Equivariant homology theory and assembly maps

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Motivation for the Baum-Connes Conjecture for torsionfree groups

The aim is to compute the *topological K-theory*

$$K_*(C_r^*(G))$$

of the *reduced group C^*-algebra $C_r^*(G)$* in the sense that we want to identify it with more familiar terms. The key idea comes from the observation that $K_*(C_r^*(G))$ has some *homological properties*. More precisely, if G is the amalgamated product $G = G_1 *_{G_0} G_2$ for subgroups $G_i \subseteq G$, then there is a long exact sequence due to *Pimsner-Voiculescu*

$$\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) \to K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \to K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \to K_{n-1}(C_r^*(G_1)) \oplus K_{n-1}(C_r^*(G_2)) \to \cdots .$$

If $\phi: G \to G$ is a group automorphism and $G \rtimes_\phi \mathbb{Z}$ the associated semidirect product, then there is a long exact sequence due to *Pimsner-Voiculescu*

$$\begin{aligned}\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G)) \xrightarrow{\phi_* - \mathrm{id}} K_n(C_r^*(G)) \\ \to K_n(C_r^*(G \rtimes_\phi \mathbb{Z})) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G)) \\ \xrightarrow{\phi_* - \mathrm{id}} K_{n-1}(C_r^*(G)) \to \cdots .\end{aligned}$$

Notice that the are analogous sequences in group homology

$$\begin{aligned}\cdots \xrightarrow{\partial_{n+1}} H_n(BG_0) \to H_n(BG_1) \oplus H_n(BG_2) \\ \to H_n(BG) \xrightarrow{\partial_n} H_{n-1}(BG_0) \\ \to H_{n-1}(BG_1) \oplus H_{n-1}(BG_2) \\ \to H_{n-1}(BG) \xrightarrow{\partial_{n-1}} \cdots\end{aligned}$$

and

$$\cdots \xrightarrow{\partial_{n+1}} H_n(BG) \xrightarrow{\phi_* - \mathrm{id}} H_n(BG) \\ \to H_n(B(G \rtimes_\phi \mathbb{Z})) \xrightarrow{\partial_n} H_{n-1}(BG) \\ \xrightarrow{\phi_* - \mathrm{id}} H_{n-1}(BG) \to \cdots .$$

The first naive guess $H_n(BG) = K_n(C_r^*(G))$ fails already for the trivial group. On the other hand the guess $K_n(BG) = K_n(C_r^*(G))$ works out for the trivial group. This motivates:

Conjecture 1 (Baum-Connes Conjecture for torsionfree groups). *There is an assembly map*

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G))$$

which is bijective for all $n \in \mathbb{Z}$.

Remark 2. If G is not torsionfree, the version of the Baum-Connes Conjecture above cannot hold anymore in general, Namely, for a finite group G one has

$$K_n(BG)\otimes_{\mathbb{Z}}\mathbb{Q} \cong K_n(\{\bullet\})\otimes_{\mathbb{Z}}\mathbb{Q} \cong K_n(C_r^*(\{1\}))\otimes_{\mathbb{Z}}\mathbb{Q}$$

and

$$K_n(C_r^*(G)) \;\cong\; \begin{cases} R_{\mathbb{C}}(G) & n \text{ even,} \\ \{0\} & n \text{ odd.} \end{cases}$$

To formulate the Baum-Connes Conjecture in general, some more input is needed.

The Farrell-Jones Conjecture for torsionfree groups

For algebraic K- and L-theory the situation is more complicated since there the Mayer-Vietoris sequence exist only modulo certain *Nil-terms* or *Unil-terms* as worked out by *Cappell* and *Waldhausen*. For instance the *Bass-Heller-Swan decomposition* implies

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R),$$

whereas

$$H_n(B\mathbb{Z}; \mathbf{K}(R)) \cong K_n(R) \oplus K_{n-1}(R).$$

If R is a regular ring, one can still hope for a torsionfree group G that there are isomorphisms

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{\cong} K_n(RG).$$

In L-theory one can hope for an isomorphism for a torsionfree group G

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

Here the decoration $\langle -\infty \rangle$ is forced upon us because the *Shaneson splitting*. It says

$$L_n^{\langle -j \rangle}(R\mathbb{Z}) \;\cong\; L_n^{\langle -j \rangle}(R) \oplus L_{n-1}^{\langle -j-1 \rangle}(R),$$

whereas

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle -j \rangle}(R)) \;\cong\; L_n^{\langle -j \rangle}(R) \oplus L_{n-1}^{\langle -j \rangle}(R).$$

