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Inventiones mathematicae

The relation between the Baum-Connes Conjecture and the Trace Conjecture

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Abstract. We prove a version of the L^2-index Theorem of Atiyah, which uses the universal center-valued trace instead of the standard trace. We construct for G-equivariant K-homology an equivariant Chern character, which is an isomorphism and lives over the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from the integers by inverting the orders of all finite subgroups of G. We use these two results to show that the Baum-Connes Conjecture implies the modified Trace Conjecture, which says that the image of the standard trace $K_0(C_r^*(G)) \to \mathbb{R}$ takes values in Λ^G. The original Trace Conjecture predicted that its image lies in the additive subgroup of $\mathbb{R}$ generated by the inverses of all the orders of the finite subgroups of G, and has been disproved by Roy [15].

0. Introduction and statements of results

Throughout this paper let G be a discrete group. The *Baum-Connes Conjecture for G* says that the assembly map

$$\operatorname{asmb}^G : K_0^G(\underline{E}G) \to K_0(C_r^*(G))$$

from the equivariant K-homology of the classifying space for proper G-actions $\underline{E}G$ to the topological K-theory of the reduced C^*-algebra $C_r^*(G)$ is bijective [3, page 8], [5, Conjecture 3.1]. In connection with this conjecture Baum and Connes [3, page 21] also made the sometimes so called *Trace Conjecture*. It says that the image of the composition

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

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is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs though all finite subgroups of G. Here $\mathcal{N}(G)$ is the group von Neumann algebra, i the change of rings homomorphism associated to the canonical inclusion $C_r^*(G) \to \mathcal{N}(G)$ and $\mathrm{tr}_{\mathcal{N}(G)}$ is the map induced by the standard von Neumann trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$. Roy has construced a counterexample to the Trace Conjecture in this form in [15] based on her article [16]. She constructs a group Γ, whose finite subgroups are all of order 1 or 3, together with an element in $K_0^G(\underline{E}G)$, whose image under $\mathrm{tr}_{\mathcal{N}(\Gamma)} \circ i \circ \mathrm{asmb}$ is $-\frac{1105}{9}$. The point is that $3 \cdot \frac{1105}{9}$ is not an integer. Notice that Roy's example does not imply that the Baum-Connes Conjecture does not hold for Γ. Since the group Γ contains a torsionfree subgroup of index 9 and the Trace Conjecture for torsionfree groups does follow from the Baum-Connes Conjecture, the Baum-Connes Conjecture predicts that the image of $\mathrm{tr}_{\mathcal{N}(\Gamma)} \circ i : K_0(C_r^*(\Gamma)) \to \mathbb{R}$ is contained in $\{r \in \mathbb{R} \mid 9 \cdot r \in \mathbb{Z}\}$. So one could hope that the following version of the Trace Conjecture is still true. Denote by

$$\Lambda^G := \mathbb{Z}\left[\frac{1}{|\mathcal{F}in(G)|}\right] \tag{0.1}$$

the ring $\mathbb{Z} \subset \Lambda^G \subset \mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting all the orders $|H|$ of finite subgroups of G. For Roy's group Γ this is $\{m \cdot 3^{-n} \mid m, n \in \mathbb{Z}, n \geq 0\}$ and obviously contains $-\frac{1105}{9}$.

Conjecture 0.2 (Modified Trace Conjecture for a group G). *The image of the composition*

$$K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.

The motivation for this paper is to prove

Theorem 0.3. *The image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\mathrm{id} \otimes \mathrm{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is Λ^G.

In particular the modified Trace Conjecture 0.2 holds for G, if the assembly map $\mathrm{asmb}^G : K_0^G(\underline{E}G) \to K_0(C_r^*(G))$ *appearing in the Baum-Connes Conjecture is surjective.*

In order to prove Theorem 0.3 (actually a generalization of it in Theorem 0.8), we will prove a slight generalization of Atiyah's L^2-Index Theorem and construct an equivariant Chern character for equivariant K-homology of proper G-CW-complexes, which is bijective and defined after applying $\Lambda^G \otimes_{\mathbb{Z}} -$.

Let M be a closed Riemannian manifold and $D^* = (D^*, d^*)$ be an elliptic complex of differential operators of order 1 on M. Denote by $\operatorname{index}(D^*) \in \mathbb{Z}$ its index. Let $\overline{M} \to M$ be a G-covering. Then one can lift D^* to an elliptic G-equivariant complex $\overline{D}^*$. Using the trace $\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$ Atiyah [1] defines its L^2*-index* $\operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*) \in \mathbb{R}$ and shows

$$\operatorname{index}(D^*) = \operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*).$$

Let $EG \to BG$ be the universal G-covering. The L^2-index theorem of Atiyah implies that the composition

$$K_0^G(EG) \xrightarrow{\operatorname{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

agrees with the composition

$$K_0^G(EG) \xrightarrow{\operatorname{ind}_{G\to\{1\}}} K_0(BG) \xrightarrow{K_0(\operatorname{pr})} K_0(*) \xrightarrow{\operatorname{asmb}^{\{1\}}} K_0(C_r^*(\{1\})) \xrightarrow{\dim_{\mathbb{C}}} \mathbb{Z} \hookrightarrow \mathbb{R}.$$

Since for a torsionfree group G the spaces EG and $\underline{E}G$ agree, the Baum-Connes Conjecture for a torsionfree group G does imply that the image of $K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$ is $\mathbb{Z}$ [3, Corollary 1 on page 21]. Instead of using the the standard von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$, one can use the universal center-valued trace $\operatorname{tr}^u_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G))$ to define an index

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) \in \mathcal{Z}(\mathcal{N}(G)),$$

which takes values in the center $\mathcal{Z}(\mathcal{N}(G))$ of the group von Neumann algebra $\mathcal{N}(G)$. Thus we get additional information, namely, for any element $g \in G$, whose conjugacy class (g) is finite, we get a complex number. However, it turns out that the value at classes (g) with $g \neq 1$ is zero and that the value at (1) is the index of D^*. Namely, we will show in Sect. 1

Theorem 0.4. *Under the conditions above we get in* $\mathcal{Z}(\mathcal{N}(G))$

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) = \operatorname{index}(D^*) \cdot 1_{\mathcal{N}(G)}.$$

As an illustration we discuss the special case, where G is finite, M is an oriented closed $4k$-dimensional manifold with free orientation preserving G-action and D^* is the signature operator. Then Theorem 0.4 reduces to the well-known statement that the equivariant signature

$$\operatorname{sign}^G(M) := [H_{2k}(M)^+] - [H_{2k}(M)^-] \in \operatorname{Rep}_{\mathbb{C}}(G)$$

is equal to $\text{sign}(G\backslash M) \cdot [\mathbb{C}G]$ for $\text{sign}(G\backslash M) \in \mathbb{Z}$ the (ordinary) signature of $G\backslash M$. We mention that this implies $\text{sign}(M) = |G| \cdot \text{sign}(G\backslash M)$. Theorem 0.4 is a special case of Theorem 5.4 but we will need it in the proof of Theorem 5.4 and therefore will have to prove it first.

The second ingredient is a variation of the equivariant Chern character of [13] for equivariant K-homology of proper G-CW-complexes. Recall that proper means that all isotropy groups are finite. The construction in [13] works for equivariant homology theories with a Mackey structure on the coefficient system in general, but requires to invert all primes. The construction we will give here works after applying $\Lambda^G \otimes_{\mathbb{Z}} ?$ and has a different source.

Denote for a proper G-CW-complex X by $\mathcal{F}(X)$ the set of all subgroups $H \subset G$, for which $X^H \neq \emptyset$, and by

$$\Lambda^G(X) := \mathbb{Z}\left[\frac{1}{\mathcal{F}(X)}\right] \tag{0.5}$$

the ring $\mathbb{Z} \subset \Lambda^G(X) \subset \Lambda^G$ obtained from $\mathbb{Z}$ by inverting the orders of all subgroups $H \in \mathcal{F}(X)$. Denote by

$$J^G \text{ resp. } J^G(X) \tag{0.6}$$

the set of conjugacy classes (C) of finite cyclic subgroups $C \subset G$ resp. the subset $J^G(X) \subset J^G$ of conjugacy classes (C) of finite cyclic subgroups $C \subset G$, for which X^C is non-empty. Obviously $\Lambda^G = \Lambda^G(\underline{E}G)$ and $J^G = J^G(\underline{E}G)$ since $\underline{E}G$ is characterized up to G-homotopy by the property that $\underline{E}G^H$ is contractible (and hence non-empty) for finite $H \subset G$ and empty for infinite $H \subset G$. Let $C \subset G$ be a finite cyclic subgroup. Let $C_G C$ be the centralizer and $N_G C$ be the normalizer of $C \subset G$. Let $W_G C$ be the quotient $N_G C / C_G C$. We will construct an idempotent $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$ which acts on $\Lambda^C \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$. We will see in Lemma 3.4 (b) that the cokernel of

$$\oplus_{D \subset C, D \neq C} \text{ind}_D^C : \oplus_{D \subset C, D \neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C)$$

is isomorphic to the image of the idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{C}}(C).$$

After introducing and proving some preliminary results about modules over a category and representation theory of finite groups in Sects. 2 and 3, we will prove in Sect. 4

Theorem 0.7. *Let X be a proper G-CW-complex. Put $\Lambda = \Lambda^G(X)$ and $J = J^G(X)$. Then there is for $p = 0, 1$ a natural isomorphism called* equivariant Chern character

$$\begin{aligned}\mathrm{ch}_p^G(X) : \oplus_{(C)\in J}\ \Lambda \otimes_{\mathbb{Z}} K_p\big(C_G C\backslash X^C\big) \\ \otimes_{\Lambda[W_G C]} \operatorname{im}\big(\theta_C : \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)\big) \\ \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p^G(X).\end{aligned}$$

Notice that the equivariant Chern character of Theorem 0.7 reduces to the obvious isomorphism $K_0(G\backslash X) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(\{1\}) \xrightarrow{\cong} K_0^G(X)$, if G acts freely on X. In the special case, where G is finite, X is the one-point-space $\{*\}$ and $p = 0$, the equivariant Chern character reduces to an isomorphism

$$\begin{aligned}\oplus_{(C)\in J^G} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}\left[\frac{1}{|G|}\right][W_G C]} \operatorname{im}\Bigg(\theta_C : \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\ \to \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)\Bigg) \\ \xrightarrow{\cong} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(G).\end{aligned}$$

This is a strong version of the well-known theorem of Artin that the map induced by induction

$$\oplus_{(C)\in J^G} \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(G)$$

is surjective for any finite group G. Artin's theorem is the reason why it does suffice rationally to consider all finite cyclic subgroups of G instead of all finite subgroups in Theorem 0.7. One might expect (and has to do integrally) in view of the Baum-Connes Conjecture and the fact that $\underline{E}G$ involves all finite subgroups that one has to take all finite subgroups into account.

