ESTIMATES FOR SPECTRAL DENSITY FUNCTIONS OF MATRICES OVER $\mathbb{C}[\mathbb{Z}^d]$

LÜCK, W.

Abstract. We give a polynomial bound on the spectral density function of a matrix over the complex group ring of $\mathbb{Z}^d$. It yields an explicit lower bound on the Novikov-Shubin invariant associated to this matrix showing in particular that the Novikov-Shubin invariant is larger than zero.

1. Introduction

1.1. **Summary.** The main result of this paper is that for a (m,n)-matrix A over the complex group ring of $\mathbb{Z}^d$ the Novikov-Shubin invariant of the bounded $\mathbb{Z}^d$-equivariant operator $r_A^{(2)}\colon L^2(\mathbb{Z}^d)^m \to L^2(\mathbb{Z}^d)^n$ given by right multiplication with A is larger than zero. Actually rather explicit lower bounds in terms of elementary invariants of the minors of the matrix A will be given. This is a direct consequence of a polynomial bound of the spectral density function of $r_A^{(2)}$ which is interesting in its own right. It will play a role in the forthcoming paper [1], where we will twist L^2-torsion with finite dimensional representations and it will be crucial that we allow complex coefficients and not only integral coefficients.

Novikov-Shubin invariants were originally defined analytically in [10, 11]. More information about them can be found for instance in [8, Chapter 2].

Before we state the main result, we need the following notions.

1.2. **The width and the leading coefficient.** Consider a non-zero element $p = p(z_1^{\pm 1},\dots,z_d^{\pm 1})$ in $\mathbb{C}[\mathbb{Z}^d] = \mathbb{C}[z_1^{\pm 1},\dots,z_d^{\pm 1}]$ for some integer $d \geq 1$.

There are integers n_d^- and n_d^+ and elements $q_n(z_1^{\pm 1},\dots,z_{d-1}^{\pm 1})$ in $\mathbb{C}[\mathbb{Z}^{d-1}] = \mathbb{C}[z_1^{\pm 1},\dots,z_{d-1}^{\pm 1}]$ uniquely determined by the properties that

$$\begin{aligned} n_d^- &\leq n_d^+;\\ q_{n_d^-}(z_1^{\pm 1},\dots,z_{d-1}^{\pm 1}) &\neq 0;\\ q_{n_d^+}(z_1^{\pm 1},\dots,z_{d-1}^{\pm 1}) &\neq 0;\\ p(z_1^{\pm 1},\dots,z_d^{\pm 1}) &= \sum_{n=n_d^-}^{n_d^+} q_n(z_1^{\pm 1},\dots,z_{d-1}^{\pm 1})\cdot z_d^n. \end{aligned}$$

In the sequel denote

$$\begin{aligned} w(p) &= n_d^+ - n_d^-;\\ q^+(p) &= q_{n_d^+}(z_1^{\pm 1},\dots,z_{d-1}^{\pm 1}). \end{aligned}$$

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Define inductively elements $p_i(z_1^{\pm 1}, \ldots, z_{d-i}^{\pm 1})$ in $\mathbb{C}[\mathbb{Z}^{d-i}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_{d-i}^{\pm 1}]$ and integers $w_i(p) \geq 0$ for $i = 0, 1, 2, \ldots, d$ by

$$\begin{aligned} p_0(z_1^{\pm 1}, \ldots, z_d^{\pm 1}) &:= p(z_1^{\pm 1}, \ldots, z_d^{\pm 1}); \\ p_1(z_1^{\pm 1}, \ldots, z_{d-1}^{\pm 1}) &:= q^+(p) \\ p_i &:= q^+(p_{i-1}) \quad \text{for } i = 1, 2 \ldots, d; \\ w_0(p) &:= w(p) \\ w_i(p) &:= w(p_i) \quad \text{for } i = 1, 2 \ldots, (d-1). \end{aligned}$$

Define the *width* of $p = p(z_1^{\pm 1}, \ldots, z_d^{\pm 1})$ to be

$$\text{wd}(p) = \max\{w_0(p), w_1(p), \ldots, w_{d-1}(p)\}, \tag{1.1}$$

and the leading *coefficient of* p to be

$$\text{lead}(p) \quad = \quad p_d. \tag{1.2}$$

Obviously we have

$$\begin{aligned} \text{wd}(p) \geq \text{wd}(p_1) \geq \text{wd}(p_2) \geq \cdots \geq \text{wd}(p_d) = 0; \\ \text{lead}(p) = \text{lead}(p_1) = \ldots = \text{lead}(p_0) \neq 0. \end{aligned}$$

Notice that p_i, $\text{wd}(p)$ and $\text{lead}(p)$ do depend on the ordering of the variables $z_1, \ldots, z_d$.

