# THE $L^2$-ALEXANDER TORSION IS SYMMETRIC

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ABSTRACT. We show that the $L^2$-Alexander torsion of a 3-manifold is symmetric. This can be viewed as a generalization of the symmetry of the Alexander polynomial of a knot.

## 1. INTRODUCTION

An *admissible triple* $(N, \phi, \gamma)$ consists of an irreducible, orientable, compact 3–manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary, a non-zero class $\phi \in H^1(N; \mathbb{Z}) = \operatorname{Hom}(\pi_1(N), \mathbb{Z})$ and a homomorphism $\gamma \colon \pi_1(N) \to G$ such that $\phi$ factors through $\gamma$.

In [DFL14a, DFL14b] we used the $L^2$–torsion (see e.g. [Lü02]) to associate to an admissible triple $(N, \phi, \gamma)$ the $L^2$–Alexander torsion $\tau^{(2)}(N, \phi, \gamma)$ which is a function

$$\tau^{(2)}(N, \phi, \gamma) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$$

that is well-defined up to multiplication by a function of the type $t \mapsto t^m$ for some $m \in \mathbb{Z}$. We recall the definition in Section 6.1.

The goal of this paper is to show that the $L^2$-Alexander torsion is symmetric. In order to state the symmetry result we need to recall that given a 3-manifold $N$ the *Thurston norm* [Th86] of some $\phi \in H^1(N; \mathbb{Z}) = \operatorname{Hom}(\pi_1(N), \mathbb{Z})$ is defined as

$$x_N(\phi) := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface $S$ with connected components $S_1 \cup \cdots \cup S_k$, we define its complexity as

$$\chi_-(S) = \sum_{i=1}^{k} \max\{-\chi(S_i), 0\}.$$

Thurston [Th86] showed that $x_N$ is a, possibly degenerate, norm on $H^1(N; \mathbb{Z})$. We can now formulate the main result of this paper.

**Theorem 1.1.** *Let $(N, \phi, \gamma)$ be an admissible triple. Then for any representative $\tau$ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi) \mod 2$ such that*

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

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It is worth looking at the case that $N = S^3 \setminus \nu K$ is the complement of a tubular neighborhood $\nu K$ of an oriented knot $K \subset S^3$. We denote by $\phi_K \colon \pi_1(S^3 \setminus \nu K) \to \mathbb{Z}$ the epimorphism sending the oriented meridian to 1. Let $\gamma \colon \pi_1(N) \to G$ be a homomorphism such that $\phi_K$ factors through $\gamma$. We define

$$\tau^{(2)}(K, \gamma) := \tau^{(2)}(S^3 \setminus \nu K, \phi_K, \gamma).$$

If we take $\gamma = \mathrm{id}$ to be the identity, then we showed in [DFL14b] that

$$\tau^{(2)}(K, \mathrm{id}) = \Delta_K^{(2)}(t) \cdot \max\{1, t\},$$

where $\Delta_K^{(2)}(t) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ denotes the $L^2$-Alexander invariant which was first introduced by Li–Zhang [LZ06a, LZ06b, LZ08] and which was also studied in [DW10, DW13, BA13a, BA13b].

If we take $\gamma = \phi_K$, then we showed in [DFL14b] that the $L^2$-Alexander torsion $\tau^{(2)}(K, \phi_K)$ is fully determined by the Alexander polynomial $\Delta_K(t)$ of $K$ and that in turn $\tau^{(2)}(K, \phi_K)$ almost determines the Alexander polynomial $\Delta_K(t)$. In this sense the $L^2$-Alexander torsion can be viewed as a 'twisted' version of the Alexander polynomial, and at least morally it is related to the twisted Alexander polynomial of Lin [Li01] and Wada [Wa94] and to the higher-order Alexander polynomials of Cochran [Co04] and Harvey [Ha05]. We refer to [DFL14a] for more on the relationship and similarities between the various twisted invariants.

If $K$ is a knot, then any Seifert surface is dual to $\phi_K$ and it immediately follows that $x(\phi_K) \leq \max\{2 \cdot \mathrm{genus(K)} - 1, 0\}$. In fact an elementary argument shows that for any *non-trivial* knot we have the equality $x(\phi_K) = 2 \cdot \mathrm{genus(K)} - 1$. It follows in particular that the Thurston norm of $\phi_K$ is odd. We thus obtain the following corollary to Theorem 1.1.

**Theorem 1.2.** *Let $K \subset S^3$ be an oriented non-trivial knot and let $\gamma \colon \pi_1(N) \to G$ be a homomorphism such that $\phi_K$ factors through $\gamma$. Then there exists an odd $n$ with*

$$\tau^{(2)}(K, \gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(K, \gamma)(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

The proof of Theorem 1.1 has many similarities with the proof of the main theorem in [FKK12] which in turn builds on the ideas of Turaev [Tu86, Tu90, Tu01, Tu02]. In an attempt to keep the proof as short as possible we will on several occasions refer to [FKK12] and [Tu90] for definitions and results.

Our symmetry theorem can be viewed as a variation on the theme that the ordinary Alexander polynomials of knots and links are symmetric, and that the twisted Reidemeister torsions corresponding to unitary representations are symmetric. We refer to [Ki96, FKK12, HSW10] for proofs of such symmetry results.