Remark 3 (Consequence for $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$). *Let G be a torsionfree group. Then the Farrell-Jones Conjecture predicts*

$$H_n(BG, \mathbf{K}(\mathbb{Z})) \;\cong\; K_n(\mathbb{Z}G).$$

Since $K_n(\mathbb{Z}) = \pi_n(\mathbf{K}(\mathbb{Z})) = 0$ for $n \leq -1$ and $\widetilde{K}_n(\mathbb{Z}) = 0$ for $n = 0, 1$, an easy application of the Atiyah-Hirzebruch spectral sequence implies

$$\begin{aligned} \widetilde{K}_0(\mathbb{Z}G) &= \{0\}; \\ \mathrm{Wh}(G) &= \{0\}. \end{aligned}$$

As in the Baum-Connes Conjecture the version of the Farrell-Jones Conjecture formulated above cannot extend in general to groups with torsion.

Equivariant homology theories

Definition 4 (G-homology theory).
A G-homology theory $\mathcal{H}_*^G$ is a covariant functor $\mathcal{H}_*^G$ from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded R-modules together with natural transformations

$$\partial_n^G(X, A) : \mathcal{H}_n^G(X, A) \to \mathcal{H}_{n-1}^G(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- *G-homotopy invariance;*
- *Long exact sequence of a pair;*
- *Excision;*
- *Disjoint union axiom.*

Definition 5 (Equivariant homology theory). *An* equivariant homology theory $\mathcal{H}_*^?$ *consists of a G-homology theory $\mathcal{H}_*^G$ for every group G together with the following so called* induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms*

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \;\to\; \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying

- *Bijectivity*
  If $\ker(\alpha)$ *acts freely on X, then* $\operatorname{ind}_\alpha$ *is a bijection;*

- *Compatibility with the boundary homomorphisms*

- *Functoriality in α*

- *Compatibility with conjugation*

Example 6. Here are some examples for equivariant homology theories $\mathcal{H}_*^?$:

- Quotients
  Let $\mathcal{K}_*$ be a non-equivariant homology theory. Define $\mathcal{H}_*^?$ by
  $$\mathcal{H}_*^G(X) \ := \ \mathcal{K}_*(G\backslash X).$$

- Borel homology
  Let $\mathcal{K}_*$ be a non-equivariant homology theory. Define $\mathcal{H}_*^?$ by
  $$\mathcal{H}_*^G(X) \ := \ \mathcal{K}_*(EG \times_G X).$$

- Equivariant bordism
  Let X be a proper G-CW-complex. Define the n-th G-bordism group $\Omega_n^G(X)$ by the G-bordism classes of proper cocompact smooth G-manifolds M with G-reference maps to X.

A spectrum $\mathbf{E}$ defines a homology theory by sending a space X to $\pi_*^s(X_+ \wedge \mathbf{E})$. This generalizes to the equivariant setting as follows.

Theorem 7 (Equivariant homology theories and spectra). *Consider a covariant functor*

$$\mathbf{E}\colon \text{GROUPOIDS} \to \text{SPECTRA}$$

sending equivalences of groupoids to weak equivalences of spectra.

Then there exists an equivariant homology theory $\mathcal{H}_*^?(-;\mathbf{E})$ with the property that for every group G, subgroup $H \subseteq G$ and $n \in \mathbb{Z}$

$$\mathcal{H}_n^G(G/H) \;=\; \mathcal{H}_n^H(\{\bullet\}) \;=\; \pi_n^s(\mathbf{E}(H)).$$

Proof. Given a group G and a G-set S we obtain a groupoid $\mathcal{G}^G(S)$ whose objects are elements in S and whose morphisms from s_1 to s_2 are the elements $g \in G$ with $gs_1 = s_2$. Composition comes from the group structure on G. Thus we obtain a covariant functor

$$\mathbf{E}^G\colon \mathrm{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \mathbf{E}(\mathcal{G}^G(G/H)).$$

A G-CW-complex X defines a contravariant functor

$$X^?\colon \mathrm{Or}(G) \to \mathsf{SPACES}, \quad G/H \mapsto X^H.$$

We obtain a spectrum $X^?_+ \wedge_{\mathrm{Or}(G)} \mathbf{E}^G$ by the balanced smash product. Its n-th stable homotopy group is $\mathcal{H}_n^G(X;\mathbf{E})$. $\square$

The next result is due to *Davis-Lück*.

