ON THE GROWTH OF BETTI NUMBERS IN p-ADIC ANALYTIC TOWERS

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Abstract. We study the asymptotic growth of Betti numbers in tower of finite covers and provide simple proofs of approximation results, which were previously obtained by Calegari-Emerton [6, 7], in the generality of arbitrary p-adic analytic towers of covers. Further, we also obtain partial results about arbitrary pro-p towers.

1. Introduction and statement of results

This paper is mainly concerned with the asymptotic growth of Betti numbers in a tower of finite covers of a compact space X associated to a chain of subgroups of the fundamental group of X which gives rise to a p-adic analytic group. Both Betti numbers with coefficients in $\mathbb{Q}$ and $\mathbb{F}_p$ are considered. Especially the case of $\mathbb{F}_p$-coefficients received a lot of attention in recent years. We only name here the work of Calegari-Emerton [6, 7], which is motivated by the p-adic Langlands program, and the work of Lackenby [21, 22] in group theory, which is connected to property τ and 3-manifold theory.

1.1. **Global setup.** With the exception of section 4, we retain the following setup throughout this paper. Let X be a connected compact CW-complex with fundamental group Γ. Let p be a prime, let n be a positive integer, and let $\phi : \Gamma \to \mathrm{GL}_n(\mathbb{Z}_p)$ be a homomorphism. The closure of the image of ϕ, which is denoted by G, is a p-adic analytic group admitting an exhausting filtration by open normal subgroups:

$$G_i = \ker\left(G \to \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z})\right).$$

Set $\Gamma_i = \phi^{-1}(G_i)$, and let X_i be the corresponding finite cover of X. Let $\overline{X}$ be the cover of X corresponding to the kernel of ϕ and $\overline{\Gamma} = \Gamma/\mathrm{Ker}(\phi)$; note that $\overline{\Gamma}$ acts properly and freely on $\overline{X}$ with quotient $\overline{\Gamma}\backslash\overline{X} = X$. Our main concern is the growth of the Betti numbers

$$b_k(X_i) = \dim_{\mathbb{Q}} H_k(X_i, \mathbb{Q});$$
$$b_k(X_i, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H_k(X_i, \mathbb{F}_p),$$

with coefficients in $\mathbb{Q}$ and $\mathbb{F}_p$ as functions of i.

1.2. **Growth of Betti numbers in a p-adic analytic tower.** W. Lück proved that for each integer k the sequence $b_k(X_i)/[\Gamma : \Gamma_i]$ always converges as $i \to \infty$, and the limit equals the k-th *L^2-Betti number* $\beta_k(\overline{X}, \overline{\Gamma})$ of the action of $\overline{\Gamma}$ on $\overline{X}$. In that context we obtain the following result on the rate of convergence in terms of the dimension of G as a p-adic analytic group. We refer to [12, Theorem 8.36 on p. 201] for equivalent characterizations of the dimension of G.

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Theorem 1.1. *Let $d = \dim(G)$. Then, for any integer k and as i tends to infinity, we have:*

$$b_k(X_i) = \beta_k(\overline{X}, \overline{\Gamma})[\Gamma : \Gamma_i] + O\left([\Gamma : \Gamma_i]^{1-1/d}\right).$$

The novelty of Theorem 1.1 is obviously the error term. For more general covers, this has already been studied by Sarnak and Xue [28] and by Clair and Whyte [8] but they obtain much weaker results, in particular their results don't apply when 0 occurs in the L^2-spectrum of $\overline{X}$.[1]

Theorem 1.1 generalizes in the case of trivial coefficients the main theorem of Calegari-Emerton [6] which deals with arithmetic locally symmetric spaces. After this paper had been put on the ArXiv Frank Calegari informed us that Theorem 1.1 can be deduced from the method of [6]. In fact both proofs rest on a theorem of Harris, see Theorem 2.1 below, but we believe that our method of proof is somewhat simpler. We refer to Section 3 for more details on the relation to the work of Calegari-Emerton.

1.3. **Growth of $\mathbb{F}_p$-Betti numbers in a p-adic analytic tower.** Homological algebra over Iwasawa algebras and the theory of p-adic analytic groups provide important tools to study the asymptotic growth of Betti numbers in a p-adic analytic tower of covers. Whilst Iwasawa algebras are hidden in the proof of Theorem 1.1, they are essential even in the formulation of a corresponding result for $\mathbb{F}_p$-Betti numbers. The *Iwasawa algebra* of G over $R = \mathbb{F}_p$ or $\mathbb{Z}_p$ is the completion of the group algebra $R[G]$:

$$R[[G]] = \varprojlim R[G/G_i].$$

The Iwasawa algebra is a right and left Noetherian domain. Further, if G is torsion-free, then $R[[G]]$ does not contain zero divisors and its non-zero elements satisfy the Ore condition, see [16, §6]. This means that the ring of fractions $Q(R[[G]])$ is a skew field, the *Ore localization* of $R[[G]]$. Hence there is a notion of *rank*:

Definition 1.2. If G is torsion-free, we define the *rank* of a left $R[[G]]$-module M as

$$\operatorname{rank}_{R[[G]]}(M) = \dim_{Q(R[[G]])}\big(Q(R[[G]]) \otimes_{R[[G]]} M\big).$$

For general G we define the *rank* of M as

$$\operatorname{rank}_{R[[G]]}(M) = \frac{1}{[G : G_0]} \operatorname{rank}_{R[[G_0]]}(M),$$

where $G_0 < G$ is any uniform, hence torsion-free, subgroup, and M is regarded as an $R[[G_0]]$-module by restriction.

Using the above rank, we define an analog of L^2-Betti numbers in characteristic p. For a CW-complex Y the cellular chain complex will always be denoted by $C_*(Y)$. It is a consequence of the proof of Theorem 1.1 (see (2.10)) that, if you replace $\mathbb{F}_p$ by $\mathbb{Z}_p$ in the definition below, you obtain the L^2-Betti numbers of $\bar{X}$.

Definition 1.3. The *mod p L^2-Betti numbers* of the $\bar{\Gamma}$-space $\bar{X}$ are defined as

$$\beta_k(\overline{X}, \overline{\Gamma}; \mathbb{F}_p) = \operatorname{rank}_{\mathbb{F}_p[[G]]}(H_k(\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[\bar{\Gamma}]} C_*(\overline{X}, \mathbb{F}_p))),$$

where $\mathbb{F}_p[[G]]$ is regarded as a right $\mathbb{F}_p[[\bar{\Gamma}]]$-module via $\phi : \Gamma \to G$.

[1] We should note however that in the special case of lattices in $\mathrm{SU}(2,1)$ Sarnak and Xue produce a better exponent of $\frac{7}{12}$ in the error term for the first Betti number.