Theorem 0.7 gives a computation of $\Lambda^G \otimes K_0^G(\underline{E}G)$, namely

$$\begin{aligned}\oplus_{(C)\in J^G}\ \Lambda^G \otimes_{\mathbb{Z}} K_p(B(C_G C)) \\ \otimes_{\Lambda^G[W_G C]} \operatorname{im}\big(\theta_C : \Lambda^G \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda^G \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)\big) \\ \xrightarrow{\cong} \Lambda^G \otimes_{\mathbb{Z}} K_p^G(\underline{E}G).\end{aligned}$$

Another construction of an equivariant Chern character using completely different methods can be found in [4]. However, it works only after applying $\mathbb{C} \otimes_{\mathbb{Z}} -$ and therefore cannot be used for our purposes here.

In Theorem 5.4 we will identify the composition of the Chern character of Theorem 0.7 with the map

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\mathrm{id}\otimes \mathrm{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id}\otimes i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

with an easier to understand and to calculate homomorphism, whose image is obvious from its definition. This will immediately imply

Theorem 0.8. *Let* Λ^G *resp.* J^G *be the ring resp. set introduced in* (0.1) *resp.* (0.6). *Then the image of the composition*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0^G(\underline{E}G) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} \mathrm{asmb}^G} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

is the image of the map given by induction

$$\bigoplus_{(C) \in J^G} \mathrm{id} \otimes \mathrm{ind}_C^G : \bigoplus_{(C) \in J^G} \Lambda^G \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \to \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

Now Theorem 0.3 follows from Theorem 0.8.

The change of rings and K-theory map $l : K_0(\mathbb{C}G) \to K_0(C_r^*(G))$ from the algebraic K_0-group of the complex group ring $\mathbb{C}G$ to the topological K_0-group of $C_r^*(G)$ is in general far from being surjective. There is some evidence that it is injective after applying $\Lambda \otimes_{\mathbb{Z}} ?$ (see [13, Theorem 0.1]). Theorem 0.8 gives some evidence for the conjecture that the image of $\Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$ agrees with the image of the composition $\Lambda^G \otimes_{\mathbb{Z}} K_0(\mathbb{C}G) \xrightarrow{l} \Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \xrightarrow{\mathrm{id} \otimes_{\mathbb{Z}} i} \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$.

Alain Valette pointed out to the author that the Modified Trace Conjecture 0.2 implies the following conjecture of Farkas [8, p. 593]

Conjecture 0.9 (Farkas). *If the rational number* m/n *is in the image of the composition*

$$K_0(\mathbb{C}G) \to K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

and the prime p *divides* n *but not* m, *then* G *has an element of order* p.

Notice that the Modified Trace Conjecture 0.2 implies that the image of the composition

$$K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in $\mathbb{Q}$ which is not known to be true in general. Some evidence for this claim comes from the theorem of Zalesskii that the image of the composition

$$K_0(\mathbb{C}G) \to K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in $\mathbb{Q}$. For its proof see [6, Sect. 3], [19].

The paper is organized as follows

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1. The L^2-index theorem

In this section we prove a slight generalization of the L^2-index theorem of Atiyah [1]. Let $\overline{M}$ be a Riemannian manifold (without boundary) together with a cocompact free proper action of G by isometries. In other words, $M = G\backslash\overline{M}$ is a closed Riemannian manifold, the projection $p : \overline{M} \to M$ is a G-covering and $\overline{M}$ is equipped with the Riemannian metric induced by the one of M. Let $D^* = (D^*, d^*)$ be an elliptic complex of differential operators $d^p : D^p \to D^{p+1}$ of order 1 acting on the space of sections $D^p = C^\infty(E^p)$ of vector bundles $E^p \to M$. Define $\overline{E}^p$ by p^*E^p and $\overline{D}^p$ by $L^2C^\infty(\overline{E}_p)$. Then G acts on $\overline{E}^p$ and $\overline{D}^p$. Since differential operators are local operators, there is a unique lift of each operator d^p to a G-equivariant differential operator $\widehat{d}^p : C^\infty(\overline{E}^p) \to C^\infty(\overline{E}^{p+1})$. We obtain an elliptic G-complex $(C^\infty(\overline{E}^*), \widehat{d}^*)$. Let $\overline{d}^p : \overline{D}^p \to \overline{D}^{p+1}$ be the minimal closure of $\widehat{d}^p$ which is the same as its maximal closure [1, Proposition 3.1].

Since D^* is elliptic, each cohomology module $H^p(D^*) := \ker(d^p)/\operatorname{im}(d^{p-1})$ is a finitely generated $\mathbb{C}$-module. Hence we can define the *index* of the elliptic complex D^* by

$$\operatorname{index}(D^*) := \sum_{p\geq 0}(-1)^p \cdot \dim_{\mathbb{C}}(H^p(D^*)) \in \mathbb{Z}. \tag{1.1}$$

Next we want to define an analogous invariant for the lifted complex $\overline{D}^*$. The group von Neumann algebra $\mathcal{N}(G)$ of G is the $*$-algebra $\mathcal{B}(l^2(G))^G$ of all bounded G-equivariant operators $l^2(G) \to l^2(G)$, where we equip $l^2(G)$ with the obvious left G-action. Let

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C} \tag{1.2}$$

be the *standard von Neumann trace*, which sends $f \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ to $\langle f(e), e\rangle_{l^2(G)}$, where e denotes the element in $l^2(G)$ given by the unit element in $G \subset l^2(G)$. Denote by $\mathcal{Z}(\mathcal{N}(G))$ the center of $\mathcal{N}(G)$. There is the universal center-valued trace [9, Theorem 7.1.12 on p. 462, Proposition 7.4.5 on p. 483, Theorem 8.2.8 on p. 517, Proposition 8.3.10 on p. 525, Theorem 8.4.3 on p. 532]

$$\operatorname{tr}^u_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G)) \tag{1.3}$$

which is uniquely determined by the following two properties:

(a) tr^u is a trace with values in the center, i.e. tr^u is $\mathbb{C}$-linear, for $a \in \mathcal{N}(G)$ with $a \geq 0$ we have $\mathrm{tr}^u(a) \geq 0$ and $\mathrm{tr}^u(ab) = \mathrm{tr}^u(ba)$ for all $a, b \in \mathcal{N}(G)$;
(b) $\mathrm{tr}^u(a) = a$ for all $a \in Z(\mathcal{N}(G))$.

The map tr^u has the following further properties:

(c) tr^u is faithful;
(d) tr^u is normal. Equivalently, tr^u is continuous with respect to the ultraweak topology on $\mathcal{N}(G)$;
(e) $||\,\mathrm{tr}^u(a)|| \leq ||a||$ for $a \in \mathcal{N}(G)$;
(f) $\mathrm{tr}^u(ab) = a\,\mathrm{tr}^u(b)$ for all $a \in Z(\mathcal{N}(G))$ and $b \in \mathcal{N}(G)$;
(g) Let p and q be projections in $\mathcal{N}(G)$. Then p and q are equivalent, i.e. $p = vv^*$ and $q = v^*v$, if and only if $\mathrm{tr}^u(p) = \mathrm{tr}^u(q)$;
(h) Any linear functional $f : \mathcal{N}(G) \to \mathbb{C}$, which is continuous with respect to the norm topology on $\mathcal{N}(G)$ and which is central, i.e. $f(ab) = f(ba)$ for all $a, b \in \mathcal{N}(G)$, factorizes as

$$\mathcal{N}(G) \xrightarrow{\mathrm{tr}^u} Z(\mathcal{N}(G)) \xrightarrow{f|_{Z(\mathcal{N}(G))}} \mathbb{C}.$$

In particular $\mathrm{tr}_{\mathcal{N}(G)} \circ \mathrm{tr}^u_{\mathcal{N}(G)} = \mathrm{tr}_{\mathcal{N}(G)}$.

A *Hilbert $\mathcal{N}(G)$-module* V is a Hilbert space V together with a G-action by isometries such that there exists a Hilbert space H and a G-equivariant projection $p : H \otimes l^2(G) \to H \otimes l^2(G)$ with the property that V and $\mathrm{im}(p)$ are isometrically G-linearly isomorphic. Here $H \otimes l^2(G)$ is the tensor product of Hilbert spaces and G acts trivially on H and on $l^2(G)$ by the obvious left multiplication. Notice that p is not part of the structure, only its existence is required. We call V *finitely generated* if H can be chosen to be finite-dimensional.

Our main examples of Hilbert $\mathcal{N}(G)$-modules are the Hilbert spaces $\overline{D}^p$ which are isometrically G-isomorphic to $L^2(C^\infty(E^p)) \otimes l^2(G)$. This can be seen using a fundamental domain $\mathcal{F}$ for the G-action on $\overline{M}$ which is from a measure theory point of view the same as M. A morphism $f : V \to W$ of Hilbert $\mathcal{N}(G)$-modules is a densely defined closed G-equivariant operator. The differentials $\overline{d}^p$ are morphisms of Hilbert $\mathcal{N}(G)$-modules.

Let $f : V \to V$ be a *morphism of Hilbert $\mathcal{N}(G)$-modules* which is positive. Choose a G-projection $p : H \otimes l^2(G) \to H \otimes l^2(G)$ and an isometric invertible G-equivariant operator $u : \mathrm{im}(p) \to V$. Let $\{b_i \mid i \in I\}$ be a Hilbert basis for H. Let $\overline{f}$ be the composition

$$H \otimes l^2(G) \xrightarrow{p} \mathrm{im}(p) \xrightarrow{u} V \xrightarrow{f} V \xrightarrow{u^{-1}} \mathrm{im}(p) \hookrightarrow H \otimes l^2(G).$$

Define the *von Neumann trace* of $f : V \to V$ by

$$\mathrm{tr}_{\mathcal{N}(G)}(f) := \sum_{i \in I} \langle \overline{f}(b_i \otimes e), b_i \otimes e \rangle_{H \otimes l^2(G)} \qquad \in [0, \infty]. \quad (1.4)$$

This is indeed independent of the choice of p, u and the Hilbert basis $\{b_i \mid i \in I\}$. If V is finitely generated and f is bounded, then $\operatorname{tr}_{\mathcal{N}(G)}(f) < \infty$ is always true. Define the von Neumann dimension of a Hilbert $\mathcal{N}(G)$-module V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(\mathrm{id}: V \to V) \qquad \in [0, \infty]. \tag{1.5}$$

If V is a finitely generated Hilbert $\mathcal{N}(G)$-module, we define the *universal center-valued von Neumann dimension*

$$\dim^u_{\mathcal{N}(G)}(V) := \operatorname{tr}^u_{\mathcal{N}(G)}(\mathrm{id}: V \to V) \in \mathcal{Z}(\mathcal{N}(G)) \tag{1.6}$$

analogously to $\dim_{\mathcal{N}(G)}(V)$ replacing $\operatorname{tr}_{\mathcal{N}(G)}$ by $\operatorname{tr}^u_{\mathcal{N}(G)}$. Given a finitely generated Hilbert $\mathcal{N}(G)$-module V, we have $\operatorname{tr}_{\mathcal{N}(G)}(\dim^u_{\mathcal{N}(G)}(V)) = \dim_{\mathcal{N}(G)}(V)$.