Theorem 8 (K- and L-Theory Spectra over Groupoids). *Let R be a ring (with involution). There exist covariant functors*

$$\begin{aligned}\mathbf{K}_R\colon \text{GROUPOIDS} &\to \text{SPECTRA};\\ \mathbf{L}_R^{\langle j\rangle}\colon \text{GROUPOIDS} &\to \text{SPECTRA};\\ \mathbf{K}^{\text{top}}\colon \text{GROUPOIDS}^{\text{inj}} &\to \text{SPECTRA}\end{aligned}$$

with the following properties:

- *They send equivalences of groupoids to weak equivalences of spectra;*

- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned}\pi_n(\mathbf{K}_R(G)) &\cong K_n(RG);\\ \pi_n(\mathbf{L}_R^{\langle j\rangle}(G)) &\cong L_n^{\langle j\rangle}(RG);\\ \pi_n(\mathbf{K}^{\text{top}}(G)) &\cong K_n(C_r^*(G)).\end{aligned}$$

Classifying spaces of families of subgroups

Definition 9 (Family of subgroups). *A family $\mathcal{F}$ of subgroups of the group G is a set of subgroups of G which is closed under conjugation and taking subgroups.*

Examples for families are

$\{1\}$	trivial subgroup
$\mathcal{FIN}$	finite subgroups
$\mathcal{VCY}$	virtually cyclic subgroups
$\mathcal{ALL}$	all subgroups

Definition 10 (Classifying space of a family). *Let $\mathcal{F}$ be a family of subgroups of G. A model for the classifying space of the family $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}(G)$ such that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$.*

If $\mathcal{F}$ is $\mathcal{FIN}$ or $\mathcal{VCY}$, we also write $\underline{E}G$ and $\underline{\underline{E}}G$. Sometimes $\underline{E}G$ is called the classifying space for proper G-actions.

Theorem 11. *The G-CW-complex $E_{\mathcal{F}}(G)$ is characterized uniquely up to G-homotopy by the property that for every G-CW-complex X whose isotropy groups belong to $\mathcal{F}$ there is up to G-homotopy precisely one G-map $X \to E_{\mathcal{F}}(G)$.*

Obviously $E_{\{1\}}(G) = EG$ and $E_{\mathcal{ALL}}(G) = G/G$.

Remark 12 (Models for $\underline{E}G$). The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small. For instance

- *Rips complex* for word hyperbolic groups;
- *Teichmüller space* for mapping class groups;
- *Outer space* for the group of outer automorphisms of free groups;
- L/K for a connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;
- *CAT(0)-spaces* with proper isometric G-actions, e.g., Riemannian manifolds with non-positive sectional curvature or trees.

Formulations of the conjectures in general

Definition 13 (Assembly map). *Let $\mathcal{H}_*^G$ be a G-homology theory. Let $\mathcal{F}$ be a family of subgroups. Let $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ be the projection. The associated* assembly map *is*

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G).$$

Definition 14 (Meta-Isomorphism-Conjecture). *The* Meta-Isomorphism Conjecture *for a G-homology theory $\mathcal{H}_*^G$ and a family $\mathcal{F}$ says that the assembly map*

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

is bijective for all $n \in \mathbb{Z}$.

Definition 15 (Baum-Connes-Conjecture). *The* Baum-Connes Conjecture *is the Meta-Isomorphism Conjecture for $\mathcal{H}_*^G(-,\mathbf{K}^{\mathrm{top}}) = K_*^G(-)$ and the family $\mathcal{FIN}$, i.e., it predicts the bijectivity of*

$$K_n^G(\underline{E}G) \to K_n(C_r^*(G)).$$

Definition 16 (Farrell-Jones-Conjecture). *The* Farrel-Jones Conjecture *is the Meta-Isomorphism Conjecture for* $\mathcal{H}_*^G(-,\mathbf{K}_R)$ *or* $\mathcal{H}_*^G(-,\mathbf{L}_R^{\langle -\infty\rangle})$ *and the family* $\mathcal{VCY}$*, i.e., it predicts the bijectivity of the assembly maps*

$$\begin{aligned} H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) &\to K_n(RG); \\ H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) &\to L_n^{\langle -\infty\rangle}(RG). \end{aligned}$$

