A Survey on classifying spaces for families

Wolfgang Lück*
Fachbereich Mathematik und Informatik
Westfälische Wilhelms-Universität
Münster
Einsteinstr. 62
48149 Münster
Germany
lueck@math.uni-muenster.de
http://www.math.uni-muenster.de/u/lueck

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1. The G-CW-version

Group means always locally compact Hausdorff topological group with a countable base for its topology.

Definition 1. (G-CW-complex) *A* G-CW-**complex** X *is a* G*-space together with a* G*-invariant filtration*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X *carries the colimit topology with respect to this filtration (i.e. a set* $C \subseteq X$ *is closed if and only if* $C \cap X_n$ *is closed in* X_n *for all* $n \geq 0$*) and* X_n *is obtained from* X_{n-1} *for each* $n \geq 0$ *by attaching equivariant* n*-dimensional cells, i.e. there exists a* G*-pushout*

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow[\coprod_{i \in I_n} Q_i^n]{} & X_n \end{array}$$

Remark 2. (Proper G-CW-complexes) A G-space X is called **proper** if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is compact. A G-CW-complex X is proper if and only if all its isotropy groups are compact.

Definition 3. (Family of subgroups) A **family $\mathcal{F}$ of subgroups** of G is a set of (closed) subgroups of G which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCY} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{COM} & = & \{\text{compact subgroups}\}; \\ \mathcal{COMOP} & = & \{\text{compact open subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition 4. (Classifying G-CW-complex for a family of subgroups) *Let $\mathcal{F}$ be a family of subgroups of G. A model $E_{\mathcal{F}}(G)$ for the* **classifying G-CW-complex for the family $\mathcal{F}$ of subgroups** *is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:*

1. *All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;*

2. *For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.*

We abbreviate $\underline{E}G := E_{\mathcal{COM}}(G)$ and call it the **universal G-CW-complex for proper G-actions**. *We also write $EG = E_{\mathcal{TR}}(G)$.*

The notion of a classifying space for a family is due to tom Dieck.

Theorem 5. (Homotopy characterization of $E_{\mathcal{F}}(G)$) *Let $\mathcal{F}$ be a family of subgroups.*

1. *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*

2. *Two model for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*

3. *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is weakly contractible.*

2. The numerable version

Definition 6. ($\mathcal{F}$-numerable G-space) A **$\mathcal{F}$-numerable G-space** is a G-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by G-subspaces such that there is for each $i \in I$ a G-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$ and there is a locally finite partition of unity $\{e_i \mid i \in I\}$ subordinate to $\{U_i \mid i \in I\}$ by G-invariant functions $e_i \colon X \to [0, 1]$.

Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable G-space belong to $\mathcal{F}$. If $f \colon X \to Y$ is a G-map and Y is $\mathcal{F}$-numerable, then X is also $\mathcal{F}$-numerable.

Lemma 7. *Let $\mathcal{F}$ be a family. Then a G-CW-complex is $\mathcal{F}$-numerable if each isotropy group is a subgroup of some element in $\mathcal{F}$.*

Proof. This follows from the Slice Theorem and the fact that $G\backslash X$ is a CW-complex and hence paracompact. $\square$

Definition 8. (Classifying numerable G-space for a family of subgroups) *Let $\mathcal{F}$ be a family of subgroups of G. A model $J_{\mathcal{F}}(G)$ for the* **classifying numerable G-space for the family $\mathcal{F}$ of subgroups** *is a G-space which has the following properties:*

1. *$J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable;*

2. *For any $\mathcal{F}$-numerable G-space X there is up to G-homotopy precisely one G-map $X \to J_{\mathcal{F}}(G)$.*

We abbreviate $\underline{J}G := J_{\mathcal{COM}}(G)$ and call it the **universal numerable G-space for proper G-actions**, *or briefly* **the universal space for proper G-actions**. *We also write $JG = J_{\mathcal{TR}}(G)$*

Remark 9. (Proper G-spaces) A $\mathcal{COM}$-numerable G-space X is proper. Not every proper G-space is $\mathcal{COM}$-numerable. But a G-CW-complex X is proper if and only if it is $\mathcal{COM}$-numerable.

