APPROXIMATING THE FIRST L^2-BETTI NUMBER OF RESIDUALLY FINITE GROUPS

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Abstract. We show that the first L^2-betti number of a finitely generated residually finite group can be estimated from below by using ordinary first betti numbers of finite index normal subgroups. As an application we construct a finitely generated infinite residually finite torsion group with positive first L^2-betti number.

1. Introduction

Let G be a finitely generated residually finite group G and let $\{N_i\}_{i\in\mathbb{N}}$ be a sequence of finite index normal subgroups of G such that

$$N_1 \geq N_2 \geq \dots, \quad \text{and} \quad \bigcap_{i=1}^{\infty} N_i = \{1\}. \tag{1}$$

The approximation theorem proved by the first author in [5] implies that if G is finitely presented, then

$$b_1^{(2)}(G) = \lim_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]}. \tag{2}$$

where $b_1^{(2)}(G)$ is the first L^2-betti number of G. In particular the limit in (2) exists and is independent of the choice of the sequence of normal subgroups satisfying (1).

The question of whether (2) holds for any finitely generated group was open until now. It is partially motivated by some other open problems in group theory. For instance, the affirmative answer would imply the existence of a non-residually finite hyperbolic group [9] and would disprove either the cost vs first L^2-betti number conjecture or the fixed price conjecture (see [10] for details).

In this paper we first prove the following.

Theorem 1.1. *Let G be a finitely generated residually finite group. For every sequence of finite index normal subgroups $\{N_i\}_{i\in\mathbb{N}}$ of G satisfying (1), we have*

$$b_1^{(2)}(G) \geq \limsup_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]}. \tag{3}$$

Unfortunately Theorem 1.1 is not sufficient for the above-mentioned applications. In fact, the opposite inequality would be sufficient, but our next result shows that it does not hold. Observe that the the right side of (2) equals 0 for any sequence $\{N_i\}$ whenever G is a torsion group.

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Theorem 1.2. *For every prime p, there exists a finitely generated infinite residually finite p-group with positive first L^2-betti number.*

Our interest in groups constructed in Theorem 1.2 is also motivated by von-Neumann-type problems. Recall that the original von Neumann problem (sometimes referred to as the von Neumann–Day problem) asks whether there exist non-amenable groups without non-abelian free subgroups. This question was first answered affirmatively by Olshanskii in [7] and since then many other examples have been constructed including residually finite [3] and finitely presented ones [8]. Similar problems were considered for groups satisfying other conditions close to non-amenability (see, e.g., [2], [9]). Interestingly, a result of Lackenby [4, Theorem 1.6] implies that if a finitely presented infinite group has positive first L^2-betti number (which can be thought of as an extreme form of non-amenability) and is residually p-finite, then it does contain non-abelian free subgroups. Our Theorem 1.2 shows that this result can not be extended to all finitely generated groups.

2. Estimating the first L^2-betti number from below

In this section we prove Theorem 1.1. The main ingredient of the proof is the result of the first author stating that the approximation conjecture holds for groups from a class $\mathcal{G}$, which includes in particular all residually finite groups. We refer to [6, Chapter 13] for more details.

Proof of Theorem 1.1. Choose a presentation

$$G = \langle s_1, s_2, \ldots s_g \mid R_1, R_2, \ldots \rangle,$$

where the number of generators is finite. For a natural number j, let G_j be the finitely presented group given by the presentation

$$G = \langle s_1, s_2, \ldots s_g \mid R_1, R_2, \ldots R_j \rangle.$$

Let $\psi_i \colon G_i \to G$ and $\varphi_{j,k} \colon G_j \to G_k$ for $j \leq k$ be the obvious projections. We have

$$G = \operatorname{colim}_j G_j.$$

The system of group homomorphisms

$$G_0 \xrightarrow{\varphi_{0,1}} G_1 \xrightarrow{\varphi_{1,2}} G_2 \xrightarrow{\varphi_{2,3}} \cdots$$

induces a system of maps of CW-complexes

$$BG_0 \xrightarrow{\psi_{0,1}} BG_1 \xrightarrow{\psi_{1,2}} BG_2 \xrightarrow{\psi_{2,3}} \cdots$$

We can arrange that the 2-skeleton of BG_j is finite for all $j \geq 0$ since each G_j is finitely presented. Let X be the infinite mapping telescope of this system. It is a CW-complex. Since we have for $k \geq 0$

$$\pi_k(X) = \operatorname{colim}_j \pi_k(BG_j)$$

we conclude

$$\pi_k(X) = \begin{cases} \{1\} & k \geq 2; \\ G & k = 1. \end{cases}$$

Hence X is a model for BG.