Remark 1.3 (Leading coefficient)**.** The name "leading coefficient" comes from the following alternative definition. Equip $\mathbb{Z}^d$ with the lexicographical order, i.e., we put $(m_1, \ldots, m_d) < (n_1, \ldots, n_d)$, if $m_d < n_d$, or if $m_d = n_d$ and $m_{d-1} < n_{d-1}$, or if $m_d = n_d$, $m_{d-1} = n_{d-1}$ and $m_{d-2} < n_{d-2}$, or if $\ldots$, or if $m_i = n_i$ for $i = d, (d-1), \ldots, 2$ and $m_1 < n_1$. We can write p as a finite sum with complex coefficients $a_{n_1,\ldots,n_d}$

$$p(z_1^{\pm}, \ldots, z_d^{\pm}) = \sum_{(n_1,\ldots,n_d) \in \mathbb{Z}^d} a_{n_1,\ldots,n_d} \cdot z_1^{n_1} \cdot z_2^{n_2} \cdot \cdots \cdot z_d^{n_d}.$$

Let $(m_1, \ldots m_d) \in \mathbb{Z}^d$ be maximal with respect to the lexicographical order among those elements $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ for which $a_{n_1,\ldots,n_d} \neq 0$. Then the leading coefficient of p is $a_{m_1,\ldots,m_d}$.

1.3. **The L^1-norm of a matrix.** For an element $p = \sum_{g \in \mathbb{Z}^d} \lambda_g \cdot g \in \mathbb{C}[\mathbb{Z}^d]$ define $||p||_1 := \sum_{g \in G} |\lambda_g|$. For a matrix $A \in M_{m,n}(\mathbb{C}[\mathbb{Z}^d])$ define

$$||A||_1 \quad = \quad \max\{||a_{i,j}||_1 \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{1.4}$$

The main purpose of this notion is that it gives an a priori upper bound on the norm $r_A^{(2)} \colon L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)$, namely, we get from [8, Lemma 13.33 on page 466]

$$||r_A^{(2)}|| \quad \leq \quad m \cdot n \cdot ||A||_1. \tag{1.5}$$

1.4. **The spectral density function.** Given $A \in M_{m,n}(\mathbb{C}[\mathbb{Z}^d])$, multiplication with A induces a bounded $\mathbb{Z}^d$-equivariant operator $r_A^{(2)} \colon L^2(\mathbb{Z}^d)^m \to L^2(\mathbb{Z}^d)^n$. We will denote by

$$F\big(r_A^{(2)}\big) \colon [0, \infty) \quad \to \quad [0, \infty) \tag{1.6}$$

its *spectral density function* in the sense of [8, Definition 2.1 on page 73], namely, the von Neumann dimension of the image of the operator obtained by applying the functional calculus to the characteristic function of $[0, \lambda^2]$ to the operator $(r_A^{(2)})^* r_A^{(2)}$. In the special case $m = n = 1$, where A is given by an element $p \in \mathbb{C}[\mathbb{Z}^d]$, it can be computed in terms of the Haar measure μ_{T^d} of the d-torus T^d see [8, Example 2.6 on page 75]

$$F\big(r_A^{(2)}\big)(\lambda) \quad = \quad \mu_{T^d}\big(\{(z_1,\ldots,z_d) \in T^d \mid |p(z_1,\ldots,z_d)| \le \lambda\}\big). \tag{1.7}$$

1.5. **The main result.** Our main result is:

Theorem 1.8 (Main Theorem)**.** *Consider any natural numbers d, m, n and a non-zero matrix $A \in M_{m,n}(\mathbb{C}[\mathbb{Z}^d])$. Let B be a quadratic submatrix of A of maximal size k such that the corresponding minor $p = \det_{\mathbb{C}[\mathbb{Z}^d]}(B)$ is non-trivial. Then:*

(1) *If $\operatorname{wd}(p) \ge 1$, the spectral density function of $r_A^{(2)} \colon L^2(\mathbb{Z}^d)^m \to L^2(\mathbb{Z}^d)^n$ satisfies for all $\lambda \ge 0$*

$$F\big(r_A^{(2)}\big)(\lambda) - F\big(r_A^{(2)}\big)(0) \le \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot k \cdot d \cdot \operatorname{wd}(p) \cdot \left(\frac{k^{2k-2} \cdot (||B||_1)^{k-1} \cdot \lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d \cdot \operatorname{wd}(p)}}.$$

If $\operatorname{wd}(p) = 0$, then $F\big(r_A^{(2)}\big)(\lambda) = 0$ for all $\lambda < |\operatorname{lead}(p)|$ and $F\big(r_A^{(2)}\big)(\lambda) = 1$ for all $\lambda \ge |\operatorname{lead}(p)|$;

(2) *The Novikov-Shubin invariant of $r_A^{(2)}$ is ∞ or ∞^+ or a real number satisfying*

$$\alpha\big(r_A^{(2)}\big) \ge \frac{1}{d \cdot \operatorname{wd}(p)},$$

and is in particular larger than zero.

It is known that the Novikov-Shubin invariants of $r_A^{(2)}$ for a matrix A over the integral group ring of $\mathbb{Z}^d$ is a rational numbers larger than zero unless its value is ∞ or ∞^+. This follows from Lott [5, Proposition 39]. (The author of [5] informed us that his proof of this statement is correct when $d = 1$ but has a gap when $d > 1$. The nature of the gap is described in [6, page 16]. The proof in this case can be completed by the same basic method used in [5].) This confirms a conjecture of Lott-Lück [7, Conjecture 7.2] for $G = \mathbb{Z}^d$. The case of a finitely generated free group G is taken care of by Sauer [12].

Virtually finitely generated free abelian groups and virtually finitely generated free groups are the only cases of finitely generated groups, where the positivity of the Novikov-Shubin invariants for all matrices over the complex group ring is now known. In this context we mention the preprints [2, 3], where examples of groups G and matrices $A \in M_{m,n}(\mathbb{Z}G)$ are constructed for which the Novikov-Shubin invariant of $r_A^{(2)}$ is zero, disproving a conjecture of Lott-Lück [7, Conjecture 7.2].