Theorem 10. (Homotopy characterization of $J_{\mathcal{F}}(G)$) *Let $\mathcal{F}$ be a family of subgroups.*

1. *For any family $\mathcal{F}$ there exists a model for $J_{\mathcal{F}}(G)$ whose isotropy groups belong to $\mathcal{F}$;*

2. *Two models for $J_{\mathcal{F}}(G)$ are G-homotopy equivalent;*

3. *For $H \in \mathcal{F}$ the H-fixed point set $J_{\mathcal{F}}(G)^H$ is contractible.*

3. Comparision of the two versions

There is always a G-map

$$\phi_{\mathcal{F}} \colon E_{\mathcal{F}}(G) \to J_{\mathcal{F}}(G)$$

which is unique up to G-homotopy.

Example 11. *Let G be totally disconnected. Then*

$$\phi_{\mathcal{TR}} \colon EG \to JG$$

is a G-homotopy equivalence if and only if G is discrete.

Theorem 12. ($\underline{E}G = \underline{J}G$) *The G-map $\phi_{\mathcal{F}}$ is a G-homotopy equivalence if $\mathcal{F} = \mathcal{COM}$, i.e. we get a G-homotopy equivalence*

$$\phi_{\mathcal{COM}} \colon \underline{E}G \xrightarrow{\simeq} \underline{J}G.$$

Lemma 13. *Let G be a totally disconnected group Then the following square commutes up to G-homotopy and consists of G-homotopy equivalences*

$$\begin{array}{ccc}
E_{\mathcal{COMOP}}(G) & \longrightarrow & J_{\mathcal{COMOP}}(G) \\
\downarrow & & \downarrow \\
\underline{E}G & \longrightarrow & \underline{J}G
\end{array}$$

4. Special models

There are interesting special models

- Operator theoretic models;

- G/K for an almost connected group G with $K \subseteq G$ maximal compact subgroup;

- Actions on CAT(0)-spaces;

- Actions on affine buildings;

- The Rips complex for word-hyperbolic groups;

- The Borel-Serre compactification and arithmetic groups;

- Mapping class groups and Teichmüller space;

- Out(F_n) and outer space.

4.1. Operator Theoretic Model

Let $C_0(G)$ be the Banach space of complex valued functions of G vanishing at infinity with the supremum-norm. The group G acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$. Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions f such that f is not identically zero and has non-negative real numbers as values.

Theorem 14. (Operator theoretic model) *The G-space $PC_0(G)$ is a model for $\underline{J}G$.*

Example 15. Let G be discrete. Another model for $\underline{J}G$ is the space

$$X_G = \{f\colon G \to [0,1] \mid f \text{ has finite support}, \sum_{g \in G} f(g) = 1\}$$

with the topology coming from the supremum norm.

Remark 16. (Simplicial Model) Let G be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements g_i in G. This also a model for $\underline{E}G$.

The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies. The identity map induces a (continuous) G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism

4.2. Almost Connected Groups

The following result is due to Abels.

Theorem 17. Almost connected groups) *Let G be a (locally compact Hausdorff) topological group. Suppose that G is almost connected, i.e. the group G/G^0 is compact for G^0 the component of the identity element. Then G contains a maximal compact subgroup K which is unique up to conjugation. The G-space G/K is a model for $\underline{J}G$.*

Theorem 18. (Discrete subgroups of almost connected Lie groups) *Let L be a Lie group with finitely many path components. Then L contains a maximal compact subgroup K which is unique up to conjugation. The L-space L/K is a model for $\underline{E}L$.*

If $G \subseteq L$ is a discrete subgroup of L, then L/K with the obvious left G-action is a finite dimensional G-CW-model for $\underline{E}G$.

4.3. Actions on CAT(0)-spaces

Theorem 19. (Actions on CAT(0)-spaces) *Let G be a (locally compact Hausdorff) topological group. Let X be a proper G-CW-complex. Suppose that X has the structure of a complete* CAT(0)*-space for which G acts by isometries. Then X is a model for $\underline{E}G$.*

Remark 20. This result contains as special case isometric G actions on simply-connected complete Riemannian manifolds with non-positive sectional curvature and G-actions on trees.