The paper is structured as follows. In Section 2 we recall the notion of an Euler structure which is due to Turaev. We proceed in Section 3 with the definition of the $L^2$-torsion of a manifold corresponding to a representation over a group ring. We prove our main technical duality theorem, which holds for manifolds of any dimension,

in Section 4. In Section 5 we relate the $L^2$-torsion of a 3-manifold to a relative $L^2$-torsion. Finally, in Section 6 we introduce the $L^2$-Alexander torsion of an admissible triple and we prove Theorem 1.1.

**Conventions.** All manifolds are assumed to be smooth, connected, orientable and compact. All CW-complexes are assumed to be finite and connected. If $G$ is a group then we equip $\mathbb{C}[G]$ with the usual involution given by complex conjugation and by $\overline{g} := g^{-1}$ for $g \in G$. We extend this involution to matrices over $\mathbb{C}[G]$ by applying the involution to each entry.

Given a ring $R$ we will view all modules as left $R$-modules, unless we say explicitly otherwise. Furthermore, given a matrix $A \in M_{m,n}(R)$, by a slight abuse of notation, we denote by $A \colon R^m \to R^n$ the $R$-homomorphism of left $R$-modules obtained by right multiplication with $A$ and thinking of elements in $R^m$ as the only row in a $(1, m)$-matrix.

## 2. EULER STRUCTURES

In this section we recall the notion of an Euler structure of a pair of CW-complexes and manifolds which is due to Turaev. We refer to [Tu90, Tu01, FKK12] for full details. Throughout this paper, given a space $X$, we denote by $\mathcal{H}_1(X)$ the first integral homology group viewed as a multiplicative group.

2.1. **Euler structures on CW-complexes.** Let $X$ be a finite CW-complex of dimension $m$ and let $Y$ be a proper subcomplex. We denote by $p \colon \widetilde{X} \to X$ the universal covering of $X$ and we write $\widetilde{Y} := p^{-1}(Y)$. An *Euler lift* $c$ is a set of cells in $\widetilde{X}$ such that each cell of $X \setminus Y$ is covered by precisely one of the cells in the Euler lift.

Using the canonical left action of $\pi = \pi_1(X)$ on $\widetilde{X}$ we obtain a free and transitive action of $\pi$ on the set of cells of $\widetilde{X} \setminus \widetilde{Y}$ lying over a fixed cell in $X \setminus Y$. If $c$ and $c'$ are two Euler lifts, then we can order the cells such that $c = \{c_{ij}\}$ and $c' = \{c'_{ij}\}$ and such that for each $i$ and $j$ the cells $c_{ij}$ and $c'_{ij}$ lie over the same $i$-cell in $X \setminus Y$. In particular there exist unique $g_{ij} \in \pi$ such that $c'_{ij} = g_{ij} \cdot c_{ij}$. We now write $\mathcal{H} = \mathcal{H}_1(X)$ and we denote the projection map $\pi \to \mathcal{H}$ by $\Psi$. We define

$$c'/c := \prod_{i=0}^{m} \prod_{j} \Psi(g_{ij})^{(-1)^i} \in \mathcal{H}.$$

We say that $c$ and $c'$ are *equivalent* if $c'/c \in \mathcal{H}$ is trivial. An equivalence class of Euler lifts will be referred to as an *Euler structure*. We denote by $\mathrm{Eul}(X, Y)$ the set of Euler structures. If $Y = \emptyset$ then we will also write $\mathrm{Eul}(X) = \mathrm{Eul}(X, Y)$.

Given $g \in \mathcal{H}$ and $e \in \mathrm{Eul}(X, Y)$ we define $g \cdot e \in \mathrm{Eul}(X, Y)$ as follows: pick a representative $c$ for $e$ and pick $\widetilde{g} \in \pi_1(X)$ which represents $g$, then act on one $i$-cell of $c$ by $g^{(-1)^i}$. The resulting Euler lift represents an element in $\mathrm{Eul}(X, Y)$ which is independent of the choice of the cell. We denote by $g \cdot e$ the Euler structure represented by this new Euler lift. This defines a free and transitive $\mathcal{H}$-action on $\mathrm{Eul}(X, Y)$, with $(g \cdot e)/e = g$.

If $(X', Y')$ is a cellular subdivision of $(X, Y)$, then there exists a canonical $\mathcal{H}_1(X)$-equivariant bijection

$$\sigma\colon \mathrm{Eul}(X, Y) \to \mathrm{Eul}(X', Y')$$

which is defined as follows: Let $e \in \mathrm{Eul}(X, Y)$ and pick an Euler lift for $(X, Y)$ which represents $e$. There exists a unique Euler lift for $(X', Y')$ such that the cells in the Euler lift of $(X', Y')$ are contained in the cells of the Euler lift of $(X, Y)$. We then denote by $\sigma(e)$ the Euler structure represented by this Euler lift. This map agrees with the map defined by Turaev [Tu90, Section 1.2].

2.2. **Euler structures of smooth manifolds.** Now we will quickly recall the definition of Euler structures on smooth manifolds. Let $N$ be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of $\partial N$ such that $\chi(N, \partial_0 N) = 0$. We write $\mathcal{H} = \mathcal{H}_1(N)$. A *triangulation* of $N$ is a pair $(X, t)$ where $X$ is a simplicial complex and $t\colon |X| \to N$ is a homeomorphism. Note that $t^{-1}(\partial_0 N)$ is a simplicial subspace of $X$. Throughout this section we write $Y := t^{-1}(\partial_0 N)$. For the most part we will suppress $t$ from the notation. Following [Tu90, Section I.4.1] we consider the projective system of sets $\{\mathrm{Eul}(X, Y)\}_{(X,t)}$ where $(X, t)$ runs over all $C^1$-triangulations of $N$ and where the maps are the $\mathcal{H}$-equivariant bijections between these sets induced either by $C^1$-subdivisions or by smooth isotopies in $N$.