1.6. **Example.** Consider the case $d = 2$, $m = 3$ and $n = 2$ and the $(3,2)$-matrix over $\mathbb{C}[\mathbb{Z}^2]$

$$A = \begin{pmatrix} z_1^3 & -1 & 1 \\ 2 \cdot z_1 \cdot z_2^2 - 16 & z_2 & z_1 z_2 \end{pmatrix}$$

Let B be the $(2,2)$-submatrix obtained by deleting third column. Then $k = 2$,

$$B = \begin{pmatrix} z_1^3 & -1 \\ 2 \cdot z_1 \cdot z_2^2 - 16 & z_2 \end{pmatrix}$$

and we get

$$p := \det_{\mathbb{C}[\mathbb{Z}^2]}(B) = z_1^3 \cdot z_2 + 2 \cdot z_1 \cdot z_2^2 - 16.$$

Using the notation of Section 1.2 one easily checks $p_1(z_1) = 2 \cdot z_1$, $\operatorname{wd}(p) = 2$, and $\operatorname{lead}(p) = 2$. Obviously $||A||_1 = \max\{|1|, |-1|, |2| + |16|, |1|\} = 18$. Hence

Theorem 1.8 implies for all $\lambda \geq 0$

$$\begin{aligned} F\big(r_A^{(2)}\big)(\lambda) - F\big(r_A^{(2)}\big)(0) &\leq \frac{192 \cdot \sqrt{2}}{\sqrt{47}} \cdot \lambda^{\frac{1}{4}}. \\ \alpha\big(r_A^{(2)}\big) &\geq \frac{1}{4}. \end{aligned}$$

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2. The case $m = n = 1$

The main result of this section is the following

Proposition 2.1. *Consider an non-zero element p in $\mathbb{C}[\mathbb{Z}^d] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$. If $\operatorname{wd}(p) = 0$, then $F\big(r_A^{(2)}\big)(\lambda) = 0$ for all $\lambda < |\operatorname{lead}(p)|$ and $F\big(r_A^{(2)}\big)(\lambda) = 1$ for all $\lambda \geq |\operatorname{lead}(p)|$. If $\operatorname{wd}(p) \geq 1$, we get for the spectral density function of $r_p^{(2)}$ for all $\lambda \geq 0$*

$$F\big(r_p^{(2)}\big)(\lambda) \leq \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot d \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d \cdot \operatorname{wd}(p)}}.$$

For the case $d = 1$ and p a monic polynomial, a similar estimate of the shape $F\big(r_p^{(2)}\big)(\lambda) \leq C_k \cdot \lambda^{\frac{1}{k-1}}$ can be found in [4, Theorem 1], where the $k \geq 2$ is the number of non-zero coefficients, and the sequence of real numbers $(C_k)_{k \geq 2}$ is recursively defined and satisfies $C_k \geq k - 1$.

2.1. **Degree one.** In this subsection we deal with Proposition 2.1 in the case $d = 1$.

We get from the Taylor expansion of $\cos(x)$ around 0 with the Lagrangian remainder term that for any $x \in \mathbb{R}$ there exists $\theta(x) \in [0, 1]$ such that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{\cos(\theta(x) \cdot x)}{4!} \cdot x^4.$$

This implies for $x \neq 0$ and $|x| \leq 1/2$

$$\left|\frac{2 - 2\cos(x)}{x^2} - 1\right| = \left|\frac{2 \cdot \cos(\theta(x) \cdot x)}{4!} \cdot x^2\right| \leq \left|\frac{2 \cdot \cos(\theta(x) \cdot x)}{4!}\right| \cdot |x|^2 \leq \frac{1}{12} \cdot \frac{1}{4} = \frac{1}{48}.$$

Hence we get for $x \in [-1/2, 1/2]$

$$\frac{47}{48} \cdot x^2 \leq 2 - 2\cos(x). \tag{2.2}$$

Lemma 2.3. *For any complex number $a \in \mathbb{Z}$ we get for the spectral density function of $(z - a) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$*

$$F\big(r_{z-a}^{(2)}\big)(\lambda) \leq \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot \lambda \quad \text{for } \lambda \in [0, \infty).$$

Proof. We compute using (1.7), where $r := |a|$,

$$\begin{aligned} F\big(r_{z-a}^{(2)}\big)(\lambda) &= \mu_{S^1}\{z \in S^1 \mid |z - a| \leq \lambda\} \\ &= \mu_{S^1}\{z \in S^1 \mid |z - r| \leq \lambda\} \\ &= \mu_{S^1}\{\phi \in [-1/2, 1/2] \mid |\cos(\phi) + i\sin(\phi) - r| \leq \lambda\} \\ &= \mu_{S^1}\{\phi \in [-1/2, 1/2] \mid |\cos(\phi) + i\sin(\phi) - r|^2 \leq \lambda^2\} \\ &= \mu_{S^1}\{\phi \in [-1/2, 1/2] \mid (\cos(\phi) - r)^2 + \sin(\phi)^2 \leq \lambda^2\} \\ &= \mu_{S^1}\{\phi \in [-1/2, 1/2] \mid r \cdot (2 - 2\cos(\phi) + (r-1)^2 \leq \lambda^2\}. \end{aligned}$$