Remark 17 (Assembling from subgroups). The idea behind all of these conjectures is that for a certain functor from groups to spectra its value on a group G can be assembled by its values on all subgroups occuring in $\mathcal{F}$. These conjectures predict a kind of *induction theorem*. However, degrees are mixed in the sense that $K_n(RG)$ is computable in terms of and affected by $K_m(RH)$ for all $H \in \mathcal{VCY}$ and all $m \in \mathbb{Z}$, $m \leq n$.

Remark 18 (Status). These conjectures are known for many groups but are open in general. The methods of proofs are different depending on the groups or the conjecture considered. They mainly use techniques from *controlled topology, homotopy theory, K-theory, operator theory and geometry*. For a *survey* of the status, method of proofs, the history and the applications of the Baum-Connes Conjecture and the Farrell-Jones Conjecture we refer to the article by *Lück and Reich* in the handbook of K-theory (2005).

We mention the recent result of *Bartels, Lück and Reich*.

Theorem 19. *The Farrell-Jones Conjecture for algebraic K-theory $K_n(RG)$ holds for all $n \in \mathbb{Z}$, all coefficient rings R and all word-hyperbolic groups G.*

Remark 20 (Applications). These conjectures are very deep. They give a lot of structural insight and imply a variety of classical prominent conjectures such as

- *Bass Conjecture*
  Values of Hattori-Stalling ranks.

- *Borel Conjecture*
  Topological rigidity of aspherical manifolds.

- *Stable Gromov-Lawson-Rosenberg Conjecture*
  Obstruction for positive scalar curvature metrics.

- *Kadison Conjecture*
  Idempotents in $C_r^*(G)$

- *Novikov Conjecture*
  Homotopy invariance of higher signatures.

- *generalized Trace Conjecture*
  Values of the trace maps.

Remark 21 (Interpretations of the assembly maps). This implications and also the proofs of the Baum-Connes Conjecture and the Farrell-Jones Conjecture are often consequences of a good *geometric interpretation* of the assembly maps, e.g., in terms of index theory (*Kasparov*), controlled topology (*Quinn*) or in terms of surgery theory (*Quinn, Ranicki*).

These two conjectures are also the main tool in computations, where the target of the assembly map is the object of interest and the source of the assembly map is tractable for computations. The latter is due to the *homotopic theoretic description* of the assembly map due to *Davis-Lück* since well-known tools like spectral sequences and Chern characters can be generalized to the equivariant setting and then successfully applied.

We will discuss both aspects.

Remark 22 (The Meta Conjecture and other theories). The Meta Conjecture applies also to other theories. *Farrell-Jones* study the *pseudo-isotopy functor*. In a project by *Lück-Reich-Rognes-Varisco* the version of the Meta-Conjecture is treated for *topological Hochschild homology* and for *topological cyclic homology*, where the relevant family consists of cyclic subgroups. An important feature of the homotopy theoretic assembly map is that it is *natural in the theories* which are plugged in. Thus for instance construction such as the cyclotomic trace or change of rings homomorphism or the passage from algebraic to topological K-theory yield transformations between the relevant assembly maps

Index-theoretic interpretation of the Baum-Connes assembly map

The Baum-Connes assembly map

$$K_0^G(\underline{E}G) \to K_0(C_r^*(G))$$

has the following interpretation. Elements in $K_0^G(\underline{E}G)$ are represented by pairs (M, P^*) consisting of a cocompact proper smooth G-manifold M with G-invariant Riemannian metric and a G-equivariant elliptic complex P^* of differential operators of order 1. The Baum-Connes assembly map assigns to this pair its *G-index* in the sense of *Mishchenko-Fomenko*. So the surjectivity of the Baum-Connes map says that every element in $K_0(C_r^*(G))$ can be realized as an index of a pair (M, P^*). The injectivity says that the index decides when two such cycles (M, P^*) and (N, Q^*) are homologous.

