TOPOLOGICAL K-(CO-)HOMOLOGY OF CLASSIFYING SPACES OF DISCRETE GROUPS

MICHAEL JOACHIM AND WOLFGANG LÜCK

Abstract. Let G be a discrete group. We give methods to compute for a generalized (co-)homology theory its values on the Borel construction $EG \times_G X$ of a proper G-CW-complex X satisfying certain finiteness conditions. In particular we give formulas computing the topological K-(co)homology $K_*(BG)$ and $K^*(BG)$ up to finite abelian torsion groups. They apply for instance to arithmetic groups, word hyperbolic groups, mapping class groups and discrete cocompact subgroups of almost connected Lie groups. For finite groups G these formulas are sharp. The main new tools we use for the K-theory calculation are a Cocompletion Theorem and Equivariant Universal Coefficient Theorems which are of independent interest. In the case where G is a finite group these theorems reduce to well-known results of Greenlees and Bökstedt.

0. Introduction

One of our main goals in this paper is to compute for a discrete group G, a proper G-CW-complex X and a generalized cohomology theory $\mathcal{H}^*$ and a generalized homology theory $\mathcal{H}_*$ the groups $\mathcal{H}^*(EG \times_G X)$ and $\mathcal{H}_*(EG \times_G X)$. In particular the case is interesting, where X can be chosen to be non-equivariantly contractible because then $EG \times_G X$ is a model for the classifying space BG. The main results are Theorem 3.6 and Theorem 4.1. In the introduction we will concentrate on the case, where $\mathcal{H}^*$ and $\mathcal{H}_*$ are topological K-theory K^* and K_* and on classifying spaces BG.

Throughout the paper $H_*(Y;M)$ and $H^*(Y;M)$ denote singular (co-)-homology of Y with coefficients in the abelian group M, and we omit M from the notation in the case $M = \mathbb{Z}$. Let $\widehat{\mathbb{Z}_p}$ be the ring of p-adic integers, which is the inverse limit of the inverse system of projections $\mathbb{Z}/p \xleftarrow{\mathrm{pr}_2} \mathbb{Z}/p^2 \xleftarrow{\mathrm{pr}_3} \mathbb{Z}/p^3 \xleftarrow{\mathrm{pr}_4} \ldots$. Denote by $\mathbb{Z}/p^\infty$ the quotient $\mathbb{Z}[1/p]/\mathbb{Z}$ which is isomorphic to the colimit of the directed system of inclusions $\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \ldots$. The proof of the following result will be given in Section 5.

Theorem 0.1 (Topological K-theory of classifying spaces)**.** *Let G be a discrete group. Let X be a finite proper G-CW-complex. Suppose for every $k \in \mathbb{Z}$ that $\widetilde{H}_k(X) = 0$ vanishes. Given a prime number p and $k \in \mathbb{Z}$, define the natural number*

$$r_p^k(G) \quad := \sum_{(g) \in \mathrm{con}_p(G)} \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} \left(H^{k+2i}(BC_G\langle g\rangle;\mathbb{Q})\right),$$

where $\mathrm{con}_p(G)$ denotes the set of conjugacy classes of non-trivial elements of p-power order and $C_G\langle g\rangle$ is the centralizer in G of the cyclic subgroup $\langle g\rangle$ generated by g. Let $\mathcal{P}(G)$ be the set of primes p which divide the order H of some finite subgroup $H \subseteq G$. Then:

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(i) There is an exact sequence

$$0 \to A \to K^k(G\backslash X) \to K^k(BG) \to B \times \prod_{p\in\mathcal{P}(G)} (\widehat{\mathbb{Z}_p})^{r_p^k(G)} \to C \to 0,$$

where A, B and C are finite abelian groups with

$$A \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = B \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = C \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = 0;$$

(ii) Dually there is an exact sequence

$$0 \to C' \to \prod_{p\in\mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r_p^{k+1}(G)} \times B' \to K_k(BG) \to K_k(G\backslash X) \to A' \to 0,$$

where A', B' and C' are finite abelian groups with

$$A' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = B' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = C' \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] = 0;$$

(iii) If we invert all primes in $\mathcal{P}(G)$, then we obtain isomorphisms

$$K^k(BG) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] \cong K^k(G\backslash X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] \times \prod_{p\in\mathcal{P}(G)} (\widehat{\mathbb{Q}_p})^{r_p^k(\underline{E}G)};$$

$$K_k(BG) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right] \cong K_k(G\backslash X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right].$$

Under the conditions appearing in Theorem 0.1 the sets $\mathrm{con}_p(G)$ and $\mathcal{P}(G)$ are finite and the dimension of $BC_G\langle g\rangle$ is bounded by the finite dimension of X. Hence the numbers $r_p^k(X)$ are well-defined.

The exact sequences of assertions (i) and (ii) of Theorem 0.1 are in fact dual to each other when working in the category of topological groups. Namely, recall that skeletal filtration on BG imposes the structure of a pro-discrete group on $K^*(BG)$, while the p-adic integers can be equipped with the pro-finite topology. With these topologies the exact sequence of Theorem 0.1 (ii) is the Pontryagin dual of the exact sequence of Theorem 0.1 (i) and vice versa. Moreover both exact sequence of Theorem 0.1 are also exact sequences in the category of topological groups in the sense of [39].

A model $\underline{E}G$ for the *classifying space for proper G-actions* is a proper G-CW-complex, whose H-fixed point sets are contractible for all finite subgroups $H \subseteq G$. It is unique up to G-homotopy. It is a good candidate for X in Theorem 0.1, provided that there exists a finite G-CW-complex model for $\underline{E}G$. Examples, for which this is true, are arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group [11], [32], mapping class groups (see [29]), groups which are hyperbolic in the sense of Gromov (see[27], [28]), virtually poly-cyclic groups, and groups which are cocompact discrete subgroups of Lie groups with finitely many path components [1, Corollary 4.14]. On the other hand, for any CW-complex Y there exists a group G such that Y and $G\backslash\underline{E}G$ are homotopy equivalent (see [17]). More information about these spaces $\underline{E}G$ can be found for instance in [8], [19], [36, Section I.6].

In order to apply Theorem 0.1 one needs to understand the CW-complex $G\backslash X$. This is often possible using geometric input, in particular in the case $X = \underline{E}G$ for the groups mentioned above. Notice that q-torsion in $K^k(BG)$ and $K_k(BG)$ for a prime number q which does not belong to $\mathcal{P}(G)$ must come from the q-torsion in $K^k(G\backslash X)$ and $K_k(G\backslash X)$.

The rational version of our formula for K-cohomology has already been proved using equivariant Chern characters in [20, Theorem 0.1], see also [3, Theorem 6.3].

If G is finite, a model for $\underline{E}G$ is $\{\bullet\}$ and one gets a complete answer integrally, see for instance [20, Theorem 0.3].

We will recall the Completion Theorem 2.4 of [21, Theorem 6.5]) and deduce from it in Section 2.

Theorem 0.2 (Cocompletion Theorem)**.** *Let G be a discrete group. Let X be a finite proper G-CW-complex and let L be a finite dimensional proper G-CW-complex whose isotropy subgroups have bounded order. Fix a G-map $f\colon X \to L$ and regard $K_G^*(X)$ as a module over $\mathbb{K}_G(L)$. Moreover, let $I = \mathbb{I}_G(L)$ be the augmentation ideal (see Definition 2.1).*

Then there is a short exact sequence

$$0 \to \underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(K_G^{*+1}(X)/I^n \cdot K_G^{*+1}(X), \mathbb{Z}) \to K_*(EG \times_G X) \\ \to \underrightarrow{\operatorname{colim}}_{n\geq 1} \hom_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z}) \to 0.$$

When working in the category of topological abelian groups and continuous homomorphisms the sequence can be written in the following more compact form

$$0 \to \operatorname{ext}^1_{\mathrm{cts}}(K_G^{*+1}(X)\widehat{_I}, \mathbb{Z}) \to K_*(EG \times_G X) \to \hom_{\mathrm{cts}}(K_G^*(X)\widehat{_I}, \mathbb{Z}) \to 0.$$

Theorem 0.2 is closely related to the local cohomology approach to equivariant K-homology of Greenlees [15] (see Remark 2.8 below).

In Section 5 we prove

Theorem 0.3 (Equivariant Universal Coefficient Theorem for K-theory)**.** *Let G be a discrete group. Let X be a finite proper G-CW-complex X.*

Then there are short exact sequences, natural in X,

$$0 \to \operatorname{ext}_{\mathbb{Z}}(K^G_{*-1}(X), \mathbb{Z}) \to K_G^*(X) \to \hom_{\mathbb{Z}}(K^G_*(X), \mathbb{Z}) \to 0; \tag{0.4}$$

$$0 \to \operatorname{ext}_{\mathbb{Z}}(K_G^{*+1}(X), \mathbb{Z}) \to K^G_*(X) \to \hom_{\mathbb{Z}}(K_G^*(X), \mathbb{Z}) \to 0, \tag{0.5}$$

where the homomorphism on the right hand sides are given by (5.19) *and* (5.20) *respectively. The sequence splits unnaturally.*

Theorem 0.3 reduces for finite groups to the corresponding results of Bökstedt [10] as in explained in Remark 5.21.

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1. Some preliminaries about pro-modules

It will be crucial to handle pro-systems and pro-isomorphisms and not to pass directly to inverse limits. Otherwise we would loose important information which is for instance needed in order to pass from K-cohomology to K-homology using universal coefficients theorems. In this section we fix our notation for handling pro-R-modules for a commutative ring R, where ring always means associative ring with unit. For the definitions in full generality see for instance [5, Appendix] or [7, Section 2]. This exposition agrees with the one in [20, Section 2] and is repeated for the reader's convenience.

For simplicity, all pro-R-modules dealt with here will be indexed by the positive integers. We write $\{M_n, \alpha_n\}$ or briefly $\{M_n\}$ for the inverse system

$$M_0 \xleftarrow{\alpha_1} M_1 \xleftarrow{\alpha_2} M_2 \xleftarrow{\alpha_3} M_3 \xleftarrow{\alpha_4} \cdots .$$

and also write $\alpha_n^m := \alpha_{m+1} \circ \cdots \circ \alpha_n\colon G_n \to G_m$ for $n > m$ and put $\alpha_n^n = \mathrm{id}_{G_n}$. For the purposes here, it will suffice (and greatly simplify the notation) to work with "strict" pro-homomorphisms $\{f_n\}\colon \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$, i.e., a collection of homomorphisms $f_n\colon M_n \to N_n$ for $n \geq 1$ such that $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$ holds for each $n \geq 2$. Kernels and cokernels of strict homomorphisms are defined in the obvious way.

A pro-R-module $\{M_n, \alpha_n\}$ will be called *pro-trivial* if for each $m \geq 1$, there is some $n \geq m$ such that $\alpha_n^m = 0$. A strict homomorphism $f\colon \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ is a *pro-isomorphism* if and only if $\ker(f)$ and $\operatorname{coker}(f)$ are both pro-trivial, or, equivalently, for each $m \geq 1$ there is some $n \geq m$ such that $\operatorname{im}(\beta_n^m) \subseteq \operatorname{im}(f_m)$ and $\ker(f_n) \subseteq \ker(\alpha_n^m)$. A sequence of strict homomorphisms

$$\{M_n, \alpha_n\} \xrightarrow{\{f_n\}} \{M'_n, \alpha'_n\} \xrightarrow{g_n} \{M''_n, \alpha''_n\}$$

will be called *exact* if the sequences of R-modules $M_n \xrightarrow{f_n} N_n \xrightarrow{g_n} M''_n$ is exact for each $n \geq 1$, and it is called *pro-exact* if $g_n \circ f_n = 0$ holds for $n \geq 1$ and the pro-R-module $\{\ker(g_n)/\operatorname{im}(f_n)\}$ is pro-trivial.

The following results will be needed later.