Let X_j be the mapping telescope of the finite system

$$BG_0 \xrightarrow{\psi_{0,1}} BG_1 \xrightarrow{\psi_{1,2}} BG_2 \xrightarrow{\psi_{2,3}} \cdots \xrightarrow{\psi_{j-1,j}} BG_j$$

Then the 2-skeleton of X_j is finite and we have the nested sequence of CW-subcomplexes

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

of X satisfying $X = \bigcup_j X_j$. Let $\widetilde{X} \to X$ be the universal covering of X. This is a model for the universal principal G-bundle $G \to EG \to BG$. Let $\widetilde{X_n}$ be its restriction to X_n. We obtain a nested sequence of free G-CW-subcomplexes

$$\widetilde{X}_0 \subseteq \widetilde{X}_1 \subseteq \widetilde{X}_2 \subseteq \cdots$$

of $\widetilde{X}$ satisfying $\widetilde{X} = \bigcup_j \widetilde{X}_j$.

Since taking homology and taking tensor products are compatible with directed colimits, we get

$$H_1\big(\widetilde{X_n};\mathcal{N}(G)\big) = \operatorname{colim}_j H_1\big(\widetilde{X};\mathcal{N}(G)\big).$$

Since the G-CW-complex $\widetilde{X}_j$ has only finitely many orbits of cells of dimension at most 2 for all $j \geq 0$, we get $\dim_{\mathcal{N}(G)}\big(H_1(\widetilde{X}_j;\mathcal{N}(G)\big) < \infty$ for all $j \geq 0$. Each map $\varphi_{j-1,j}\colon G_{j-1} \to G_j$ is surjective. Hence each map $\psi_{j-1,j}\colon BG_{j-1} \to BG_j$ is 1-connected. Therefore each inclusion $\widetilde{X}_{j-1} \to \widetilde{X}_j$ is 1-connected. This implies that the induced map $H_1\big(\widetilde{X}_{j-1},\mathcal{N}(G)\big) \to H_1\big(\widetilde{X}_j,\mathcal{N}(G)\big)$ is surjective. Hence we get for $j \geq 1$

$$\dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X}_{j-1};\mathcal{N}(G)\big)\Big) \geq \dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X}_j;\mathcal{N}(G)\big)\Big).$$

We conclude from [6, Theorem 6.13 (2) on page 243]

$$\dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X};\mathcal{N}(G)\big)\Big) = \lim_j \dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X}_j;\mathcal{N}(G)\big)\Big). \tag{4}$$

Consider a nested sequence $G = N_0 \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of normal subgroups of finite index such that $\bigcap_{i=0}^{\infty} N_i = \{1\}$. Since G is residually finite it satisfies the Approximation Conjecture (see [6, Conjecture 13.1]), and we obtain

$$\dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X}_j;\mathcal{N}(G)\big)\Big) = \lim_i \frac{b_1(N_i\backslash\widetilde{X}_j)}{[G:N_i]}. \tag{5}$$

Indeed this follows from [6, Theorem 13.3 on page 454] applied to the 2-skeleton of $\widetilde{X}_j$ (which has only finitely many orbits of cells) and the observation that both sides of (5) only depend on the 2-skeleton.

Since $\widetilde{X}_j \to \widetilde{X}$ is 1-connected, the map $N_i\backslash\widetilde{X}_j \to N_i\backslash\widetilde{X}$ is 1-connected and hence $b_1(N_i\backslash\widetilde{X}_j) \geq b_1(N_i\backslash\widetilde{X})$ holds for all $j \geq 0$. This implies for all $j \geq 0$

$$\lim_i \frac{b_1(N_i\backslash\widetilde{X}_j)}{[G:N_i]} \geq \limsup_i \frac{b_1(N_i\backslash\widetilde{X})}{[G:N_i]}. \tag{6}$$