Example 23 (Kadison Conjecture). Let G be a torsionfree group. Then the following diagram commutes by *Atiyah's L^2-index theorem*

$$\begin{array}{ccccc} K_0(BG) = K_0^G(\underline{E}G) & \longrightarrow & K_0(C_r^*(G)) & \xrightarrow{\text{tr}} & \mathbb{R} \\ \downarrow {\scriptstyle K_0(\text{pr})} & & & & \uparrow \\ K_0(\{\bullet\}) & & \xrightarrow[\cong]{} & & \mathbb{Z} \end{array}$$

Hence a conclusion of the Baum-Connes Conjecture is the *Trace Conjecture* which predicts that the image of the trace map tr is the integers. This implies the *Kadison Conjecture* which says that every idempotent in $C_r^*(G)$ is 0 or 1.

Wall's finiteness obstruction

Given a finitely dominated CW-complex X *Wall* defines its *finiteness obstruction*

$$\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}\pi)$$

for $\pi = \pi_1(X)$ by

$$\widetilde{o}(X) \; := \; \sum_{n \geq 0} (-1)^n \cdot [P_n]$$

where P_* is any finite projective $\mathbb{Z}\pi$-chain complex P_* which is $\mathbb{Z}\pi$-homotopy equivalent to $C_*(\widetilde{X})$. It is zero if and only X is homotopy equivalent to a finite CW-complex. For a finitely presented group G any element in $\widetilde{K}_0(\mathbb{Z}G)$ can be realized as $\widetilde{o}(X)$ of a finitely dominated CW-complex for $G \cong \pi_1(X)$. This implies:

Theorem 24 (Wall). *Let G be a finitely presented group. Then $\widetilde{K}_0(\mathbb{Z}G)$ vanishes if and only if every finitely dominated CW-complex X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*

s-Cobordism Theorem

We conclude from the *s-Cobordism Theorem* due to *Barden, Kirby, Mazur, Siebemann, Smale, Stallings* for every finitely presented group G and $n \geq 6$ that the following statements are equivalent

- Wh(G) vanishes.

- Every n-dimensional compact s-cobordism W with $\pi_1(W) \cong G$ is trivial;

Since Wh$(\{1\})$ vanishes, this implies the *Poincaré Conjecture* in dimension ≥ 5.

The Borel Conjecture

Conjecture 25 (Borel Conjecture). *Let G be a finitely presented group. The* Borel Conjecture for G *says:*

1. *Let X be an aspherical finitely dominated Poincaré complex with $G \cong \pi_1(X)$. Then X is homotopy equivalent to a closed manifold.*

2. *Let M and N be two aspherical closed manifolds with $G \cong \pi_1(M) \cong \pi_1(N)$. Then any homotopy equivalence $M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic.*

The Borel Conjecture may be viewed as a topological version of *Mostow rigidity*.

Remark 26 (The Farrell-Jones Conjecture implies the Borel Conjecture). The Borel Conjecture can be reformulated in the language of surgery theory to the statement that the *topological structure set* $\mathcal{S}^{\mathrm{top}}(M)$ of an aspherical closed topological manifold M consists of a single point. This set is the set of equivalence classes of homotopy equivalences $f\colon M' \to M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0\colon M_0 \to M$ and $f_1\colon M_1 \to M$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic.

The *surgery sequence* of a closed orientable topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \ldots \to \mathcal{N}_{n+1}(M \times [0,1], M \times \{0,1\}) \\ \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\mathrm{top}}(M) \\ \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) It is attributed to *Browder, Kirby, Novikov, Siebenmann, Sullivan and Wall*. An *algebraic version* has been developed by *Ranicki*. The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L-theory assembly map where one works with the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1\rangle$ of $\mathbf{L}^s(\mathbb{Z})$. The map

$$H_k(M;\mathbf{L}^s(\mathbb{Z})\langle 1\rangle) \to H_k(M;\mathbf{L}^s(\mathbb{Z}))$$

is injective for $k = n$ and an isomorphism for $k > n$. Because of the K-theoretic assumptions we can replace the s-decoration with the $\langle -\infty\rangle$-decoration. Since $BG = M$, the Farrell-Jones Conjecture implies that the maps $\sigma\colon \mathcal{N}_n(M) \to L_n^s(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M\times[0,1], M\times\{0,1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that

$\mathcal{S}^{\text{top}}(M)$ is a point and hence the Borel Conjecture holds for M.

