The K-theoretic Farrell-Jones Conjecture for hyperbolic groups
on the occasion of
Beno Eckmann's 90-th birthday

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Outline and goal

- Explain the K-theoretic and L-theoretic Farrell-Jones Conjecture.
- Discuss applications and the potential of these conjectures.
- Relate it to Beno Eckmann's work and to the work of other fameous Swizz mathematicians.
- State our main theorem which is joint work with Arthur Bartels and Holger Reich.
- Further options
  - Link the Farrell-Jones Conjecture to the Baum-Connes Conjecture.
  - Make a few comments about the proof.

Algebraic K-theory

Conjecture (Farrell-Jones)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.
- Example $G = \mathbb{Z}$: $K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R)$.

Definition (Projective class group $K_0(R)$)

Let R be an (associative) ring (with unit). Define its *projective class group*

$$K_0(R)$$

to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphisms classes of finitely generated projective R-modules under direct sum.
- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let G be a finite group and F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$.
- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- $K_0(R)$ measures the deviation of finitely generated projective R-module from being (stably) finitely generated free.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.

- A CW-complex X is finitely dominated if there is a finite CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.
- A finitely dominated CW-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

  called its finiteness obstruction.
- A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes (C.T.C. Wall).

Definition (K_1-group $K_1(R)$)

Define the *K_1-group of a ring R*

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$;
- $[\mathrm{id}_P] = 0$.

- This is the same as $GL(R)/[GL(R), GL(R)]$.
- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ in $\widetilde{K}_1(R) = K_1(R)/\{\pm 1\} = \operatorname{cok}(K_1(\mathbb{Z}) \to K_1(R))$.
- The assignment $A \mapsto [A] \in K_1(R)$ is the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Definition (*h*-cobordism)

An *h*-cobordism over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem (Barden, Mazur, Stallings, Kirby-Siebenmann))

Let M_0 be a closed manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M_0)$. Let $(W; M_0, M_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(G)$ vanishes.

- The *s*-Cobordism Theorem implies the Poincaré Conjecture in dimension ≥ 5.
- It is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.

- In order to illustrate the depth of the Farrell-Jones Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the K-theoretic Farrell-Jones Conjecture for the coefficient ring R.

Lemma

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*

Lemma

Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$.
Then the Whitehead group $\mathrm{Wh}(G)$ *is trivial.*

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG;\mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \le -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG;\mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$0 \to H_0(BG;\mathbf{K}_{\mathbb{Z}}) = \{\pm 1\} \to H_1(BG;\mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G,G] \to 1.$$

  This implies

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0.$$

In particular we get for a torsionfree group $G \in \mathcal{FJ}(\mathbb{Z})$

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism $W = (W; M_0, M_1)$ of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial.
- If G belongs to $\mathcal{FJ}(\mathbb{Z})$, then it is of type FF if and only if it is of type FP.

Conjecture (Kaplansky)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (Bartels-L.-Reich (2007))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree.*
- *G is torsionfree and sofic, e.g., residually amenable.*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Conjecture (Farrell-Jones)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $E_{\mathcal{VC}\mathrm{yc}}(G)$ is the classifying space of the family of virtually cyclic subgroups;
- $H_*^G(-; \mathbf{K}_R)$ is the G-homology theory satisfying for every $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}_R) = K_n(RH).$$

- We think of it as an advanced induction theorem (such as Artin's or Brower's induction theorem for representations of finite groups).

Theorem (Bartels-L.-Reich (2007))

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}(R)$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Suppose that $G \in \mathcal{FJ}(F)$. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

Conjecture (Bass)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- If G is finite, this is just the Theorem of Swan.
- A stronger version of the Bass Conjecture predicts that for the quotient field F of R

$$K_0(RG) \to K_0(FG)$$

factorizes as

$$K_0(RG) \to K_0(R) \to K_0(F) \to K_0(FG).$$

Theorem (Linnell-Farrell)

Let G be a group. Suppose that

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic. (This is true if $G \in \mathcal{FJ}(F)$ for every field F of prime characteristic).
Then the Bass Conjecture is satisfied for every integral domain R.

- Beno Eckmann: "Cyclic homology of groups and the Bass Conjecture", Comment. Math. Helv. 61, 193–202 (1986).
- Beno Eckmann: "Projective and Hilbert modules over group algebras, and finitely dominated spaces", Comment. Math. Helv. 71, 453–462 (1996).

Problem: Let X be a finitely dominated CW-complex with fundamental group π. Is it true that the passage

$$K_0(\mathbb{Z}\pi) \to K_0(\mathcal{N}(\pi)) \xrightarrow{\dim_{\mathcal{N}(\pi)}} \mathbb{R}$$

annihilates $\widetilde{o}(X)$, sends $o(X)$ to the L^2-Euler characteristic $\chi^{(2)}(\widetilde{X})$ and $\chi^{(2)}(\widetilde{X}) = \chi(X)$?