Lemma 1.1. *Let $0 \to \{M'_n, \alpha'_n\} \xrightarrow{\{f_n\}} \{M_n, \alpha_n\} \xrightarrow{\{g_n\}} \{M''_n, \alpha''_n\} \to 0$ be a pro-exact sequence of pro-R-modules. Then there is a natural exact sequence*

$$0 \to \varprojlim_{n\geq 1} M'_n \xrightarrow{\varprojlim_{n\geq 1} f_n} \varprojlim_{n\geq 1} M_n \xrightarrow{\varprojlim_{n\geq 1} g_n} \varprojlim_{n\geq 1} M''_n \xrightarrow{\delta}$$
$$\varprojlim^1_{n\geq 1} M'_n \xrightarrow{\varprojlim^1_{n\geq 1} f_n} \varprojlim^1_{n\geq 1} M_n \xrightarrow{\varprojlim^1_{n\geq 1} g_n} \varprojlim^1_{n\geq 1} M''_n \to 0.$$

In particular a pro-isomorphism $\{f_n\}\colon \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ induces isomorphisms

$$\begin{array}{lccc} \varprojlim_{n\geq 1} f_n\colon & \varprojlim_{n\geq 1} M_n & \xrightarrow{\cong} & \varprojlim_{n\geq 1} N_n; \\ \varprojlim^1_{n\geq 1} f_n\colon & \varprojlim^1_{n\geq 1} M_n & \xrightarrow{\cong} & \varprojlim^1_{n\geq 1} N_n. \end{array}$$

Proof. If $0 \to \{M'_n, \alpha'_n\} \xrightarrow{\{f_n\}} \{M_n, \alpha_n\} \xrightarrow{g_n} \{M''_n, \alpha''_n\} \to 0$ is exact, the construction of the six-term sequence is obvious (see for instance [34, Proposition 7.63 on page 127]). Hence it remains to show for a pro-trivial pro-R-module $\{M_n, \alpha_n\}$ that $\varprojlim_{n\geq 1} M_n$ and $\varprojlim^1_{n\geq 1} M_n$ vanish. This follows directly from the standard construction for these limits as the kernel and cokernel of the homomorphism

$$\prod_{n\geq 1} M_n \to \prod_{n\geq 1} M_n, \quad (x_n)_{n\geq 1} \mapsto (x_n - \alpha_{n+1}(x_{n+1}))_{n\geq 1}.$$

$\square$

Lemma 1.2. *Let $0 \to \{M'_n, \alpha'_n\} \xrightarrow{\{f_n\}} \{M_n, \alpha_n\} \xrightarrow{\{g_n\}} \{M''_n, \alpha''_n\} \to 0$ be a pro-exact sequence of pro-R-modules. Then there is a natural exact sequence*

$$0 \to \varinjlim_{n\geq 1} \hom_{\mathbb{Z}}(M''_n, \mathbb{Z}) \to \varinjlim_{n\geq 1} \hom_{\mathbb{Z}}(M_n, \mathbb{Z})$$
$$\to \varinjlim_{n\geq 1} \hom_{\mathbb{Z}}(M'_n, \mathbb{Z}) \to \varinjlim_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(M''_n, \mathbb{Z})$$
$$\to \varinjlim_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(M_n, \mathbb{Z}) \to \varinjlim_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(M'_n, \mathbb{Z}) \to 0.$$

In particular a pro-isomorphism $\{f_n\}\colon \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ induces isomorphisms

$$\begin{array}{rcl} \varinjlim_{n\geq 1} \hom_{\mathbb{Z}}(N_n, \mathbb{Z}) & \xrightarrow{\cong} & \varinjlim_{n\geq 1} \hom_{\mathbb{Z}}(M_n, \mathbb{Z}); \\ \varinjlim_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(N_n, \mathbb{Z}) & \xrightarrow{\cong} & \varinjlim_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(M_n, \mathbb{Z}). \end{array}$$

Proof. The proof is analogous to the one of the previous Lemma 1.1 using the fact that $\varinjlim_{n\geq 1}$ is an exact functor. $\square$

2. Completion and cocompletion theorems

Let G be a discrete group. Denote by K_G^* equivariant topological K-theory. This is a multiplicative G-cohomology theory for proper G-CW-complexes which comes with various extra structures such as induction, restriction and inflation. It is defined in terms of classifying spaces for G-vector bundles over proper G-CW-complexes (see [21, Theorem 2.7 and Section 3]).

Recall that a G-CW-complex X is proper if and only if its isotropy groups are finite [18, Theorem 1.23 on page 18] and is finite if and only if X is cocompact, i.e. $G\backslash X$ is compact. If one considers only finite proper G-CW-complexes, then $K_G^*(X)$ has other descriptions which are all equivalent. There is a construction due to Phillips [30] in terms of infinite dimensional vector bundles. In Lück and Oliver [22, Theorem 3.2] it is shown that it suffices to use finite dimensional G-vector bundles and that one can give a definition in terms of the Grothendieck group $\mathbb{K}_G(X)$ of the monoid of isomorphism classes of G-vector bundles over X. In case where G is the trivial group, we just write $\mathbb{K}(X)$. One can also define for a finite proper G-CW-complex X the equivariant topological K-theory as the topological K-theory $K_*(C_0(X)\rtimes G)$ of the crossed product C^*-algebra $C_0(X)\rtimes G$ (see [30, Theorem 6.7 on page 96]) , while equivariant topological K-homology of X (by the dual of the Green-Julg theorem [9, Theorem 20.2.7 (b)]) also can be defined as the equivariant KK-group $KK_*^G(C_0(X),\mathbb{C})$. Here $C_0(X)$ denotes the C^*-algebra of continuous function on X vanishing at infinity.

For any proper G-CW-complex X the is a natural ring homomorphism

$$\gamma_G(X)\colon \mathbb{K}_G(X)\to K_G^0(X)$$

which allows to regard $K_G^p(X)$ as a $\mathbb{K}_G(X)$-module in the sequel.

Definition 2.1 (Augmentation ideal)**.** *The augmentation ideal $\mathbb{I}_G(Y)\subseteq \mathbb{K}_G(Y)$ is given by the set of elements in $\mathbb{K}_G(Y)$ represented by virtual G-vector bundles of dimension zero on all components of Y.*

If G is trivial we just write $\mathbb{I}(Y)$.

We have the following easy but crucial lemma (cf. Lemma 4.2 in [22]).

Lemma 2.2. *Let Z be a CW-complex of dimension $n-1$. Then the n-fold product of elements in $\mathbb{I}(Z)\subseteq\mathbb{K}(Z)$ is zero.*

Now fix a finite proper G-CW-complex X, and a map $f\colon X\to L$ to a finite dimensional proper G-CW-complex L whose isotropy subgroups have bounded order. We obtain a ring homomorphism $f^*\colon \mathbb{K}_G(L)\to\mathbb{K}_G(X)$ by the pullback construction. Hence we can regard $K_G^*(X)$ as a module over the ring $\mathbb{K}_G(L)$. Put $I=\mathbb{I}_G(L)$. For any natural number $n\geq 1$, consider the composite

$$\begin{aligned} I^n\cdot K_G^*(X)\subseteq K_G^*(X)\xrightarrow{\mathrm{pr}^*} K_G^*(EG\times X)\xrightarrow{\cong} K^*(EG\times_G X)&\\ \xrightarrow{K^*(i_{n-1})} K^*((EG\times_G X)^{n-1}),&\end{aligned}$$

where $\mathrm{pr}\colon EG\times X\to X$ is the projection, $i_{n-1}\colon (EG\times_G X)^{n-1}\to EG\times_G X$ is the inclusion of the $(n-1)$-skeleton and the isomorphism $K_G^*(EG\times X)\xrightarrow{\cong} K^*(EG\times_G X)$ comes from dividing out the (free proper) G-action [21, Proposition 3.3]. This composite is trivial, since the image is contained in the ideal which is generated by the set $\mathbb{I}((EG\times_G X)^{n-1})^n$, and the latter is trivial, since $\mathbb{I}((EG\times_G X)^{n-1})^n=0$ by Lemma 2.2. The composites therefore define a pro-homomorphism

$$\lambda^{X,f}\colon \{K_G^*(X)/I^n\cdot K_G^*(X)\}\quad\to\quad\{K^*((EG\times_G X)^{n-1})\}, \tag{2.3}$$

where the structure maps on the left side are given by the obvious projections and on the right side are induced by the various inclusions of the skeletons. The following theorem is taken from Lück-Oliver [21, Theorem6.5]), [22, Theorem 4.3]).

Theorem 2.4 (Completion Theorem). *Let G be a discrete group. Let X be a finite proper G-CW-complex and let L be a finite dimensional proper G-CW-complex whose isotropy subgroups have bounded order. Fix a G-map $f\colon X \to L$ and regard $K_G^*(X)$ as a module over $\mathbb{K}_G(L)$. Moreover, let $I = \mathbb{I}_G(L)$ be the augmentation ideal.*

Then

$$\lambda^{X,f}\colon \{K_G^*(X)/I^n \cdot K_G^*(X)\} \to \{K^*((EG \times_G X)^{n-1})\}$$

is a pro-isomorphism of pro-$\mathbb{Z}$-modules.

The inverse system $\{K_G^*(X)/I^n \cdot K_G^*(X)\}$ satisfies the Mittag-Leffler condition. In particular

$$\varprojlim_{n\geq 1}^1 K^{*-1}(EG \times_G X)^n) = 0,$$

and $\lambda^{X,f}$ and the various inclusions $i_n\colon (EG \times_G X)^n \to EG \times_G X$ induce isomorphisms

$$K_G^*(X)\widehat{_I} \xrightarrow{\cong} K^*(EG \times_G X) \xrightarrow{\cong} \varprojlim_{n\geq 1} K^*((EG \times_G X)^n),$$

where $K_G^*(X)\widehat{_I} = \varprojlim_{n\geq 1} K_G^*(X)/I^n \cdot K_G^*(X)$ is the I-adic completion of $K_G^*(X)$.

Remark 2.5. In the case where G is finite and L is a point, Theorem 2.4 above coincides with the Atiyah-Segal completion theorem for finite groups (see [7, Theorem 2.1]). The classical Atiyah-Segal completion theorem is stated for compact Lie groups. However, the theorem above does not hold if G is replaced by a Lie group of positive dimension (see [22, Section 5]).

We now pass to K-homology.

Lemma 2.6. *If X is a finite proper G-CW-complex, then $EG \times_G X$ is homotopy equivalent to a CW-complex of finite type.*

Proof. We use induction over the dimension and subinduction over the number of equivariant cells of top dimension in X. The induction beginning $X = \emptyset$ is trivial. In the induction step we write X as a pushout of a diagram $G/H \times D^n \xleftarrow{i} G/H \times S^{n-1} \to Y$, where i is the inclusion, Y a G-CW-subcomplex of X and $n = \dim(X)$. We obtain a pushout of CW-complexes

$$\begin{tikzcd} EG \times_G G/H \times S^{n-1} \arrow[r] \arrow[d, "\mathrm{id}_{EG} \times_G i"] & EG \times_G Y \arrow[d] \\ EG \times_G G/H \times D^n \arrow[r] & EG \times_G X \end{tikzcd}$$

with $\mathrm{id}_{EG} \times_G i$ a cofibration. Hence $EG \times_G X$ has the homotopy type of a CW-complex of finite type if $EG \times_G G/H \times S^{n-1}$, $EG \times_G Y$ and $EG \times_G G/H \times D^n$ have this property. This is true for the first two by the induction hypothesis and for the third one since it is homotopy equivalent to BH. $\square$

Now we can give the proof of the Cocompletion Theorem 0.2

Proof of Theorem 0.2. Because of Lemma 2.6 we can choose a CW-complex Y of finite type and a cellular homotopy equivalence $f\colon Y \to EG \times_G X$. Let $f^n\colon Y^n \to (EG \times_G X)^n$ be the map induced on the n-skeletons. Notice that f^n is not necessarily a homotopy equivalence and $K^*(f^n)$ is not necessarily an isomorphism. Nevertheless, one easily checks that we obtain a pro-isomorphism of pro-$\mathbb{Z}$-modules

$$\{K^*(f^n)\}\colon \{K^*((EG \times_G X)^n)\} \to \{K^*(Y^n)\}.$$

Thus we obtain from the Completion Theorem 2.4 a pro-isomorphism of pro-$\mathbb{Z}$-modules

$$\{K^*(f^n)\} \circ \lambda^{X,f} \colon \{K_G^*(X)/I^n \cdot K_G^*(X)\} \to \{K^*(Y^n)\}.$$