For these characteristic p analogs of L^2-Betti numbers there is an approximation result similar to Theorem 1.1:

Theorem 1.4. *Let $d = \dim(G)$. Then for any integer k and as i tends to infinity, we have:*

$$b_k(X_i; \mathbb{F}_p) = \beta_k(\overline{X}, \overline{\Gamma}; \mathbb{F}_p)[\Gamma : \Gamma_i] + O\left([\Gamma : \Gamma_i]^{1-1/d}\right).$$

In particular, the limit of the sequence $b_k(X_i; \mathbb{F}_p)/[\Gamma : \Gamma_i]$ exists and is equal to $\beta_k(\overline{X}, \overline{\Gamma}; \mathbb{F}_p)$.

Here again Calegari informed us that Theorem 1.4 can be deduced from his joint ongoing work with Emerton on *completed cohomology.* In fact one key feature of their theory is to set up the right framework to determine the growth rate of (mod p) Betti numbers even if the corresponding (mod p) L^2-Betti number vanishes. Proving unconditional results seems difficult; we nevertheless point out that when X is 3-dimensional the main result of [7] implies in particular that the error term in Theorem 1.4 is the best possible in general. We also note that – in his PhD-thesis [30, Theorem 5.3.1] – Liam Wall had first constructed examples of p-adic analytic towers of covers of a finite volume hyperbolic 3-manifold which shows that one cannot replace the error term $O([\Gamma : \Gamma_i]^{1-1/d})$ by $O([\Gamma : \Gamma_i]^{1-1/d-\epsilon})$ for some $\epsilon > 0$.

We may have $\beta_k(\overline{X}, \overline{\Gamma}) \neq \beta_k(\overline{X}, \overline{\Gamma}; \mathbb{F}_p)$. An example is given in [23, Example 6.2]. One can even construct an example with X being a *manifold* (see Section 5):

Proposition 1.5. *There exists a link complement X and a sequence of p-covers X_i of X such that*

$$\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq \lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}.$$

We don't know of any example with $\overline{X}$ being aspherical.

1.4. Beyond p-adic analytic groups.

The following theorem about arbitrary pro-p towers is certainly known to some experts but we could find no proof in the literature except in degree one.

Theorem 1.6. *Let k be a field of characteristic $p > 0$. Let X be a compact connected CW-complex with Γ as fundamental group. Let $(\Gamma_i)_{i \geq 0}$ be a residual p-chain. We denote the finite cover of X associated to Γ_i by X_i. Then, for any $n \geq 0$, the sequence of normalized Betti numbers with k-coefficients*

$$\left(\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]}\right)_{i \geq 0}$$

is monotone decreasing and converges as $i \to \infty$.

We moreover prove that the limit is an integer in many situations, see Theorem 4.3 and the remark following it.

1.5. Acknowledgments.

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2. Proof of Theorem 1.1 and 1.4

In the sequel we treat the cases $R = \mathbb{Z}_p$ and $R = \mathbb{F}_p$ simultaneously. Depending on which case, we denote by $\dim_R(M)$ either the dimension of a $\mathbb{F}_p$-vector space or the $\mathbb{Z}_p$-rank of a $\mathbb{Z}_p$-module, which equals the dimension of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$.

As in the work of Calegari-Emerton and Emerton [6, 7] the following result of M. Harris [18, Theorem 1.10] for which corrections appear in [19], is crucial. Although not explicitly stated as such, a proof is also contained in the work of Farkas-Linnell [16]. We give a complete proof blending ideas from both Farkas-Linnell's and Harris' papers.

Theorem 2.1 (Harris)**.** *Let $R = \mathbb{Z}_p$ or $R = \mathbb{F}_p$. Let M be a finitely generated $R[[G]]$-module. Then*

$$\dim_R\bigl(R\widehat{\otimes}_{R[[G_i]]}M\bigr) = \operatorname{rank}_{R[[G]]}(M)[G : G_i] + O([G : G_i]^{1-1/d}). \tag{2.1}$$

Here $\widehat{\otimes}$ denotes the completed tensor product.

Proof. Passing to a finite index subgroup of G we may, and shall, assume that G is uniform and torsion-free. The proof then proceeds through a sequence of reductions.

Reduction to the case of cokernels of elements in $R[[G]]$. We first show that it suffices to show the theorem for $R[[G]]$-modules of the form

$$M = \operatorname{coker}\bigl(R[[G]] \xrightarrow{-\cdot a} R[[G]]\bigr), \tag{2.2}$$

with $a \in R[[G]]$. Let N be an arbitrary finitely generated $R[[G]]$-module. Since $R[[G]]$ is Noetherian, N is finitely presented and we can find a matrix $A \in M(r \times s, R[[G]])$ such that

$$N = \operatorname{coker}\bigl(R[[G]]^r \xrightarrow{r_A} R[[G]]^s\bigr),$$

where r_A denotes the right multiplication

$$r_A(x_1, \ldots, x_r) = (x_1, \ldots, x_r) \cdot A$$

with A. Since the Ore localization $Q(R[[G]])$ is a skew field, by row and column reduction in $M_r(Q(R[[G]]))$ one can find invertible matrices $B \in M(r \times r, Q(R[[G]]))$ and $C \in M(s \times s, Q(R[[G]]))$ such that $D := BAC$ is a block matrix of the form

$$D = \begin{pmatrix} I_\delta & 0_{\delta, s-\delta} \\ 0_{r-\delta,\delta} & 0_{r-\delta \times s-\delta} \end{pmatrix} \in M(r \times s, R[[G]]), \tag{2.3}$$

where $I_\delta \in M(\delta \times \delta, R[[G]])$ is the identity matrix with $\delta \in \{0, \ldots, s\}$ and the other blocks are suitable zero matrices. In other words, D describes the projection onto the first δ coordinates $(x_1, x_2, \ldots, x_r) \mapsto (x_1, x_2, \ldots, x_\delta, 0, 0, \ldots)$. Since B, C are invertible, we have

$$\operatorname{rank}_{R[[G]]}(N) = s - \delta.$$

There are nonzero $b, c \in R[[G]]$ such that bB, Cc are matrices over $R[[G]]$. We have

$$(bB)A(Cc) = \begin{pmatrix} bc \cdot I_\delta & 0_{\delta,s-\delta} \\ 0_{r-\delta,\delta} & 0_{\delta\times\delta} \end{pmatrix}. \tag{2.4}$$

Let A_i, B_i, and C_i be the mod G_i reductions of A, bB, and Cc. Because of $\ker(r_{A_i}) \subset \ker(r_{C_i} \circ r_{A_i}) = \ker(r_{A_iC_i})$ one obtains $\dim_R \ker(r_{A_i}) \le \dim_R \ker(r_{A_iC_i})$. Because of $\operatorname{im}(r_{B_iA_iC_i}) = \operatorname{im}(r_{A_iC_i} \circ r_{B_i}) \subset \operatorname{im}(r_{A_iC_i})$ we have $\dim_R \ker(r_{A_iC_i}) \le \dim_R \ker(r_{B_iA_iC_i})$.