Now we define $\mathrm{Eul}(N, \partial_0 N)$ by identifying the sets $\{\mathrm{Eul}(X, Y)\}_{(X,t)}$ via these bijections. We refer to $\mathrm{Eul}(N, \partial_0 N)$ as the set of Euler structures on $(N, \partial_0 N)$. Note that for a $C^1$-triangulation $X$ of $N$ we get a canonical $\mathcal{H}$-equivariant bijection $\mathrm{Eul}(X, Y) \to \mathrm{Eul}(N, \partial_0 N)$.

## 3. THE $L^2$-TORSION OF A MANIFOLD

3.1. **The Fuglede-Kadison determinant and the $L^2$-torsion of a chain complex.** Before we start with the definition of the $L^2$-Alexander torsion we need to recall some key properties of the Fuglede-Kadison determinant and the definition of the $L^2$-torsion of a chain complex of free based left $\mathbb{C}[G]$-modules. Throughout the section we refer to [Lü02] and to [DFL14b] for details and proofs.

We fix a group $G$. Let $A$ be a $k \times l$-matrix over $\mathbb{C}[G]$. Then there exists the notion of $A$ being of 'determinant class'. (To be slightly more precise, we view the $k \times l$-matrix $A$ as a homomorphism $\mathcal{N}(G)^l \to \mathcal{N}(G)^k$, where $\mathcal{N}(G)$ is the von Neumann algebra of $G$, and then there is the notion of being of 'determinant class'.) We treat this entirely as a black box, but we note that if $G$ is residually amenable, e.g., a 3-manifold group [He87] or solvable, then by [Lü94, Sc01, Cl99, ES05] any matrix

over $\mathbb{Q}[G]$ is of determinant class. If the matrix $A$ is not of determinant class then for the purpose of this paper we define $\det_{\mathcal{N}(G)}(A) = 0$. On the other hand, if $A$ is of determinant class, then we define

$$\det_{\mathcal{N}(G)}(A) := \text{Fuglede-Kadison determinant of } A \in \mathbb{R}_{>0}.$$

Note that we do not assume that $A$ is a square matrix. We will not provide a definition of the Fuglede-Kadison determinant but we summarize a few key properties in the following theorem which is basically a consequence of [Lü02, Example 3.12] and [Lü02, Theorem 3.14].

**Theorem 3.1.** (1) *If $A$ is a square matrix with complex entries such that the usual determinant $\det(A) \in \mathbb{C}$ is non-zero, then $\det_{\mathcal{N}(G)}(A) = |\det(A)|$.*
(2) *The Fuglede-Kadison determinant does not change if we swap two rows or two columns.*
(3) *Right multiplication of a column by $\pm g$ with $g \in G$ does not change the Fuglede–Kadison determinant.*
(4) *For any matrix $A$ over $\mathbb{C}[G]$ we have*

$$\det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(\overline{A}^t).$$

Note that (2) implies that when we study determinants of homomorphisms we can work with unordered bases.

Now let

$$C_* = \left( 0 \to C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \to 0 \right)$$

be a chain complex of free left $\mathbb{C}[G]$-modules. We can then consider the corresponding $L^2$-Betti numbers $b_i^{(2)}(C_*) \in \mathbb{R}_{\geq 0}$, as defined in [Lü02].

Now suppose that the chain complex is equipped with bases $B_i \subset C_i$, $i = 0, \dots, l$. If at least one of the $L^2$-Betti numbers $b_i^{(2)}(C_*)$ is non-zero or if at least one the boundary maps is not of determinant class, then we define the $L^2$-torsion $\tau^{(2)}(C_*, B_*) := 0$. Otherwise we define the $L^2$-torsion of the based chain complex $C_*$ to be

$$\tau^{(2)}(C_*, B_*) := \prod_{i=1}^{l} \det_{\mathcal{N}(G)}(A_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

where the $A_i$ denote the boundary matrices corresponding to the given bases. Note that this definition is the multiplicative inverse of the exponential of the $L^2$-torsion as defined in [Lü02, Definition 3.29].

3.2. **The twisted $L^2$-torsion of a pair of CW-complexes.** Let $(X, Y)$ be a pair of finite CW-complexes and let $e \in \text{Eul}(X, Y)$. We denote by $p\colon \widetilde{X} \to X$ the universal covering of $X$ and we write $\widetilde{Y} := p^{-1}(Y)$. Note that the deck transformation turns $C_*(\widetilde{X}, \widetilde{Y})$ naturally into a chain complex of left $\mathbb{Z}[\pi_1(X)]$-modules.

Now let $G$ be a group and let $\varphi\colon \pi(X) \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. We view elements of $\mathbb{C}[G]^d$ as row vectors. Right multiplication via $\varphi(g)$ thus turns $\mathbb{C}[G]^d$ into a right $\mathbb{Z}[\pi_1(X)]$-module. We then consider the chain complex

$$C^{\varphi}_*(X, Y; \mathbb{C}[G]^d) := \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y})$$

of left $\mathbb{C}[G]$-modules.