¿From the K-homology version of the universal coefficient theorem for topological K-theory 5.1 for finite CW-complexes and the fact that $\underrightarrow{\operatorname{colim}}_{n\geq 1}$ is an exact functor, we get the exact sequence

$$0 \to \underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(K^{*+1}(Y^n), \mathbb{Z}) \to K_*(Y) \\ \to \underrightarrow{\operatorname{colim}}_{n\geq 1} \hom_{\mathbb{Z}}(K^*(Y^n), \mathbb{Z}) \to 0.$$

The map f and the pro-isomorphism $\{K^*(f^n)\} \circ \lambda^{X,f}$ induce isomorphisms (see Lemma 1.2)

$$\begin{array}{rcl} K_*(f) \colon K_*(Y) & \xrightarrow{\cong} & K_*(EG \times_G X); \\ \underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z}) & \xrightarrow{\cong} & \underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(K^{*+1}(Y^n), \mathbb{Z}); \\ \underrightarrow{\operatorname{colim}}_{n\geq 1} \hom_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z}) & \xrightarrow{\cong} & \underrightarrow{\operatorname{colim}}_{n\geq 1} \hom_{\mathbb{Z}}(K^*(Y^n), \mathbb{Z}). \end{array}$$

Combining these isomorphisms with the exact sequence above proves the Cocompletion Theorem 0.2. □

Remark 2.7. The Cocompletion Theorem 0.2 can be formulated elegantly within the category of abelian topological groups and continuous homomorphisms. If we equip the completion $K_G^*(X)\widehat{_I}$ with the I-adic topology and $\mathbb{Z}$ with the discrete topology then the set of continuous homomorphisms $\hom_{\mathrm{cts}}(K_G^*(X)\widehat{_I}, \mathbb{Z})$ is isomorphic to $\underrightarrow{\operatorname{colim}}_{n\geq 1} \hom_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z})$. On the other hand, although the category of topological abelian groups is not exact one can introduce a notion of exact sequences (in the sense of [39, Section 1.1]) and correspondingly a notion of a group of isomorphisms classes of extensions (see [39, Corollary on page 537]). In the case at hand we get that the group of isomorphisms classes of extensions $\operatorname{ext}_{\mathrm{cts}}(K_G^*(X)\widehat{_I}, \mathbb{Z})$, the continuous ext-group in the sense of [39], is isomorphic to $\underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^1_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z})$. With these identifications the exact sequence of the Cocompletion Theorem 0.2 reads

$$0 \to \operatorname{ext}^1_{\mathrm{cts}}(K_G^{*+1}(X)\widehat{_I}, \mathbb{Z}) \to K_*(EG \times_G X) \to \hom_{\mathrm{cts}}(K_G^*(X)\widehat{_I}, \mathbb{Z}) \to 0.$$

Remark 2.8. The Cocompletion Theorem 0.2 is closely related to the local cohomology approach to equivariant K-homology due to Greenlees [15]. If G is a finite group it follows from [15, (4.2) and (5.1)] that there is a short exact sequence

$$0 \to H_I^1(K_\circ^G(X))_{*+1} \to K_*(EG \times_G X) \to H_I^0(K_\circ^G(X))_* \to 0, \tag{2.9}$$

where $H_I^k(M_\circ)$ denotes the k-th local cohomology of the graded $R(G)$-module $M_\circ$ with respect to the augmentation ideal $I \subset R(G)$. A precise definition of the local cohomology groups occuring in (2.9) can be found in [15, Section 2]. By a Theorem of Grothendieck in [16] (quoted in [15] as Theorem 2.5 (ii)) one has

$$H_I^n(K_\circ^G(X))_* \cong \underrightarrow{\operatorname{colim}}_{n\geq 1} \operatorname{ext}^n_{R(G)}(R(G)/I^n, K_*^G(X)).$$

Using exact sequence of the Equivariant Universal Coefficient Theorem for K-homology as stated in Remark 5.21, the adjunction

$$\hom_{R(G)}(M, \hom_{R(G)}(C, N)) \cong \hom_{R(G)}(M \otimes_{R(G)} C, N))$$

for $R(G)$-modules M, C, N with C being finitely generated, and the $R(G)$-module isomorphism $\operatorname{ext}^i_{R(G)}(M, R(G)) \cong \operatorname{ext}^i_{\mathbb{Z}}(M, \mathbb{Z})$ (emphasized in Remark 5.21) one can see that the exact sequence of the Cocompletion Theorem 0.2 is exact if and only if (2.9) is. In particular the Cocompletion Theorem yields an alternative proof for the exactness of (2.9).

3. Borel cohomology

Let $\mathcal{H}^*$ be a (generalized) cohomology theory with values in the category of $\mathbb{Z}$-modules which satisfies the *disjoint union axiom* for arbitrary index sets, i.e., for any family $\{X_i \mid i \in I\}$ the map

$$\prod_{i\in I} \mathcal{H}^k(j_i)\colon \mathcal{H}^k(\coprod_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} \mathcal{H}^k(X_i)$$

is an isomorphism, where $j_i\colon X_i \to \coprod_{i\in I} X_i$ is the canonical inclusion. Any such theory $\mathcal{H}^*$ is given by an Ω-spectrum $\mathbf{E}$ and, vice versa, any cohomology theory given by an Ω-spectrum satisfies the disjoint union axiom. Given a CW-complex X, let $\widetilde{\mathcal{H}}^k(X)$ be the cokernel of the map $\mathcal{H}^k(\{\bullet\}) \to \mathcal{H}^k(X)$ induced by the projection $X \to \{\bullet\}$. Our main example for $\mathcal{H}^*$ will be topological K-theory K^*. If M is an abelian group, we define the cohomology theory $\mathcal{H}^*(-;M)$ by the Ω-spectrum which is the fibrant replacement of the smash product of the spectrum associated with $\mathcal{H}^*$ with the Moore spectrum associated to M. If M is a ring R, then $\mathcal{H}^*(-;R)$ takes values in the category of R-modules.

Lemma 3.1. *Let X be a CW-complex such that its reduced singular cohomology $\widetilde{H}^k(X;\mathcal{H}^l(\{\bullet\}))$ with coefficients in the abelian group $\mathcal{H}^l(\{\bullet\})$ vanishes for all $k \geq 0$ and $l \in \mathbb{Z}$. Then*

(i) The inclusion $X^{n-1} \to X^n$ of the $(n-1)$-skeleton into the n-skeleton induces the zero-map $\widetilde{\mathcal{H}}^k(X^n) \to \widetilde{\mathcal{H}}^k(X^{n-1})$ for all $k \in \mathbb{Z}$ and $n \geq 2$. The pro-$\mathbb{Z}$-module $\{\widetilde{\mathcal{H}}^k(X^n)\}$ is pro-trivial;

(ii) We have $\widetilde{\mathcal{H}}^k(X) = 0$ for all $k \in \mathbb{Z}$.

Proof. (i) Since X^n is finite dimensional, the reduced Atiyah-Hirzebruch spectral cohomology sequence converges to $\widetilde{\mathcal{H}}^{k+l}(X^n)$. It has as E_2-term $E_2^{k,l}(X^n) = \widetilde{H}^k(X^n;\mathcal{H}^l(\{\bullet\}))$. Since $\widetilde{H}^k(X;\mathcal{H}^l(\{\bullet\})) = 0$ for all k, we have $E_2^{k,l}(X^n) = 0$ for $k \neq n$. This implies $E_\infty^{k,l}(X^n) = 0$ for $k \neq n$. We have the descending filtration $F^{k,m-k}\mathcal{H}^m(X^n)$ of $\mathcal{H}^m(X^n)$ such that

$$F^{k,l}\mathcal{H}^m(X^n)/F^{k+1,l-1}\mathcal{H}^m(X^n) \cong E_\infty^{k,l}(X^n).$$

Hence $F^{k,l}\mathcal{H}^m(X^n) = 0$ for $k \geq n$ and $F^{k,l}\mathcal{H}^m(X^n) = \mathcal{H}^m(X^n)$ for $k < n$. Since the map $\mathcal{H}^m(X^n) \to \mathcal{H}^m(X^{n-1})$ respects this filtration, it must be trivial.

(ii) Recall Milnor's exact sequence [38, Theorem 1.3 in XIII.1 on page 605]

$$0 \to \varprojlim^1_{n\to\infty} \widetilde{\mathcal{H}}^{k-1}(X^n) \to \widetilde{\mathcal{H}}^k(X) \to \varprojlim_{n\to\infty} \widetilde{\mathcal{H}}^k(X^n) \to 0.$$

Since $\{\widetilde{\mathcal{H}}^k(X^n)\}$ is pro-trivial for all $k \in \mathbb{Z}$, we conclude from Lemma 1.1

$$\begin{aligned} \varprojlim^1_{n\to\infty} \widetilde{\mathcal{H}}^{k-1}(X^n) &= 0; \\ \varprojlim_{n\to\infty} \widetilde{\mathcal{H}}^k(X^n) &= 0. \end{aligned}$$

This finishes the proof of Lemma 3.1. □

Lemma 3.2. *Let Y be a finite CW-complex Let R be a commutative associative ring which is flat over $\mathbb{Z}$. Then the canonical R-map*

$$\mathcal{H}^k(Y) \otimes_{\mathbb{Z}} R \xrightarrow{\cong} \mathcal{H}^k(Y;R)$$

is an isomorphism.

Proof. Since R is flat over $\mathbb{Z}$, the map $\mathcal{H}^k(Y) \otimes_{\mathbb{Z}} R \xrightarrow{\cong} \mathcal{H}^k(Y;R)$ is a transformation of homology theories. It is bijective for $Y = \{\bullet\}$. Hence by a Mayer-Vietoris argument it is bijective for every finite CW-complex Y. □

Lemma 3.3. *Let X be a finite proper G-CW-complex. Let $\mathcal{P}(X)$ be the set of primes p which divide the order of some isotropy group of X. Let $\mathbb{Z} \subseteq \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \subseteq \mathbb{Q}$ be the ring obtained from $\mathbb{Z}$ by inverting the elements in $\mathcal{P}(X)$. Let $q(X)\colon EG \times_G X \to G\backslash X$ be the projection. Then there is for $k \in \mathbb{Z}$ a R-map, natural in X,*

$$r^k(X)\colon \mathcal{H}^k(EG \times_G X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \to \mathcal{H}^k(G\backslash X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$$

such that $r^k(X) \circ \mathcal{H}^k(q(X)) \otimes_{\mathbb{Z}} \mathrm{id}\colon \mathcal{H}^k(G\backslash X)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \to \mathcal{H}^k(G\backslash X)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$ is an isomorphism.

Proof. Put $\Lambda = \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$. Consider the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^k(G\backslash X) \otimes_{\mathbb{Z}} \Lambda & \xrightarrow{\mathcal{H}^k(q(X)) \otimes_{\mathbb{Z}} \Lambda} & \mathcal{H}^k(EG \times_G X) \otimes_{\mathbb{Z}} \Lambda \\
\downarrow & & \downarrow \\
\mathcal{H}^k(G\backslash X;\Lambda) & \xrightarrow{\mathcal{H}^k(q(X);\Lambda)} & \mathcal{H}^k(EG \times_G X;\Lambda)
\end{array}$$

The left vertical arrow is an isomorphism by Lemma 3.2. Hence it suffices to show that the lower horizontal map is an isomorphism. Since X is finite proper, a Mayer-Vietoris argument shows that it suffices to treat the case $X = G/H$ for some finite group $H \subset G$ such that $|H|$ is invertible in Λ. Since $\widetilde{H}^k(BH;\mathcal{H}^q(\{\bullet\};\Lambda)) = 0$ vanishes for all k by [13, Corollary 10.2 in Chapter III on page 84], this follows from Lemma 3.1 (ii). $\square$

Lemma 3.4. *Let H be a finite group. Let $\mathcal{P}(H)$ be the set of primes dividing $|H|$. The canonical map of pro-$\mathbb{Z}$-modules*

$$\{\widetilde{\mathcal{H}}^k(BH^{n-1})\} \xrightarrow{\cong} \prod_{p \in \mathcal{P}(H)} \{\widetilde{\mathcal{H}}^k(BH^{n-1};\widehat{\mathbb{Z}_p})\}$$

is a pro-isomorphism for $k \in \mathbb{Z}$. The pro-module

$$\prod_{\substack{p \text{ prime} \\ p \notin \mathcal{P}(H)}} \{\widetilde{\mathcal{H}}^k(BH^{n-1};\widehat{\mathbb{Z}_p})\}$$

is pro-trivial.