Therefore: $\dim_R \ker(r_{A_i}) \leq \dim_R \ker(r_{B_iA_iC_i})$. Assuming the theorem is proved for modules as in (2.2), this implies that

$$\begin{aligned}\dim_R(R\widehat{\otimes}_{R[[G_i]]}N) = \dim_R \operatorname{coker}(r_{A_i}) &= \dim_R \ker(r_{A_i}) - (r-s)[G:G_i] \\ &\leq \dim_R \ker(r_{B_iA_iC_i}) - (r-s)[G:G_i] \\ &= \dim_R \operatorname{coker}(r_{B_iA_iC_i}) \\ &= (s-\delta)\cdot[G:G_i] + O([G:G_i]^{1-1/d}).\end{aligned} \tag{2.5}$$

To prove the assertion for N, under the assumption that it holds for modules as in (2.2), it remains to show that

$$\dim_R \operatorname{coker}(r_{A_i}) \geq (s-\delta)[G:G_i] - O([G:G_i]^{1-1/d}). \tag{2.6}$$

Let $E \in M((r-\delta)\times r, R[[G]])$ be the matrix such that

$$R[[G]]^{r-\delta} \xrightarrow{r_E} R[[G]]^r,\ (y_1,y_2,\ldots,y_{r-\delta})\cdot E = (0,\ldots,0,y_1,y_2,\ldots,y_{r-\delta}).$$

Let $F \in M((r-\delta)\times r, R[[G]])$ be the matrix $E\cdot(bB)$. The same argument as before leading to (2.5) but now applied to F shows

$$\dim_R \operatorname{coker}(r_{F_i}) \leq \delta\cdot[G:G_i] + O([G:G_i]^{1-1/d}). \tag{2.7}$$

We have $0 = r_D \circ r_E = r_{ED}$, yielding $ED = 0$ and

$$FAC = E(bB)AC = E(bI_r)BAC = (bI_r)EBAC = (bI_{r-\delta})ED = 0.$$

Note that we used $E(bI_r) = (bI_{r-\delta})E$ here. As C is invertible this implies $FA = 0$ and $r_A \circ r_F = r_{FA} = 0$. In particular, $r_{A_i} \circ r_{F_i} = 0$ and hence $\operatorname{im}(r_{F_i}) \subseteq \ker(r_{A_i})$. So $\dim_R \operatorname{im}(r_{F_i}) \leq \dim_R \ker(r_{A_i})$. We compute

$$\begin{aligned}&\dim_R \operatorname{coker}(r_{A_i}) + \dim_R \operatorname{coker}(r_{F_i}) \\ =\;& s\cdot[G:G_i] - \dim_R \operatorname{im}(r_{A_i}) + r\cdot[G:G_i] - \dim_R \operatorname{im}(r_{F_i}) \\ \geq\;& s\cdot[G:G_i] - \dim_R \operatorname{im}(r_{A_i}) + r\cdot[G:G_i] - \dim_R \ker(r_{A_i}) \\ =\;& s\cdot[G:G_i].\end{aligned}$$

Now (2.6) follows from (2.7).

Reduction to the case $R = \mathbb{F}_p$. To prove the statement for a finitely generated $\mathbb{Z}_p[[G]]$-module M we may assume that $\operatorname{rank}_{\mathbb{Z}_p[[G]]}(M) = 0$ due to the reduction to the case (2.2) and the fact that $\mathbb{Z}_p[[G]]$ has no zero-divisors. For the following reason we may, in addition, assume that M has no p-torsion: Let $T = \{m \in M \,|\, \exists_{d(m)\in\mathbb{N}}\ p^{d(m)}\cdot m = 0\}$ be its p-torsion part. Obviously, T is a $\mathbb{Z}_p[[G]]$-submodule of M. One easily sees by additivity of dimension that

$$\dim_{\mathbb{Q}_p}(\mathbb{Q}_p\widehat{\otimes}_{\mathbb{Z}_p[[G_i]]}M/T) = \dim_{\mathbb{Q}_p}(\mathbb{Q}_p\widehat{\otimes}_{\mathbb{Z}_p[[G_i]]}M) \text{ and } \operatorname{rank}_{\mathbb{Z}_p[[G]]}(M) = \operatorname{rank}_{\mathbb{Z}_p[[G]]}(M/T).$$

Hence we may assume that M has no p-torsion. We prove now that

$$\operatorname{rank}_{\mathbb{F}_p[[G]]}(M/pM) = \operatorname{rank}_{\mathbb{Z}_p[[G]]}(M), \tag{2.8}$$

hence both are zero. Since the ring $\mathbb{Z}_p[[G]]$ has finite projective dimension [3, Section 5.1] and every projective $\mathbb{Z}_p[[G]]$-module is free [32, Corollary 7.5.4 on p. 127] and $\mathbb{Z}_p[[G]]$ is

Noetherian, the finitely generated $\mathbb{Z}_p[[G]]$-module M possesses a finite resolution by finitely generated free $\mathbb{Z}_p[[G]]$-modules:

$$0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0.$$

Applying the functor $N \mapsto \mathbb{F}_p \otimes_{\mathbb{Z}} N \cong N/pN$ yields a resolution of M/pM by finitely generated, free $\mathbb{F}_p[[G]]$-modules since M has no p-torsion:

$$0 \to \mathbb{F}_p \otimes_{\mathbb{Z}} F_n \to \ldots \to \mathbb{F}_p \otimes_{\mathbb{Z}} F_0 \to M/pM \to 0.$$

Now equation (2.8) follows since the rank functions over $\mathbb{F}_p[[G]]$ and $\mathbb{Z}_p[[G]]$ are additive and the equation obviously holds for finitely generated free $\mathbb{Z}_p[[G]]$-modules.

Let $N = \mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}_p[[G_i]]} M$. Because of $\operatorname{rank}_{\mathbb{Z}_p[[G]]}(M) = 0$ it is enough to prove $\dim_{\mathbb{Z}_p}(N) = O([G : G_i]^{1-1/d})$. This follows from the $\mathbb{F}_p[[G]]$-case, $\operatorname{rank}_{\mathbb{F}_p[[G]]}(M/pM) = 0$, and the inequality

$$\dim_{\mathbb{Z}_p}(N) \leq \dim_{\mathbb{F}_p}(N/pN) = \dim_{\mathbb{F}_p}\big(\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[G_i]]} M/pM\big).$$

So we reduced the proof of the theorem to the case $R = \mathbb{F}_p$ and henceforth assume $R = \mathbb{F}_p$.