Now let $e \in \mathrm{Eul}(X, Y)$. We pick an Euler lift $\{c_{ij}\}$ which represents $e$. Throughout this paper we denote by $v_1, \dots, v_d$ the standard basis for $\mathbb{C}[G]^d$. We equip the chain complex $C^{\varphi}_*(X, Y; \mathbb{C}[G]^d)$ with the basis provided by the $v_k \otimes c_{ij}$. Therefore we can define

$$\tau^{(2)}(X, Y, \varphi, e) := \tau^{(2)}\left(C^{\varphi}_*(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\right) \in \mathbb{R}_{\geq 0}.$$

We summarize a few properties of the $L^2$-torsion in the following lemma.

**Lemma 3.2.**      (1) *The number $\tau^{(2)}(X, Y, \varphi, e)$ is well–defined, i.e., independent of the choice of the Euler lift which represents $e$.*
   (2) *If $g \in \mathcal{H}_1(X)$, then*

$$\tau^{(2)}(X, Y, \varphi, ge) = \det\nolimits_{\mathcal{N}(G)}(\varphi(g^{-1})) \cdot \tau^{(2)}(X, Y, \varphi, e).$$

   (3) *If $\delta\colon \pi_1(X) \to GL(d, \mathbb{C}[G])$ is conjugate to $\varphi$, i.e., if there exists an $A \in GL(d, \mathbb{C}[G])$ such that $\delta(g) = A\varphi(g)A^{-1}$ for all $g \in \pi_1(X)$, then*

$$\tau^{(2)}(X, Y, \delta, e) = \tau^{(2)}(X, Y, \varphi, e).$$

   (4) *If $(X', Y')$ is a cellular subdivision of $(X, Y)$ and if $e' \in \mathrm{Eul}(X', Y')$ is the Euler structure corresponding to $e$, then*

$$\tau^{(2)}(X', Y', \varphi, e') = \tau^{(2)}(X, Y, \varphi, e).$$

The proofs are completely analogous to the proofs for ordinary Reidemeister torsion as given in [Tu86, Tu01, FKK12]. In the interest of time and space we will therefore not provide the proofs.

3.3. **The $L^2$-Alexander torsion for manifolds.** Let $N$ be a manifold and let $\partial_0 N \subset \partial N$ be a union of components of $\partial N$. Let $G$ be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. Finally let $e \in \mathrm{Eul}(N, \partial_0 N)$.

Recall that for any $C^1$-triangulation $f\colon X \to N$ we get a canonical bijection $\mathrm{Eul}(X, Y) \xrightarrow{f_*} \mathrm{Eul}(N, \partial_0 N)$. Now we define

$$\tau^{(2)}(N, \partial_0 N, \varphi, e) := \tau^{(2)}(X, Y, \varphi \circ f_*, f_*^{-1}(e)).$$

By Lemma 3.2 (4) and the discussion in [Tu90] the invariant $\tau^{(2)}(N, \partial_0 N, \varphi, e) \in \mathbb{R}_{\geq 0}$ is well-defined, i.e., independent of the choice of the triangulation.

## 4. Duality for torsion of manifolds equipped with Euler structures

4.1. **The algebraic duality theorem for $L^2$-torsion.** Let $G$ be a group and let $V$ be a right $\mathbb{C}[G]$-module. We denote by $\overline{V}$ the left $\mathbb{C}[G]$-module with the same underlying abelian group together with the module structure given by $v \cdot_{\overline{V}} p := \overline{p} \cdot_V v$ for any $p \in \mathbb{C}[G]$ and $v \in V$. If $V$ is a left $\mathbb{C}[G]$-module then we can consider $\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$ the set of all left $\mathbb{C}[G]$-module homomorphisms. Note that the fact that the range $\mathbb{C}[G]$ is a $\mathbb{C}[G]$-bimodule implies that $\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$ is naturally a right $\mathbb{C}[G]$-module.

In the following let $C_*$ be a chain complex of length $m$ of left $\mathbb{C}[G]$-modules with boundary operators $\partial_i$. Suppose that $C_*$ is equipped with a basis $B_i$ for each $C_i$. We denote by $C^\#$ the *dual chain complex* whose chain groups are the $\mathbb{C}[G]$-left modules $C_i^\# := \overline{\mathrm{Hom}_{\mathbb{C}[G]}(C_{m-i}, \mathbb{C}[G])}$ and where the boundary map $\partial_i^\# : C_{i+1}^\# \to C_i^\#$ is given by $(-1)^{m-i}\partial_{m-i-1}^*$. This means that for any $c \in C_{m-i}$ and $d \in C_{i+1}^\#$ we have $\partial_i^\#(d)(c) = (-1)^{m-i}d(\partial_{m-i-1}(c))$. We denote by $B_*^\#$ the bases of $C^\#$ dual to the bases $B_*$. We have the following lemma.

**Lemma 4.1.** *If $\tau^{(2)}(C_*, B_*) = 0$, then $\tau^{(2)}(C_*^\#, B_*^\#) = 0$, otherwise we have*

$$\tau^{(2)}(C_*, B_*) = \tau^{(2)}(C_*^\#, B_*^\#)^{(-1)^{m+1}}.$$

*Proof.* We first note that by the proof of [Lü02, Theorem 1.35 (3)] the $L^2$-Betti numbers of $C_*$ vanish if and only if the $L^2$-Betti numbers of $C_*^\#$ vanish. If either does not vanish, then it thus follows that the other does not vanish, and both torsions are zero by definition.