Proof. We have the exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{i} \prod_{p \in \mathcal{P}(H)} \widehat{\mathbb{Z}_p} \to \operatorname{coker}(i) \to 0$$

where i is the product of the canonical embeddings $\mathbb{Z} \to \widehat{\mathbb{Z}_p}$. It induces a long exact sequence

$$\ldots \to \widetilde{\mathcal{H}}^{k-1}(BH^{n-1};\operatorname{coker}(i)) \to \widetilde{\mathcal{H}}^k(BH^{n-1}) \to \widetilde{\mathcal{H}}^k(BH^{n-1};\prod_{p \in \mathcal{P}(H)} \widehat{\mathbb{Z}_p}) \\ \to \widetilde{\mathcal{H}}^k(BH^{n-1};\operatorname{coker}(i)) \to \ldots$$

and thus an exact sequence of pro-$\mathbb{Z}$-modules

$$\{\widetilde{\mathcal{H}}^{k-1}(BH^{n-1};\operatorname{coker}(i))\} \to \{\widetilde{\mathcal{H}}^k(BH^{n-1})\} \to \{\widetilde{\mathcal{H}}^k(BH^{n-1};\prod_{p \in \mathcal{P}(H)} \widehat{\mathbb{Z}_p})\} \\ \to \{\widetilde{\mathcal{H}}^k(BH^{n-1};\operatorname{coker}(i))\}$$

Multiplication with the order of $|H|$ induces an isomorphism $|H| \cdot \mathrm{id}\colon \operatorname{coker}(i) \xrightarrow{\cong} \operatorname{coker}(i)$. Hence $\widetilde{H}^k(BH; \mathcal{H}^l(\{\bullet\}; \operatorname{coker}(i)))$ vanishes for all $k, l \in \mathbb{Z}$ by [13, Corollary 10.2 in Chapter III on page 84]. We conclude from Lemma 3.1 (i) that the pro-$\mathbb{Z}$-module $\{\widetilde{\mathcal{H}}^k(BH^{n-1}; \operatorname{coker}(i)\}$ is trivial. This shows that the obvious map of pro-$\mathbb{Z}$-modules

$$\{\widetilde{\mathcal{H}}^k(BH^{n-1})\} \xrightarrow{\cong} \{\widetilde{\mathcal{H}}^k(BH^{n-1}; \prod_{p \in \mathcal{P}(H)} \widehat{\mathbb{Z}_p})\}$$

is bijective. The canonical map

$$\widetilde{\mathcal{H}}^k(Y; \prod_{p \in \mathcal{P}(H)} \widehat{\mathbb{Z}_p}) \to \prod_{p \in \mathcal{P}(H)} \widetilde{\mathcal{H}}^k(Y; \widehat{\mathbb{Z}_p})$$

is a natural transformation of cohomology theories satisfying the disjoint union axiom and is an isomorphism for $Y = \{\bullet\}$ since the set $\mathcal{P}(H)$ is finite. Hence it is an isomorphism for every finite-dimensional Y and in particular for $Y = BH^{n-1}$. We conclude that the obvious map of pro-$\mathbb{Z}$-modules

$$\{\widetilde{\mathcal{H}}^k(BH^{n-1})\} \xrightarrow{\cong} \prod_{p \in \mathcal{P}(H)} \{\widetilde{\mathcal{H}}^k(BH^{n-1}; \widehat{\mathbb{Z}_p})\}$$

is bijective.

If p does not divide $|H|$, then $H^k(BH; \widetilde{\mathcal{H}}^q(\{\bullet\}; \widehat{\mathbb{Z}_p})) = 0$ by [13, Corollary 10.2 in Chapter III on page 84]. We conclude from Lemma 3.1 (i) that the map induced by the inclusion

$$\{\widetilde{\mathcal{H}}^k(BH^n; \widehat{\mathbb{Z}_p})\} \to \{\widetilde{\mathcal{H}}^k(BH^{n-1}; \widehat{\mathbb{Z}_p})\}$$

is trivial for all $k \in \mathbb{Z}$ and $n \geq 2$. This implies that

$$\prod_{p \notin \mathcal{P}(H)} \{\widetilde{\mathcal{H}}^k(BH^{n-1}; \widehat{\mathbb{Z}_p})\}$$

is pro-trivial. $\square$

Lemma 3.5. *Let X be a proper G-CW-complex. Let $\mathcal{P}$ be a set of primes containing $\mathcal{P}(X)$. Then the canonical map*

$$\mathcal{H}^k(q(X)\colon EG \times_G X \to G\backslash X) \xrightarrow{\cong} \prod_{p \in \mathcal{P}} \mathcal{H}^k(q(X)\colon EG \times_G X \to G\backslash X; \widehat{\mathbb{Z}_p})$$

is an isomorphism.

Proof. We conclude from the Milnor's exact sequence [38, Theorem 1.3 in XIII.1 on page 605] and the Five-Lemma that it suffices to treat the case, where X is finite dimensional. Using Mayer-Vietoris sequences the claim can be reduced to the case $X = G/H$ for some finite group H such that $\mathcal{P}$ contains the set $\mathcal{P}(H)$ of primes dividing the order of H. So we must show the canonical map

$$\widetilde{\mathcal{H}}^k(BH) \xrightarrow{\cong} \prod_{p \in \mathcal{P}} \widetilde{\mathcal{H}}^k(BH; \widehat{\mathbb{Z}_p})$$

is bijective for any finite group H with $(H) \subseteq \mathcal{P}$. By Milnor's exact sequence [38, Theorem 1.3 in XIII.1 on page 605] and the Five-Lemma, it remains to show the bijectivity of

$$\begin{aligned} \varprojlim_{n\to\infty} \widetilde{\mathcal{H}}^k(BH^{n-1}) &\to \varprojlim_{n\to\infty} \prod_{p \in \mathcal{P}} \widetilde{\mathcal{H}}^k(BH^{n-1}; \widehat{\mathbb{Z}_p}); \\ {\varprojlim}^1_{n\to\infty} \widetilde{\mathcal{H}}^k(BH^{n-1}) &\to {\varprojlim}^1_{n\to\infty} \prod_{p \in \mathcal{P}} \widetilde{\mathcal{H}}^k(BH^{n-1}; \widehat{\mathbb{Z}_p}). \end{aligned}$$

This follows from Lemma 1.1 and Lemma 3.4. $\square$

For a map $f\colon X \to Y$ and a cohomology theory $\mathcal{H}^*$ define $\mathcal{H}^k(f)$ to be $\mathcal{H}^k(\operatorname{cyl}(f), X)$, where $\operatorname{cyl}(f)$ is the mapping cylinder of f.

Theorem 3.6 (Cohomology of the Borel construction). *Let $\mathcal{H}^*$ be a cohomology theory which satisfies the disjoint union axiom. Let X be a proper G-CW-complex. Let $\mathcal{P}(X)$ be the set of primes p for which p divides the order $|G_x|$ of the isotropy group G_x of the some $x \in X$.*

(i) There is a natural long exact sequence

$$\ldots \to \mathcal{H}^k(G\backslash X) \xrightarrow{\mathcal{H}^k(q(X))} \mathcal{H}^k(EG \times_G X)$$
$$\to \prod_{p \in \mathcal{P}(X)} \mathcal{H}^{k+1}(q(X)\colon EG \times_G X \to G\backslash X; \widehat{\mathbb{Z}_p}) \xrightarrow{\delta^k} \mathcal{H}^{k+1}(G\backslash X) \xrightarrow{\mathcal{H}^{k+1}(q(X))} \ldots$$

(ii) Suppose that X is a finite proper G-CW-complex.
Then for all $k \in \mathbb{Z}$ the map $\mathcal{H}^k(q(X))$ appearing in assertion (i) becomes split injective after applying $-\otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$ and we obtain a natural isomorphism

$$\mathcal{H}^k(EG \times_G X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \xrightarrow{\cong}$$
$$\mathcal{H}^k(G\backslash X)\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \times \prod_{p \in \mathcal{P}(X)} \mathcal{H}^{k+1}(q(X)\colon EG\times_G X \to G\backslash X; \widehat{\mathbb{Z}_p})\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right];$$

(iii) Suppose that X is a finite proper G-CW-complex. Suppose that $\mathcal{H}^k(\{\bullet\})$ is finitely generated as abelian groups for all $k \in \mathbb{Z}$. Assume that $\widetilde{\mathcal{H}}^k(BH; \widehat{\mathbb{Z}_p})$ is a finitely generated $\widehat{\mathbb{Z}_p}$-module for all $k \in \mathbb{Z}$ and all isotropy groups H of X.
Then for all $k \in \mathbb{Z}$ the abelian group $\mathcal{H}^k(G\backslash X)$ is finitely generated, and for appropriate natural numbers $r_p^k(X)$ there is an exact sequence

$$0 \to A \to \mathcal{H}^k(G\backslash X) \to \mathcal{H}^k(EG \times_G X) \to B \times \prod_{p \in \mathcal{P}(X)} (\widehat{\mathbb{Z}_p})^{r_p^k(X)} \to C \to 0,$$

where A, B and C are finite abelian groups with

$$A \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] = B \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] = C \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] = 0.$$

Proof. (i) This follows from Lemma 3.5 and the long exact sequence associated to $q(X)\colon EG \times_G X \to G\backslash X$.

(ii) This follows from assertion (i) and Lemma 3.3.

(iii) Since by assumption X is a finite proper G-CW-complex and the $\widehat{\mathbb{Z}_p}$-module $\mathcal{H}^k(q(G/H) : EG \times_G G/H \to G\backslash(G/H); \widehat{\mathbb{Z}_p}) = \widetilde{\mathcal{H}}^k(BH; \widehat{\mathbb{Z}_p})$ is finitely generated for all $k \in \mathbb{Z}$ and all isotropy groups H of X, the $\widehat{\mathbb{Z}_p}$-module $\mathcal{H}^k(q(X) : EG \times_G X \to G\backslash X; \widehat{\mathbb{Z}_p})$ is finitely generated for all $k \in \mathbb{Z}$. Since $\mathcal{H}^k(\{\bullet\})$ is a finitely generated abelian group by assumption for all $k \in \mathbb{Z}$ and $G\backslash X$ is a finite CW-complex, the abelian group $\mathcal{H}^k(G\backslash X)$ is finitely generated for $k \in \mathbb{Z}$.