Reduction to G being standard. We finally reduce the assertion to the case that G is standard in the sense of [12, §8.4]. Being a p-adic analytic group, G has an open subgroup H which is standard with respect to the manifold structure induced from G, see [12, Theorem 8.29]. Since H is open, we have $G_i < H$ for i greater than some i_0. Being standard H has a preferred collection of open normal subgroups H_i which satisfy: $G_{i_0+i-1} \subset H_i \subset G_i$ $(i \geq 1)$; see e.g., [12, Ex. 6 p. 168].

Recall that we may assume that $\operatorname{rank}_{\mathbb{F}_p[[G]]}(M) = 0$ due to the reduction to the case (2.2) and the fact that $\mathbb{F}_p[[G]]$ has no zero-divisors.

Now if the assertion holds for H with respect to the H_i's, then it follows that for $i > i_0$ the left hand side of (2.1) is bounded by a constant times $\dim_R\big(R \widehat{\otimes}_{R[[H_{i-i_0+1}]]} M\big)$ and is therefore $O([H : H_i]^{1-1/d}) = O([G : G_i]^{1-1/d})$, so the assertion holds for G as well. We assume from now on that G is standard and that $G_i = \psi^{-1}(p^i \mathbb{Z}_p^d)$ where ψ is the global atlas of G.

The remaining argument. Let M be as in (2.2). We may assume $a \neq 0$. By a fundamental result of Lazard, the graded ring $\operatorname{gr} \mathbb{F}_p[[G]]$ with respect to the filtration $(\Delta^n)_{n \geq 0}$ by powers of the augmentation ideal $\Delta \subset \mathbb{F}_p[[G]]$ is a polynomial algebra $\mathbb{F}_p[X_1, \ldots, X_d]$ with indeterminates $X_i = x_i - 1 + \Delta^2$ [32, Theorem 8.7.10 on p. 160], where $\{x_1, \ldots, x_d\} \subset G$ is a minimal generating set. Let $I_i \subset \mathbb{F}_p[[G]]$ be the closure of the ideal generated by elements $\lambda(h-1)$ with $h \in G_i$. Note that $N/I_iN \cong \mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[G_i]]} N$ for any $\mathbb{F}_p[[G]]$-module N. Since $\mathbb{F}_p[[G]]$ is a domain, $\operatorname{rank}_{\mathbb{F}_p[[G]]}(M) = 0$. Now for each integer $i \geq 1$ (if $p > 2$) or $i \geq 2$ (if $p = 2$), the global atlas ψ of G induces an epimorphism $G_i \to p^i \mathbb{Z}_p^d / p^{i+1} \mathbb{Z}_p^d$ with kernel G_{i+1}. It therefore follows that $[G : G_i] = Cp^{id}$ for some rational constant $C > 0$ and we have to show that

$$\dim_{\mathbb{F}_p}(M/I_iM) = O(p^{(d-1)i}). \tag{2.9}$$

But it follows from [12, Lemma 7.1] that there exists a positive integer m such that $\Delta^{mp^i} \subset I_i$ for all i. It therefore suffices to show (2.9) with I_i replaced by Δ^{mp^i}. Let $s \geq 0$ be such that $a \in \Delta^s \backslash \Delta^{s+1}$. Let $a_i : \mathbb{F}_p[[G]]/\Delta^{mp^i} \to \mathbb{F}_p[[G]]/\Delta^{mp^i}$ be the map induced by right multiplication with a. We have

$$\begin{aligned}\dim_{\mathbb{F}_p}(M/\Delta^{mp^i}M) &= \dim_{\mathbb{F}_p}\operatorname{coker}(a_i)\\ &= \dim_{\mathbb{F}_p}\ker(a_i)\\ &= \dim_{\mathbb{F}_p}(\Delta^{mp^i-s}/\Delta^{mp^i}).\end{aligned}$$

The last equality follows from the fact the graded ring is a polynomial ring. For the same reason the last number equals the number of monomials in a polynomial ring with d variables each of which has total degree in the interval $[mp^i - s, mp^i)$. The number of monomials of degree $< k$ is $\binom{d+k-1}{d}$. Hence

$$\dim_{\mathbb{F}_p}(M/\Delta^{mp^i}M) = \binom{d+mp^i-1}{d} - \binom{d+mp^i-s-1}{d}.$$

As a polynomial in p^i, each binomial coefficient has leading term $(mp^i)^d$. Their difference is a polynomial in p^i with degree at most $d-1$. This implies (2.9). □

Proofs of Theorems 1.1 and 1.4. We show for both cases $R = \mathbb{Z}_p$ and $R = \mathbb{F}_p$ simultaneously that

$$b_k(X_i; R) = \operatorname{rank}_{R[[G]]}\big(H_k(R[[G]] \otimes_{R\bar{\Gamma}} C_*(\overline{X}))\big) \cdot [\Gamma : \Gamma_i] + O\left([\Gamma : \Gamma_i]^{1-1/d}\right). \tag{2.10}$$

The CW-structure on X lifts to a $\bar{\Gamma}$-equivariant CW-structure on $\bar{X}$ and to $\Gamma_i\backslash\Gamma$-equivariant CW-structures on X_i. We may also view $\bar{X}$ as a Γ-space via the quotient map $\Gamma \to \bar{\Gamma}$. Let $C_*(\bar{X})$ be the cellular chain complex of $\bar{X}$. Each chain module $C_k(\bar{X})$ is a finitely generated free $\mathbb{Z}[\bar{\Gamma}]$-module. The differentials in the chain complex $C_* = R \otimes_{\mathbb{Z}} C_*(\bar{X})$ are denoted by ∂_*. Note that $R \otimes_{R[\Gamma_i]} C_*$ is isomorphic to the cellular chain complex $R \otimes_{\mathbb{Z}} C_*(X_i)$ as an $R[\Gamma_i\backslash\Gamma]$-chain complex. In particular, we have

$$H_*(X_i, R) \cong H_*(R \otimes_{R[\Gamma_i]} C_*). \tag{2.11}$$

We write $\hat{C}_*$ and $\hat{\partial}_*$ short for $R[[G]] \otimes_{R\bar{\Gamma}} C_*$ and its differentials. We denote the cycles and boundaries in the chain complexes $\hat{C}_*$ and C_* by $\hat{Z}_*$, $\hat{B}_*$ and Z_*, B_*, respectively. Let $r_n \in \mathbb{N}_0$ be the rank of the finitely generated free $R[[G]]$-module $\hat{C}_n$. In each degree n we have the obvious exact sequence

$$0 \to \hat{Z}_n \to \hat{C}_n \xrightarrow{\hat{\partial}_n} \hat{C}_{n-1} \to \operatorname{coker}(\hat{\partial}_n) \to 0.$$