Since the G-CW-complex $\widetilde{X}$ is a model for EG, we get $b_1(N_i\backslash\widetilde{X}) = b_1(N_i)$ for all i and $b_1^{(2)}(G) = \dim_{\mathcal{N}(G)}\Big(H_1\big(\widetilde{X};\mathcal{N}(G)\big)\Big)$. We conclude from (5) and (6) for all j

$$\dim_{\mathcal{N}(G)}\big(\widetilde{X}_j;\mathcal{N}(G)\big) \geq \limsup_i \frac{b_1(N_i)}{[G:N_i]} \tag{7}$$

Now (4) and (7) imply

$$b_1^{(2)}(G) \geq \limsup_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]}.$$

□

3. Virtual deficiency of finitely presented groups

In what follows, "p-finite" always means "equal to a power of p". Given two elements x, y of a group G, we write x^y for $y^{-1}xy$. We denote by $\langle\langle S\rangle\rangle^G$ (or just $\langle\langle S\rangle\rangle$ if no confusion is possible) the normal closure of a subset S in G, i.e., the smallest normal subgroup of G containing S. Finally if G is finitely presented, $\mathrm{def}(G)$ denotes the deficiency of G, i.e., the maximum of the difference between the number of generators and the number of relations over all finite presentations of G.

Let $G = F/R$ be a finitely presented group, where $F = \langle x_1, \ldots, x_d\rangle$ is free of rank d and $R = \langle\langle R_1, \ldots, R_r\rangle\rangle^F$. Let H be a finite index subgroup of G, K the full preimage of H in F. By the Nilsen-Schreier formula K is a free group of rank $(d-1)j+1$, where $j = [F : K] = [G : H]$. It is straightforward to check that $R = \langle\langle R_i^t \mid i = 1, \ldots, r,\, t \in T\rangle\rangle^K$, where T is a set of left coset representatives of K in F. Thus $H = K/R$ has a presentation with $(d-1)j+1$ generators and $r|T| = rj$ relations. In particular,

$$\mathrm{def}(H) - 1 \geq (\mathrm{def}(G) - 1)[G : H].$$

We will refer to this property of deficiency as *supermultiplicativity.*

Definition 3.1. Let G be a finitely presented group. We define the *p-virtual deficiency* of G by

$$\mathrm{vd}_p(G) = \sup \frac{\mathrm{def}(H) - 1}{[G : H]}, \tag{8}$$

where the supremum is taken over all normal subgroups $H \leq G$ of p-finite index.

Remark 3.2. Clearly every group of positive p-virtual deficiency is infinite. Using supermultiplicativity it is easy to show that the definition does not change if we take the supremum over all (not necessarily normal) subgroups of p-finite index in G. We do not know if the supremum in (8) is always achieved and whether it is always rational. Some applications of a similar notion of p-deficiency can be found in [1].

The following lemma was proved in [10, Lemma 2.3]. (The proof is an easy exercise.)

Lemma 3.3. *Let G be a finitely presented group, N a finite index normal subgroup of G, g an element of G, m the order of gN in G/N. Let M denote the natural image of N in the quotient group $G/\langle\langle g^m\rangle\rangle$. Then*

$$\mathrm{def}(M) \geq \mathrm{def}(N) - [G : N]/m.$$

Lemma 3.3 can be used to construct nontrivial examples of groups with positive p-virtual deficiency. For a group G, we denote by $\widehat{G}_p$ the image of G in its pro-p completion. That is $\widehat{G}_p = G/R$, where R is the intersection of all normal subgroups N of p-finite index in G.

Lemma 3.4. *Let G be a finitely presented group. Suppose that the image of an element $g \in G$ in $\widehat{G}_p$ has infinite order. Then for every $\delta > 0$ there exist arbitrary large integers n such that*

$$\mathrm{vd}_p\left(G/\langle\langle g^{p^n}\rangle\rangle\right) \geq \mathrm{vd}_p(G) - \delta. \tag{9}$$