Recall that $\widehat{\mathbb{Z}_p}$ is a principal ideal domain and for each prime ideal I there is $l \in \mathbb{N}_0$ such that I is isomorphic to (p^l) and $\widehat{\mathbb{Z}_p}/I$ is isomorphic to $\mathbb{Z}/p^l$. Hence for any $k \in \mathbb{Z}$ we have an isomorphism

$$\mathcal{H}^{k+1}(q(X)\colon EG \times_G X \to G\backslash X; \widehat{\mathbb{Z}_p}) \cong B_p \times (\widehat{\mathbb{Z}_p})^{r_k^p(X)}$$

for some finite abelian p-group B_p and some natural number $r_p^k(X)$. Taking the product over the primes $p \in \mathcal{P}(X)$, we get from Lemma 3.5

$$\mathcal{H}^{k+1}(q(X)\colon EG \times_G X \to G\backslash X) \cong \prod_{p \in \mathcal{P}(X)} B_p \times (\widehat{\mathbb{Z}_p})^{r_p^k(X)}.$$

Since $\mathcal{P}(X)$ is finite,

$$B := \prod_{p \in \mathcal{P}(X)} B_p$$

is a finite abelian group which vanishes after inverting the primes in $\mathcal{P}(X)$ and we have

$$\mathcal{H}^{k+1}(q(X)\colon EG \times_G X \to G\backslash X) \cong B \times \prod_{p \in \mathcal{P}(X)} (\widehat{\mathbb{Z}_p})^{r_p^k(X)}.$$

We obtain from assertion (i) the long exact sequence

$$0 \to A \to \mathcal{H}^k(G\backslash X) \to \mathcal{H}^k(EG \times_G X) \to B \times \prod_{p \in \mathcal{P}(X)} (\widehat{\mathbb{Z}_p})^{r_k^p(X)} \to C \to 0,$$

where A and C can be identified with the image of boundary operators

$$\begin{array}{rcl} A & \cong & \operatorname{image}(\delta^{k-1}\colon \mathcal{H}^{k-1}(EG \times_G X \to G\backslash X) \to \mathcal{H}^k(G\backslash X)); \\ C & \cong & \operatorname{image}(\delta^k\colon \mathcal{H}^k(EG \times_G X \to G\backslash X) \to \mathcal{H}^{k+1}(G\backslash X)). \end{array}$$

We conclude from assertion (ii) that the image of the boundary operators vanishes after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$. Since we have already shown that the abelian group $\mathcal{H}^k(G\backslash X))$ is finitely generated for all $k \in \mathbb{Z}$, A and C are finite abelian groups which vanish after inverting all primes in $\mathcal{P}(X)$. This finishes the proof of Theorem 3.6. □

Theorem 3.7. *Let X be a finite proper G-CW-complex. If we take $\mathcal{H}^*$ to be topological K-theory K^* in Theorem 3.6, then the numbers $r_p^k(X)$ appearing in assertion (iii) of Theorem 3.6 are given by*

$$r_p^k(X) = \sum_{(g) \in \operatorname{con}_p(G)} \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} \left(H^{k+2i}(C_G\langle g\rangle\backslash X^{\langle g\rangle}; \mathbb{Q})\right).$$

Proof. Consider the equivariant cohomology theory with values in $\widehat{\mathbb{Q}_p}$ (in the sense of [20, Section 1] which is given for a proper G-CW-complex Y by

$$\mathcal{H}_G^k(Y) := K^{k+1}(q(Y)\colon EG \times Y \to G\backslash Y; \widehat{\mathbb{Z}_p}) \otimes_{\widehat{\mathbb{Z}_p}} \widehat{\mathbb{Q}_p}.$$

Notice that it does not satisfy the disjoint union axiom (for arbitrary index sets) since infinite products are not compatible with $- \otimes_{\widehat{\mathbb{Z}_p}} \widehat{\mathbb{Q}_p}$, but this will not matter since we will finally consider a finite proper G-CW-complex X. Let $\mathcal{F}(X)$ be the family of subgroups H of G with $X^H \neq \emptyset$. We obtain from [20, Theorem 4.2]) (using the notation from that paper) an isomorphism of $\widehat{\mathbb{Q}_p}$-modules

$$\mathcal{H}_G^k(X) \xrightarrow{\cong} \prod_{p+q=k} H^p_{\widehat{\mathbb{Q}_p}\mathtt{Sub}(G;\mathcal{F}(X))}(X; \mathcal{H}_G^q(G/?)). \tag{3.8}$$

We get from the Atiyah-Segal Completion Theorem (see [7, Theorem 2.1]) and [20, Theorem 3.5] (using the notation of [20, Theorem 3.5]) pro-isomorphism of pro-$\mathbb{Z}$-modules

$$\{\widetilde{K}^q(BH^{n-1})\} \xrightarrow{\cong} \begin{cases} \prod_{p \in \mathcal{P}(H)} \{\operatorname{im}(\operatorname{res}_H^{H_p})/p^n \cdot \operatorname{im}(\operatorname{res}_H^{H_p})\} & q = 0; \\ \{0\} & q = 1. \end{cases}$$

Hence we obtain from Lemma 3.4 a pro-isomorphism of pro-$\mathbb{Z}$-modules

$$\prod_{p\in\mathcal{P}(H)} \{\widetilde{K}^n(BH^{n-1};\widehat{\mathbb{Z}_p}))\} \xrightarrow{\cong} \prod_{p\in\mathcal{P}(H)} \{\operatorname{im}(\operatorname{res}_H^{H_p})/p^n\cdot \operatorname{im}(\operatorname{res}_H^{H_p})\}.$$

One easily checks that it induces for each prime $p\in\mathcal{P}(H)$ an isomorphism of pro-isomorphism of pro-$\mathbb{Z}$-modules

$$\{\widetilde{K}^0(BH^{n-1};\widehat{\mathbb{Z}_p}))\} \xrightarrow{\cong} \{\operatorname{im}(\operatorname{res}_H^{H_p})/p^n\cdot \operatorname{im}(\operatorname{res}_H^{H_p})\}.$$

This implies that the two functors from $\mathtt{Sub}(G;\mathcal{F}(X))$ to the category of $\widehat{\mathbb{Q}_p}$-modules which send a an object $H\in\mathcal{F}(X)$ to $\mathcal{H}^q(G/H)=\widetilde{K}^q(BH;\widehat{\mathbb{Z}_p})\otimes_{\widehat{\mathbb{Z}_p}}\widehat{\mathbb{Q}_p}$ and to $\operatorname{im}(\operatorname{res}_H^{H_p})\otimes_{\widehat{\mathbb{Z}_p}}\widehat{\mathbb{Q}_p}$ respectively agree for even q. For odd q the functor given by $\mathcal{H}^q(G/H)=\widetilde{K}^q(BH;\widehat{\mathbb{Z}_p})\otimes_{\widehat{\mathbb{Z}_p}}\widehat{\mathbb{Q}_p}$ is trivial. Hence we obtain from (3.8) the $\widehat{\mathbb{Q}_p}$-isomorphism

$$\mathcal{H}^k_G(X) \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} H^{k+2i}_{\widehat{\mathbb{Q}_p}\mathtt{Sub}(G)}(X;\operatorname{im}(\operatorname{res}_?^{?_p})\otimes_{\widehat{\mathbb{Z}_p}}\widehat{\mathbb{Q}_p}). \tag{3.9}$$

Now one shows analogous to the argument in [20, Section 4] using [20, Theorem 5.2 (c) and Example 5.3] that there is an isomorphism of $\widehat{\mathbb{Q}_p}$-modules

$$\begin{aligned} H^{k+2i}_{\widehat{\mathbb{Q}_p}\mathtt{Sub}(G)}&(X;\operatorname{im}(\operatorname{res}_?^{?_p})\otimes_{\widehat{\mathbb{Z}_p}}\widehat{\mathbb{Q}_p}) \\ &\cong \prod_{(g)\in\operatorname{con}_p(G)}\prod_{i\in\mathbb{Z}} H^{k+2i}(C_G\langle g\rangle\backslash X^{\langle g\rangle};\widehat{\mathbb{Q}_p})). \end{aligned}\tag{3.10}$$

Now we conclude from (3.9) and (3.10).

$$\begin{aligned}\dim_{\widehat{\mathbb{Q}_p}}(\mathcal{H}_G(X)) &= \sum_{(g)\in\operatorname{con}_p(G)}\sum_{i\in\mathbb{Z}}\dim_{\widehat{\mathbb{Q}_p}}\left(H^{k+2i}(C_G\langle g\rangle\backslash X^{\langle g\rangle};\widehat{\mathbb{Q}_p})\right)\\ &= \sum_{(g)\in\operatorname{con}_p(G)}\sum_{i\in\mathbb{Z}}\dim_{\mathbb{Q}}\left(H^{k+2i}(C_G\langle g\rangle\backslash X^{\langle g\rangle};\mathbb{Q})\right).\end{aligned}$$

Since $\mathcal{H}^k_G(X)=K^{k+1}(q(Y)\colon EG\times X\to G\backslash X;\widehat{\mathbb{Z}_p})$ is $\widehat{\mathbb{Z}_p}$-isomorphic to $(\widehat{\mathbb{Z}_p})^{r^k_p(X)}$ by definition of $r^k_p(X)$, Theorem 3.7 follows. □

4. Borel homology

The material of Section 3 has analogues for Borel homology. We begin with the analogue of Theorem 3.6. Let $\mathcal{H}_*$ be a (generalized) homology theory with values in the category of $\mathbb{Z}$-modules which satisfies the *disjoint union axiom* for arbitrary index sets, i.e., for any family $\{X_i\mid i\in I\}$ the map

$$\bigoplus_{i\in I}\mathcal{H}_k(j_i)\colon \bigoplus_{i\in I}\mathcal{H}_k(X_i)\xrightarrow{\cong}\mathcal{H}_k(\coprod_{i\in I}X_i)$$

is an isomorphism, where $j_i\colon X_i\to\coprod_{i\in I}X_i$ is the canonical inclusion. Given a CW-complex X, let $\widetilde{\mathcal{H}}_k(X)$ be the kernel of the map $\mathcal{H}_k(X)\to\mathcal{H}_k(\{\bullet\})$ induced by the projection $X\to\{\bullet\}$. We define for any abelian group A

$$\mathcal{H}_k(X;A):=\mathcal{H}_{k-d}(X\times M(A,d),X\times\{\bullet\})$$

where d is some positive integer and $M(A,d)$ is the Moore space associated to A in degree d.

Theorem 4.1 (Homology of the Borel construction)**.** *Let $\mathcal{H}_*$ be a homology theory which satisfies the disjoint union axiom. Let X be a proper G-CW-complex. Let $\mathcal{P}(X)$ be the set of primes p for which p divides the order $|G_x|$ of the isotropy group G_x of the some $x\in X$. Then:*

(i) There is a long exact sequence

$$\ldots \to \mathcal{H}_{k+1}(G\backslash X) \to \bigoplus_{p\in\mathcal{P}(X)} \mathcal{H}_{k+1}(q(X)\colon EG\times_G X \to G\backslash X;\mathbb{Z}/p^\infty) \to$$
$$\to \mathcal{H}_k(EG\times_G X) \xrightarrow{\mathcal{H}_k(q(X))} \mathcal{H}_k(G\backslash X) \to \ldots;$$

(ii) The map $\mathcal{H}_k(q(X))$ appearing in assertion (i) induces after applying $-\otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$ a natural isomorphism

$$\mathcal{H}_k(q(X))\otimes_{\mathbb{Z}} \mathrm{id}\colon \mathcal{H}_k(EG\times_G X)\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \xrightarrow{\cong} \mathcal{H}_k(G\backslash X)\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right];$$

(iii) Suppose that X is a finite proper G-CW-complex. Suppose that $\mathcal{H}_k(\{\bullet\})$ is finitely generated as abelian groups for all $k\in\mathbb{Z}$. Assume that $\widetilde{\mathcal{H}}_k(BH;\mathbb{Z}/p^\infty)$ can be embedded into $(\mathbb{Z}/p^\infty)^r$ for some $r=r(k,H)$ for all $k\in\mathbb{Z}$ and all isotropy groups H of X.