By additivity of $\operatorname{rank}_{R[[G]]}$ we obtain that

$$\begin{aligned}\operatorname{rank}_{R[[G]]}\big(H_k(\hat{C}_*)\big) &= \operatorname{rank}_{R[[G]]}\big(\hat{Z}_k/\hat{B}_k\big)\\ &= \operatorname{rank}_{R[[G]]}(\hat{Z}_k) - \operatorname{rank}_{R[[G]]}(\hat{B}_k)\\ &= r_k - r_{k-1} + \operatorname{rank}_{R[[G]]}\big(\operatorname{coker}(\hat{\partial}_k)\big) - \operatorname{rank}_{R[[G]]}(\hat{B}_k)\\ &= r_k - r_{k-1} + \operatorname{rank}_{R[[G]]}\big(\operatorname{coker}(\hat{\partial}_k)\big) - \big(r_k - \operatorname{rank}_{R[[G]]}\big(\operatorname{coker}(\hat{\partial}_{k+1})\big)\big)\\ &= \operatorname{rank}_{R[[G]]}\big(\operatorname{coker}(\hat{\partial}_k)\big) + \operatorname{rank}_{R[[G]]}\big(\operatorname{coker}(\hat{\partial}_{k+1})\big) - r_{k-1}.\end{aligned} \tag{2.12}$$

By (2.11), a similar argument as above, and right-exactness of the tensor product, we obtain that

$$\begin{aligned}\dim_R\bigl(H_k(X_i,R)\bigr) &= \dim_R\bigl(H_k(R\otimes_{R[\Gamma_i]} C_*)\bigr)\\ &= \dim_R\bigl(\operatorname{coker}(R\otimes_{R[\Gamma_i]}\partial_k)\bigr)+\dim_R\bigl(\operatorname{coker}(R\otimes_{R[\Gamma_i]}\partial_{k+1})\bigr)-[\Gamma:\Gamma_i]\cdot r_{k-1}\\ &= \dim_R\bigl(R\otimes_{R[\Gamma_i]}\operatorname{coker}(\partial_k)\bigr)+\dim_R\bigl(R\otimes_{R[\Gamma_i]}\operatorname{coker}(\partial_{k+1})\bigr)-[\Gamma:\Gamma_i]\cdot r_{k-1}.\end{aligned}$$

Hence,

$$\frac{b_k(X_i,R)}{[\Gamma:\Gamma_i]}=\frac{\dim_R\bigl(R\otimes_{R[\Gamma_i]}\operatorname{coker}(\partial_k)\bigr)}{[\Gamma:\Gamma_i]}+\frac{\dim_R\bigl(R\otimes_{R[\Gamma_i]}\operatorname{coker}(\partial_{k+1})\bigr)}{[\Gamma:\Gamma_i]}-r_{k-1}. \tag{2.13}$$

The natural map

$$R\otimes_{R[\Gamma_i]} R[\Gamma]\xrightarrow{\cong} R\widehat{\otimes}_{R[[G_i]]}R[[G]]$$

induced by $\phi:\Gamma\to G$ is a right $R[\Gamma]$-module isomorphism (recall that we regard $R[[G]]$ as a right $R[\Gamma]$-module via ϕ). The inverse is obtained as follows: Since $\Gamma_i\backslash\Gamma\cong G_i\backslash G$, there is a natural continuous homomorphism from G to the invertible elements of the R-algebra $R\otimes_{R[\Gamma_i]}R[\Gamma]$. By the universal property of the completed group algebra there is a continuous homomorphism $R[[G]]\to R\otimes_{R[\Gamma_i]}R[\Gamma]$ which descends to the desired inverse. As a consequence we get isomorphisms

$$\begin{aligned}R\widehat{\otimes}_{R[[G_i]]}\operatorname{coker}(\hat\partial_k) &\cong R\otimes_{R[[G_i]]}R[[G]]\otimes_{R[\bar\Gamma]}\operatorname{coker}(\partial_k)\\ &\cong R\otimes_{R[[G_i]]}R[[G]]\otimes_{R[\Gamma]}\operatorname{coker}(\partial_k)\\ &\cong R\otimes_{R[\Gamma_i]}\operatorname{coker}(\partial_k).\end{aligned}$$

and, thus,

$$\frac{b_k(X_i,R)}{[G:G_i]}=\frac{\dim_R\bigl(R\widehat{\otimes}_{R[[G_i]]}\operatorname{coker}(\hat\partial_k)\bigr)}{[G:G_i]}+\frac{\dim_R\bigl(R\widehat{\otimes}_{R[[G_i]]}\operatorname{coker}(\hat\partial_{k+1})\bigr)}{[G:G_i]}-r_{k-1}. \tag{2.14}$$

Now (2.10) follows from (2.12), (2.14), and Theorem 2.1. Note that (2.10) is exactly the statement of Theorem 1.4 in the case $R=\mathbb{F}_p$. Next we explain how Theorem 1.1 follows from (2.10) when $R=\mathbb{Z}_p$. Since $\mathbb{Q}_p$ has characteristic zero, we have $b_k(X_i)=b_k(X_i,\mathbb{Q}_p)=b_k(X_i,\mathbb{Z}_p)$. Since $b_k(X_i)/[\Gamma:\Gamma_i]\to\beta_k(\overline{X},\overline{\Gamma})$ as $i\to\infty$ [24], we conclude

$$\beta_k(\overline{X},\overline{\Gamma})=\operatorname{rank}_{\mathbb{Z}_p[[G]]}\bigl(H_k(\mathbb{Z}_p[[G]]\otimes_{\mathbb{Z}_p\bar\Gamma}C_*)\bigr). \tag{2.15}$$ □

3. Relation with the completed homology

Calegari and Emerton [5, 7] have introduced the completed homology groups:

$$\widetilde{H}_k=\varprojlim H_k(X_i,\mathbb{Z}_p)\quad\text{and}\quad \widetilde{H}_k(\mathbb{F}_p)=\varprojlim H_k(X_i,\mathbb{F}_p).$$

These modules carry continuous actions of G and may therefore be considered as $\mathbb{Z}_p[[G]]$-modules or $\mathbb{F}_p[[G]]$-modules, respectively. In this section we want to clarify the relation of completed cohomology to (mod p) L^2-Betti numbers.