We now suppose that the $L^2$-Betti numbers of $C_*$ vanish. We denote by $A_i$ the matrices of the boundary maps of $C_*$ with respect to the given basis. It follows easily from the definitions that the boundary matrices of the chain complex $C_*^\#$ with respect to the basis $B_*^\#$ are given by $(-1)^{m-i}\overline{A_i}^\dagger$. The lemma is now an immediate consequence of the definitions and of Theorem 3.1 (2). $\qquad\square$

4.2. **The duality theorem for manifolds.** Before we state our main technical duality theorem we need to introduce two more definitions.

(1) Let $G$ be a group and let $\varphi \colon \pi \to \mathrm{GL}(d, \mathbb{C}[G])$ be a representation. We denote by $\varphi^\dagger$ the representation which is given by $g \mapsto \overline{\varphi(g^{-1})}^t$.

(2) Let $N$ be an $m$-manifold and let $e \in \mathrm{Eul}(N, \partial N)$. We pick a triangulation $X$ for $N$. We denote by $Y$ the subcomplex corresponding to $\partial N$. Let $X^\dagger$ be the CW-complex which is given by the cellular decomposition of $N$ dual to $X$. We pick an Euler lift $\{c_{ij}\}$ which represents $e \in \mathrm{Eul}(X, Y) = \mathrm{Eul}(N, \partial N)$. For any $i$-cell $c$ in $\widetilde{X}$ we denote by $c^\dagger$ the unique oriented $(m-i)$-cell in $\widetilde{X^\dagger}$ which has intersection number $+1$ with $c_{ij}$. The Euler lift $\{c_{ij}^\dagger\}$ defines an element in $\mathrm{Eul}(X^\dagger) = \mathrm{Eul}(N)$ that we denote by $e^\dagger$. We refer to [Tu01, Section 1.4] and [FKK12, Section 4] for details.

In this section we will prove the following duality theorem.

**Theorem 4.2.** *Let $N$ be an $m$-manifold. Let $G$ be a group and let $\varphi \colon \pi(N) \to GL(d, \mathbb{C}[G])$ be a representation. Let $e \in \mathrm{Eul}(N, \partial N)$. Then either both $\tau^{(2)}(N, \partial N, \varphi, e)$ and $\tau^{(2)}(N, \varphi^\dagger, e^\dagger)$ are zero, or the following equality holds:*

$$\tau^{(2)}(N, \partial N, \varphi, e) = \tau^{(2)}(N, \varphi^\dagger, e^\dagger)^{(-1)^{m+1}}.$$

*Proof.* As above we pick a triangulation $X$ for $N$ and we denote by $Y$ the subcomplex corresponding to $\partial N$. Let $X^\dagger$ be the CW-complex which is given by the cellular decomposition of $N$ dual to $X$. In the following we make the identification $\pi = \pi_1(X) = \pi_1(N) = \pi_1(X^\dagger)$.

For the remainder of this section we pick an Euler lift $\{c_{ij}\}$ which represents $e \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$. We denote by $c_{ij}^\dagger$ the corresponding dual cells. Theorem 4.2 follows immediately from the definitions and the following claim.

*Claim.* Either both $\tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\})$ and $\tau^{(2)}(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\})$ are zero, or the following equality holds:

$$\tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}\big) = \tau^{(2)}\big(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\}\big)^{(-1)^{m+1}}.$$

In order to prove the claim we first note that there is a unique, non-singular intersection $\mathbb{Z}$-linear pairing

$$C_{m-i}(\widetilde{X}, \widetilde{Y}) \times C_i(\widetilde{X^\dagger}) \to \mathbb{Z}$$

with the property that $a \cdot b^\dagger = \delta_{ab}$ for any cell $a$ of $\widetilde{X} \setminus \widetilde{Y}$ and any cell $b$ of $\widetilde{X}$. We then consider the following pairing:

$$
\begin{array}{rcl}
C_{m-i}(\widetilde{X}, \widetilde{Y}) \times C_i(\widetilde{X^\dagger}) & \to & \mathbb{Z}[\pi] \\
(a, b) & \mapsto & \langle a, b \rangle := \sum_{g \in \pi} (a \cdot gb) g^{-1}.
\end{array}
$$

Note that this pairing is sesquilinear in the sense that for any $a \in C_{m-i}(\widetilde{X}, \widetilde{Y})$, $b \in C_i(\widetilde{X^\dagger})$ and $p, q \in \mathbb{Z}[\pi]$ we have $\langle pa, qb \rangle = q \langle a, b \rangle \overline{p}$. It is furthermore straightforward to see that the pairing is non-singular. This pairing has the property (see e.g. [Tu01, Claim 14.4]) that the following diagram commutes:

$$
\begin{array}{ccccc}
C_{i+1}(\widetilde{X}, \widetilde{Y}) & \times & C_{m-i-1}(\widetilde{X^\dagger}) & \to & \mathbb{Z}[\pi] \\
\downarrow \partial_i & & \uparrow (-1)^{i+1} \partial_{m-i-1} & & \downarrow = \\
C_i(\widetilde{X}, \widetilde{Y}) & \times & C_{m-i}(\widetilde{X^\dagger}) & \to & \mathbb{Z}[\pi].
\end{array}
$$