Then, for appropriate natural numbers $r_k^p(X)$, there is an exact sequence

$$0\to C\to \bigoplus_{p\in\mathcal{P}(X)} (\mathbb{Z}/p^\infty)^{r_k^p(X)}\times B\to$$
$$\to \mathcal{H}_k(EG\times_G X)\xrightarrow{\mathcal{H}_k(q(X))} \mathcal{H}_k(G\backslash X)\to A\to 0,$$

where A, B and C are finite abelian groups with

$$A\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] = B\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] = C\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]=0.$$

Proof. (i) There is a canonical isomorphism

$$\bigoplus_{p\in\mathcal{P}(X)}\mathbb{Z}/p^\infty \xrightarrow{\cong} \bigoplus_{p\in\mathcal{P}(X)}\mathbb{Z}[1/p]/\mathbb{Z}\xrightarrow{\cong}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]/\mathbb{Z}.$$

Thus we obtain a short exact sequence of abelian groups

$$1\to\mathbb{Z}\to\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]\to\bigoplus_{p\in\mathcal{P}(X)}\mathbb{Z}/p^\infty\to 1.$$

The boundary of the associated Bockstein sequence yields a natural transformation of equivariant homology theories for proper G-CW-complexes satisfying the disjoint union axiom

$$\partial_k(X)\colon \bigoplus_{p\in\mathcal{P}(X)}\mathcal{H}_k(q(X)\colon EG\times_G X\to G\backslash X;\mathbb{Z}/p^\infty)$$
$$\to \mathcal{H}_{k-1}(q(X)\colon EG\times_G X\to G\backslash X).$$

If H is a finite subgroup and $\mathcal{P}$ a set of primes which contains all primes dividing the order of H, then

$$\widetilde{H}_k(BH,M)\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}}\right] = \widetilde{H}_k\left(BH, M\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}}\right]\right)=0$$

holds for all $k\in\mathbb{Z}$ and all abelian groups M [13, Corollary 10.2 in Chapter III on page 84]. By the Atiyah-Hirzebruch spectral sequence we conclude that

$$\widetilde{\mathcal{H}}_{k-1}\left(BH;\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]\right))\cong\mathcal{H}_k\left(EG\times_G G/H\to G\backslash(G/H);\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]\right)$$

vanishes for all $k\in\mathbb{Z}$ and all subgroups $H\subseteq$ that appear as isotropy group of X. This implies that $\mathcal{H}_k\left(q(X)\colon EG\times_G X\to G\backslash X;\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]\right)$ vanishes for all $k\in\mathbb{Z}$. Hence $\partial_k(X)$ is an isomorphism.

(ii) Notice that the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]$ is exact. We have already shown that

$$\mathcal{H}_k(q(X)\colon EG\times_G X \to G\backslash X)\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right] \cong \mathcal{H}_k\left(EG\times_G X \to G\backslash X; \mathbb{Z}\left[\frac{1}{\mathcal{P}(X)}\right]\right)$$

vanishes for all $k\in\mathbb{Z}$. Now the claim follow from the long exact homology sequence associated to $q(X\colon EG\times_G X \to G\backslash X)$.

(iii) The proof is analogous to the proof of Theorem 4.1 (iii), as long one has Lemma 4.2 available which we explain next.

Fix a prime p. For an abelian group A define $A^* := \hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$. If A is a finite abelian p-group, then A and A^* are abstractly isomorphic as abelian groups. If A is an abelian p-group, then A is the colimit $\operatorname{colim}_{n\to\infty}\ker(p^n\cdot \mathrm{id}_A\colon A\to A)$ and hence $A^* = \lim_{n\to\infty}(\ker(p^n\cdot\mathrm{id}_A)^*$. Hence A^* is the inverse limit of a system of finite abelian p-groups and therefore carries a canonical $\widehat{\mathbb{Z}_p}$-module structure and a canonical structure of a totally disconnected compact topological Hausdorff group. $\square$

Lemma 4.2.

(i) The following assertions are equivalent for an abelian p-group A:
   (a) A can be embedded into $(\mathbb{Z}/p^\infty)^r$ for some $r=r(A)$;
   (b) $A\cong(\mathbb{Z}/p^\infty)^r\times T$ for some $r=r(A)$ and some finite abelian p-group T;
   (c) A^* is a finitely generated $\widehat{\mathbb{Z}_p}$-module;

(ii) The category of groups A, which embed into $(\mathbb{Z}/p^\infty)^r$ for some $r=r(A)$, form a Serre category, i.e., given an exact sequence of abelian groups $0\to A\to B\to C\to 0$, then B belongs to the category if and only if both A and C do.

Proof. Since the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible and hence injective, an exact sequence $0\to A\xrightarrow{i} B\xrightarrow{p} C\to 0$ of abelian p-groups induces an exact sequence of totally disconnected compact topological groups $0\to C^*\xrightarrow{p^*} B^*\xrightarrow{i^*} A^*\to 0$.

For a topological group G let $\widehat{G}$ be its Pontryagin dual, i.e., the abelian group of continuous homomorphisms from G to S^1. Given an exact sequence $0\to G\xrightarrow{j} H\xrightarrow{q} K\to 0$ of totally disconnected compact topological Hausdorff groups, we obtain an induced exact sequence $0\to\widehat{K}\xrightarrow{\widehat{q}}\widehat{H}\xrightarrow{\widehat{j}}\widehat{G}$. For an abelian p-group A we obtain a canonical homomorphism $\phi_A\colon A\to\widehat{A^*}$ sending a to the continuous group homomorphism $\hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})\to S^1,\ \ f\mapsto\exp(2\pi i f(a))$. Thus we obtain for any short exact sequence $0\to A\xrightarrow{i} B\xrightarrow{p} C\to 0$ of abelian p-groups and commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\phi_A"] & B \arrow[r, "p"] \arrow[d, "\phi_B"] & C \arrow[r] \arrow[d, "\phi_C"] & 0 \\
0 \arrow[r] & \widehat{A^*} \arrow[r, "\widehat{i^*}"] & \widehat{B^*} \arrow[r, "\widehat{p^*}"] & \widehat{C^*} &
\end{tikzcd}$$

One easily checks that for a finite abelian p-group A the canonical map ϕ_A is bijective. Next we show that for any abelian p-group B the map ϕ_B is injective. Consider any element $b\in B$. Let $A\subseteq B$ be the finite cyclic subgroup generated by b. We conclude from the commutative diagram above that $\phi_B(b)=0$ implies that $\phi_A(b)=0$ and hence $b=0$.

Now suppose that $0\to A\xrightarrow{i} B\xrightarrow{p} C\to 0$ is an exact sequence of abelian p-groups. Recall that $\widehat{\mathbb{Z}_p}$ is a principal ideal domain. Hence B^* is a finitely generated $\widehat{\mathbb{Z}_p}$-module, if and only if both A^* and C^* are finitely generated $\widehat{\mathbb{Z}_p}$-modules. This shows that the category of abelian p-groups B for which B^* is a finitely generated $\widehat{\mathbb{Z}_p}$-module is a Serre category.

Let A be an abelian group which can be embedded into $B = (\mathbb{Z}/p^\infty)^r$. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be the exact sequence of abelian p-groups with $B = (\mathbb{Z}/p^\infty)^r$ and $C = B/A$. We want to show that $\widehat{p^*}$ is surjective. Recall that $\widehat{\mathbb{Z}_p}$ is a principal ideal domain and the prime ideals different from $\{0\}$ and $\widehat{\mathbb{Z}_p}$ look like (p^n) for n running through the positive integers. We can find isomorphisms of $\widehat{\mathbb{Z}_p}$-modules $B^* \cong (\widehat{\mathbb{Z}_p})^a \bigoplus (\widehat{\mathbb{Z}_p})^b$ and $C^* \cong (\widehat{\mathbb{Z}_p})^a$ such that the map $p^* \colon C^* \to B^*$ looks under these identifications like

$$(\widehat{\mathbb{Z}_p})^a \to (\widehat{\mathbb{Z}_p})^a \bigoplus (\widehat{\mathbb{Z}_p})^b, \quad (x_1, x_2, \ldots, x_a) \mapsto (p^{n_1}x_1, p^{n_2}x_2, \ldots, p^{n_a}x_a, 0, \ldots, 0)$$

for appropriate positive integers n_1, n_2, $\ldots n_a$. Hence it suffices to show for each positive integers n that the map $p^n \cdot \mathrm{id} \colon \widehat{\mathbb{Z}_p} \to \widehat{\mathbb{Z}_p}$ induces epimorphism between the Pontryagin duals. This induced map on the Pontryagin duals can be identified with $p^n \cdot \mathrm{id} \colon \mathbb{Z}/p^\infty \to \mathbb{Z}/p^\infty$ which is indeed surjective. Hence we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\phi_A"] & B \arrow[r, "p"] \arrow[d, "\phi_B"] & C \arrow[r] \arrow[d, "\phi_C"] & 0 \\
0 \arrow[r] & \widehat{A^*} \arrow[r, "\widehat{i^*}"] & \widehat{B^*} \arrow[r, "\widehat{q^*}"] & \widehat{C^*} \arrow[r] & 0
\end{tikzcd}$$

Since ϕ_B is bijective for $B = (\mathbb{Z}/p^\infty)^r$ and ϕ_C is injective, we conclude that ϕ_A is bijective. Since A^* is a finitely generated $\widehat{\mathbb{Z}_p}$ module, $\widehat{A^*}$ and hence A is isomorphic to $(\mathbb{Z}/p^\infty)^r \times T$ for some finite abelian p-group T and integer $r \geq 0$.

If A is isomorphic to $(\mathbb{Z}/p^\infty)^r \times T$ for some finite abelian p-group and integer $r \geq 0$, then A^* is a finitely generated $\widehat{\mathbb{Z}_p}$-module.

Suppose that A is an abelian p-group such that A^* is a finitely generated $\widehat{\mathbb{Z}_p}$-module, then A embeds into $\widehat{A^*}$ which is isomorphic to $\mathbb{Z}/p^\infty)^s \times T$ for an integer $s \geq 0$ and a finite abelian p-group T. Hence A embeds into $\mathbb{Z}/p^\infty)^r$ for some integer $r \geq 0$. This finishes the proof of Lemma 4.2 and thus of Theorem 4.1. □

5. Universal coefficient theorems for K-theory

A proof of the following Universal Coefficients Theorem can be found in [4] and [40, (3.1)], the homological version then follows from [2, Note 9 and 15].

Theorem 5.1 (Universal Coefficient Theorem for K-theory)**.** *For any CW-complex X there is a short exact sequence*

$$0 \to \mathrm{ext}_{\mathbb{Z}}(K_{*-1}(X), \mathbb{Z}) \to K^*(X) \to \hom_{\mathbb{Z}}(K_*(X), \mathbb{Z}) \to 0.$$

If X is a finite CW-complex, there is also the K-homological version

$$0 \to \mathrm{ext}_{\mathbb{Z}}(K^{*+1}(X), \mathbb{Z}) \to K_*(X) \to \hom_{\mathbb{Z}}(K^*(X), \mathbb{Z}) \to 0.$$

Corollary 5.2. *For any G-CW-complex X there is a short exact sequence*

$$0 \to \mathrm{ext}_{\mathbb{Z}}(K_{*-1}(EG \times_G X), \mathbb{Z}) \to K^*(EG \times_G X) \\ \to \hom_{\mathbb{Z}}(K_*(EG \times_G X), \mathbb{Z}) \to 0.$$

Also the homological version of the Universal Coefficient Theorem has an equivariant counterpart.

Remark 5.3. Recall that the Completion Theorem 2.4 for a finite proper G-CW-complex X yields an isomorphism $K^*(EG \times_G X) \xrightarrow{\cong} \varprojlim_{n \geq 1} K^*((EG \times_G X)^n)$. Hence $K^*(EG \times_G X)$ can be regarded as a pro-discrete group. Thus it carries a topology, the so-called skeletal topology. In terms of topological abelian groups the main statement of the Completion Theorem 2.4 says that there is a canonical isomorphism of topological groups $K^*(EG\times_G X) \cong K^*_G(X)\widehat{_I}$, where $K^*_G(X)\widehat{_I}$ carries the I-adic topology. The exact sequence introduced in Remark 2.7 then provides an Equivariant Universal Coefficient Theorem for K-homology for finite proper G-CW-complexes X, which says that the following sequence is exact

$$0 \to \operatorname{ext}^1_{\mathrm{cts}}(K^{*+1}(EG \times_G X), \mathbb{Z}) \to K_*(EG\times_G X) \to \hom_{\mathrm{cts}}(K^*(EG \times_G X), \mathbb{Z}) \to 0.$$

The following lemma will be needed for the proof of Theorem 0.1, which we will give below.

Lemma 5.4. *Suppose that X is a finite proper G-CW-complex.*

Then the two numbers $r^k_p(X)$ and $r^p_{k-1}(X)$ defined in Theorem 3.6 (iii) and Theorem 4.1 (iii) coincide for all primes p and all $k \in \mathbb{Z}$.