Proposition 3.1. *Retaining the setup in section 1.1 we have:*

$$\beta_k(\overline{X},\overline{\Gamma};\mathbb{F}_p)=\operatorname{rank}_{\mathbb{F}_p[[G]]}(\widetilde{H}_k(\mathbb{F}_p)).$$

Proof. Here again we may reduce to the case where G is torsion-free. Write $C_* = C_*(\bar{X}; \mathbb{F}_p)$. The claim is equivalent to

$$\widetilde{H}_k(\mathbb{F}_p) = \varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)$$

and

$$H_k(\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p\bar{\Gamma}} C_*) = H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*))$$

having the same $\mathbb{F}_p[[G]]$-rank. So the statement is equivalent to:

$$Q(\mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[\Gamma]]} H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)) \cong Q(\mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[\Gamma]]} \left(\varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)\right).$$

Since $\mathbb{F}_p \otimes_{\mathbb{Z}_p[\Gamma_i]} C_*$ is a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition, by [31, Theorem 3.5.8] there is a short exact sequence

$$0 \to \varprojlim{}^1 H_{k+1}(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) \to H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)) \to \varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) \to 0.$$

Moreover, since towers of finite dimensional vector spaces over a *field* satisfy the Mittag-Leffler condition, we conclude that

$$\varprojlim{}^1 H_{k+1}(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) = 0,$$

which yields the proposition. □

It follows from works of Calegari and Emerton that a similar result with $\mathbb{F}_p$ replaced by $\mathbb{Z}_p$ holds as well, that is,

$$\beta_k(\overline{X}, \overline{\Gamma}; \mathbb{Z}_p) = \mathrm{rank}_{\mathbb{Z}_p[[G]]}(\widetilde{H}_k), \tag{3.1}$$

and we want to indicate why. In [6, Theorem 3.2] it is shown for arithmetic congruence covers X_i of symmetric spaces that the so-called co-rank r_k of the completed cohomology $\widetilde{H}^k$ satisfies the equality in Theorem 1.1 with $\beta_k(\bar{X}, \bar{\Gamma})$ replaced by r_k. The proof in [6] is a consequence of a general result of Emerton [14, Theorem 2.1.5] and their Lemma 2.2. Both these results hold not only for arithmetic congruence covers but in our generality. Since the co-rank of $\widetilde{H}^k$ is the same as the rank of $\widetilde{H}_k$ [5, Theorem 1.1 (3)], this implies (3.1).

It is somewhat harder to work with completed homology, see [6, 7]. We nevertheless want to emphasize that the latter contains a lot more information. It should be the right framework to determine the growth rate of (mod p) Betti numbers even if the corresponding (mod p) L^2-Betti number vanishes. However it seems hard to extract the necessary information from completed homology, the only exception that we are aware of is in case X is 3-dimensional, see [7].

4. Approximation results for pro-p towers that are not p-adic analytic

The proof of Theorem 1.6 relies on the following well-known lemma (see also [15] for a proof).

Lemma 4.1. *Let k be a field of characteristic $p > 0$. Let Λ be a normal subgroup in a group Γ whose index is a p-power. Then*

$$\dim_k(k \otimes_{k[\Lambda]} M) \leq [\Gamma : \Lambda] \cdot \dim_k(k \otimes_{k[\Gamma]} M).$$

Proof. Because of the isomorphism

$$k \otimes_{k[\Gamma]} M \cong k \otimes_{k[\Gamma/\Lambda]} (k \otimes_{k[\Lambda]} M)$$

it suffices to prove the case where Λ is trivial and Γ is a finite p-group. Let

$$k[\Gamma]^m \xrightarrow{f} k[\Gamma]^n \to M \to 0$$

be a presentation of M. Then $\bar{f} = k \otimes_{k[\Gamma]} f$ is a presentation of $k \otimes_{k[\Gamma]} M$. Since $\dim_k(M) = |\Gamma| \cdot n - \dim_k(\mathrm{im}(f))$ and $\dim_k(k \otimes_{k[\Gamma]} M) = n - \dim_k(\mathrm{im}(\bar{f}))$, we have to show that

$$\dim_k(\mathrm{im}(f)) \geq |\Gamma| \cdot \dim_k(\mathrm{im}(\bar{f})).$$

Extend a k-basis $\{u_1, \dots, u_s\}$ of $\mathrm{im}(\bar{f})$ to a k-basis $\{u_1, \dots, u_n\}$ of $k^n = k \otimes_{k[\Gamma]} k[\Gamma]^n$. Let $x_1, \dots, x_n$ be lifts of the u_i to $k[\Gamma]^n$ such that $\{x_1, \dots, x_s\} \subset \mathrm{im}(f)$. Since $k[\Gamma]$ is a local ring with the augmentation ideal as the unique maximal ideal [32, Proposition 7.5.3], Nakayama's lemma implies that $\{x_1, \dots, x_n\}$ generates $k[\Gamma]^n$ as a $k[\Gamma]$-module. Since the k-dimension of the $k[\Gamma]$-submodule generated by x_i is at most $|\Gamma|$ and $\dim_k k[\Gamma]^n = |\Gamma| \cdot n$, the k-dimension of the $k[\Gamma]$-module generated by $\{x_1, \dots, x_i\}$ is $i|\Gamma|$. Because of $\{x_1, \dots, x_s\} \subset \mathrm{im}(f)$,

$$\dim_k(\mathrm{im}(f)) \geq |\Gamma| \cdot s = |\Gamma| \cdot \dim_k(\mathrm{im}(\bar{f}))$$

follows. □

Proof of Theorem 1.6. It follows from Lemma 4.1 that, for any finitely presented $k[\Gamma]$-module M, the sequence $(\dim_k(k \otimes_{k[\Gamma_i]} M)/[\Gamma : \Gamma_i])_{i \geq 0}$ is monotone decreasing. Let r_n be the number of n-cells in X. Let ∂_* denote the differentials in the $k[\Gamma]$-complex $C_*(\widetilde{X}; k)$. Exactly as in (2.13), one has

$$\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]} = \frac{\dim_k(k \otimes_{k[\Gamma_i]} \mathrm{coker}(\partial_n))}{[\Gamma : \Gamma_i]} + \frac{\dim_k(k \otimes_{k[\Gamma_i]} \mathrm{coker}(\partial_{n+1}))}{[\Gamma : \Gamma_i]} - r_{n-1},$$

from which monotonicity follows. Since Betti numbers are non-negative, the sequence converges. □

In the remainder of this section we study the question how we can express the limit

$$\Big(\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]}\Big)_{i \geq 0}$$

by some algebraic expression in very specific situations. For that we recall the notions of ordered group, the Malcev-Neumann power series ring, and the division closure.

An *ordered group* is a group with a strict total ordering of its elements which is invariant under left and right translations. A group which has such an ordering is called *orderable*. For example, residually torsion-free nilpotent groups are orderable [10, Proposition 1.2 on p. 274].