We estimate using (2.2) for $\phi \in [-1/2, 1/2]$

$$r \cdot (2 - 2\cos(\phi)) + (r-1)^2 \geq r \cdot (2 - 2\cos(\phi)) \geq \frac{47}{48} \cdot \phi^2.$$

This implies for $\lambda \geq 0$

$$\begin{aligned}
F\big(r_{z-a}^{(2)}\big)(\lambda) &= \mu_{S^1}\{\phi \in [-1/2,1/2] \mid r \cdot (2 - 2\cos(\phi) + (r-1)^2 \leq \lambda^2\} \\
&\leq \mu_{S^1}\{\phi \in [-1/2,1/2] \mid \frac{47}{48} \cdot \phi^2 \leq \lambda^2\} \\
&= \mu_{S^1}\left\{\phi \in [-1/2,1/2] \;\middle|\; |\phi| \leq \sqrt{\frac{48}{47}} \cdot \lambda\right\} \\
&\leq 2 \cdot \sqrt{\frac{48}{47}} \cdot \lambda \\
&= \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot \lambda.
\end{aligned}$$

□

Lemma 2.4. *Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-zero element. If $\operatorname{wd}(p) = 0$, then $F\big(r_p^{(2)}\big)(\lambda) = 0$ for all $\lambda < |\operatorname{lead}(p)|$ and $F\big(r_p^{(2)}\big)(\lambda) = 1$ for all $\lambda \geq |\operatorname{lead}(p)|$. If $\operatorname{wd}(p) \geq 1$, we get*

$$F\big(r_p^{(2)}\big)(\lambda) \leq \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{\operatorname{wd}(p)}} \quad \textit{for } \lambda \in [0, \infty).$$

Proof. If $\operatorname{wd}(p) = 0$, then p is of the shape $C \cdot z^n$, and the claim follows directly from (1.7). Hence we can assume without loss of generality that $\operatorname{wd}(p) \geq 1$. We can write $p(z)$ as a product

$$p(z) = \operatorname{lead}(p) \cdot z^k \cdot \prod_{i=1}^{r} (z - a_i)$$

for an integer $r \geq 0$, non-zero complex numbers $a_1, \ldots, a_r$ and an integer k.

Since for any polynomial p and complex number $c \neq 0$ we have for all $\lambda \in [0, \infty)$

$$F\big(r_{c \cdot p}^{(2)}\big)(\lambda) = F\big(r_p^{(2)}\big)\left(\frac{\lambda}{|c|}\right),$$

we can assume without loss of generality $\operatorname{lead}(p) = 1$. If $r = 0$, then $p(z) = z^k$ for some $k \neq 0$ and the claim follows by a direct inspection. Hence we can assume without loss of generality $r \geq 1$. Since the width, the leading coefficient and the spectral density functions of $p(z)$ and $z^{-k} \cdot p(z)$ agree, we can assume without loss of generality $k = 0$, or equivalently, that $p(z)$ has the form for some $r \geq 1$

$$p(z) = \prod_{i=1}^{r} (z - a_i).$$

We proceed by induction over r. The case $r = 1$ is taken care of by Lemma 2.3. The induction step from $r - 1 \geq 1$ to r is done as follows.

Put $q(z) = \prod_{i=1}^{r-1} (z - a_i)$. Then $p(z) = q(z) \cdot (z - a_r)$. The following inequality for elements $q_1, q_2 \in \mathbb{C}[z, z^{-1}]$ and $s \in (0, 1)$ is a special case of [8, Lemma 2.13 (3) on page 78]

$$F\big(r_{q_1 \cdot q_2}^{(2)}\big)(\lambda) \leq F\big(r_{q_1}^{(2)}\big)(\lambda^{1-s}) + F\big(r_{q_2}^{(2)}\big)(\lambda^s). \tag{2.5}$$

We conclude from (2.5) applied to $p(z) = q(z) \cdot (z - a_r)$ in the special case $s = 1/r$

$$F\big(r_p^{(2)}\big)(\lambda) \leq F\big(r_q^{(2)}\big)(\lambda^{\frac{r-1}{r}}) + F\big(r_{z-a_r}^{(2)}\big)(\lambda^{1/r}).$$

We conclude from the induction hypothesis for $\lambda \in [0,\infty)$

$$\begin{aligned} F\big(r_q^{(2)}\big)(\lambda) &\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(r-1)\cdot\lambda^{\frac{1}{r-1}}; \\ F\big(r_{z-a_r}^{(2)}\big)(\lambda) &\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot\lambda. \end{aligned}$$

This implies for $\lambda \in [0,\infty)$

$$\begin{aligned} F\big(r_p^{(2)}\big)(\lambda) &\le F\big(r_q^{(2)}\big)(\lambda^{\frac{r-1}{r}}) + F\big(r_{z-a_r}^{(2)}\big)(\lambda^{1/r}) \\ &\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(r-1)\cdot\Big(\lambda^{\frac{r-1}{r}}\Big)^{\frac{1}{r-1}} + \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot\lambda^{\frac{1}{r}} \\ &\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(r-1)\cdot\lambda^{\frac{1}{r}} + \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot\lambda^{\frac{1}{r}} \\ &= \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot r\cdot\lambda^{\frac{1}{r}}. \end{aligned}$$

□

2.2. **The induction step.** Now we finish the proof of Proposition 2.1 by induction over d. If $\mathrm{wd}(p) = 0$, then p is of the shape $C\cdot z_1^{n_1}\cdot z_2^{n_2}\cdot\dots\cdot z_d^{n_d}$, and the claim follows directly from (1.7). Hence we can assume without loss of generality that $\mathrm{wd}(p)\ge 1$. The induction beginning $d = 1$ has been taken care of by Lemma 2.4, the induction step from $d-1$ to $d\ge 2$ is done as follows.