Yes, if $\pi \in \mathcal{FJ}_K(\mathbb{Z})$ or if the Bass Conjecture holds for π and $R = \mathbb{Z}$. Yes already over $C_r^*(\pi)$, $l^1(\pi)$ or $\mathbb{Q}\pi$ instead of $\mathcal{N}(G)$ if $\pi \in \mathcal{FJ}_K(\mathbb{Z})$.

Conjecture (L.)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of volume to hyperbolic groups whose L^2-Betti numbers all vanish.

Theorem (L. (2002))

Suppose that $G \in \mathcal{FJ}(\mathbb{Z})$. Then G satisfies the Conjecture above.

- Deninger can define a p-adic Fuglede-Kadison determinant for a group G and relate it to p-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
- The surjectivity of the map

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

plays a role (33 %) in a program to prove the Atiyah Conjecture. It predicts that for a closed Riemannian manifold M with torsionfree fundamental group the p-th L^2-Betti number of its universal covering

$$b_p^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}\left(e^{-t\widetilde{\Delta_p}}(\widetilde{x},\widetilde{x})\right) d\mathrm{vol}_{\widetilde{M}}$$

is an integer.

- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in R.

Theorem (Bartels-L.-Reich (2007))

- *Every hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}(R)$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}(R)$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}(R)$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}(R)$, then $H \in \mathcal{FJ}(R)$.*

- We emphasize that this result holds for all rings R.
- The groups above are certainly wild in Bridson's universe.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, belong to $\mathcal{FJ}_K(R)$ for all R.
- If G is a torsionfree hyperbolic group and R any ring, then we get an isomorphism

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C),C\subseteq G, C\neq 1\\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG).$$

- Bartels and L. have a program to prove $G \in \mathcal{FJ}_K(R)$ if G acts properly and cocompact on a CAT(0)-space.
- This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

Algebraic L-theory

Conjecture (Farrell-Jones)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$.
- Let $\mathcal{FJ}_L(R)$ be the class of groups which satisfy the L-theoretic Farrell-Jones Conjecture for the coefficient ring R.

Conjecture (Novikov)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

Conjecture (Borel)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The L-theoretic Farrell-Jones Conjecture for a group G in the case $R = \mathbb{Z}$ implies the Novikov Conjecture in dimension ≥ 5.
- If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.
- As in the case of algebraic K-theory there is also an analogous version of the L-theoretic Farrell-Jones Conjecture for arbitrary groups G.
- Bartels and L. have a program to extend our result for the K-theoretic Farrell-Jones Conjecture also to the L-theoretic version.
- Bartels and L. have a program to prove $G \in \mathcal{FJ}_L(R)$ if G acts properly and cocompact on a CAT(0)-space. This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

Topological K-theory

Conjecture (Baum-Connes)

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G;
- There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G;\mathbb{R})).$$

- There is also a version for arbitrary groups

$$K_n^G(E_{\mathcal{F}\mathrm{in}}(G)) \to K_n(C_r^*(G)).$$

- The Bost Conjecture is the analogue for $l^1(G)$, i.e., it concerns the assembly map.

$$K_n^G(E_{\mathcal{F}\mathrm{in}}(G)) \to K_n(l^1(G)).$$

  Its composition with the canonical map $K_n(l^1(G)) \to K_n(C_r^*(G))$ is the Baum-Connes assembly map.
- Both Conjectures have versions, where coefficients in a G-C^*-algebra are allowed.
- Berrick, Chatterji and Mislin have related the Bost Conjecture to the Bass Conjecture.

Theorem (Bartels-L.-Echterhoff (2007))

Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of hyperbolic groups G_i (with not necessarily injective structure maps).
Then G satisfies the Bost Conjecture with coefficients.

- The proof uses the deep result of Lafforgue that the Bost Conjecture with coefficients is true for every hyperbolic group.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, do satisfy the Bost Conjecture with coefficients.
  So the failure of the Baum-Connes Conjecture with coefficients says that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$ is not bijective.
  The underlying problem with the Baum-Connes Conjecture is the lack of functoriality of the reduced group C^*-algebra.

$$
\begin{array}{ccc}
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{\mathbb{Z}}[1/2] & \xrightarrow{\cong} & L^p_n(\mathbb{Z}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{\mathbb{R}}[1/2] & \xrightarrow{\cong} & L^p_n(\mathbb{R}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{C_r^*(?;\mathbb{R})}[1/2] & \xrightarrow{\cong} & L^p_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{C}}[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G))[1/2]
\end{array}
$$

Comments on the proof

Here are the basic steps of the proof of the main Theorem.

Step 1: Interprete the assembly map as a forget control map.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}(R)$ holds for all rings R if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- Z-set-condition

  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;

- Long thin covers

  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:

  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^{\beta} \times \{x\} \subset U$. Here g^{β} denotes the β-ball around g in G with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above.