If Γ is an ordered group, then the set of formal power series $\sum_{\gamma \in \Gamma} a_\gamma \gamma$ with coefficients a_γ in a skew field k whose support $\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is well-ordered becomes a skew field with the obvious ring structure extending the one of the group ring $k[\Gamma]$ [9, Corollary 15.10 on p. 95] which is called the *Malcev-Neumann power series ring*. We denote it by $k((\Gamma))$[2].

Suppose R is a subring of a skew field K. Then $D(R, K)$ will denote the *division closure* of R in K, that is the smallest skew subfield of K that contains R. If M_1 and M_2 are the Malcev-Neumann power series rings of $k[\Gamma]$ with respect to two different orders and D_1, D_2 the

[2] We suppress the order in the notation for reasons to be seen below.

division closures of $k[\Gamma]$ in M_1, M_2, respectively, then there is a ring isomorphism $D_1 \cong D_2$ which is the identity on $k[\Gamma]$. This follows from the next theorem. As a consequence, the dimension of the $k((\Gamma))$-vector space

$$k((\Gamma)) \otimes_{k[\Gamma]} M \cong k((\Gamma)) \otimes_{D(k[\Gamma],k((\Gamma)))} \big(D(k[\Gamma],k((\Gamma))) \otimes_{k[\Gamma]} M\big)$$

for a $k[\Gamma]$-module M does not depend on the choice of the order on Γ.

Theorem 4.2. *Let k be a skew field, let Γ be an orderable group, and let M_1, M_2 be Malcev-Neumann power series rings for $k[\Gamma]$. Set $D_1 = D(k[\Gamma], M_1)$ and $D_2 = D(k[\Gamma], M_2)$. Then there is a ring isomorphism $\theta\colon D_1 \to D_2$ such that θ is the identity on $k[\Gamma]$.*

Proof. Recall [26, Corollary 1.4] that, being orderable, the group Γ is locally indicable, i.e., each finitely generated subgroup H surjects onto $\mathbb{Z}$. Let $J \lhd H$ be the kernel of this surjection and pick $t \in H$ so that $H = \langle J, t\rangle$. Let N_1 and N_2 denote the subrings of M_1 and M_2 respectively consisting of power series with supports in J. Then $D(kJ, D_1) \subseteq N_1$ and $D(kJ, D_2) \subseteq N_2$. Clearly $\{t^n \mid n \in \mathbb{N}\}$ is linearly independent over N_i in M_i for $i = 1, 2$. Theorem 4.2 therefore follows from a slight generalization of Hughes' theorem [11, 7.1 Theorem]; all that one needs to do is to modify the proof of [11, 7.2 Lemma]; this is done explicitly in [27, Hughes' Theorem I 6.3]. □

Theorem 4.3. *Let k be a field of characteristic $p > 0$. Let Γ be a finitely generated group, and let $\Gamma = \Gamma_0 > \Gamma_1 > \cdots$ be a descending sequence of normal subgroups such that $\bigcap_i \Gamma_i = \{1\}$ and each Γ/Γ_i is torsion-free nilpotent. Set $H_i = \Gamma_i\Gamma^{p^i}$ (thus Γ/H_i is a finite p-group). If X is a compact connected CW-complex with fundamental group Γ and X_i the finite cover corresponding to H_i, then[3], for every n,*

$$\dim_{k((\Gamma))}\big(H_n(k((\Gamma)) \otimes_{k[\Gamma]} C_*(\widetilde{X}, k))\big) = \lim_{i\to\infty} \frac{b_n(X_i; k)}{[\Gamma : H_i]}. \tag{4.1}$$

In particular, the limit on the right hand side is an integer.

Remark 4.4. A large collection of groups are known to be residually torsion-free nilpotent (RTFN): Carl Droms has proved in his PhD thesis [13, Theorem 1.1 in Chapter III on page 58] (see also [1]) that graph groups are RTFN. But this property is inherited by subgroups. It therefore follows from [17] and [4] that free groups, surface groups, reflection groups, right-angled Artin groups and arithmetic hyperbolic groups defined by quadratic forms are virtually RTFN. It follows from the recent breakthrough of Agol [2] and Wise [33] that the fundamental group of a closed hyperbolic 3-manifold is virtually RTFN.We finally note that the proof of [1, Corollary 2.3] implies that direct and free products of RTFN groups are RTFN.

One deduces the preceding theorem from the following Proposition 4.5 by a similar, even easier, argument as used in the deduction of Theorem 1.1 from Theorem 2.1. More precisely: Similarly as in (2.12) and (2.13) one expresses the left and right hand side of (4.1) by the cokernels of the n-th and $(n+1)$-th differential of the complexes $k((\Gamma)) \otimes_{k[\Gamma]} C_*(\widetilde{X}, k)$ and $C_*(X_i, k)$, respectively; then one applies the proposition below.

Proposition 4.5. *Let k be a field of characteristic $p > 0$. Let Γ be a finitely generated group, and let $\Gamma = \Gamma_0 > \Gamma_1 > \cdots$ be a descending sequence of normal subgroups such that*

[3]Recall that the left hand side of (4.1) does not depend on the order we choose to define $k((\Gamma))$.

$\bigcap_i \Gamma_i = \{1\}$ and each Γ/Γ_i is torsion-free nilpotent. Set $H_i = \Gamma_i \Gamma^{p^i}$. If M is a finitely presented $k[\Gamma]$-module, then

$$\dim_{k((\Gamma))}(k((\Gamma)) \otimes_{k[\Gamma]} M) = \lim_{i\to\infty} \frac{\dim_k(k \otimes_{k[H_i]} M)}{[\Gamma : H_i]}.$$

In particular, the limit on the right hand side is an integer.

A free group is residually torsion-free nilpotent. But even for a free group we cannot say anything for arbitrary residual p-chains. In particular, the following question remains open.

Question 4.6. *Let F be a finitely generated free group, let k be a field of characteristic $p > 0$, and let $F = F_0 > F_1 > \cdots$ be a descending sequence of normal subgroups with F/F_i a finite p-group for all i and $\bigcap_{i\in\mathbb{N}} F_i = 1$. Let M be a finitely presented kG-module. Can $\lim_{i\to\infty} |F/F_i|^{-1} \dim_k(k \otimes_{k[F_i]} M)$ be transcendental?*

In the remainder of this section we are concerned with the proof of Proposition 4.5 for which we need the following lemma.