Since $F\big(r_p^{(2)}\big)(\lambda)\le 1$, the claim is obviously true for $\frac{\lambda}{|\operatorname{lead}(p)|}\ge 1$. Hence we can assume in the sequel $\frac{\lambda}{|\operatorname{lead}(p)|}\le 1$.

We conclude from (1.7) and Fubini's Theorem applied to $T^d = T^{d-1}\times S^1$, where χ_A denotes the characteristic function of a subset A and $p_1(z_1^{\pm},\dots,z_{d-1}^{\pm 1})$ has been defined in Subsection 1.2

(2.6)

$$\begin{aligned}
&F\bigl(r_p^{(2)}\bigr)(\lambda) \\
&= \mu_{T^d}\bigl(\{(z_1,\dots,z_d)\in T^d \mid \; |p(z_1,\dots,z_d)|\le\lambda\}\bigr) \\
&= \int_{T^d} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{T^n} \\
&= \int_{T^{d-1}} \left(\int_{S^1} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{S^1}\right) d\mu_{T^{d-1}} \\
&= \int_{T^{d-1}} \chi_{\{(z_1,\dots,z_{d-1})\in T^{d-1} \mid |p_1(z_1,\dots,z_{d-1})\le |\operatorname{lead}(p)|^{1/d}\cdot\lambda^{(d-1)1/d}\}} \\
&\qquad \cdot \left(\int_{S^1} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{S^1}\right) d\mu_{T^{d-1}} \\
&\qquad\quad + \int_{T^{d-1}} \chi_{\{(z_1,\dots,z_{d-1})\in T^{d-1} \mid |p_1(z_1,\dots,z_{d-1})>|\operatorname{lead}(p)|^{1/d}\cdot\lambda^{(d-1))/d}\}} \\
&\qquad\qquad \cdot \left(\int_{S^1} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{S^1}\right) d\mu_{T^{d-1}} \\
&\le \int_{T^{d-1}} \chi_{(z_1,\dots,z_{d-1})\mid |p_1(z_1,\dots,z_{d-1})|\le|\operatorname{lead}(p)|^{1/d}\cdot\lambda^{(d-1)1/d}\}} + \\
&\qquad \max\left\{\int_{S^1} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{S^1} \,\middle|\, (z_1,\dots,z_{d-1})\in T^{d-1}\right. \\
&\qquad\qquad\qquad \left.\text{with } |p_1(z_1,\dots,z_{d-1})| > |\operatorname{lead}(p)|^{1/d}\cdot\lambda^{(d-1)/d}\right\}.
\end{aligned}$$

We get from the induction hypothesis applied to $p_1(z_1,\dots,z_{d-1})$ and (1.7) since $\frac{\lambda}{|\operatorname{lead}(p)|}\le 1$, $\operatorname{wd}(p_1)\le\operatorname{wd}(p)$ and $\operatorname{lead}(p)=\operatorname{lead}(p_1)$

(2.7)

$$\begin{aligned}
&\int_{T^{d-1}} \chi_{(z_1,\dots,z_{d-1})\mid |p_1(z_1,\dots,z_{d-1})|\le|\operatorname{lead}(p)|^{1/d}\cdot\lambda^{(d-1)1/d}\}} \\
&\quad= \int_{T^{d-1}} \chi_{(z_1,\dots,z_{d-1})\mid |p_1(z_1,\dots,z_{d-1})|\le|\operatorname{lead}(p_1)|^{1/d}\cdot\lambda^{(d-1)1/d}\}} \\
&\quad= F\bigl(r_{p_1}^{(2)}\bigr)\bigl(|\operatorname{lead}(p_1)|^{1/d}|\cdot\lambda^{(d-1)/d}\bigr) \\
&\quad\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(d-1)\cdot\operatorname{wd}(p_1)\cdot\left(\frac{|\operatorname{lead}(p_1)|^{1/d}\cdot\lambda^{(d-1)/d}}{|\operatorname{lead}(p_1)|}\right)^{\frac{1}{(d-1)\cdot\operatorname{wd}(p_1)}} \\
&\quad= \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(d-1)\cdot\operatorname{wd}(p_1)\cdot\left(\frac{\lambda}{|\operatorname{lead}(p_1)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p_1)}} \\
&\quad= \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(d-1)\cdot\operatorname{wd}(p_1)\cdot\left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p_1)}} \\
&\quad\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(d-1)\cdot\operatorname{wd}(p)\cdot\left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p_1)}} \\
&\quad\le \frac{8\cdot\sqrt{3}}{\sqrt{47}}\cdot(d-1)\cdot\operatorname{wd}(p)\cdot\left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p)}}.
\end{aligned}$$