Define the L^2-cohomology $H^p_{(2)}(\overline{D}^*)$ to be $\ker(\overline{d}^p)/\operatorname{clos}(\operatorname{im}(\overline{d}^{p-1}))$, where $\operatorname{clos}(\operatorname{im}(\overline{d}^{p-1}))$ is the closure of the image of $\overline{d}^{p-1}$. Define the p-th Laplacian by $\overline{\Delta}_p = (\overline{d}^p)^*\overline{d}^p + \overline{d}^{p-1}(\overline{d}^{p-1})^*$. By the L^2-Hodge-deRham Theorem we get a G-equivariant isometric isomorphism $\ker(\overline{\Delta}_p) \xrightarrow{\cong} H^p_{(2)}(\overline{D}^*)$. Thus $H^p_{(2)}(\overline{D}^*)$ inherits the structure of a Hilbert $\mathcal{N}(G)$-module. Moreover, it turns out to be a finitely generated Hilbert $\mathcal{N}(G)$-module. This can be deduced from the results of [14], where an index already over $C^*_r(G)$ is defined and the problem of getting finitely generated modules over $C^*_r(G)$ is treated. Namely, one can deduce from [14] after passing to the group von Neumann algebra, that there are finitely generated Hilbert $\mathcal{N}(G)$-modules U_1, U_2, V_1 and V_2 and Hilbert $\mathcal{N}(G)$-modules W_1 and W_2 together with a morphism $v : V_1 \to V_2$ and isomorphisms of Hilbert $\mathcal{N}(G)$-modules $w : W_1 \xrightarrow{\cong} W_2$, $u_1 : \overline{D}^p \oplus U_1 \xrightarrow{\cong} V_1 \oplus W_1$ and $u_2 : \overline{D}^p \oplus U_2 \xrightarrow{\cong} V_2 \oplus W_2$ such that $u_2 \circ (\Delta_p \oplus 0) = (v \oplus w) \circ u_1$. Obviously the kernel of v and hence the kernel of $\overline{\Delta}_p$ are finitely generated Hilbert $\mathcal{N}(G)$-modules.

Define the *center-valued L^2-index* and the *L^2-index*

$$\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*) := \sum_{p \geq 0} (-1)^p \cdot \dim^u_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right) \in \mathcal{Z}(\mathcal{N}(G)); \tag{1.7}$$

$$\operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*) := \sum_{p \geq 0} (-1)^p \cdot \dim_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right) \in \mathbb{R}. \tag{1.8}$$

The rest of this section is devoted to the proof of Theorem 0.4

Notation 1.9. *Denote by* $\operatorname{con}(G)_{cf}$ *the set of conjugacy classes* (g) *of elements* $g \in G$ *such that the set* (g) *is finite, or, equivalently, the centralizer* $C_G(g) = \{g' \in G \mid g'g = gg'\}$ *has finite index in* G*. For* $c \in \operatorname{con}(G)_{cf}$ *let* N_c *be the element* $\sum_{g \in c} g \in \mathbb{C}G$*. In the sequel* L_c *resp.* L_g *denotes left multiplication with* N_c *resp.* g *for* $c \in \operatorname{con}(G)_{cf}$ *resp.* $g \in G$*.*

Notice for the sequel that $N_c \in \mathcal{Z}(\mathcal{N}(G))$ and L_c is G-equivariant and commutes with all G-operators.

Lemma 1.10. *Consider $a \in \mathcal{Z}(\mathcal{N}(G))$. Then we have $a = 0$ if and only if $\operatorname{tr}_{\mathcal{N}(G)}(N_c a) = 0$ holds for any $c \in \operatorname{con}_{cf}(G)$.*

Proof. Consider $a \in \mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ which belongs to $\mathcal{Z}(\mathcal{N}(G))$. Write $a(e) = \sum_{g \in G} \lambda_g \cdot g \in l^2(G)$. Since $aR_g = R_g a$ holds for $g \in G$ and $R_g : l^2(G) \to l^2(G)$ given by right multiplication with $g \in G$, we get $\lambda_g = \lambda_{hgh^{-1}}$ for $g, h \in G$. This implies that $\lambda_g = 0$ if the conjugacy class (g) is infinite. On easily checks for an element g with finite (g)

■ Do you mean On or One?

$$|(g)| \cdot \lambda_g \;=\; \operatorname{tr}_{\mathcal{N}(G)}(N_{(g^{-1})} a).$$

□

Lemma 1.11. *We get under the conditions above.*

$$\operatorname{tr}_{\mathcal{N}(G)}\left(\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) = \operatorname{index}(D^*).$$

Proof. The L^2-index theorem of Atiyah [1, (1.1)] says

$$\operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*) \;=\; \operatorname{index}(D^*).$$

We have

$$\begin{aligned}
\operatorname{tr}_{\mathcal{N}(G)}\left(\operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) &= \operatorname{tr}_{\mathcal{N}(G)}\left(\sum_{p \geq 0} (-1)^p \dim^u_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right)\right) \\
&= \sum_{p \geq 0} (-1)^p \operatorname{tr}_{\mathcal{N}(G)}\left(\dim^u_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right)\right) \\
&= \sum_{p \geq 0} (-1)^p \dim_{\mathcal{N}(G)}\left(H^p_{(2)}(\overline{D}^*)\right) \\
&= \operatorname{index}_{\mathcal{N}(G)}(\overline{D}^*).
\end{aligned}$$

□

Next we want to prove

Lemma 1.12. *Consider an element $c \in \operatorname{con}(G)_{cf}$ with $c \neq (1)$. Then*

$$\operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \;=\; 0.$$

Proof. In the sequel we denote by $\overline{\operatorname{pr}}_p : \overline{D}^p \to \overline{D}^p$ the projection onto the kernel of the p-th Laplacian $\overline{\Delta}_p = (\overline{d}^p)^* \overline{d}^p + \overline{d}^{p-1}(\overline{d}^{p-1})^*$. By the L^2-Hodge-deRham Theorem we get a G-equivariant isometric isomorphism $\operatorname{im}(\overline{\operatorname{pr}}_p) \xrightarrow{\cong} H^p_{(2)}(\overline{D}^*)$. This implies

$$
\begin{aligned}
&\operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{tr}^u_{\mathcal{N}(G)}\left(\operatorname{id} : H^p_{(2)}(\overline{D}^*) \to H^p_{(2)}(\overline{D}^*)\right)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c : H^p_{(2)}(\overline{D}^*) \to H^p_{(2)}(\overline{D}^*)\right)\\
&= \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ \overline{\operatorname{pr}}_p : \overline{D}^p \to \overline{D}^p\right).
\end{aligned} \tag{1.13}
$$

The operator $e^{-t\overline{\Delta}_p} : \overline{D}^p \to \overline{D}^p$ is a bounded G-equivariant operator and has a smooth kernel $e^{-t\overline{\Delta}_p}(\overline{x},\overline{y}) : \overline{E}^p_{\overline{x}} \to \overline{E}^p_{\overline{y}}$ for $\overline{x},\overline{y} \in \overline{M}$. Thus $e^{-t\overline{\Delta}_p}(\omega)$ applied to a section ω is given at $\overline{y} \in \overline{M}$ by $\int_{\overline{M}} e^{-t\overline{\Delta}_p}(\overline{x},\overline{y})(\omega(\overline{x}))d\mathrm{vol}_{\overline{x}}$. The operator $L_c \circ e^{-t\overline{\Delta}_p}$ is also a bounded G-equivariant operator and has a smooth kernel $(L_c \circ e^{-t\overline{\Delta}_p})(\overline{x},\overline{y})$ satisfying

$$
\left(L_c \circ e^{-t\overline{\Delta}_p}\right)(\overline{x},\overline{y}) \;=\; \sum_{g\in c} L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{y}).
$$

If $\mathcal{F}$ is a fundamentl domain for the G-action, then [1, Proposition 4.6].

$$
\begin{aligned}
\operatorname{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) &= \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(\left(L_c \circ e^{-t\overline{\Delta}_p}\right)(\overline{x},\overline{x})\right) d\mathrm{vol}_{\overline{x}};\\
&= \sum_{g\in c}\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})\right) d\mathrm{vol}_{\overline{x}}.
\end{aligned} \tag{1.14}
$$

where $\operatorname{tr}_{\mathbb{C}}$ is the trace of an endomorphism of a finite-dimensional complex vector space. We have

$$
\lim_{t\to 0}\; \sup\left\{||e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})|| \mid \overline{x} \in \mathcal{F}\right\} = 0, \tag{1.15}
$$

where $||e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})||$ is the operator norm of the linear map $e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})$ of finite-dimensional Hilbert spaces. This follows from the finite propagation speed method of [7]. There only the standard Laplacian on 0-forms is treated, but the proof presented there carries over to the Laplacian $\overline{\Delta}_p$ associated to the lift $\overline{D}^*$ to the G-covering $\overline{M}$ of an elliptic complex D^* of differential operators of order 1 on a closed Riemannian manifold M in any dimension p. The point is that $\overline{M}$ has bounded geometry, $\overline{\Delta}_p$ is essentially selfadjoint and positive so that $\sqrt{\overline{\Delta}_p}$ makes sense, and $\frac{\partial^2}{\partial t^2} + \overline{\Delta}_p$ is strictly hyperbolic. Now one applies the results of [7, Sect. 1] and uses the estimate in [11, p. 475], where the special case of D^* being the deRham complex is treated.

Since

$$\left|\mathrm{tr}_{\mathbb{C}}\left(L_g \circ e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})\right)\right| \;\leq\; \dim_{\mathbb{C}}(E^p) \cdot ||e^{-t\overline{\Delta}_p}(\overline{x}, g^{-1}\overline{x})||$$

and $\mathcal{F}$ is relative compact, we conclude from (1.14) and (1.15)

$$\lim_{t\to 0} \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) = 0. \tag{1.16}$$

Since the trace $\mathrm{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous and $\lim_{t\to\infty} e^{-t\overline{\Delta}_p} = \overline{\mathrm{pr}}_p$ in the weak topology, we get

$$\lim_{t\to\infty} \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ e^{-t\overline{\Delta}_p}) = \mathrm{tr}_{\mathcal{N}(G)}(L_c \circ \overline{\mathrm{pr}}_p). \tag{1.17}$$

We conclude from (1.13) and (1.17)

$$\begin{aligned} &\mathrm{tr}_{\mathcal{N}(G)}\left(N_c \cdot \mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \\ &\qquad = \lim_{t\to\infty} \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ e^{-t\overline{\Delta}_p}\right). \end{aligned} \tag{1.18}$$

We have

$$\begin{aligned} &\frac{d}{dt} \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \to \overline{D}^p\right) \\ &= \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ \frac{d}{dt} e^{-t\overline{\Delta}_p}\right) \\ &= \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ (-\overline{\Delta}_p) \circ e^{-t\overline{\Delta}_p}\right) \\ &= -\sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p}\right) \\ &\qquad - \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^p)^* \circ \overline{d}^p \circ e^{-t\overline{\Delta}_p}\right) \\ &= -\sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p}\right) \\ &\qquad - \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^p)^* \circ e^{-t\overline{\Delta}_{p+1}} \circ \overline{d}^p\right) \\ &= -\sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ \overline{d}^{p-1} \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p}\right) \\ &\qquad + \sum_{p\geq 0} (-1)^p \cdot \mathrm{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p} \circ \overline{d}^{p-1}\right) \\
&\qquad\qquad + \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1}\right) \\
&= -\sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1}\right) \\
&\qquad\qquad + \sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ (\overline{d}^{p-1})^* \circ e^{-t\overline{\Delta}_p} \circ \overline{d}^{p-1}\right) \\
&= 0. \qquad (1.19)
\end{aligned}
$$