Lemma 4.7. *Let k be an arbitrary skew field, let M be a finitely presented $k[\Gamma]$-module, and let $\Gamma = \Gamma_0 > \Gamma_1 > \cdots$ be a descending sequence of normal subgroups with Γ/Γ_i torsion-free nilpotent for all i with $\bigcap_{i\in\mathbb{N}} \Gamma_i = 1$. Let D_i denote the skew field of fractions of $k[\Gamma/\Gamma_i]$, which is an Ore localization. Then there exists $m \in \mathbb{N}$ such that*

$$\dim_{D_i}(D_i \otimes_{k[\Gamma]} M) = \dim_{k((\Gamma))}(k((\Gamma)) \otimes_{k[\Gamma]} M) \text{ for all } i \geq m.$$

Proof. Choose a nonprincipal ultrafilter ω on $\mathbb{N}$ and let D denote the ultraproduct of the D_i with respect to the ultrafilter ω. Then D is a skew field and $k[\Gamma]$ embeds in D. Consider a nontrivial finitely generated subgroup H of Γ. Let n be the least positive integer such that H is not contained in Γ_n. Then $H/H \cap \Gamma_n$ is a nontrivial finitely generated torsion-free nilpotent group, so there exists $N \lhd H$ such that H/N is infinite cyclic. Choose $t \in H \setminus N$ so that $H = \langle N, t\rangle$. Then $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $k((N))$ and it follows that $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $D(k[N], k((\Gamma)))$. Next let E_i denote the skew field of fractions of $k[N/N \cap \Gamma_i]$ and form the ultraproduct E of the E_i with respect to ω. Then E is a skew field contained in D, and $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over E. Therefore $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $D(k[N], D)$ and we deduce from [27, Hughes' Theorem I 6.3] that there is an isomorphism $\theta \colon D(k[\Gamma], k((\Gamma))) \to D(k[\Gamma], D)$ such that θ is the identity on $k[\Gamma]$. Therefore $\dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M = \dim_D D \otimes_{k[\Gamma]} M$. Also $\dim_D D \otimes_{k[\Gamma]} M = \lim_\omega \dim_{D_i} D_i \otimes_{k[\Gamma]} M$ and the result follows. □

Proof of Proposition 4.5. For each $i \in \mathbb{N}$, let D_i denote the skew field of fractions of $k[\Gamma/\Gamma_i]$. By Lemma 4.7, there exists $m_0 \in \mathbb{N}$ such that $\dim_{D_m} D_m \otimes_{k[\Gamma]} M = \dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M$ for all $m \geq m_0$. For $i, m \geq m_0$, set $K_i^m = \Gamma_m \Gamma^{p^i}$. For any $m \geq m_0$ we have

$$\lim_{i\to\infty} |\Gamma/K_i^m|^{-1} \dim_k(k \otimes_{k[K_i^m]} M) = \dim_{D_m}(D_m \otimes_{k[\Gamma]} M) = \dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M$$

by [23, Theorem 6.3]. The result follows from the monotonicity lemma 4.1. □

5. Link complements

Suppose that $\Gamma \twoheadrightarrow \mathbb{Z}^d$. We embed $\mathbb{Z}^d \hookrightarrow \mathbb{Z}_p^d =: G$ and consider the homomorphism $\phi : \Gamma \to G$. The $\mathbb{Z}[\mathbb{Z}^d]$-module $H_1(\overline{X}, \mathbb{Z})$ is the Alexander invariant of X. There is a natural map $\mathbb{Z}[\mathbb{Z}^d] \to \mathbb{F}_p[[\mathbb{Z}_p^d]]$. Since $\mathbb{Z}^d$ is amenable, the group ring $\mathbb{F}_p[\mathbb{Z}^d]$ is an Ore domain and it follows from [23] that

$$\begin{aligned}\beta_k(\overline{X}, \overline{\Gamma}; \mathbb{F}_p) &= \operatorname{rank}_{\mathbb{F}_p[[\mathbb{Z}_p^d]]} H_k(\overline{X}, \mathbb{F}_p[[\mathbb{Z}_p^d]]) \\ &= \operatorname{rank}_{\mathbb{F}_p[\mathbb{Z}^d]} H_k(\overline{X}, \mathbb{F}_p[\mathbb{Z}^d]) \\ &= \operatorname{rank}_{\mathbb{F}_p[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]} H_k(\overline{X}, \mathbb{F}_p[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]).\end{aligned}$$

This last expression is zero if and only if the (first) Alexander polynomial Δ of X is non zero modulo p.

The above remark in particular applies when X is a 3-manifold with boundary a union of d tori. If X is knot complement (in which case $d = 1$) the Alexander polynomial Δ is nonzero and its coefficients are relatively prime. Hence

$$\beta_1(\overline{X}, \overline{\Gamma}; \mathbb{F}_p) = \operatorname{rank}_{\mathbb{F}_p[[\mathbb{Z}_p^d]]} H_1(\overline{X}, \mathbb{F}_p[[\mathbb{Z}_p^d]]) = 0 \tag{5.1}$$

and (see also [29, Corollary 4.4])

$$\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = 0.$$

In that case one may even deduce from (5.1) that $H_1(X_i, \mathbb{F}_p) = \mathbb{F}_p$ for all i, see [7, Lemma 5.4].

We may as well consider the case of a link complement $l = l_1 \cup \ldots \cup l_d$ in $\mathbb{S}^3$. Recall that, in the case $d = 2$, $\Delta(1)$ is equal to the linking number $\operatorname{Lk}(l_1, l_2)$ of the two components of the link. The same proof as in the knot case then shows that if p is a prime that does not divide $\operatorname{Lk}(l_1, l_2)$, then

$$\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = 0.$$

One may wonder if the proof may be extended to show that $H_1(X_i, \mathbb{F}_p) = \mathbb{F}_p^2$ for all i as is true according to [29, Theorem 5.11].

We conclude this note by the proof of Proposition 1.5.

Proof of Proposition 1.5. It follows from [20] that there exists a link l with 2 components such that $\Delta(t, t) = p$. Now note that $\Delta(t, t)$ is the Alexander polynomial associated to the abelian cover of X corresponding to the map $\Gamma \to \mathbb{Z}^2 \to \mathbb{Z}^2/\langle a - b \rangle \cong \mathbb{Z}$. Since $\Delta(t, t)$ is non zero,

$$\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]} = 0.$$

But since $\Delta(t, t)$ is zero modulo p, we have:

$$\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq 0. \qquad \square$$

Other examples of closed finite CW-complexes with a chain $\pi_1(X) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$ of in $\pi_1(X)$ normal subgroups of finite index such that $\lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq \lim_{i \to +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}$ holds can be found in [15] and [25]. One can additionally arrange $X = B\Gamma$ or $\bigcap_{i \geq 0} \Gamma_i = \{1\}$. However, the problem is still open to find an example with both $X = B\Gamma$ and $\bigcap_{i \geq 0} \Gamma_i = \{1\}$.

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