Fix $(z_1,\dots,z_{d-1}) \in T^{d-1}$ with $|p_1(z_1,\dots,z_{d-1})| > \operatorname{lead}(p)^{1/d} \cdot \lambda^{(d-1)/d}$. Consider the element $f(z_d^{\pm 1}) := p(z_1,\dots z_{d-1}, z_d^{\pm}) \in \mathbb{C}[z_d^{\pm}]$. It has the shape

$$f(z_d^{\pm}) = \sum_{n=n^-}^{n^+} q_n(z_1,\dots,z_{d-1}) \cdot z_d^n.$$

The leading coefficient of $f(z_d^{\pm 1})$ is $p_1(z_1,\dots z_{d-1}) = q_{n_+}(z_1,\dots,z_{d-1})$. Hence we get from Lemma 2.4 applied to $f(z_d^{\pm 1})$ and (1.7) since $\frac{\lambda}{|\operatorname{lead}(p)|} \le 1$, $\operatorname{wd}(f) \le \operatorname{wd}(p)$ and $|\operatorname{lead}(f)| = |p_1(z_1,\dots z_{d-1}))| > |\operatorname{lead}(p)|^{1/d} \cdot \lambda^{(d-1)/d}$

(2.8)

$$\begin{aligned}
&\int_{S^1} \chi_{\{(z_1,\dots,z_d)\in T^d \mid |p(z_1,\dots,z_d)|\le\lambda\}}\, d\mu_{S^1}\\
&= \int_{S^1} \chi_{\{z_d\in S^1 \mid |f(z_d)|\le\lambda\}}\, d\mu_{S^1}\\
&= \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(f) \cdot \left(\frac{\lambda}{\operatorname{lead}(f)}\right)^{\frac{1}{\operatorname{wd}(f)}}\\
&\le \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(f) \cdot \left(\frac{\lambda}{\operatorname{lead}(p)^{1/d}\cdot\lambda^{(d-1)/d}}\right)^{\frac{1}{\operatorname{wd}(f)}}\\
&= \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(f) \cdot \left(\frac{\lambda}{\operatorname{lead}(p)}\right)^{\frac{1}{d\cdot\operatorname{wd}(f)}}\\
&\le \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{\operatorname{lead}(p)}\right)^{\frac{1}{d\cdot\operatorname{wd}(p)}}.
\end{aligned}$$

Combining (2.6), (2.7) and (2.8) yields for λ with $\frac{\lambda}{|\operatorname{lead}(p)|} \le 1$

$$\begin{aligned}
F\big(r_p^{(2)}\big)(\lambda) &\le \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot (d-1) \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p)}}\\
&\qquad + \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p)}}\\
&= \frac{8\cdot\sqrt{3}}{\sqrt{47}} \cdot d \cdot \operatorname{wd}(p) \cdot \left(\frac{\lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d\cdot\operatorname{wd}(p)}}.
\end{aligned}$$

This finishes the proof of Proposition 2.1.

3. Proof of the main Theorem 1.8

Now we can complete the proof of our Main Theorem 1.8. We need the following preliminary result

Lemma 3.1. *Consider $B \in M_{k,k}(\mathbb{C}[\mathbb{Z}^d])$ such that $p := \det_{\mathbb{C}[\mathbb{Z}^d]}(B)$ is non-trivial. Then we get for all $\lambda \ge 0$*

$$F\big(r_B^{(2)}\big)(\lambda) \le k \cdot F\big(r_p^{(2)}\big)\big(||r_B^{(2)}||^{k-1}\cdot\lambda\big).$$

Proof. In the sequel we will identify $L^2(\mathbb{Z}^d)$ and $L^2(T^d)$ by the Fourier transformation. We can choose a unitary $\mathbb{Z}^d$-equivariant operator $U\colon L^2(\mathbb{Z}^d)^k \to L^2(\mathbb{Z}^d)^k$ and functions $f_1, f_2, \dots, f_k\colon T^d \to \mathbb{R}$ such that $0 \le f_1(z) \le f_2(z) \le \dots \le f_k(z)$ holds for all $z \in T^d$ and we have the following equality of bounded $\mathbb{Z}^d$-equivariant operators $L^2(\mathbb{Z}^d)^k = L^2(T^d)^k \to L^2(\mathbb{Z}^d)^k = L^2(T^d)^k$, see [9, Lemma 2.2]

$$(r_B^{(2)})^* \circ r_B^{(2)} = U \circ \begin{pmatrix} r_{f_1}^{(2)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_{f_2}^{(2)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_{f_3}^{(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_{f_{k-1}}^{(2)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & r_{f_k}^{(2)} \end{pmatrix} \circ U^*. \tag{3.2}$$

Since $p \neq 0$ holds by assumption and hence the rank of B over $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ is maximal, we conclude from [8, Lemma 1.34 on page 35] that $r_B^{(2)}$ and hence $r_{f_i}^{(2)}$ for each $i = 1, 2, \ldots, k$ are weak isomorphisms, i.e., they are injective and have dense images. We conclude from [8, Lemma 2.11 (11) on page 77 and Lemma 2.13 on page 78]

$$F(r_B^{(2)})(\lambda) = F\left((r_B^{(2)})^* \circ r_B^{(2)}\right)(\lambda^2) = \sum_{i=1}^k F(r_{f_i}^{(2)})(\lambda^2).$$