Put differently, the maps

$$
\begin{array}{rcl}
C_i(\widetilde{X}, \widetilde{Y}) & \to & \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-i}(\widetilde{X^\dagger}), \mathbb{Z}[\pi])} \\
a & \mapsto & (b \mapsto \langle a, b \rangle)
\end{array}
$$

define an isomorphism of based chain complexes of right $\mathbb{Z}[\pi]$-modules. In fact it follows easily from the definitions that the maps define an isomorphism

$$(C_*(\widetilde{X}, \widetilde{Y}), \{c_{ij}\}) \to \overline{(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X^\dagger}), \mathbb{Z}[\pi])}, \{(c_{ij}^\dagger)^*\})$$

of based chain complexes of left $\mathbb{Z}[\pi]$-modules. Tensoring these chain complexes with $\mathbb{C}[G]^d$ we obtain an isomorphism

$$\left( \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), \{v_k \otimes c_{ij}\} \right)$$

$$\to \left( \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\widetilde{X^\dagger})^*, \mathbb{Z}[\pi])}, \{(c_{ij}^\dagger)^* \otimes v_k\} \right)$$

of based chain complexes of $\mathbb{C}[G]$-modules. Furthermore the maps

$$\mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_i(\widetilde{X^\dagger}), \mathbb{Z}[\pi])} \;\to\; \overline{\mathrm{Hom}_{\mathbb{C}[G]}\left( C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \mathbb{C}[G] \right)}$$

$$v \otimes f \;\mapsto\; \left( \begin{array}{rcl} C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d) & \to & \mathbb{C}[G] \\ w \otimes \sigma & \mapsto & v\varphi\overline{(f(\sigma))}\,\overline{w}^t \end{array} \right)$$

induce an isomorphism

$$\left( C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\} \right) \to \left( C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d)^\#, \{(v_k \otimes c_{ij}^\dagger)^\#\} \right)$$

of based chain complexes of left $\mathbb{C}[G]$-modules. The claim is now an immediate consequence of Lemma 4.1. $\qquad\square$

## 5. Twisted $L^2$-torsion of 3-manifolds

Now we are heading towards the proof of Theorem 1.1. Therefore we are turning towards the study of $L^2$-torsions of 3-manifolds. In order to turn Theorem 4.2 into the desired symmetry result we will need to relate the $L^2$-torsions of a 3-manifold $N$ and the relative $L^2$-torsions of the pair $(N, \partial N)$. Henceforth we will restrict ourselves to one-dimensional representations since these are precisely the ones which we will need in the proof of Theorem 1.1.

### 5.1. Canonical structures on tori. 
Let $T$ be a torus. We equip $T$ with a CW-structure with one 0-cell $p$, two 1-cells $x$ and $y$ and one 2-cell $s$. We write $\pi = \pi_1(T, p)$ and by a slight abuse of notation we denote by $x$ and $y$ the elements in $\pi$ represented by $x$ and $y$. We denote by $\widetilde{T}$ the universal cover of $T$. Then there exist lifts of the cells such that the chain complex $C_*(\widetilde{T})$ of left $\mathbb{Z}[\pi]$-modules with respect to the bases given by these lifts is of the form

$$(1) \qquad 0 \to \mathbb{Z}[\pi] \xrightarrow{(y-1 \;\; 1-x)} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} 1-x \\ 1-y \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

We refer to the corresponding Euler structure of $T$ as the *canonical Euler structure on $T$*. This definition is identical to the definition provided by Turaev [Tu02, p. 10].

Given a group $G$ we say that a representation $\varphi\colon \pi \to \mathrm{GL}(1,\mathbb{C}[G])$ is *monomial* if for any $x \in \pi$ we have $\varphi(x) = zg$ for some $z \in \mathbb{C}$ and $g \in G$. We now have the following lemma.

**Lemma 5.1.** *Let $T$ be the torus and let $\varphi\colon \pi(T) \to \mathrm{GL}(1,\mathbb{C}[G])$ be a monomial representation such that $b_*^{(2)}(T;\mathbb{C}[G]^d) = 0$. Let $e$ be the canonical Euler structure on $T$. Then*
$$\tau^{(2)}(T,\varphi,e) = 1.$$

*Proof.* In [DFL14b] we used the canonical Euler structure (even though we did not call it that way) to compute $\tau^{(2)}(T,\varphi) = 1$.  □

### 5.2. **Chern classes on 3-manifolds with toroidal boundary.** Let $N$ be a compact, orientable 3-manifold with toroidal incompressible boundary and let $e \in \mathrm{Eul}(N,\partial N)$.

Let $X$ be a triangulation for $N$. We denote the subcomplexes corresponding to the boundary components of $N$ by $S_1 \cup \cdots \cup S_b$. We denote by $p\colon \widetilde{X} \to X$ and $p_i\colon \widetilde{S}_i \to S_i, i = 1,\ldots,b$ the universal covering maps of $X$ and $S_i, i = 1,\ldots,b$. For each $i$ we identify a component of $p^{-1}(S_i)$ with $\widetilde{S}_i$

We pick an Euler lift $c$ which represents $e$. For each boundary torus $S_i$ we pick an Euler lift $\widetilde{s}_i$ to $\widetilde{S}_i \subset p^{-1}(S_i) \subset \widetilde{X}$ which represents the canonical Euler structure. The set of cells $\{\widetilde{s}_1,\ldots,\widetilde{s}_b,c\}$ defines an Euler structure $K(e)$ for $N$, which only depends on $e$. Put differently, we just defined a map $K\colon \mathrm{Eul}(N,\partial N) \to \mathrm{Eul}(N)$ which is easily seen to be $\mathcal{H}_1(N)$-equivariant.