Here are some justifications for the calculation above. Recall that L_c is a bounded G-operator and commutes with any G-equivariant operator. We can commute $\operatorname{tr}_{\mathcal{N}(G)}$ and $\frac{d}{dt}$ since $\operatorname{tr}_{\mathcal{N}(G)}$ is ultraweakly continuous. We conclude $e^{-t\overline{\Delta}_{p+1}} \circ \overline{d}^p = \overline{d}^p \circ e^{-t\overline{\Delta}_p}$ from the fact that $\overline{\Delta}_{p+1} \circ \overline{d}^p = \overline{d}^p \circ \overline{\Delta}_p$ holds on $C^\infty(\overline{E}^{p-1})$. We have used at several places the typical trace relation $\operatorname{tr}_{\mathcal{N}(G)}(AB) = \operatorname{tr}_{\mathcal{N}(G)}(BA)$ which is in each case justified by [1, section 4]. In order to be able to apply this trace relation we have splitted $e^{-t\overline{\Delta}_p}$ into $e^{-\frac{t}{2}\overline{\Delta}_p} \circ e^{-\frac{t}{2}\overline{\Delta}_p}$ in the calculation above.

Hence $\sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \to \overline{D}^p\right)$ is independent of t and we conclude from (1.18)

$$
\begin{aligned}
&\operatorname{tr}_{\mathcal{N}(G)}\left(N_c \cdot \operatorname{index}^u_{\mathcal{N}(G)}(\overline{D}^*)\right) \\
&\qquad = \lim_{t\to 0}\sum_{p\geq 0}(-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)}\left(L_c \circ e^{-t\overline{\Delta}_p} : \overline{D}^p \to \overline{D}^p\right). \qquad (1.20)
\end{aligned}
$$

Now Lemma 1.12 follows from (1.16) (1.20). □

Finally Theorem 0.4 follows from Lemma 1.10, Lemma 1.11 and Lemma 1.12.

2. Modules over a category

In this section we recall some facts about modules over the category $\mathcal{S}ub = \mathcal{S}ub(G;\mathcal{F}(X))$ for a proper G-CW-complex X as far as needed here. For more information about modules over a category we refer to [12].

Let $\mathcal{S}ub := \mathcal{S}ub(G;\mathcal{F}(X))$ be the following category. Objects are the elements of the set $\mathcal{F}(X)$ of subgroups $H \subset G$, for which $X^H \neq \emptyset$. For two finite subgroups H and K in $\mathcal{F}(X)$ denote by $\operatorname{conhom}_G(H, K)$ the set of group homomorphisms $f : H \to K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that f is given by conjugation with g, i.e. $f = c(g) : H \to K, \quad h \mapsto ghg^{-1}$. Notice that

$c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $C_G H = \{g \in G | gh = hg$ for all $h \in H\}$ of H in G. The group of inner automorphisms of K acts on $\operatorname{conhom}_G(H, K)$ from the left by composition. Define the set of morphisms $\operatorname{mor}_{\mathcal{Sub}}(H, K)$ by $\operatorname{Inn}(K)\backslash \operatorname{conhom}_G(H, K)$. Let $N_G H$ be the normalizer $\{g \in G \mid gHg^{-1} = H\}$ of H. Define $H \cdot C_G H = \{h \cdot g \mid h \in H, g \in C_G H\}$. This is a normal subgroup of $N_G H$ and we define $W_G H := N_G H/(H \cdot C_G H)$. One easily checks that $W_G H$ is a finite group and that there is an isomorphism from $W_G H$ to $\operatorname{aut}_{\mathcal{Sub}}(H)$ which sends $g(H \cdot C_G H) \in W_G H$ to the automorphism of H represented by $c(g) : H \to H$. Notice that there is a morphism from H to K if and only if H is subconjugated to K. There is an isomorphism from H to K if and only if H and K are conjugated. The category $\mathcal{Sub}$ is a so called EI-category, i.e. any endomorphism in $\mathcal{Sub}$ is an isomorphism.

Many constructions in equivariant topology of proper G-spaces are carried out over the orbit category $\mathcal{Or}(G; \mathcal{F}in)$. It has as objects homogeneous spaces G/H for finite subgroups $H \subset G$. Morphisms are G-maps. Notice that $\mathcal{Sub}(G; \mathcal{F}in)$ is a quotient category of $\mathcal{Or}(G; \mathcal{F}in)$. The decisive difference between $\mathcal{Or}(G; \mathcal{F}in)$ and $\mathcal{Sub}(G; \mathcal{F}in)$ is that the automorphism group of G/H in $\mathcal{Or}(G; \mathcal{F}in)$ is $N_G H/H$ which is not finite in general, whereas the automorphism group of H in $\mathcal{Sub}(G; \mathcal{F}in)$ is $W_G H := N_G H/(H \cdot C_G H)$ which always is finite. We can work with $\mathcal{Sub}$ instead of the orbit category since we have induction homomorphisms for equivariant K-homology.

Let R be a commutative associative ring with unit. A *covariant resp. contravariant $R\mathcal{Sub}$-module M* is a covariant resp. contravariant functor from $\mathcal{Sub}$ to the category of R-modules. Morphisms are natural transformations. The structure of an abelian category on the category of R-modules carries over to the category of $R\mathcal{Sub}$-modules. In particular the notion of a projective $R\mathcal{Sub}$-module is defined. Given a contravariant $R\mathcal{Sub}$-module M and a covariant $R\mathcal{Sub}$-module N, one can define a R-module, their *tensor product over $\mathcal{Sub}$*

$$M \otimes_{R\mathcal{Sub}} N = \oplus_{H \in \mathcal{F}(X)} M(H) \otimes_R N(H)/\sim,$$

where $\sim$ is the typical tensor relation $mf \otimes n = m \otimes fn$, i.e. for each morphism $f : H \to K$ in $\mathcal{Sub}$, $m \in M(K)$ and $n \in N(H)$ we introduce the relation $M(f)(m) \otimes n - m \otimes N(f)(n) = 0$.

Given a left $R[W_G H]$-module N for $H \in \mathcal{F}(X)$, define a covariant $R\mathcal{Sub}$-module $E_H M$ by

$$(E_H M)(K) := R \operatorname{mor}_{\mathcal{Sub}}(H, K) \otimes_{R[W_G H]} N \qquad \text{for } K \subset G, |K| < \infty, \tag{2.1}$$

where $R \operatorname{mor}_{\mathcal{Sub}}(H, K)$ is the free R-module generated by the set $\operatorname{mor}_{\mathcal{Sub}}(H, K)$. Given a covariant $R\mathcal{Sub}$-module M and $H \in \mathcal{F}(X)$, define $M(H)_s$ to be the left R-submodule of $M(H)$, which is spanned by the images of all R-maps $M(f): M(K) \to M(H)$, where f runs through all morphisms $f: K \to H$ in $\mathcal{Sub}$, which have H as target and are not isomorphisms. Obviously $M(H)_s$ is an $R[W_G H]$-submodule of $M(H)$. Define a left $R[W_G H]$-module $S_H M$ by

$$S_H M := M(H)/M(H)_s. \tag{2.2}$$

Both functors E_H and S_H respect direct sums and the property finitely generated and the property projective. Given a left $R[W_G H]$-module M, $S_K \circ E_H M$ is M, if $H = K$ and is 0, if H and K are not conjugated in G.

Let M be a covariant $R\mathcal{Sub}$-module. We want to check whether it is projective or not. A necessary (but not sufficient) condition is that $S_H M$ is a projective $R[W_G H]$-module. Assume that $S_H M$ is $R[W_G H]$-projective for all objects H in $\mathcal{Sub}$. We can choose a $R[W_G H]$-splitting $\sigma_H : S_H M \to M(H)$ of the canonical projection $M(H) \to S_H M = M(H)/M(H)_s$. For a finite subgroup $H \subset G$ define the morphism of covariant $R\mathcal{Sub}$-modules

$$i_H M : E_H(M(H)) \to M$$

by $(i_H M)(K)((f : H \to K) \otimes_{R[W_G H]} m) = M(f)(m)$. We obtain after a choice of representatives $H \in (H)$ for any conjugacy class (H) of subgroups $H \in \mathcal{F}(X)$ a morphism of covariant $R\mathcal{Sub}$-modules

$$\begin{aligned} T : \oplus_{(H), H \in \mathcal{F}(X)} E_H S_H M &\xrightarrow{\oplus_{(H), H \in \mathcal{F}(X)} E_H(\sigma_H)} \\ \oplus_{(H), H \in \mathcal{F}(X)} E_H(M(H)) &\xrightarrow{\oplus_{(H), H \in \mathcal{F}(X)} i_H M} M. \end{aligned} \tag{2.3}$$

We get as a special case of [13, Theorem 2.11]

Theorem 2.4. *The morphism T is always surjective. It is bijective if and only if M is a projective $R\mathcal{Sub}$-module.*

3. Some representation theory for finite groups

Denote for a finite group H by $\operatorname{Rep}_{\mathbb{Q}}(H)$ resp. $\operatorname{Rep}_{\mathbb{C}}(H)$ the ring of finite dimensional H-representations over the field $\mathbb{Q}$ resp. $\mathbb{C}$. Recall for the sequel that these are finitely generated free abelian groups. Given an inclusion of finite groups $H \subset G$, we denote by $\operatorname{ind}_H^G : \operatorname{Rep}_{\mathbb{Q}}(H) \to \operatorname{Rep}_{\mathbb{Q}}(G)$ and $\operatorname{res}_G^H : \operatorname{Rep}_{\mathbb{Q}}(G) \to \operatorname{Rep}_{\mathbb{Q}}(H)$ the induction and restriction homomorphism and similar for $R \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}$, $\operatorname{Rep}_{\mathbb{C}}$ and $R \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}$ for a commutative ring R with $\mathbb{Z} \subset R$. Let $\operatorname{con}_{\mathbb{Q}}(H)$ be the set of $\mathbb{Q}$-conjugacy classes of elements in H, where h and h' are called $\mathbb{Q}$-conjugated if the cyclic subgroups $\langle h \rangle$ and $\langle h' \rangle$ are conjugated in G. Let $\operatorname{con}(G)$ be the set of conjugacy classes of elements in G. Denote by $\operatorname{class}_{\mathbb{Q}}(H)$ resp. $\operatorname{class}_{\mathbb{C}}(H)$ the rational resp.

complex vector space of functions $\operatorname{con}_{\mathbb{Q}}(H) \to \mathbb{Q}$ resp. $\operatorname{con}(G) \to \mathbb{C}$. Character theory yields isomorphisms [17, p. 68 and Theorem 29 on p. 102]

$$\chi_{\mathbb{Q}} : \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(H) \xrightarrow{\cong} \operatorname{class}_{\mathbb{Q}}(H);$$
$$\chi_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H) \xrightarrow{\cong} \operatorname{class}_{\mathbb{C}}(H).$$

For a finite cyclic group C denote by $\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$ the element whose character $\chi_{\mathbb{Q}}(\theta_C)$ sends $c \in C$ to 1, if c generates C, and to 0 otherwise.