Proof. For an abelian group A let $A\widehat{_p}$ be its p-adic completion, i.e., the inverse limit $\varprojlim_{n\to\infty} A/p^nA$. There is a canonical $\mathbb{Z}\widehat{_p}$-module structure on $A\widehat{_p}$. Define

$$\dim\widehat{_p}(A) := \dim_{\mathbb{Q}\widehat{_p}}(A\widehat{_p} \otimes_{\mathbb{Z}\widehat{_p}} \mathbb{Q}\widehat{_p}).$$

One easily checks

$$\dim\widehat{_p}(\mathbb{Z}) = 1; \tag{5.5}$$

$$\dim\widehat{_p}(\mathbb{Z}\widehat{_p}) = 1; \tag{5.6}$$

$$\dim\widehat{_p}(\mathbb{Z}\widehat{_q}) = 0, \quad \text{if } p \neq q; \tag{5.7}$$

$$\dim\widehat{_p}(A) = 0, \quad \text{if } A \text{ is finite}. \tag{5.8}$$

Next want to show for an exact sequence of abelian groups $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$

$$\dim\widehat{_p}(B) = \dim\widehat{_p}(A) + \dim\widehat{_p}(C). \tag{5.9}$$

We obtain an exact sequence of $\mathbb{Z}\widehat{_p}$-modules $0 \to A\widehat{_p} \xrightarrow{i\widehat{_p}} B\widehat{_p} \xrightarrow{p\widehat{_p}} C\widehat{_p} \to 0$ from [6, Corollary 10.3]. The induced sequence of $\mathbb{Q}\widehat{_p}$-modules $0 \to A\widehat{_p} \otimes_{\mathbb{Z}\widehat{_p}} \mathbb{Q}\widehat{_p} \xrightarrow{i\widehat{_p}\otimes_{\mathbb{Z}\widehat{_p}}\mathrm{id}} B\widehat{_p} \otimes_{\mathbb{Z}\widehat{_p}} \mathbb{Q}\widehat{_p} \xrightarrow{p\widehat{_p}\otimes_{\mathbb{Z}\widehat{_p}}\mathrm{id}} C\widehat{_p} \otimes_{\mathbb{Z}\widehat{_p}} \mathbb{Q}\widehat{_p} \to 0$ is exact. Now (5.9) follows. We conclude from (5.5), (5.8) and (5.9) for any finitely generated abelian group A

$$\dim\widehat{_p}(A) = \dim_{\mathbb{Z}}(A), \tag{5.10}$$

where $\dim_{\mathbb{Z}}(A)$ is the dimension of the rational vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let C_* be a finite dimensional chain complex of abelian groups such that $\dim\widehat{_p}(C_k)$ is finite and $H_k(C_*)$ is finite for each $k \in \mathbb{Z}$. Then we conclude from (5.8) and (5.9)

$$\sum_{k\in\mathbb{Z}} (-1)^k \cdot \dim\widehat{_p}(C_k) = 0. \tag{5.11}$$

Theorem 3.6 (iii) implies that there is a 2-dimensional chain complex

$$\ldots \to 0 \to K^k(G\backslash X) \to K^k(EG \times_G X) \to \prod_{p\in\mathcal{P}(X)} (\mathbb{Z}\widehat{_p})^{r^k_p(X)} \to 0 \to \ldots,$$

whose homology is finite, and that $K^k(G\backslash X)$ is a finitely generated abelian group. We conclude from (5.10) and (5.11) for any prime p

$$\dim\widehat{_p}(K^k(EG \times_G X)) = r^k_p(X) + \dim_{\mathbb{Z}}(K^k(G\backslash X)), \tag{5.12}$$

where $r^k_p(X)$ is defined to be 0 for $p \notin \mathcal{P}(X)$.

We conclude from Theorem 4.1 (iii) that there is a 2-dimensional chain complex with finite homology

$$\begin{aligned} \ldots 0 \to \operatorname{ext}_{\mathbb{Z}}(K_k(G\backslash X), \mathbb{Z}) \to \operatorname{ext}_{\mathbb{Z}}(K_k(EG \times_G X, \mathbb{Z}) \\ \to \operatorname{ext}_{\mathbb{Z}}(\bigoplus_{p \in \mathcal{P}(X)} (\mathbb{Z}/p^\infty)^{r_k^p(X)}, \mathbb{Z}) \to 0 \to 0 \to \ldots \end{aligned}$$

Since $K_k(G\backslash X), \mathbb{Z})$ is finitely generated abelian, $\operatorname{ext}_{\mathbb{Z}}(K_k(G\backslash X), \mathbb{Z})$ is finite. The $\mathbb{Z}\widehat{_p}$-module $\operatorname{ext}(\mathbb{Z}/p^\infty, \mathbb{Z}) = \operatorname{ext}(\mathbb{Z}/p^\infty, \mathbb{Z})\widehat{_p}$ is isomorphic to $\mathbb{Z}\widehat{_p}$ (see [37, Example 3.3.3 on page 73]). We conclude from (5.8) and (5.11) for any prime p

$$\dim\widehat{_p}(\operatorname{ext}_{\mathbb{Z}}(K_k(G\backslash X), \mathbb{Z})) \quad = \quad r_k^p(X). \tag{5.13}$$

Theorem 4.1 (iii) implies that the map

$$\hom_{\mathbb{Z}}(K_k(q(X)), \mathbb{Z})\colon \hom_{\mathbb{Z}}(K_k(G\backslash X), \mathbb{Z}) \to \hom_{\mathbb{Z}}(K_k(EG \times_G X), \mathbb{Z})$$

is injective and has finite cokernel. Since $K_k(G\backslash X)$ is a finitely generated abelian group, we conclude from (5.9), (5.10) and the Universal Coefficient Theorem for non-equivariant K-theory 5.1 applied to $G\backslash X$

$$\begin{aligned} \dim\widehat{_p}(\hom_{\mathbb{Z}}(K_k(EG \times_G X, \mathbb{Z})) &= \dim_{\mathbb{Z}}(K_k(G\backslash X)) \\ &= \dim_{\mathbb{Z}}(K^k(G\backslash X)). \end{aligned} \tag{5.14}$$

We conclude from Theorem 5.2 and equations (5.9), (5.13) and (5.14)

$$\dim\widehat{_p}(K^k(EG \times_G X)) \quad = \quad r_{k-1}^p(X) + \dim_{\mathbb{Z}}(K^k(G\backslash X). \tag{5.15}$$

Now (5.12) and (5.15) imply $r_p^k(X) = r_{k-1}^p(X)$. This finishes the proof of Lemma 5.4. $\square$

Lemma 5.16. *Let $f\colon X \to Y$ be a map of CW-complexes which induces isomorphisms $H_n(X;\mathbb{Z}) \xrightarrow{\cong} H_n(Y;\mathbb{Z})$ for all $n \geq 0$. Then for any cohomology theory $\mathcal{H}^*$ and any homology theory $\mathcal{H}_*$ (satisfying the disjoint union axiom) and any $n \in \mathbb{Z}$, the maps $\mathcal{H}^n(f)\colon \mathcal{H}^n(Y) \to \mathcal{H}^n(X)$ and $\mathcal{H}_n(f)\colon \mathcal{H}_n(X) \to \mathcal{H}_n(Y)$ are isomorphisms.*

Proof. Because of the excision and the suspension axiom for the cohomology theory $\mathcal{H}^*$ if suffices to establish the result for the twofold suspension of f. Since the twofold suspension of a CW-complex is simply-connected it follows from the Hurewicz theorem that the twofold suspension of f is a homotopy equivalence. The claim then follows from the homotopy invariance axiom for the cohomology theory $\mathcal{H}^*$. $\square$

Now we can give the proof of Theorem 0.1.

Proof of Theorem 0.1. This follows from Theorem 3.6, Theorem 3.7, Theorem 4.1, Lemma 5.4 and Lemma 5.16, as soon as we have shown that for every prime p, every element $g \in G$ of p-power order and every $k \in \mathbb{Z}$ we have an isomorphism of $\mathbb{Q}$-modules

$$H^k(BC_G\langle g\rangle;\mathbb{Q}) \quad \cong \quad H^k(X^{\langle g\rangle}/C_G\langle g\rangle;\mathbb{Q}). \tag{5.17}$$

We have $\widetilde{H}_k(X;\mathbb{Z}) = 0$ for all $k \in \mathbb{Z}$ by assumption. Hence we get $\widetilde{H}_k(X;\mathbb{F}_p) = 0$ for all $k \in \mathbb{Z}$, where $\mathbb{F}_p$ is the field with p elements. By Smith theory (see [12, Theorem 5.2 in III.5 on page 130]) we conclude that $X^{\langle g\rangle}$ is non-empty and $\widetilde{H}_k(X^{\langle g\rangle};\mathbb{F}_p) = 0$ for all $k \in \mathbb{Z}$. Hence $\widetilde{H}_k(X^{\langle g\rangle};\mathbb{Q}) = 0$ for all $k \in \mathbb{Z}$ and $\langle g\rangle$ is subconjugated to an isotropy group of X. Since X is a finite G-CW-complex, we conclude that the sets $\operatorname{con}_p(G)$ and $\mathcal{P}(G)$ are finite.

Let Y be any proper $C_G\langle g\rangle$-CW-complex with $\widetilde{H}_k(Y;\mathbb{Q}) = 0$ for all $k \in \mathbb{Z}$. Choose a $C_G\langle g\rangle$-map $f\colon Y \to \underline{E}C_G\langle g\rangle$. The $\mathbb{Q}[C_G\langle g\rangle]$-chain map

$$C_*(f) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}} \colon C_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \to C_*(\underline{E}C_G\langle g\rangle) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a $\mathbb{Q}[C_G\langle g\rangle]$-chain map of projective $\mathbb{Q}[C_G\langle g\rangle]$-chain complexes which induces an isomorphism on homology. Hence it is a $\mathbb{Q}[C_G\langle g\rangle]$-chain homotopy equivalence. It induces a chain homotopy equivalence of $\mathbb{Q}$-chain complexes $C_*(f) \otimes_{\mathbb{Z}[G\langle g\rangle]} \mathbb{Q}$. Hence we obtain a $\mathbb{Q}$-isomorphism $H^k(Y/C_G\langle g\rangle;\mathbb{Q}) \xrightarrow{\cong} H^k(\underline{E}C_G\langle g\rangle)/C_G\langle g\rangle;\mathbb{Q})$. If we apply this to $Y = X$ and $Y = EC_G\langle g\rangle$, we get (5.17). This finishes the proof of Theorem 0.1. □

Next we deal with the equivariant universal coefficient theorem. For a finite proper G-CW-complex its equivariant K-homology $K_k^G(X)$ can be identified with the expression $K_k(\mathbb{C}, C_0(X) \rtimes G)$ given by Kasparov's KK-theory. The Kasparov intersection pairing yields for a finite proper G-CW-complex a pairing

$$K_G^k(X) \otimes K_k^G(X) \quad \to \quad KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}. \tag{5.18}$$

Taking adjoints gives homomorphisms

$$K_G^*(X) \quad \to \quad \hom_{\mathbb{Z}}(K_*^G(X),\mathbb{Z}); \tag{5.19}$$

$$K_*^G(X) \quad \to \quad \hom_{\mathbb{Z}}(K_G^*(X),\mathbb{Z}). \tag{5.20}$$

Now we give the proof of the Equivariant Universal Coefficient Theorem for K-theory 0.3

Proof of Theorem 0.3. Assume first that X is a proper orbit G/H. Green's imprimitivity theorem [14, §2] in this case says that $C_0(G/H) \rtimes G$ and $\mathbb{C}H$ are Morita equivalent, and therefore they are also KK-equivalent. Hence $K_G^*(G/H) \cong K_*(C_0(G/H)\rtimes G) \cong K_*(\mathbb{C}H)$, and $K_*^G(G/H) \cong KK(C_0(G/H)\rtimes G,\mathbb{C}) \cong KK(\mathbb{C}H,\mathbb{C})$. Since $\mathbb{C}H$ is a finite C^*-algebra $K_G^*(G/H)$ and $K_*^G(G/H)$ are both finitely generated (projective) $\mathbb{Z}$-modules it follows by induction over the dimension and a subinduction over the number of cells of top dimension that $K_G^*(X)$ and $K_*^G(X)$ are finitely generated for all finite proper G-CW-complexes X. In particular (0.4) is exact if and only if (0.5) is.