Given $e \in \mathrm{Eul}(N)$ there exists a unique element $g \in \mathcal{H}_1(N)$ such that $e = g \cdot K(e^\dagger)$. Following Turaev [Tu02, p. 11] we define $c_1(e) := g \in H_1(N;\mathbb{Z})$ and we refer to $c_1(e)$ as the *Chern class of $e$*.

### 5.3. **Torsions of 3-manifolds.** Let $\pi$ and $G$ be groups and let $\varphi\colon \pi \to \mathrm{GL}(1,\mathbb{C}[G])$ be a monomial representation. It follows from the multiplicativity of the Fuglede-Kadison determinant, see [Lü02, Theorem 3.14], that given $g \in \pi$ the invariant $\det_{\mathcal{N}(G)}(\varphi(g))$ only depends on the homology class of $g$. Put differently, $\det_{\mathcal{N}(G)} \circ \varphi\colon \pi \to \mathbb{R}_{\geq 0}$ descends to a map $\det_{\mathcal{N}(G)} \circ \varphi\colon H_1(\pi;\mathbb{Z}) \to \mathbb{R}_{\geq 0}$. We can now formulate the following theorem.

**Theorem 5.2.** *Let $N$ be a 3-manifold which is either closed or which has toroidal and incompressible boundary. Let $G$ be a group and let $\varphi\colon \pi(N) \to \mathrm{GL}(1,\mathbb{C}[G])$ be a monomial representation. Suppose that $b_*^{(2)}(\partial N;\mathbb{C}[G]) = 0$. Then for any $e \in \mathrm{Eul}(N,\partial N)$ we have*
$$\tau^{(2)}(N,\partial N,\varphi,e^\dagger) = \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N,\varphi,e).$$

*Proof.* The assumption that $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$ together with the proof of [Lü02, Theorem 1.35 (2)] implies that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$ if and only if $b_*^{(2)}(N, \partial N; \mathbb{C}[G]) = 0$. If both are non-zero, then both torsions $\tau^{(2)}(N, \partial N, \varphi, e^\dagger)$ and $\tau^{(2)}(N, \varphi, e)$ are zero. For the remainder of this proof we now assume that $b_*^{(2)}(N; \mathbb{C}[G]) = 0$.

We pick a triangulation $X$ for $N$. As usual we denote by $Y$ the subcomplex corresponding to $\partial N$. Let $e \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$. We pick an Euler lift $c_*$ which represents $e^\dagger$. We denote the components of $Y$ by $Y_1 \cup \cdots \cup Y_b$ and we pick $\tilde{s}_*^1, \ldots, \tilde{s}_*^b$ as in the previous section. We write $\tilde{s}_* = \tilde{s}_*^1 \cup \cdots \cup \tilde{s}_*^b$. We denote by $\{\tilde{s}_* \cup c_*\}$ the resulting Euler lift for $X$. Recall that this Euler lift represents $K(e)$. We have the following claim.

*Claim.*
$$\tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}\big).$$

In order to prove the claim we consider the following short exact sequence of chain complexes
$$0 \to \bigoplus_{i=1}^{b} C_*^\varphi(Y_i; \mathbb{C}[G]) \to C_*^\varphi(X; \mathbb{C}[G]) \to C_*^\varphi(X, Y; \mathbb{C}[G]) \to 0,$$
with the bases
$$\{s_*^i\}_{i=1,\ldots,b}, \ \{\tilde{s}_* \cup c_*\} \text{ and } \{c_*\}.$$
Note that these bases are in fact compatible, in the sense that the middle basis is the image of the left basis together with a lift of the right basis. By Lemma 5.1 we have $\tau^{(2)}(C_*^\varphi(Y_i; \mathbb{C}[G]), \{\tilde{s}_*^i\}) = 1$ for $i = 1, \ldots, b$. Now it follows from the multiplicativity of torsion, see [Lü02, Theorem 3.35], that
$$\tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) = \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{c_* \cup \tilde{s}_*\}\big).$$
Here we used that the complexes are acyclic. This concludes the proof of the claim.

Finally it follows from this claim, the definitions and Lemma 3.2 that
$$
\begin{aligned}
\tau^{(2)}(N, \partial N, \varphi, e^\dagger) &= \tau^{(2)}\big(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}\big) \\
&= \tau^{(2)}\big(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}\big) \\
&= \tau^{(2)}(N, \varphi, K(e^\dagger)) \\
&= \tau^{(2)}(N, \varphi, c_1(e)^{-1}e) = \det\nolimits_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e).
\end{aligned}
$$
$\square$

## 6. THE SYMMETRY OF THE $L^2$-ALEXANDER TORSION

### 6.1. The $L^2$-Alexander torsion for 3-manifolds.
Let $(N, \phi, \gamma\colon \pi_1(N) \to G)$ be an admissible triple and let $e \in \mathrm{Eul}(N)$. Given $t \in \mathbb{R}_{>0}$ we consider the representation
$$
\begin{aligned}
\gamma_t\colon \pi_1(N) &\to \mathrm{GL}(1, \mathbb{C}[G]) \\
g &\mapsto (t^{\phi(g)}\gamma(g)).
\end{aligned}
$$