Proof. Let us fix any $K > 0$. We wish to find $n \geq K$ that satisfies (9). Without loss of generality we can assume that

$$\frac{1}{p^K} \leq \delta/2. \tag{10}$$

Since g has infinite order in $\widehat{G}_p$, there is a p-finite quotient Q of G such that the order of the image of g in Q is at least p^K. Let H be a normal subgroup of p-finite index in G such that

$$\frac{\operatorname{def}(H)-1}{[G:H]} \geq \operatorname{vd}_p(G) - \delta/2 \tag{11}$$

and let $N = \operatorname{Ker}(G \to Q) \cap H$. Clearly N is normal of p-finite index in G. Since $N \leq \operatorname{Ker}(G \to Q)$, the order of gN in G/N is p^n for some $n \geq K$. Let M denote the image of N in $G_1 = G/\langle\langle g^{p^n}\rangle\rangle$. Note that $[G_1 : M] = [G : N]$. Using subsequently Lemma 3.3, supermultiplicativity, (10), and (11) we obtain

$$\frac{\operatorname{def}(M)-1}{[G_1:M]} \geq \frac{\operatorname{def}(N)-1}{[G:N]} - \frac{1}{p^n} \geq \frac{\operatorname{def}(H)-1}{[G:H]} - \frac{1}{p^K} \geq \operatorname{vd}_p(G) - \delta.$$

□

The next two lemmas allow us to estimate the first betti and L^2-betti numbers of some (not necessarily finitely presented) residually finite groups.

Lemma 3.5. *For any finitely presented group G, we have $b_1(\widehat{G}_p) \geq \operatorname{def}(G)$.*

Proof. If G has a presentation with d generators and r relations, then $G/[G,G]$ is a quotient of $\mathbb{Z}^d$ by a subgroup of rank at most r. It is well-known that $G/[G,G]$ maps onto $\mathbb{Z}^{d-r}$ in this case. Since free abelian groups are residually p-finite, $\widehat{G}_p$ also maps onto $\mathbb{Z}^{d-r}$ and hence $b_1(\widehat{G}_p) \geq d - r$. □

Lemma 3.6. *For any finitely presented group G, we have*

$$b_1^{(2)}(\widehat{G}_p) \geq \operatorname{vd}_p(G). \tag{12}$$

Proof. Fix some $\varepsilon > 0$. Let H be a subgroup of p-finite index in G that satisfies (11). Let K be the image of H in $\widehat{G}_p$ and let M be any subgroup of p-finite index in K. We denote by N be the full preimage of M in G (see the diagram below).

$$\begin{array}{ccccc} M & \triangleleft & K & \triangleleft & \widehat{G}_p \\ \uparrow & & \uparrow & & \uparrow \\ N & \triangleleft & H & \triangleleft & G \end{array}$$

Using supermultiplicativity of deficiency and (11), we obtain

$$\operatorname{def}(N) - 1 \geq (\operatorname{def}(H)-1)[H:N] \geq (\operatorname{vd}_p(G) - \varepsilon)[G:H][H:N]. \tag{13}$$

Since $[G:N]$ is a power of p, $\widehat{N}_p \cong M$. By (13) and Lemma 3.5 we have

$$b_1(M) \geq \operatorname{def}(N) > (\operatorname{vd}_p(G) - \varepsilon)[G:H][H:N] = (\operatorname{vd}_p(G) - \varepsilon)[\widehat{G}_p : K][K:M]. \tag{14}$$

Since K is residually p-finite and (14) holds for any subgroup $M \leq K$ of p-finite index, we obtain

$$b_1^{(2)}(K) \geq (\operatorname{vd}_p(G) - \varepsilon)[\widehat{G}_p : K]$$

by Theorem 1.1. Using multiplicativity of $b_1^{(2)}$ and letting $\varepsilon \to 0$, we get (12). □

4. Residually finite p-groups with positive first L^2-betti number

We are now ready to prove Theorem 1.2. In fact, we prove a stronger result.

Theorem 4.1. *For any prime p, any integer $n \geq 2$, and any $\varepsilon > 0$, there exists a finitely generated infinite residually finite p-group Q generated by n elements such that $b_1^{(2)}(Q) \geq n-1-\varepsilon$.*