Let $C \subset H$ be a cyclic subgroup of the finite group H. Then we get for $h \in H$

$$\begin{aligned} \frac{1}{[H:C]} \cdot \chi_{\mathbb{Q}}\left(\operatorname{ind}_C^H \theta_C\right)(h) &= \frac{1}{[H:C]} \cdot \frac{1}{|C|} \cdot \sum_{l \in H, l^{-1}hl \in C} \chi_{\mathbb{Q}}(\theta_C)(l^{-1}hl) \\ &= \frac{1}{|H|} \cdot \sum_{l \in H, \langle l^{-1}hl \rangle = C} 1. \end{aligned}$$

Denote by $[\mathbb{Q}] \in \operatorname{Rep}_{\mathbb{Q}}(H)$ the class of the trivial H-representation $\mathbb{Q}$. Notice that $\chi_{\mathbb{Q}}([\mathbb{Q}])$ is the constant function with values 1. We get in $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(H)$

$$1 \otimes_{\mathbb{Z}} [\mathbb{Q}] = \sum_{C \subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C, \tag{3.1}$$

since for any $l \in H$ and $h \in H$ there is precisely one cyclic subgroup $C \subset H$ with $C = \langle l^{-1}hl \rangle$ and $\chi_{\mathbb{Q}}$ is bijective. In particular we get for a finite cyclic group C in $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$

$$\theta_C = 1 \otimes_{\mathbb{Z}} [\mathbb{Q}] - \sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D. \tag{3.2}$$

Now one easily checks by induction over the order of the finite cyclic subgroup C that the element θ_C satisfies

$$\theta_C \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C). \tag{3.3}$$

Obviously θ_C is an idempotent in $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$. By the obvious change of rings homomorphism, $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$ becomes a $\mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$-module. Hence multiplication with θ_C defines an idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

It is natural with respect to group automorphisms of C, since θ_C is invariant under group automorphisms of C.

Lemma 3.4. *(a) For a finite group H the map*

$$\bigoplus_{C\subset H, C \text{ cyclic}} \operatorname{ind}_C^H : \bigoplus_{C\subset H, C \text{ cyclic}} \mathbb{Z}\left[\frac{1}{|H|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|H|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$$

is surjective;

(b) Let C be a finite cyclic group. Then the image resp. cokernel of

$$\bigoplus_{D\subset C, D\neq C} \operatorname{ind}_D^C : \bigoplus_{D\subset C, D\neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$$

is equal resp. isomorphic to the kernel resp. image of the idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C);$$

(c) Let C be a finite cyclic group. The image of the idempotent endomorphism

$$\theta_C : \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C);$$

is a projective $\mathbb{Z}[\frac{1}{|C|}][\operatorname{aut}(C)]$*-module, where the* $\operatorname{aut}(C)$*-operation comes from the obvious* $\operatorname{aut}(C)$*-operation on C.*

Proof. (a) follows from the following calculation for $x \in \mathbb{Z}[\frac{1}{|H|}] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$ based on (3.1)

$$\begin{aligned} x &= (1 \otimes_{\mathbb{Z}} [\mathbb{Q}]) \cdot x \\ &= \left(\sum_{C\subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C \right) \cdot x \\ &= \sum_{C\subset H, C \text{ cyclic}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \left(\theta_C \cdot \operatorname{res}_H^C x\right). \end{aligned}$$

(b) follows from the following two calculations based on (3.2) for $x \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$

$$\begin{aligned} x - \theta_C \cdot x &= (1 \otimes [\mathbb{Q}] - \theta_C) \cdot x \\ &= \left(\sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D \right) \cdot x \\ &= \sum_{D \subset C, D \neq C} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \left(\theta_D \cdot \operatorname{res}_C^D x \right) \end{aligned}$$

and for $D \subset C$, $D \neq C$ and $y \in \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D)$

$$\theta_C \cdot \operatorname{ind}_D^C y = \operatorname{ind}_D^C \left(\operatorname{res}_C^D \theta_C \cdot y \right) = \operatorname{ind}_D^C (0 \cdot y) = 0.$$

(c) Put $\Lambda = \mathbb{Z}\left[\frac{1}{|C|}\right]$. Let C_p be the p-Sylow subgroup of C for a prime p. There are canonical isomorphisms

$$\begin{aligned} C &\cong \prod_p C_p; \\ \operatorname{aut}(C) &\cong \prod_p \operatorname{aut}(C_p); \\ P : \otimes_p \operatorname{Rep}_{\mathbb{C}}(C_p) &\cong \operatorname{Rep}_{\mathbb{C}}(C), \end{aligned}$$

where p runs through the prime numbers diving $|C|$. The isomorphism P assigns to $\otimes_p [V_p]$ for C_p-representations V_p the class of the C-representation $\otimes_p V_p$ with the factorwise action of $\operatorname{aut}(C) = \prod_p \operatorname{aut}(C_p)$. The following diagram commutes

$$\begin{array}{ccc} \otimes_p \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C_p) & \xrightarrow{P} & \operatorname{Rep}_{\mathbb{C}}(C) \\ \downarrow{\scriptstyle \otimes_p \theta_{C_p}} & & \downarrow{\scriptstyle \theta_C} \\ \otimes_p \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C_p) & \xrightarrow[P]{} & \operatorname{Rep}_{\mathbb{C}}(C) \end{array}$$

Thus we obtain an isomorphism of $\Lambda[\operatorname{aut}(C)]$-modules

$$\otimes_p \operatorname{im}(\theta_{C_p}) \xrightarrow{\cong} \operatorname{im}(\theta_C),$$

where $\operatorname{aut}(C) = \prod_p \operatorname{aut}(C_p)$ acts factorwise on the source. Hence the claim for C follows if we know it for C_p for all primes p. Therefore it remains to treat the case $C = \mathbb{Z}/p^n$ for some prime number p and positive integer n. Notice that then $\Lambda = \mathbb{Z}\left[\frac{1}{p}\right]$.

In the sequel we abbreviate $A(n) = \operatorname{aut}(\mathbb{Z}/p^n)$. This is isomorphic to the multiplicative group of units $\mathbb{Z}/p^{n\times}$ in $\mathbb{Z}/p^n$ and hence an abelian group of order $p^{n-1} \cdot (p-1)$. Denote by $A(n)_p$ the p-Sylow subgroup and by $A(n)'_p$

the subgroup $\{a \in A(n) \mid a^{p-1} = 1\}$ which is cyclic of order $(p-1)$. We get a canonical isomorphism

$$A(n) \cong A(n)_p \times A(n)'_p$$

Notice that $\mathbb{Z}/p^n$ has precisely one subgroup of order p^m for $0 \le m \le n$ which will be denoted by $\mathbb{Z}/p^m$. These subgroups are characteristic and hence restriction to these subgroups yields homomorphisms $A(n) \to A(n-1) \to \ldots \to A(1)$. They induce epimorphisms $A(m)_p \to A(m-1)_p$ and isomorphisms $A(m)'_p \xrightarrow{\cong} A(m-1)'_p$. Using these isomorphisms we will identify

$$A(n)'_p = A(n-1)'_p = \ldots = A(1)'_p = \mathbb{Z}/p^{\times}.$$

Thus we get canonical decompositions

$$A(n) = A(n)_p \times \mathbb{Z}/p^{\times}.$$

Let M be a $\Lambda[A(n)]$-module. Let $\operatorname{res} M$ be the $\Lambda[\mathbb{Z}/p^{\times}]$-module obtained by restriction. The following maps are $\Lambda[A(n)]$-homomorphisms

$$q : \Lambda[A(n)_p] \otimes_{\Lambda} \operatorname{res} M \to M, \qquad a \otimes m \mapsto am;$$

$$s : M \to \Lambda[A(n)_p] \otimes_{\Lambda} \operatorname{res} M, \qquad m \mapsto \frac{1}{|A(n)_p|} \cdot \sum_{a \in A(n)_p} a \otimes a^{-1}m,$$

where $A(n) = A(n)_p \times \mathbb{Z}/p^{\times}$ acts factorwise on $\Lambda[A(n)_p] \otimes_{\Lambda} \operatorname{res} M$. They satisfy $q \circ s = \mathrm{id}$. Obviously $\Lambda[A(n)_p] \otimes_{\Lambda} \operatorname{res} M$ is $\Lambda[A(n)]$-projective if $\operatorname{res} M$ is $\Lambda[\mathbb{Z}/p^{\times}]$-projective. This shows that M is $\Lambda[A(n)]$-projective if its restriction $\operatorname{res} M$ to a $\Lambda[\mathbb{Z}/p^{\times}]$-module is projective. Therefore it suffices to show that $\operatorname{im}(\theta_C)$ is $\Lambda[\mathbb{Z}/p^{\times}]$-projective.

The composition of the induction homomorphism $\operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1}) \to \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^n)$ with the restriction homomorphism $\operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^n) \to \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1})$ is $p \cdot \mathrm{id} : \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1}) \to \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1})$. We conclude from Lemma 3.4 (b) applied with $C = \mathbb{Z}/p^n$ that the $\Lambda[\mathbb{Z}/p^{\times}]$-module $\operatorname{im}(\theta_C)$ is isomorphic to the kernel of the surjective restriction homomorphism $\operatorname{res} : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^n) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1})$. Hence there is an exact sequence of $\Lambda[\mathbb{Z}/p^{\times}]$-modules

$$0 \to \operatorname{im}(\theta_C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^n) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1}) \to 0.$$

It induces an exact sequence of $\Lambda[\mathbb{Z}/p^{\times}]$-modules

$$0 \to \operatorname{im}(\theta_C) \to \ker\bigl(\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^n) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\})\bigr)$$
$$\to \ker\bigl(\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^{n-1}) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\})\bigr) \to 0$$

whose central and final terms are augmentation ideals. Hence it suffices to show that the $\Lambda[\mathbb{Z}/p^{\times}]$-module $\ker\bigl(\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^m) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\})\bigr)$ is projective for $m = 1, 2 \ldots n$.