Then we denote by $\tau^{(2)}(N, \phi, \gamma, e)$ the function

$$\tau^{(2)}(N, \phi, \gamma, e)\colon \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$$
$$t \mapsto \tau^{\overline{(2)}}(N, \gamma_t, e).$$

Note that for a different Euler class $e'$ we have $e' = ge$ for some $g \in \mathcal{H}_1(N)$ and it follows from Lemma 3.2 that

$$\tau^{(2)}(N, \phi, \gamma, ge)(t) = t^{-\phi(g)} \tau^{(2)}(N, \phi, g, e)(t) \text{ for all } t \in \mathbb{R}_{>0}.$$

Put differently, the functions $\tau^{(2)}(N, \phi, \gamma, e)$ and $\tau^{(2)}(N, \phi, \gamma, ge)$ are equivalent. We denote by $\tau^{(2)}(N, \phi, \gamma)$ the equivalence class of the functions $\tau^{(2)}(N, \phi, \gamma, e)$ and we refer to $\tau^{(2)}(N, \phi, \gamma)$ as the $L^2$-Alexander torsion of $(N, \phi, \gamma)$.

6.2. **Proof of Theorem 1.1.** For the reader's convenience we recall the statement of the proof of Theorem 1.1.

**Theorem 1.1.** *Let $(N, \phi, \gamma)$ be an admissible triple. Then for any representative $\tau$ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi)$ mod 2 such that*

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

*Proof.* Let $e \in \mathrm{Eul}(N)$. We write $\tau = \tau^{(2)}(N, \gamma, \phi, e)$. Let $t \in \mathbb{R}_{>0}$. It follows easily from the definitions that $(\gamma_t)^\dagger = \gamma_{t^{-1}}$. Using Theorems 4.2 and 5.2 we see that the following equalities hold:

$$
\begin{aligned}
\tau(t) = \tau^{(2)}(N, \gamma, \phi, e) = \tau^{(2)}(N, \gamma_t, e) &= \tau^{(2)}(N, \partial N, (\gamma_t)^\dagger, e^\dagger) \\
&= \tau^{(2)}(N, \partial N, \gamma_{t^{-1}}, e^\dagger) \\
&= \det\nolimits_{\mathcal{N}(G)}(\gamma_{t^{-1}}(c_1(e))) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\
&= \det\nolimits_{\mathcal{N}(G)}(t^{-\phi(c_1(e))} c_1(e)) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\
&= t^{-\phi(c_1(e))} \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\
&= t^{-\phi(c_1(e))} \cdot \tau(t^{-1}).
\end{aligned}
$$

Now it suffices to prove the following claim:

*Claim.* For any $\phi \in H^1(N; \mathbb{Z})$ we have

$$\phi(c_1(e)) = x_N(\phi) \text{ mod } 2.$$

Let $S$ be a Thurston norm minimizing surface which is dual to $\phi$. Since $N$ is irreducible and since $N \neq S^1 \times D^2$ we can arrange that $S$ has no disk components. Therefore we have

$$x_N(\phi) \equiv \chi_-(S) \equiv b_0(\partial S) \mod 2\mathbb{Z}.$$

On the other hand, by [Tu02, Lemma VI.1.2] and [Tu02, Section XI.1] we have that

$$b_0(\partial S) \equiv c_1(e) \cdot S \mod 2\mathbb{Z}$$

where $c_1(e) \cdot S$ is the intersection number of $c_1(e) \in H_1(N) = \mathcal{H}_1(N)$ with $S$. Since $S$ is dual to $\phi$, we obtain that

$$\phi(c_1(e)) \equiv c_1(e) \cdot S \equiv b_0(\partial S) \equiv \chi_-(S) \equiv x_N(\phi) \bmod 2\mathbb{Z}.$$

This concludes the proof of the claim.

$\square$

6.3. **Extending the main result to real cohomology classes.** A *real admissible triple* $(N, \phi, \gamma)$ consists of an irreducible, orientable, compact 3–manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary, a non-zero class $\phi \in H^1(N; \mathbb{R}) = \mathrm{Hom}(\pi_1(N), \mathbb{R})$ and a homomorphism $\gamma \colon \pi_1(N) \to G$ such that $\phi$ factors through $\gamma$. Verbatim the same definition as in Section 6.1 associates to $(N, \phi, e)$ a function $\tau^{(2)}(N, \phi, e) \colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ that is well-defined up to multiplication by a function of the form $t \mapsto t^r$ for some $r \in \mathbb{R}$. Furthermore, verbatim the same argument as in the proof of Theorem 1.1 gives us the following result.

**Theorem 6.1.** *Let $(N, \phi, \gamma)$ be a real admissible triple. Then for any representative $\tau$ of $\tau^{(2)}(N, \phi, \gamma)$ there exists an $r \in \mathbb{R}$ such that*

$$\tau(t^{-1}) = t^r \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

The only difference to Theorem 1.1 is that for real cohomology classes $\phi \in H^1(N; \mathbb{R})$ we can not relate the exponent $r$ to the Thurston norm of $\phi$.

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