For $i = 1, 2, \ldots, k$ we have $f_1(z) \leq f_i(z)$ for all $z \in T^d$ and hence $F\bigl(r_{f_i}^{(2)}\bigr)(\lambda) \leq F\bigl(r_{f_1}^{(2)}\bigr)(\lambda)$ for all $\lambda \geq 0$. This implies

$$F\bigl(r_B^{(2)}\bigr)(\lambda) \leq k \cdot F\bigl(r_{f_1}^{(2)}\bigr)(\lambda^2). \tag{3.3}$$

Let $B^* \in M_{k,k}(\mathbb{C}[\mathbb{Z}^d)$ be the matrix obtain from B by transposition and applying to each entry the involution $\mathbb{C}[\mathbb{Z}^d] \to \mathbb{C}[\mathbb{Z}^d|$ sending $\sum_{g \in G} \lambda_g \cdot g$ to $\sum_{g \in G} \overline{\lambda_g} \cdot g^{-1}$. Then $\bigl(r_B^{(2)}\bigr)^* = r_{B^*}^{(2)}$. Since $(r_B^{(2)})^* \circ r_B^{(2)} = r_{BB^*}^{(2)}$ and $\det_{\mathbb{C}[\mathbb{Z}^d]}(BB^*) = \det_{\mathbb{C}[\mathbb{Z}^d]}(B) \cdot \det_{\mathbb{C}[\mathbb{Z}^d]}(B^*) = p \cdot p^*$ holds, we conclude from (3.2) the equality of functions $T^d \to [0, \infty]$

$$pp^* = \prod_{i=1}^k f_i.$$

Since $\sup\{|f_i(z)| \mid z \in T^d\}$ agrees with the operatornorm $||r_{f_i}^{(2}||$ and we have $||r_B^{(2)}||^2 = ||(r_B^{(2)})^* r_B^{(2)}|| = \max\bigl\{||r_{f_i}^{(2)}|| \bigm| i = 1, 2, \ldots, k\bigr\} = ||r_{f_k}^{(2)}||$, we obtain the inequality of functions $T^d \to [0, \infty]$

$$pp^* \leq \left(\prod_{i=2}^k ||r_{f_i}^{(2)}||\right) \cdot f_1 \leq \bigl(||r_B^{(2)}||^2\bigr)^{k-1} \cdot f_1.$$

Hence we get for all $\lambda \geq 0$

$$\begin{aligned} F\bigl(r_{pp^*}^{(2)}\bigr)\left(\bigl(||r_B^{(2)}||^{k-1}\lambda\bigr)^2\right) &= F\bigl(r_{pp^*}^{(2)}\bigr)\left(\bigl(||r_B^{(2)}||^2\bigr)^{k-1}\lambda^2\right) \\ &\geq F\left(\bigl(||r_B^{(2)}||^2\bigr)^{k-1} \cdot r_{f_1}^{(2)}\right)\left(\bigl(||r_B^{(2)}||^2\bigr)^{k-1} \cdot \lambda^2\right) \\ &= F\bigl(r_{f_1}^{(2)}\bigr)(\lambda^2). \end{aligned}$$

This together with (3.3) and [8, Lemma 2.11 (11) on page 77] implies

$$\begin{aligned} F\bigl(r_B^{(2)}\bigr)(\lambda) &\leq k \cdot F\bigl(r_{f_1}^{(2)}\bigr)(\lambda^2) \\ &\leq k \cdot F\bigl(r_{pp^*}^{(2)}\bigr)\left(\bigl(||r_B^{(2)}||^{k-1}\lambda\bigr)^2\right) \\ &\leq k \cdot F\bigl(r_p^{(2)}\bigr)\bigl(||r_B^{(2)}||^{k-1}\lambda\bigr). \end{aligned}$$

□

Proof of the Main Theorem 1.8. (1) In the sequel we denote by $\dim_{\mathcal{N}(G)}$ the von Neumann dimension, see for instance [8, Subsection 1.1.3]. The rank of the matrices A and B over the quotient field $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ is k. The operator $r_B^{(2)} \colon L^2(\mathbb{Z}^d)^k \to L^2(\mathbb{Z}^d)^k$ is a weak isomorphism, and $\dim_{\mathcal{N}(\mathbb{Z}^d)}(\overline{\operatorname{im}(r_A^{(2)})}) = k$ because of [8, Lemma 1.34 (1) on page 35]. In particular we have $F\bigl(r_B^{(2)}\bigr)(0) = 0$.

Let $i^{(2)} \colon L^2(\mathbb{Z}^d)^k \to L^2(\mathbb{Z}^d)^m$ be the inclusion corresponding to $I \subseteq \{1, 2, \ldots, m\}$ and let $\operatorname{pr}^{(2)} \colon L^2(\mathbb{Z}^d)^n \to L^2(\mathbb{Z}^d)^k$ be the projection corresponding to $J \subseteq \{1, 2, \ldots, n\}$, where I and J are the subsets specifying the submatrix B. Then $r_B^{(2)} \colon L^2(\mathbb{Z}^d)^k \to L^2(\mathbb{Z}^d)^k$ agrees with the composite

$$r_B^{(2)} \colon L^2(\mathbb{Z}^d)^k \xrightarrow{i^{(2)}} L^2(\mathbb{Z}^d)^m \xrightarrow{r_A^{(2)}} L^2(\mathbb{Z}^d)^n \xrightarrow{\operatorname{pr}^{(2)}} L^2(\mathbb{Z}^d)^k.$$