Recall that $\mathbb{Z}/p^{\times}$ is a subgroup of $A(m) = \operatorname{aut}(\mathbb{Z}/p^m)$ and thus acts on $\mathbb{Z}/p^m - \{\overline{0}\}$ in the obvious way. Denote for $k \in \mathbb{Z}$ by $\mathbb{C}_k$ the one-dimensional $\mathbb{Z}/p^m$-representation for which $\overline{b} \in \mathbb{Z}/p^m$ acts by multiplication with $\exp(\frac{2\pi ikb}{p^m})$. We obtain a $\Lambda[\mathbb{Z}/p^{\times}]$-homomorphism

$$Q : \Lambda[\mathbb{Z}/p^m - \{\overline{0}\}] \to \ker\left(\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^m) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\})\right)$$

by sending $\overline{k}$ to $[\mathbb{C}_k] - \frac{1}{p^m} \cdot [\mathbb{C}[\mathbb{Z}/p^m]]$. This is the composition of the inclusion $\Lambda[\mathbb{Z}/p^m - \{\overline{0}\}] \to \Lambda[\mathbb{Z}/p^m]$, the isomorphism $\Lambda[\mathbb{Z}/p^m] \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}(\mathbb{Z}/p^m)$ sending $\overline{k}$ to $[\mathbb{C}_k]$ and the split epimorphism $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}(\mathbb{Z}/p^m) \to \ker\left(\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/p^m) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(\{1\})\right)$ sending $[V]$ to $[V] - \frac{\dim(V)}{p^m} \cdot [\mathbb{C}[\mathbb{Z}/p^m]]$. One easily checks that Q is an isomorphism of $\Lambda[\mathbb{Z}/p^{\times}]$-modules. Hence it remains to show that $\mathbb{Z}/p^{\times}$-acts freely on $\mathbb{Z}/p^m - \{\overline{0}\}$ because then $\Lambda[\mathbb{Z}/p^m - \{\overline{0}\}]$ is a free $\Lambda[\mathbb{Z}/p^{\times}]$-module.

Consider $x \in \mathbb{Z}/p^m$ with $x \neq \overline{0}$. We have to show for $a \in \mathbb{Z}/p^{\times} = A(m)'_p \subset A(m)$ that $a(x) = x$ implies $a = \operatorname{id}$. Since x is non-zero, x generates a cyclic subgroup $\mathbb{Z}/p^l$ for some $l \in \{1, 2, \ldots m\}$. Then $a \in A(m)$ restricted to $A(l)$ is an automorphism $\mathbb{Z}/p^l \to \mathbb{Z}/p^l$ which sends a generator to itself. Hence this automorphism of $\mathbb{Z}/p^l$ is the identity. This implies that a is the identity in $A(l)'_p = \mathbb{Z}/p^{\times}$. This finishes the proof of Lemma 3.4. □

The next result is analogous to [13, Lemma 7.4] but we have to go through its proof again because here we want to invert only the orders of finite subgroups of G, whereas in [13] we have considered everything over $\mathbb{Q}$.

Theorem 3.5. *Let G be a group and $\Lambda = \Lambda^G(X)$ as defined in* (0.5). *Consider the covariant $\Lambda\mathcal{S}ub$-module $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ which sends a finite subgroup group $H \subset G$ to $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H)$. Then*

(a) *$S_H\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is trivial if the finite subgroup $H \subset G$ is not cyclic. For a finite cyclic subgroup $C \subset G$, the $\Lambda[W_GC]$-module $S_C\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is isomorphic to the image of the idempotent $\Lambda[W_GC]$-homomorphism*

$$\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

The isomorphism is given by the composition of the obvious inclusion $\operatorname{im}(\theta_C) \to \operatorname{Rep}_{\mathbb{C}}(C)$ with the obvious projection $\operatorname{Rep}_{\mathbb{C}}(C) \to S_C\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$;

(b) *$\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ is a projective $\Lambda\mathcal{S}ub$-module;*

(c) *Let M be a contravariant $\Lambda\mathcal{S}ub$-module. There is a natural isomorphism of Λ-modules*

$$\bigoplus_{(C), C \text{ cyclic}, C \in \mathcal{F}(X)} M(C) \otimes_{\Lambda[W_GC]} \operatorname{im}\left(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\right) \cong M \otimes_{\Lambda\mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?);$$

(d) $\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(H)$ *is a flat* $\Lambda\mathcal{S}ub$*-module, i.e. for an exact sequence* $0 \to M_0 \to M_1 \to M_2 \to 0$ *of contravariant* $\Lambda\mathcal{S}ub$*-modules the induced sequence of R-modules* $0 \to M_0 \otimes_{\Lambda\mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \to M_1 \otimes_{\Lambda\mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \to M_2 \otimes_{\Lambda\mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \to 0$ *is exact.*

Proof. (a) We conclude from Lemma 3.4 (a) that $S_H\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?)$ is trivial if H is not cyclic. If $H = C$ for a finite cyclic subgroup $C \subset G$, the assertion follows from Lemma 3.4 (b).

(b) Notice that $N_G H/C_G H$ is a subgroup of $\mathrm{aut}(H)$ and all $W_G H$-operations are induced by the obvious $\mathrm{aut}(H)$-operations. We conclude from Lemma 3.4 (c) and assertion (a) that $S_H\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?)$ is a projective $\Lambda[W_G H]$-module for all $H \in \mathcal{F}(X)$. Because of Theorem 2.4 it suffices to show for the morphism T for $\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?)$ defined in (2.3) that $T(K)$ is injective for any given element $K \in \mathcal{F}(X)$.

Consider an element u in the kernel of $T(K)$. Put $J(H) = \mathrm{mor}_{\mathcal{S}ub}(H, K)/(W_G H)$ for $H \in \mathcal{F}(X)$ and put $I = \{(H) \mid H \in \mathcal{F}(X)\}$. Choose for any $(H) \in I$ a representative $H \in (H)$. Then fix for any element $\overline{f} \in J(H)$ a representative $f : H \to K$ in $\mathrm{mor}_{\mathcal{S}ub}(H, K)$. For the remainder of the proof of assertion (b) we abbreviate $L(?) := \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?)$. We can find elements $x_{H,f} \in S_H L$ for $(H) \in I$ and $\overline{f} \in J(H)$ such that only finitely many of the $x_{H,f}$-s are different from zero and u can be written as

$$u = \sum_{(H)\in I} \sum_{\overline{f}\in J(H)} (f : H \to K) \otimes_{\Lambda[W_G H]} x_{H,f}.$$

We want to show that all elements $x_{H,f}$ are zero. Suppose that this is not the case. Let (H_0) be maximal among those elements $(H) \in I$ for which there is $\overline{f} \in J(H)$ with $x_{H,f} \neq 0$, i.e. if for $(H) \in I$ the element $x_{H,f}$ is different from zero for some morphism $f : H \to K$ in $\mathcal{S}ub$ and there is a morphism $H_0 \to H$ in $\mathcal{S}ub$, then $(H_0) = (H)$. In the sequel we choose for any of the morphisms $f : H \to K$ in $\mathcal{S}ub$ a group homomorphism denoted in the same way $f : H \to K$ representing it. Recall that $f : H \to K$ is given by conjugation with an appropriate element $g \in G$. Fix $f_0 : H_0 \to K$ with $x_{H_0 \cdot f_0} \neq 0$. We claim that the composition

$$\begin{aligned} A : \oplus_{(H)\in I} E_H \circ S_H(L(K)) &\xrightarrow{T(K)} L(K) \xrightarrow{\mathrm{res}_K^{\mathrm{im}(f_0)}} L(\mathrm{im}(f_0)) \\ &\xrightarrow{\mathrm{ind}_{f_0^{-1}:\mathrm{im}(f_0)\to H_0}} L(H_0) \xrightarrow{\mathrm{pr}_{H_0}} S_{H_0} L \end{aligned}$$

maps u to $m \cdot x_{H_0,f_0}$ for some integer $m > 0$ which is invertible in Λ. This would lead to a contradiction because of $T(K)(u) = 0$ and $x_{H_0,f_0} \neq 0$.

Consider $(H) \in I$ and $\overline{f} \in J(H)$. It suffices to show that $A\big((f : H \to K) \otimes_{\Lambda[W_G H]} x_{H,f}\big)$ is $[K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot x_{H,f}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and is zero otherwise. One easily checks that $A((f : H \to K) \otimes_{\Lambda[W_G H]} x_{H,f})$ is the image of $x_{H,f}$ under the composition

$$a(H, f) : S_H L \xrightarrow{\sigma_H} L(H) \xrightarrow{\operatorname{ind}_{f : H \to \operatorname{im}(f)}} L(\operatorname{im}(f)) \xrightarrow{\operatorname{ind}_{\operatorname{im}(f)}^K} L(K)$$
$$\xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} L(\operatorname{im}(f_0)) \xrightarrow{\operatorname{ind}_{f_0^{-1} : \operatorname{im}(f_0) \to H_0}} L(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0} L,$$

where σ_H is a $\Lambda[W_G H]$-splitting of the canonical projection $L(H) \to S_H L$. It exists because $S_H L$ is a projective $\Lambda[W_G H]$-module by assertion (a).

The Double Coset formula implies

$$\operatorname{res}_K^{\operatorname{im}(f_0)} \circ \operatorname{ind}_{\operatorname{im}(f)}^K = \sum_{k \in \operatorname{im}(f_0) \backslash K / \operatorname{im}(f)} \operatorname{ind}_{c(k) : \operatorname{im}(f) \cap k^{-1} \operatorname{im}(f_0) k \to \operatorname{im}(f_0)} \circ \operatorname{res}_{\operatorname{im}(f)}^{\operatorname{im}(f) \cap k^{-1} \operatorname{im}(f_0) k}.$$

The composition $\operatorname{pr}_{H_0} \circ \operatorname{ind}_{f_0^{-1} : \operatorname{im}(f_0) \to H_0} \circ \operatorname{ind}_{c(k) : \operatorname{im}(f) \cap k^{-1} \operatorname{im}(f_0) k \to \operatorname{im}(f_0)}$ is trivial, if $c(k) : \operatorname{im}(f) \cap k^{-1} \operatorname{im}(f_0) k \to \operatorname{im}(f_0)$ is not an isomorphism. This follows from the definition of $S_{H_0} L$ (see (2.2)). Suppose that $c(k) : \operatorname{im}(f) \cap k^{-1} \operatorname{im}(f_0) k \to \operatorname{im}(f_0)$ is an isomorphism. Then $k^{-1} \operatorname{im}(f_0) k \subset \operatorname{im}(f)$. Since H_0 has been chosen maximal among the H for which $x_{H,f} \neq 0$ for some morphism $f : H \to K$, this implies either that $x_{H,f} = 0$ or that $k^{-1} \operatorname{im}(f_0) k = \operatorname{im}(f)$. Suppose $k^{-1} \operatorname{im}(f_0) k = \operatorname{im}(f)$. Then $(H) = (H_0)$ which implies $H = H_0$. Moreover, the homomorphisms in $\mathcal{S}ub$ represented by f_0 and f agree. Hence the group homomorphisms f_0 and f agree themselves and we get $k \in N_G \operatorname{im}(f_0) \cap K$. This implies that $a(H, f) = [K \cap N_G \operatorname{im}(f_0) : \operatorname{im}(f_0)] \cdot \operatorname{id}$ if $(H) = (H_0)$ and $\overline{f} = \overline{f_0}$, and that otherwise $a(H, f) = 0$ or $x_{H,f} = 0$ holds. Hence the map T is injective.