By a result of Rosenberg and Schochet [31, Theorem 1.17 and Theorem 7.10] the sequence (0.4) is exact and splits unnaturally if $C_0(X) \rtimes G$ is contained in the category of C^*-algebras $\mathcal{N}$, introduced in [31]. We shall prove that $C_0(X) \rtimes G \in \mathcal{N}$ by induction over the number of cells of X.

Since $C_0(G/H) \rtimes G$ is KK-equivalent to $\mathbb{C}H$ and the category $\mathcal{N}$ contains all C^*-algebras which are KK-equivalent to finite C^*-algebras (c.f. [9, 22.3.5 (b)]) we conclude that the C^*-algebras $C_0(G/H) \rtimes G$ for all proper orbits G/H are in $\mathcal{N}$. A second property of the category $\mathcal{N}$ is that it is closed under extensions [9, 22.3.4 (N3)]. If X is a finite proper G-CW-complex for which $C_0(X)\rtimes G$ is in $\mathcal{N}$, and Y is obtained from X by attaching a proper n-dimensional G-cell $Z = G/H \times e_n$ we obtain an exact sequence $0 \to C_0(Z) \to C_0(Y) \to C_0(X) \to 0$. Taking the crossed product with G we obtain an exact sequence

$$0 \to C_0(Z) \rtimes G \to C_0(Y) \rtimes G \to C_0(X) \rtimes G \to 0.$$

The extension property of $\mathcal{N}$ yields that $C_0(Y) \rtimes G$ is in $\mathcal{N}$. □

Remark 5.21. Let G be a finite group. The isomorphism of abelian groups

$$\mu\colon R(G) \to \hom_{\mathbb{Z}}(R(G),\mathbb{Z})$$

which sends $[V]$ to the map $\mu(V)\colon R(G) \to \mathbb{Z}, [W] \mapsto \dim_{\mathbb{C}}(V \otimes_{\mathbb{C}} W)^G)$, is the special case $X = \{\bullet\}$ of the Equivariant Universal Coefficient Theorem for K-theory 0.3 It is in fact an isomorphism of $R(G)$-modules and we get for any $R_{\mathbb{C}}(G)$-module M

a natural isomorphism of $R_{\mathbb{C}}(G)$-modules $\operatorname{ext}^i_{R_{\mathbb{C}}(G)}(M, R_{\mathbb{C}}(G)) \xrightarrow{\cong} \operatorname{ext}^i_{\mathbb{Z}}(M, \mathbb{Z})$ for $i \geq 0$ (see [26, 2.5 and 2.10]). Using change of ring isomorphisms, the Equivariant Universal Coefficient Theorem for K-theory 0.3 is equivalent to the exactness of the short exact sequences

$$0 \to \operatorname{ext}_{R(G)}(K^G_{*-1}(X), R(G)) \to K^*_G(X) \to \hom_{R(G)}(K^G_*(X), R(G)) \to 0,$$
$$0 \to \operatorname{ext}_{R(G)}(K^{*+1}_G(X), R(G)) \to K^G_*(X) \to \hom_{R(G)}(K^*_G(X), R(G)) \to 0.$$

The exactness of these sequences has been proved by Bökstedt [10] using the concept of Anderson duality. Bökstedt's technique also can be used to prove the Equivariant Universal Coefficient Theorem for K-theory 0.3 for a finite group G.

6. Examples

Example 6.1. Consider the group $G = SL_3(\mathbb{Z})$. We conclude from [33, Corollary on page 8] that for $G = SL_3(\mathbb{Z})$ the quotient space $G\backslash\underline{E}G$ is contractible. Hence the long exact sequence in Theorem 3.6 (i) reduces to an isomorphism

$$\widetilde{\mathcal{H}}^k(BG) \xrightarrow{\cong} \prod_{p \in \mathcal{P}(G)} \widetilde{\mathcal{H}}^k(BG; \widehat{\mathbb{Z}_p})$$

and the one of Theorem 4.1 (i) to the isomorphism

$$\bigoplus_{p \in \mathcal{P}(G)} \widetilde{\mathcal{H}}_{k+1}(BG; \mathbb{Z}/p^\infty) \xrightarrow{\cong} \widetilde{\mathcal{H}}_k(BG).$$

¿From the classification of finite subgroups of $SL_3(\mathbb{Z})$ we see that $SL_3(\mathbb{Z})$ contains up to conjugacy four subgroups of order 2 and two cyclic subgroups of order 3. The cyclic subgroups of order 3 have finite normalizers and the action of the normalizer on each of this group is non-trivial. There are no cyclic subgroups of order p for a prime p different from 2 and 3. Hence we see that $\operatorname{con}_2(G)$ contains four elements and $\operatorname{con}_3(G)$ contains two elements. The rational homology of all the centralizers of elements in $\operatorname{con}_2(G)$ and $\operatorname{con}_3(G)$ agree with the one of the trivial group (see [3, Example 6.6]). We get in the notation of Theorem 0.1 that $r^0_2(G) = 4$, $r^0_3(G) = 2$, $r^1_2(G) = 0$, $r^0_3(G) = 1$ and $r^k_p(G) = 0$ for $p \neq 2, 3$ and all k. We conclude from Theorem 0.1 that there is an exact sequence

$$0 \to \widetilde{K}^0(BG) \to (\widehat{\mathbb{Z}_2})^4 \oplus (\widehat{\mathbb{Z}_3})^2 \oplus B_0 \to C_0 \to 0$$

and an isomorphism

$$\widetilde{K}^1(BG) \cong D_1$$

for finite abelian groups B_0, C_0 and D_1 which vanish after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{6}\right]$. Actually the computation using Brown-Petersen cohomology and the Conner-Floyd relation in [35] show that one can choose the groups B_0, C_0 and D_1 to be zero.

The next result shall illustrate that the knowledge of the spaces $\underline{E}G$ allows to reduce the computation of the (co-)homology of BG to the one of its finite subgroups. Let G be a discrete group. Let $\mathcal{MFIN}$ be the subfamily of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following assertions concerning G:

(M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

(NM) $M \in \mathcal{MFIN} \to NM = M$;

For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq Q$. By attaching free G-cells we get an inclusion of G-CW-complexes $j_1\colon \coprod_{i\in I} G\times_{M_i} EM_i \to EG$, The we obtain by [24, Corollary 2.11] a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I} G\times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle f_1} \\ \coprod_{i\in I} G/M_i & \xrightarrow{k_1} & \underline{E}G \end{array} \tag{6.2}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

Here are some examples of groups Q which satisfy conditions (M) and (NM):

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$.

  The conditions (M), (NM) are satisfied by [23, Lemma 6.3]. There are models for $\underline{E}G$ whose underlying space is $\mathbb{R}^n$. The quotient $G\backslash\underline{E}G$ looks like the quotient of T^n by a finite group.
- Fuchsian groups F

  The conditions (M), (NM) are satisfied. (see for instance [23, Lemma 4.5]). In [23] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated. The quotients $G\backslash\underline{E}G$ are closed orientable surfaces.
- One-relator groups G

  Let G be a one-relator group. Let $G = \langle (q_i)_{i\in I} \mid r\rangle$ be a presentation with one relation. We only have to consider the case, where Q contains torsion. Let F be the free group with basis $\{q_i \mid i \in I\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in Q$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $q \in Q$ the implication $q^{-1}Cq \cap C \neq 1 \Rightarrow q \in C$ holds. These claims follows from [25, Propositions 5.17, 5.18 and 5.19 in II.5 on pages 107 and 108]. Hence Q satisfies (M) and (NM). There are explicit two-dimensional models for $\underline{E}G$ with one 0-cell $G/C\times D^0$, as many free 1-cells $G\times D^1$ as there are elements in I and one free 2-cell $G\times D^2$ (see [13, Exercise 2 (c) II. 5 on page 44]).

Theorem 6.3. *Suppose that the discrete group G satisfies conditions (M) and (NM). Let $p\colon BG \to G\backslash\underline{E}G$ be the map induced by the canonical G-map $EG \to \underline{E}G$ and $Bj_i\colon BM_i \to BG$ be the map induced by the inclusion $j_i\colon M_i \to G$. Then*

(i) Let $\mathcal{H}^*$ be a cohomology theory satisfying the disjoint union axiom. Then there is a long exact sequence

$$\ldots \xrightarrow{\prod_{i\in I}\widetilde{\mathcal{H}}^{k-1}(Bj_i)} \prod_{i\in I}\widetilde{\mathcal{H}}^{k-1}(BM_i) \xrightarrow{\delta^{k-1}} \widetilde{\mathcal{H}}^k(G\backslash\underline{E}G)$$
$$\xrightarrow{\widetilde{\mathcal{H}}^k(p)} \widetilde{\mathcal{H}}^k(BG) \xrightarrow{\prod_{i\in I}\widetilde{\mathcal{H}}^k(Bj_i)} \prod_{i\in I}\widetilde{\mathcal{H}}^k(BM_i) \xrightarrow{\delta^k} \widetilde{\mathcal{H}}^{k+1}(G\backslash\underline{E}G) \to \ldots$$

The map $\widetilde{\mathcal{H}}^k(p)$ is split injective after applying $-\otimes_{\mathbb{Z}}\mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right]$, provided that I is finite;

(ii) Let $\mathcal{H}_*$ be a homology theory satisfying the disjoint union axiom. Then there is a long exact sequence

$$\ldots \xrightarrow{\widetilde{\mathcal{H}}_{k+1}(p)} \widetilde{\mathcal{H}}_{k+1}(G\backslash \underline{E}G) \xrightarrow{\partial_{k+1}} \bigoplus_{i\in I} \widetilde{\mathcal{H}}_k(BM_i) \xrightarrow{\bigoplus_{i\in I}\widetilde{\mathcal{H}}_k(Bj_i)} \widetilde{\mathcal{H}}_k(BG)$$
$$\xrightarrow{\widetilde{\mathcal{H}}_k(p)} \widetilde{\mathcal{H}}_k(G\backslash \underline{E}G) \xrightarrow{\partial_k} \bigoplus_{i\in I} \widetilde{\mathcal{H}}_{k-1}(BM_i) \xrightarrow{\bigoplus_{i\in I}\widetilde{\mathcal{H}}_{k-1}(Bj_i)} \ldots$$

The map $\widetilde{\mathcal{H}}_k(p)$ is split surjective after applying $-\otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right]$.

Proof. These long exact sequences come from the Mayer-Vietoris sequences associated to the pushout which is obtained from the G-pushout (6.2) by dividing out the G-action. For the splitting after applying $-\otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\mathcal{P}(G)}\right]$ see Lemma 3.3 and its obvious homological version. □

Example 6.4. Let F be a cocompact Fuchsian group with presentation

$F = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_t \mid c_1^{\gamma_1} = \ldots = c_t^{\gamma_t} = c_1^{-1}\cdots c_t^{-1}[a_1,b_1]\cdots[a_g,b_g] = 1\rangle$

for integers $g, t \geq 0$ and $\gamma_i > 1$. Then $G\backslash\underline{E}G$ is an orientable closed surface S_g of genus g. Since S_g is stably a wedge of spheres, we have

$$\mathcal{H}^n(S_g) \cong \mathcal{H}^n(\{\bullet\}) \oplus \mathcal{H}^{n-1}(\{\bullet\})^{2g} \oplus \mathcal{H}^{n-2}(\{\bullet\}).$$

If we suppose that $\mathcal{H}^n(\{\bullet\})$ is torsionfree for all $n \in \mathbb{Z}$, then we obtain from Theorem 6.3 (i) for every $n \in \mathbb{Z}$ the exact sequence

$$0 \to \mathcal{H}^n(S_g) \to \widetilde{\mathcal{H}}^n(BG) \to \prod_{i=1}^{t} \widetilde{\mathcal{H}}^n(B\mathbb{Z}/\gamma_i) \to 0.$$

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Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany

Rheinische Wilhelms-Universität Bonn, Mathematisches Institut, Endenicher Allee 62, 53115 Bonn, Germany

E-mail address: `joachim@math.uni-muenster.de`

URL: `http://www.math.uni-muenster.de/u/joachim/`

E-mail address: `wolfgang.lueck@him.uni-bonn.de`

URL: `http://www.him.uni-bonn.de/lueck/`