Proof. We use a modification of the main construction from [10]. From now on, let us fix p and denote $\widehat{G}_p$ simply by $\widehat{G}$ for a group G. Let F be the free group of rank n with basis X. We enumerate all elements of $F = \{1 = f_0, f_1, \dots\}$ and construct inductively a commutative diagram of quotient groups of F and epimorphisms

$$\begin{array}{ccccc} G_0 & \stackrel{\alpha_1}{\longrightarrow} & G_1 & \stackrel{\alpha_2}{\longrightarrow} & \dots \\ \downarrow & & \downarrow & & \\ \widehat{G}_0 & \stackrel{\hat{\alpha}_1}{\longrightarrow} & \widehat{G}_1 & \stackrel{\hat{\alpha}_2}{\longrightarrow} & \dots \\ \triangledown & & \triangledown & & \\ N_0 & & N_1 & & \dots, \end{array} \tag{15}$$

where the vertical arrows are natural maps from groups to their images in pro-p completions and for every $k \in \mathbb{N} \cup \{0\}$ the following conditions hold. For simplicity we use the same notation for elements $f_0, f_1, \dots$ and their images in quotients of F.

(a) The order of f_k in $\widehat{G}_k$ is p-finite.
(b) G_k is finitely presented and $\mathrm{vd}_p(G_k) > n - 1 - \varepsilon$.
(c) N_k is of p-finite index in $\widehat{G}_k$ and the shortest nontrivial element of N_k has length at least k in the word metric on $\widehat{G}_k$ corresponding to the natural image of X.
(d) If $k \geq 1$, then $\mathrm{Ker}(\hat{\alpha}_k) \leq N_{k-1}$ and $N_k \leq \hat{\alpha}_k(N_{k-1})$.

Properties (a)-(c) are easy to verify for $G_0 = F$ and $N_0 = \widehat{G}_0$. For $k > 0$ we consider two cases.

Case 1. Suppose that the order of f_k in $\widehat{G}_{k-1}$ is finite. Note that it is p-finite in this case as $\widehat{G}_{k-1}$ is residually p-finite. Let $G_k = G_{k-1}$, $\alpha_k = \mathrm{id}$, and $\hat{\alpha}_k = \mathrm{id}$. Then (a) and (b) obviously hold. Since $\widehat{G}_k$ is residually p-finite, there exists a subgroup $N_k \lhd \widehat{G}_k$ of p-finite index that satisfies (c). We can always find such N_k inside $\hat{\alpha}_k(N_{k-1})$ so that (d) is satisfied as well.

Case 2. Suppose that the order of f_k in $\widehat{G}_{k-1}$ is infinite. Let p^m be the order of f_k in $\widehat{G}_{k-1}/N_{k-1}$. By Lemma 3.4 and the inductive assumption, we can choose $n \geq m$ such that the quotient group $G_k = G_{k-1}/\langle\langle f_k^{p^n} \rangle\rangle$ satisfies (b). Let α_k be the natural epimorphism $G_{k-1} \to G_k$ and let $\hat{\alpha}_k$ be the epimorphism induced by α_k. As $n \geq m$, $f_k^{p^n} \in N_{k-1}$ in the group $\widehat{G}_{k-1}$ and hence $\mathrm{Ker}(\hat{\alpha}_k) \leq N_{k-1}$. Again since $\widehat{G}_k$ is residually finite, there exists a normal subgroup $N_k \leq \alpha_k(N_{k-1})$ of p-finite index in $\widehat{G}_k$ that satisfies (c). This completes the inductive step.

Let now Q be the (co)limit of the second row in (15). Condition (a) obviously implies that Q is a torsion p-group. By (c) and (d) Q is residually finite. Indeed if $q \in Q$ is a nontrivial element of length k with respect to the word metric corresponding to the natural image of X and g is a shortest preimage of q in $\widehat{G}_{k+1}$, then $g \notin N_{k+1}$ by (c). Note that (d) implies $\mathrm{Ker}(\widehat{G}_{k+1} \to Q) \leq N_{k+1}$. Hence Q maps onto $\widehat{G}_{k+1}/N_{k+1}$ and q is taken to $g \neq 1$ by this map. Observe also that by (b) all groups $\widehat{G}_k$ are infinite and hence Q is infinite as well.

Finally we note that Lemma 3.6 and property (b) imply that $b_1^{(2)}(\widehat{G}_k) \geq n-1-\varepsilon$. By semi-continuity of the first L^2-Betti number (see [11, Theorem 1]) we obtain

$$b_1^{(2)}(Q) \geq \lim_{k\to\infty} \sup b_1^{(2)}(\widehat{G}_k) \geq n-1-\varepsilon.$$

□

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