(c) follows from assertion (a) and the bijectivity of the isomorphism T for $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$ defined in (2.3) because there is a natural isomorphism

$$M \otimes_{\Lambda \mathcal{S}ub} E_H S_H \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \xrightarrow{\cong} M(H) \otimes_{\Lambda[W_G H]} S_H \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?). \tag{3.6}$$

Now (d) follows from (c) and the fact that the $\Lambda[W_G H]$-module $S_H \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?) \cong \operatorname{im}(\theta_C)$ is projective. This finishes the proof of Theorem 3.5. □

4. The construction of the Chern character

In this section we want to prove Theorem 0.7. There are similarities with the construction in [13]. The main difference is that here we want to give a construction, where we only have to invert the orders of elements in $\mathcal{F}(X)$, whereas in [13] we have worked over the rationals. In [13] we have used the Hurewicz homomorphism from stable homotopy to singular homology, which is only an isomorphism after inverting all primes. We will use the multiplicative structure of K_*^G instead and work with a different source for the equivariant Chern character, which allows us to invert only the orders of finite subgroups of G.

In the sequel we denote by $K_p^G(X)$ the equivariant K-homology of a proper G-CW-complex X. It is defined by $\operatorname{colim}_{Y\subset X} KK_G^p(C_0(Y),\mathbb{C})$, where Y runs over all cocompact G-subcomplexes of X and $KK_G^p(C_0(Y),\mathbb{C})$ denotes equivariant KK-theory of the G-C^*-algebra $C_0(X)$ of continuous functions $X\to\mathbb{C}$, which vanish at infinity, and the C^*-algebra $\mathbb{C}$ with the trivial G-action. Given a homomorphism $\phi : H\to G$ of groups and a proper H-CW-complex, then $\operatorname{ind}_\phi X := G\times_\phi X$ is a proper G-CW-complex and there is an induction homomorphism

$$\operatorname{ind}_\phi : K_0^H(X)\to K_0^G(\operatorname{ind}_\phi X).$$

If the kernel of ϕ acts freely on X, then ind_ϕ is bijective. In particular we get for a proper G-CW-complex X a homomorphism

$$K_p^G(X)\xrightarrow{\operatorname{ind}_{G\to\{1\}}} K_p(G\backslash X),$$

which is bijective if G acts freely on X. There is an external product

$$\mu : K_p^G(X)\times K_q^{G'}(X')\to K_{p+q}^{G\times G'}(X\times X')$$

for groups G and G', a proper G-CW-complex X and a proper G'-CW-complex X'. External products and induction are compatible. For more information about equivariant K-homology and KK-theory we refer to [10] and in particular for the induction homomorphisms to [18].

Let X be a proper G-CW-complex. We have introduced the ring $\Lambda = \Lambda^G(X)$ in (0.5). We want to construct for $H\in\mathcal{F}(X)$ and $p=0,1$ a Λ-homomorphism

$$\begin{aligned}\underline{\operatorname{ch}}_p^G(X)(H) : \Lambda\otimes_{\mathbb{Z}} K_p\bigl(C_GH\backslash X^H\bigr)\otimes_{\mathbb{Z}}\operatorname{Rep}_{\mathbb{C}}(H)\\ \to \Lambda\otimes_{\mathbb{Z}} K_p^G(X),\end{aligned}\tag{4.1}$$

where $K_p(C_GH\backslash X^H)$ is the (non-equivariant) K-homology of the CW-complex $C_GH\backslash X^H$. The map will be defined by the following composition

$$
\begin{array}{c}
\Lambda \otimes_{\mathbb{Z}} K_p\big(C_G H\backslash X^H\big) \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H) \\
\Big\uparrow \scriptstyle{\mathrm{id}\otimes_{\mathbb{Z}} K_p(\mathrm{pr}_1)\otimes_{\mathbb{Z}}\mathrm{id}} \ \cong \\
\Lambda \otimes_{\mathbb{Z}} K_p\big(EG \times_{C_G H} X^H\big) \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(H) \\
\Big\uparrow \scriptstyle{\mathrm{id}\otimes_{\mathbb{Z}} \mathrm{ind}_{C_G H\to\{1\}}\otimes j} \ \cong \\
\Lambda \otimes_{\mathbb{Z}} K_p^{C_G H}(EG \times X^H) \otimes_{\mathbb{Z}} K_0^H(*) \\
\Big\downarrow \scriptstyle{\mu} \\
\Lambda \otimes_{\mathbb{Z}} K_p^{C_G H\times H}(EG \times X^H) \\
\Big\downarrow \scriptstyle{\mathrm{ind}_{m_H}} \ \cong \\
\Lambda \otimes_{\mathbb{Z}} K_p^G\big(\mathrm{ind}_{m_H} EG \times X^H\big) \\
\Big\downarrow \scriptstyle{\mathrm{id}\otimes_{\mathbb{Z}} K_p^G(\mathrm{ind}_{m_H}\mathrm{pr}_2)} \\
\Lambda \otimes_{\mathbb{Z}} K_p^G\big(\mathrm{ind}_{m_H} X^H\big) \\
\Big\downarrow \scriptstyle{\mathrm{id}\otimes_{\mathbb{Z}} K_p^G(v_H)} \\
K_p^G(X)
\end{array}
$$

Some explanations are in order. We have a left $C_G H$-action on $EG \times X^H$ by $g(e,x) = (ge, gx)$ for $g \in C_G H$, $e \in EG$ and $x \in X^H$. It extends to a $C_G H \times H$-action by letting the factor H act trivially. The map $\mathrm{pr}_1 : EG \times_{C_G H} X^H \to C_G H\backslash X^H$ is the canonical projection. It induces an isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(\mathrm{pr}_1) : \Lambda \otimes_{\mathbb{Z}} K_p\big(EG \times_{C_G H} X^H\big) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p\big(C_G H\backslash X^H\big)$$

since each isotropy group of the $C_G H$-space X^H is finite and for any finite group L the projection induces an isomorphism $\Lambda \otimes_{\mathbb{Z}} H_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} H_p(*)$ and hence by the Atiyah-Hirzebruch spectral sequence an isomorphism $\Lambda \otimes_{\mathbb{Z}} K_p(BL) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_p(*)$ for all p. The isomorphism $j : K_0^H(*) \xrightarrow{\cong} \operatorname{Rep}_{\mathbb{C}}(H)$ is the canonical isomorphism. The group homomorphism $m_H : C_G H \times H \to G$ sends (g,h) to gh. Since its kernel acts freely on $EG \times X^H$, the map ind_{m_H} is bijective. We denote by $\mathrm{pr}_2 : EG \times X^H \to X^H$ the canonical projection. The G-map $v_H : \mathrm{ind}_{m_H} X^H = G \times_{m_H} X^H \to X$ sends (g,x) to gx.

Notice that we obtain a contravariant $\Lambda\mathcal{S}ub$-module $\Lambda \otimes_{\mathbb{Z}} K_p(C_G?\backslash X^?)$ by assigning to a finite subgroup $H \subset G$ the Λ-module $\Lambda \otimes_{\mathbb{Z}} K_p(C_G H\backslash X^H)$. We have already introduced the covariant Λ-module $\Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(?)$. Analogously to [13] one checks that the various maps $\underline{\mathrm{ch}}_p^G(X)(H)$ defined above induce a map of Λ-modules

$$\mathrm{ch}_p^G(X) : \Lambda \otimes_{\mathbb{Z}} K_p(C_G?\backslash X^?) \otimes_{\Lambda \mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \to \Lambda \otimes_{\mathbb{Z}} K_p^G(X). \tag{4.2}$$

Notice that for $L \in \mathcal{F}(X)$ and $X = G/L$ the $\Lambda\mathcal{S}ub$-module $\Lambda \otimes_{\mathbb{Z}} K_0(C_G?\backslash(G/L)^?)$ is isomorphic to the $\Lambda\mathcal{S}ub$-module $\Lambda\,\mathrm{mor}_{\mathcal{S}ub}(?, L)$, which sends a finite subgroup $H \subset G$ to the free Λ-module with base $\mathrm{mor}_{\mathcal{S}ub}(H, L)$. By the Yoneda Lemma one obtaines a canonical isomorphism

$$\Lambda \otimes_{\mathbb{Z}} K_p(C_G?\backslash(G/L)^?) \otimes_{\Lambda \mathcal{S}ub} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(?) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(L).$$

One easily checks that under this identification $\mathrm{ch}_0^G(G/L)$ becomes the canonical identification of $\Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(L)$ with $\Lambda \otimes_{\mathbb{Z}} K_0^G(G/L)$. Notice that $K_1(C_G?\backslash(G/L)^?)$ and $K_1^G(G/L)$ are both trivial. Hence $\mathrm{ch}_p^G(G/L)$ is bijective for all $L \in \mathcal{F}(X)$ and $p = 0, 1$. Because of Theorem 3.5 (d) the source of ch_*^G is an equivariant homology theory on proper G-CW-complexes Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. One easily checks that ch_*^G is compatible with the Mayer-Vietoris sequences. By induction over the number of equivariant cells and the Five-Lemma $\mathrm{ch}_p^G(Y)$ is bijective for any finite proper G-CW-complex Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$. Notice that $K_p^G(Y)$ is the colimit $\mathrm{colim}_{Z \subset Y} K_p^G(Z)$, where Z runs through all finite G-CW-subcomplexes Z of Y. The analogous statement holds for the source of ch_*^G. Hence $\mathrm{ch}_p^G(Y)$ is bijective for all proper G-CW-complexes Y with $\mathcal{F}(Y) \subset \mathcal{F}(X)$ and $p = 0, 1$. Now Theorem 0.7 follows from Theorem 3.5 (c). □

5. The Baum-Connes Conjecture and the Trace Conjecture

In the sequel we denote for a proper G-CW-complex X by

$$\mathrm{asmb}^G : K_0^G(X) \to K_0(C_r^*(G)) \tag{5.1}$$

the assembly map which essentially assigns to an element in $K_0^G(X)$ represented by an equivariant Kasparov cycle its index. Given a homomorphism $\phi : H \to G$ of groups with finite kernel, there is an induction homomorphism $\mathrm{ind}_\phi : K_p(C_r^*(H)) \to K_p(C_r^*(G))$ such that the following diagram commutes [18, Theorem 1]

$$\begin{array}{ccc} K_0^H(X) & \xrightarrow{\mathrm{asmb}^H} & K_0(C_r^*(H)) \\ \downarrow{\scriptstyle \mathrm{ind}_\phi} & & \downarrow{\scriptstyle \mathrm{ind}_\phi} \\ K_0^G(\mathrm{ind}_\phi X) & \xrightarrow{\mathrm{asmb}^G} & K_0(C_r^*(G)) \end{array}$$

These induction homomorphisms, the assembly maps and the change of rings homomorphisms associated to the passage from $C_r^*(G)$ to $\mathcal{N}(G)$ are compatible with the external products

$$\mu : K_p^G(X) \times K_q^{G'}(X') \to K_{p+q}^{G\times G'}(X \times X');$$
$$\mu : K_p(C_r^*(G)) \times K_q(C_r^*(G')) \to K_{p+q}(C_r^*(G \times G'));$$
$$\mu : K_p(\mathcal{N}(G)) \times K_q(\mathcal{N}(G')) \to K_{p+q}(\mathcal{N}(G \times G'))$$

for groups G and G', a proper G-CW-complex X and a proper G'-CW-complex X'. We will use in the sequel the elementary fact that for any G-map $f : X \to Y$ of proper G-CW-complexes the composition $K_0^G(X) \xrightarrow{K_0^G(f)} K_0^G(Y) \xrightarrow{\operatorname{asmb}^G} K_0(C_r^*(G))$ is $\operatorname{asmb}^G : K_0^G(X) \to K_0(C_r^*(G))$. In the sequel the letter i denotes change of rings homomorphism for the canonical map $C_r^*(G) \to \mathcal{N}(G)$.

