde{X})$ and $\mathrm{el}(r_{y-1} \colon \mathbb{Z}G \to \mathbb{Z}G)$ are L^2-acyclic, the complex $\Sigma\,\mathrm{el}(r_{\frac{\partial R}{\partial x}})$ is also L^2-acyclic. Hence we get in $\mathrm{Wh}^w(G)$

$$\rho_u^{(2)}(C_*(\widetilde{X})) = -[r_{\frac{\partial R}{\partial x}} \colon \mathbb{Z}G \to \mathbb{Z}G] + [r_{y-1} \colon \mathbb{Z}G \to \mathbb{Z}G].$$

Waldhausen [38, page 249-250] has proved that $\mathrm{Wh}(G)$ vanishes. Hence $\rho_u^{(2)}(C_*(\widetilde{X}))$ depends only on the homotopy type of X and hence is an invariant of G as a group which we denote by $\rho_u^{(2)}(G)$. Hence $\rho_u^{(2)}(G)$ is independent of the presentation and satisfies

$$\rho_u^{(2)}(G) = -\left[r_{\frac{\partial R}{\partial x}} \colon \mathbb{Z}G \to \mathbb{Z}G\right] + [r_{y-1} \colon \mathbb{Z}G \to \mathbb{Z}G].$$

Now suppose that G satisfies the Atiyah Conjecture. Then we have the polytope homomorphism

$$\mathbb{P}\colon \operatorname{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z}}^{\operatorname{Wh}}(H_1(G)_f)$$

introduced in (3.11). For an element $u = \sum_{g\in G} r_g \cdot g \in \mathbb{Z}G$ define its support by

$$\operatorname{supp}_G(u) = \{g \in G \mid r_g \neq 0\} \subseteq G.$$

Let $P(u) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f$ be the integral polytope which is the convex hull of the subset $\operatorname{pr}(\operatorname{supp}_G(u)) \subseteq H_1(G)_f$. Then we conclude from the definitions

$$\mathbb{P}\big(-\rho_u^{(2)}(G)\big) \;=\; \left[P\left(\frac{\partial R}{\partial x}\right)\right] - [P(y-1)]. \tag{4.4}$$

The polytope $P(y-1)$ is the convex hull of the two points 0 and $\operatorname{pr}(y) \in H_1(G)$. Consider the example $G = \mathbb{Z}^2 = \langle x, y | xyx^{-1}y^{-1}\rangle$. We compute

$$\frac{\partial xyx^{-1}y^{-1}}{\partial x} \;=\; 1-y$$

and hence we get in $\mathcal{P}_{\mathbb{Z}}^{\operatorname{Wh}}(\mathbb{Z}^2)$

$$\rho_u^{(2)}(\mathbb{Z}^2) = -[P(1-y)] + [P(y-1)] = 0.$$

This is consistent with Example 2.7, where we have shown $\rho_u^{(2)}(\widetilde{T^2}; \mathcal{N}(\mathbb{Z}^2)) = 0$.

Remark 4.5. With the same notation as above, the first author and Tillmann [12] assigned to such a presentation $\pi = \langle x, y \mid R\rangle$ in an elementary way a polytope $P(\pi)$ in $H_1(G;\mathbb{R})$. If G satisfies the Atiyah Conjecture it follows from (4.4) and [12, Proposition 3.5] that $P(\pi)$ and $P(G)$ represent the same element of $\mathcal{P}_{\mathbb{Z}}^{\operatorname{Wh}}(H_1(G)_f)$. In particular this shows that $P(\pi)$ is an invariant of the underlying group G and not just of the presentation. This proves [12, Conjecture 1.2], provided that the Atiyah Conjecture holds for G.

If π is furthermore the fundamental group of a 3-manifold, then Theorem 3.37 says in particular that $2 \cdot P(\pi) = [T(M)^*]$, this recovers the main theorem of [11].

4.3. **Group endomorphisms.** Let $f\colon G \to G$ be a monomorphism of a group G. Suppose that G admits a finite model for BG. Then we can consider the mapping torus T_{Bf} of the induced map $Bf\colon BG \to BG$ induced by f. It is straightforward to see that it is a finite model for the classifying spaces of the HNN-extension $G*_f$ associated to f. (Indeed, the only statement that needs verification is that the higher homotopy groups of T_{Bf} are zero. We denote by $\widetilde{T_{Bf}}$ the obvious infinite cyclic cover. It suffices to show that its higher homotopy groups are zero. But each map $S^k \to \widetilde{T_{Bf}}$ lies in a compact subset of $\widetilde{T_{Bf}}$, in particular it lies in a subspace given by finitely many mapping tori of $Bf\colon BG \to BG$ glued together, but such subspaces are homotopy equivalent to BG and hence aspherical. See also [22, Section 2].) By [22, Theorem 2.1] the space T_{Bf} is L^2-acyclic. Since the simple homotopy type of T_{Bf} is independent of the choice of BG and Bf, see [4, (22.1)], we get well-defined invariants.

$$\rho_u^{(2)}(f) \;:=\; \rho_u^{(2)}(\widetilde{T_{Bf}}) \quad \in \operatorname{Wh}^w(G*_f); \tag{4.6}$$

$$P(f) \;:=\; P(\widetilde{T_{Bf}}) \quad \in \mathcal{P}_{\mathbb{Z}}^{\operatorname{Wh}}(H_1(G*_f)), \tag{4.7}$$

where for (4.7) we have to assume that $G*_f$ satisfies the Atiyah Conjecture. (This is for example always satisfied if G is a free group, see [14, Section 2.9] for details.)

We expect that $P(f)$ is a useful invariant, already for automorphisms of free groups $P(f)$ should contain some interesting information.

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