For the question whether a Poincaré complex is homotopy equivalent to a closed manifold, is answered by the *total surgery obstruction* of *Ranicki*.

Computational aspects

Next we want to indicate how one can try to compute the source of the assembly map what is in general much easier then to compute the target.

There are basically the following general tools available:

- Analysis of nice models for $\underline{E}G$ *(Leary, Lück, Nucinkis, Soulé, . . .)*
- Splitting of Nil-terms *(Bartels)*
- Analysis of Nil-terms *(Bass, Connolly, Grunewald, Kozniewsky, Prassidis, Weibel)*

- Analysis of UNil-terms *(Banagl, Brookman, Cappell, Connolly, Davis, Qayum Khan, Ranicki)*

- Equivariant Atiyah-Hirzebruch spectral sequence (*Davis-Lück)*

- p-chain spectral sequence *(Davis-Lück)*

- Equivariant Chern characters *(Lück)*

Example 27 (Infinite dihedral group). Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$ be the *infinite dihedral group*. It acts on the tree $\mathbb{R}$ by letting $\mathbb{Z}$ acting by translation and $\mathbb{Z}/2$ by reflecting in 0. This is a model for $\underline{E}D_\infty$. It consists of two equivariant 0-cells $D_\infty/\mathbb{Z}/2 \times D^0$ and one equivariant 1-cell $D_\infty \times D^1$. If $\mathcal{H}_*^?$ is an equivariant homology theory, we obtain a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{H}_n^{\{1\}}(\{\bullet\}) \to \mathcal{H}_n^{\mathbb{Z}/2}(\{\bullet\}) \oplus \mathcal{H}_n^{\mathbb{Z}/2}(\{\bullet\}) \\ \to \mathcal{H}_n^{D_\infty}(\underline{E}D_\infty) \to \mathcal{H}_{n-1}^{\{1\}}(\{\bullet\}) \\ \to \mathcal{H}_{n-1}^{\mathbb{Z}/2}(\{\bullet\}) \oplus \mathcal{H}_{n-1}^{\mathbb{Z}/2}(\{\bullet\}) \cdots \end{aligned}$$

In the case of the Baum-Connes Conjecture which is known to be true for D_∞ we obtain the short split exact sequence

$$\begin{aligned} 0 \to R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(\mathbb{Z}/2) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \\ \to K_0(C_r^*(D_\infty)) \to 0 \end{aligned}$$

and $K_1(C_r^*(D_\infty)) \cong \{0\}$.

Notice that the Farrell-Jones Conjecture makes no predictions since D_∞ is virtually cyclic.

Remark 28 (Integral versus rational computations). Integral computations seem only to be possible in special cases. One cannot hope for a general formula for $K_n(C_r^*(G))$. The computation of $K_n(\mathbb{Z}G)$ and $L_n(\mathbb{Z}G)$ is even harder since one has to deal with virtually cyclic subgroups instead of finite subgroups, which comes from the appearance of Nil-terms and UNil-terms.

If one is interested only in rational information, the situation improves a lot thanks to *equivariant Chern characters*. They predict that in all cases of interest the equivariant Atiyah-Hirzebruch spectral sequence collapses in the strongest sense.

Instead of going through its construction and the proofs we mention one consequence which follows from the Baum-Connes Conjecture, the Farrell-Jones Conjecture, the naturality of the assembly maps in the theory considered and the equivariant Chern character.

Theorem 29 (Lück). *Let G be a (discrete) group. Suppose that the Baum-Connes Conjecture for G and the Farrell-Jones Conjecture for $K_n(\mathbb{C}G)$ are true. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram*

$$\begin{array}{ccc}
\bigoplus_{\substack{p+q=n \\ (g)\in T}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G) \\
\downarrow & & \downarrow \\
\bigoplus_{\substack{p+q=n \\ (g)\in T}} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n^{\mathrm{top}}(C_r^*(G))
\end{array}$$

where $C_G\langle g\rangle$ is the centralizer of the cyclic group generated by g in G and the vertical arrows come from the obvious change of ring and of K-theory maps $K_q(\mathbb{C}) \to K_q^{\mathrm{top}}(\mathbb{C})$ and $K_n(\mathbb{C}G) \to K_n^{\mathrm{top}}(C_r^*(G))$.