Let X be a proper G-CW-complex. We have introduced $J = J^G(X)$ in (0.6). Define the homomorphism

$$\begin{aligned}\xi_1 : \oplus_{(C)\in J}\ &\Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C)\\ &\otimes_{\Lambda[W_GC]} \operatorname{im}\big(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\big)\\ &\to K_0(\mathcal{N}(G))\end{aligned} \tag{5.2}$$

by the composition of the equivariant Chern character of Theorem 0.7

$$\begin{aligned}\operatorname{ch}_0^G(X) : \oplus_{(C)\in J}\ &\Lambda \otimes_{\mathbb{Z}} K_0(C_GC\backslash X^C)\\ &\otimes_{\Lambda[W_GC]} \operatorname{im}\big(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\big)\\ &\xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_0^G(X),\end{aligned}$$

the assembly map

$$\operatorname{id} \otimes \operatorname{asmb}^G : \Lambda \otimes_{\mathbb{Z}} K_0^G(X) \to \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G))$$

and the change of rings homomorphism

$$\operatorname{id} \otimes i : \Lambda \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \to \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)).$$

This is the homomorphism which we want to understand. In particular we are interested in its image. We will identify it with a second easier to compute homomorphism

$$\begin{aligned}\xi_2 : \oplus_{(C)\in J}\ &\Lambda \otimes_{\mathbb{Z}} K_0\big(C_GC\backslash X^C\big)\\ &\otimes_{\Lambda[W_GC]} \operatorname{im}\big(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\big)\\ &\to \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)),\end{aligned} \tag{5.3}$$

which is defined as follows. Let $l : \operatorname{im}(\theta_C) \to \operatorname{Rep}_{\mathbb{C}}(C)$ be the inclusion. Let $K_0(\mathrm{pr}) : K_0(C_GC\backslash X^C) \to K_0(*)$ be induced by the projection from $C_GC\backslash X^C$ to the one-point space $*$. We obtain a map

$$(i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr})) \otimes l : K_0\big(C_GC\backslash X^C\big) \otimes \operatorname{im}(\theta_C) \\ \to K_0(\mathcal{N}(\{1\})) \otimes \operatorname{Rep}_{\mathbb{C}}(C).$$

Define

$$\alpha : K_0(\mathcal{N}(\{1\})) \otimes \operatorname{Rep}_{\mathbb{C}}(C) \to \operatorname{Rep}_{\mathbb{C}}(C) \\ [U] \otimes [W] \mapsto \dim_{\mathbb{C}}(U) \cdot [W].$$

Notice that α is essentially given by the external product and $K_0(\mathcal{N}(H)) = \operatorname{Rep}_{\mathbb{C}}(H)$ holds by definition for any finite group H. Induction yields a map

$$\operatorname{ind}_C^G : K_0(\mathcal{N}(C)) \to K_0(\mathcal{N}(G)).$$

The composition of these three maps above induces for any finite cyclic subgroup $C \subset G$ a homomorphism

$$\xi_2(C) : \Lambda \otimes_{\mathbb{Z}} K_0\big(C_GC\backslash X^C\big) \\ \otimes_{\Lambda[W_GC]} \operatorname{im}\big(\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)\big) \\ \to K_0(\mathcal{N}(G)).$$

Define ξ_2 to be the direct sum $\bigoplus_{(C)\in J} \xi_2(C)$ after the choice of a representative $C \in (C)$ for each $(C) \in J$.

Theorem 5.4. *Let X be a proper G-CW-complex. Then the maps ξ_1 of* (5.2) *and ξ_2 of* (5.3) *agree.*

Proof. In the sequel maps denoted by the letter μ will be given by external products and pr denotes the projection from a space to the one-point space $*$. Fix a cyclic subgroup $C \in \mathcal{F}(X)$. Notice that the homomorphism $m_C : C_GC \times C \to G \quad (g, c) \mapsto gc$ has a finite kernel so that induction is defined also on the level of the reduced group C^*-algebra and the group von Neumann algebra. Denote by $\nu : \Lambda \otimes_{\mathbb{Z}} K_0^{C_GC\times C}(EG \times X^C) \to \Lambda \otimes_{\mathbb{Z}} K_0^G(X)$ the composition of the maps $\mathrm{id} \otimes K_0^G(v_C)$, $\mathrm{id} \otimes K_0^G(\operatorname{ind}_{m_C} \mathrm{pr}_2)$ and ind_{m_C} appearing in the definition of $\underline{\mathrm{ch}}_0(X)(C)$. Then the following diagram commutes

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\mathrm{id} \otimes i \circ \mathrm{asmb}^{C_GC} \otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC)) \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \\
\mu \downarrow & & \mu \downarrow \\
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC\times C}(EG \times X^C) & \xrightarrow{\mathrm{id} \otimes i \circ \mathrm{asmb}^{C_GC\times C}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC \times C)) \\
\nu \downarrow & & \operatorname{ind}_{m_C} \downarrow \\
\Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\mathrm{id} \otimes i \circ \mathrm{asmb}^G} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))
\end{array}$$

For any group G the map induced by the center-valued von Neumann dimension

$$\dim^u_{\mathcal{N}(G)} : K_0(\mathcal{N}(G)) \to \mathcal{Z}(\mathcal{N}(G))$$

is injective. Given a CW-complex Z and an element $\eta \in K_0(Z)$, there is a closed manifold M with a map $f : M \to Z$ and an elliptic complex D^* of differential operators of order 1 over M such that $K_0(f) : K_0(M) \to K_0(Z)$ maps the class $[D^*] \in K_0(M)$ to η [2]. In the case $Z = BG$ the composition

$$K_0(M) \xrightarrow{K_0(f)} K_0(BG) \xrightarrow{(\mathrm{ind}_{G\to\{1\}})^{-1}} K_0^G(EG) \xrightarrow{\mathrm{asmb}^G} K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim^u_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G))$$

resp. the composition

$$K_0(M) \xrightarrow{K_0(\mathrm{pr})} K_0(*) \xrightarrow{\mathrm{asmb}^{\{1\}}} K_0(C^*(\{1\}) \xrightarrow{i} K_0(\mathcal{N}(\{1\})) \xrightarrow{\mathrm{ind}^G_{\{1\}}} K_0(\mathcal{N}(G)) \xrightarrow{\dim^u_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G))$$

maps $[D^*]$ to the element $\mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)$ resp. $\mathrm{index}(D^*) \cdot 1_{\mathcal{N}(G)}$, where $\mathrm{index}^u_{\mathcal{N}(G)}(\overline{D}^*)$ resp. $\mathrm{index}(D^*)$ has been defined in (1.7) resp. (1.1). We conclude from Theorem 0.4 and the injectivity of the map $\dim^u_{\mathcal{N}(G)}$ of (1.6) that the following diagram commutes

$$\begin{array}{ccc} K_0^G(EG) & \xrightarrow{i\circ\mathrm{asmb}^G} & K_0(\mathcal{N}(G)) \\ {\scriptstyle K_0(\mathrm{pr})\circ\mathrm{ind}^{-1}_{G\to\{1\}}}\downarrow & & \uparrow{\scriptstyle \mathrm{ind}^G_{\{1\}}} \\ K_0(*) & \xrightarrow{i\circ\mathrm{asmb}^{\{1\}}} & K_0(\mathcal{N}(\{1\})) \end{array}$$

Since there is a C_GC-map $EG \times X^C \to EC_GC$, we conclude from the diagram above applied to the case $G = C_GC$ that the following diagram commutes

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}} K_0^{C_GC}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{\mathrm{id}\otimes i\circ\mathrm{asmb}^{C_GC}\otimes j} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(C_GC)) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\ \downarrow{\scriptstyle \mathrm{id}\otimes(K_0(\mathrm{pr})\circ\mathrm{ind}_{C_GC\to\{1\}})\otimes j} & & \uparrow{\scriptstyle \mathrm{id}\otimes\mathrm{ind}^{C_GC}_{\{1\}}\otimes\mathrm{id}} \\ \Lambda \otimes_{\mathbb{Z}} K_0(*) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id}\otimes i\circ\mathrm{asmb}^{\{1\}}\otimes\mathrm{id}} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\}) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \end{array}$$

The composition

$$K_0(\mathcal{N}(\{1\}) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\mathrm{ind}^{C_GC}_{\{1\}}\otimes\mathrm{id}} K_0(\mathcal{N}(C_GC)) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\mu} K_0(\mathcal{N}(C_GC \times C)) \xrightarrow{\mathrm{ind}_{m_C}} K_0(\mathcal{N}(G))$$

agrees with the composition

$$K_0(\mathcal{N}(\{1\})) \otimes \mathrm{Rep}_{\mathbb{C}}(C) \xrightarrow{\alpha} \mathrm{Rep}_{\mathbb{C}}(C) = K_0(\mathcal{N}(C)) \xrightarrow{\mathrm{ind}_C^G} K_0(\mathcal{N}(G)).$$

We conclude that the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0^{C_GC}(EG \times X^C) \otimes_{\mathbb{Z}} K^C(*) & \xrightarrow{(\mathrm{id} \otimes i \circ \mathrm{asmb}^G) \circ \nu \circ \mu} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G)) \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \left(i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr}) \circ \mathrm{ind}_{C_GC \to \{1\}}\right) \otimes j} & & \Big\uparrow {\scriptstyle \mathrm{id} \otimes \mathrm{ind}_C^G} \\
\Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(\{1\})) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id} \otimes \alpha} & \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C)
\end{array}$$

Hence the following diagram commutes for any cyclic subgroup $C \in \mathcal{F}(X)$

$$\begin{array}{ccc}
\Lambda \otimes_{\mathbb{Z}} K_0(C_GC \backslash X^C) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) & \xrightarrow{\mathrm{id} \otimes (\alpha \circ (i \circ \mathrm{asmb}^{\{1\}} \circ K_0(\mathrm{pr})) \otimes \mathrm{id})} & \Lambda \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \\
{\scriptstyle \mathrm{id} \otimes \underline{\mathrm{ch}}_0^G(X)(C)} \Big\downarrow & & {\scriptstyle \mathrm{id} \otimes \mathrm{ind}_C^G} \Big\downarrow \\
\Lambda \otimes_{\mathbb{Z}} K_0^G(X) & \xrightarrow{\mathrm{id} \otimes (i \circ \mathrm{asmb}^G)} & \Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))
\end{array}$$

Now Theorem 5.4 (and hence also Theorem 0.8) follow. □

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