Let $p^{(2)} \colon L^2(\mathbb{Z}^d)^m \to \ker(r_A^{(2)}))^{\perp}$ be the orthogonal projection onto the orthogonal complement $\ker(r_A^{(2)})^{\perp} \subseteq L^2(G)^m$ of the kernel of $r_A^{(2)}$. Let $j^{(2)} \colon \overline{\operatorname{im}(r_A^{(2)})} \to L^2(G)^n$ be the inclusion of the closure of the image of $r_A^{(2)}$. Let $(r_A^{(2)})^{\perp} \colon \ker(r_A^{(2)})^{\perp} \to \overline{\operatorname{im}(r_A^{(2)})}$ be the $\mathbb{Z}^d$-equivariant bounded operator uniquely determined by

$$r_A^{(2)} \quad = \quad j^{(2)} \circ (r_A^{(2)})^{\perp} \circ p^{(2)}.$$

The operator $(r_A^{(2)})^{\perp}$ is a weak isomorphism by construction. We have the decomposition of the weak isomorphism

$$r_B^{(2)} = \operatorname{pr}^{(2)} \circ r_A^{(2)} \circ i^{(2)} = \operatorname{pr}^{(2)} \circ j^{(2)} \circ (r_A^{(2)})^{\perp} \circ p^{(2)} \circ i^{(2)}. \tag{3.4}$$

This implies that the morphism $p^{(2)} \circ i^{(2)} \colon L^2(\mathbb{Z}^d)^k) \to \ker(r_A^{(2)})^{\perp}$ is injective and the morphism $\operatorname{pr}^{(2)} \circ j^{(2)} \colon \overline{\operatorname{im}(r_A^{(2)})} \to L^2(\mathbb{Z}^d)^k$ has dense image. Since we already know $\dim_{\mathcal{N}(G)}\bigl(\overline{\operatorname{im}(r_A^{(2)})}\bigr) = k = \dim_{\mathcal{N}(G)}\bigl(L^2(\mathbb{Z}^d)^k\bigr)$, the operators $p^{(2)} \circ i^{(2)} \colon L^2(\mathbb{Z}^d)^k \to \ker(r_A^{(2)})^{\perp}$ and $\operatorname{pr}^{(2)} \circ j^{(2)} \colon \overline{\operatorname{im}(r_A^{(2)})} \to L^2(\mathbb{Z}^d)$ are weak isomorphisms. Since the operatornorm of $\operatorname{pr}^{(2)} \circ j^{(2)}$ and of $p^{(2)} \circ i^{(2)}$ is less or equal to 1, we conclude from [8, Lemma 2.13 on page 78] and (3.4)

$$\begin{aligned}
& F\bigl(r_A^{(2)}\bigr)(\lambda) - F\bigl(r_A^{(2)}\bigr)(0) \\
& \quad = \quad F\bigl((r_A^{(2)})^{\perp}\bigr)(\lambda) \\
& \quad \leq \quad F\bigl(\operatorname{pr}^{(2)} \circ j^{(2)} \circ (r_A^{(2)})^{\perp} \circ p^{(2)} \circ i^{(2)}\bigr)\bigl(||\operatorname{pr}^{(2)} \circ j^{(2)}|| \cdot ||p^{(2)} \circ i^{(2)}|| \cdot \lambda\bigr) \\
& \quad = \quad F\bigl(r_B^{(2)}\bigr)\bigl(||\operatorname{pr}^{(2)} \circ j^{(2)}|| \cdot ||p^{(2)} \circ i^{(2)}|| \cdot \lambda\bigr) \\
& \quad \leq \quad F\bigl(r_B^{(2)}\bigr)(\lambda).
\end{aligned}$$

Put $p = \det_{\mathbb{C}[\mathbb{Z}^d]}(B)$. If $\operatorname{wd}(p) = 0$, the claim follows directly from Proposition 2.1. It remains to treat the case $\operatorname{wd}(p) \geq 1$. The last inequality together with (1.5) applied to B, Proposition 2.1 applied to p and Lemma 3.1 applied to B yields for $\lambda \geq 0$

$$\begin{aligned}
& F\bigl(r_A^{(2)}\bigr)(\lambda) - F\bigl(r_A^{(2)}\bigr)(0) \\
& \quad \leq \quad F\bigl(r_B^{(2)}\bigr)(\lambda) \\
& \quad \leq \quad k \cdot F\bigl(r_p^{(2)}\bigr)\bigl(||r_B^{(2)}||^{k-1} \cdot \lambda\bigr) \\
& \quad \leq \quad k \cdot F\bigl(r_p^{(2)}\bigr)\bigl((k^2 \cdot ||B||_1)^{k-1} \cdot \lambda\bigr) \\
& \quad \leq \quad \frac{8 \cdot \sqrt{3}}{\sqrt{47}} \cdot k \cdot d \cdot \operatorname{wd}(p) \cdot \left(\frac{k^{2k-2} \cdot (||B||_1)^{k-1} \cdot \lambda}{|\operatorname{lead}(p)|}\right)^{\frac{1}{d \cdot \operatorname{wd}(p)}}.
\end{aligned}$$

This finishes the proof of assertion (1). Assertion (2) is a direct consequence of assertion (1) and the definition of the Novikov-Shubin invariant. This finishes the proof of Theorem 1.8. □

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Mathematicians Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: wolfgang.lueck@him.uni-bonn.de

URL: http://www.him.uni-bonn.de/lueck