4.4. Affine Buildings

Theorem 21. *(***Affine buildings***) Let G be a totally disconnected group. Suppose that G acts on the affine building by simplicial automorphisms such that each isotropy group is compact. Then Σ is a model for both $J_{\mathcal{COMOP}}(G)$ and $\underline{J}G$ and the barycentric subdivision Σ' is a model for both $E_{\mathcal{COMOP}}(G)$ and $\underline{E}G$.*

Example 22 (Bruhat-Tits building). An important example is the case of a reductive p-adic algebraic group G and its associated affine Bruhat-Tits building $\beta(G)$. Then $\beta(G)$ is a model for $\underline{J}G$ and $\beta(G)'$ is a model for $\underline{E}G$ by Theorem 21.

4.5. The Rips Complex of a Word-Hyperbolic Group

The **Rips complex** $P_d(G,S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realization of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$. The obvious G-action by simplicial automorphisms on $P_d(G,S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G,S)'$

Theorem 23. (Rips complex) *Let G be a (discrete) group with a finite symmetric set of generators. Suppose that (G,S) is δ-hyperbolic for the real number $\delta \geq 0$. Let d be a natural number with $d \geq 16\delta + 8$. Then the barycentric subdivision of the Rips complex $P_d(G,S)'$ is a finite G-CW-model for $\underline{E}G$.*

4.6. Arithmetic Groups

Arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group possess finite models for $\underline{E}G$. Namely, let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$ and let $K \subseteq G(\mathbb{R})$ a maximal compact subgroup. If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left A-action is a model for $E_{\mathcal{FIN}}(A)$ as already explained in Theorem 18. The A-space $G(\mathbb{R})/K$ is not necessarily cocompact.

Theorem 24. Borel-Serre compactification) *The Borel-Serre completion of $G(\mathbb{R})/K$ is a finite A-CW-model for $E_{\mathcal{FIN}}(A)$.*

4.7. Mapping Class groups

Let $\Gamma_{g,r}^s$ be the **mapping class group** of an orientable compact surface F of genus g with s punctures and r boundary components. We will always assume that $2g+s+r>2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative. It is well-known that the associated **Teichmüller space** $\mathcal{T}_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly. Actually

Theorem 25. *(***Teichmüller space***) The $\Gamma_{g,r}^s$-space $\mathcal{T}_{g,r}^s$ is a model for $E_{\mathcal{FIN}}(\Gamma_{g,r}^s)$.*

4.8. Outer Automorphism Groups of Free groups

Let F_n be the free group of rank n. Denote by Out(F_n) the group of outer automorphisms of F_n, i.e. the quotient of the group of all automorphisms of F_n by the normal subgroup of inner automorphisms. Culler and Vogtmann have constructed a space X_n called **outer space** on which Out(F_n) acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.

The space X_n contains a **spine** K_n which is an Out(F_n)-equivariant deformation retraction. This space K_n is a simplicial complex of dimension $(2n-3)$ on which the Out(F_n)-action is by simplicial automorphisms and cocompact. Actually the group of simplicial automorphisms of K_n is Out(F_n) by results due to Bridson and Vogtman.

Theorem 26. *The barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\,\mathrm{Out}(F_n)$.*

5. Relevance and Applications of Classifying Spaces for Families

5.1. Baum-Connes Conjecture

The goal of the Baum-Connes Conjecture is the computation of the topological K-theory $K_n(C_r^*(G))$ of the reduced group C^*-algebra of G.

Conjecture 27 (Baum-Connes Conjecture). *The assembly map defined by taking the equivariant index*

$$\operatorname{asmb}\colon K_n^G(\underline{J}G) \xrightarrow{\cong} K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

5.2. Farrell-Jones Conjecture

Let G be a discrete group. Let R be a associative ring with unit. The goal of the Farrell-Jones Conjecture is to compute the algebraic K-groups $K_n(RH)$ and the algebraic L-groups $L_n^{-\infty}(RG)$.

Conjecture 28 (Farrell-Jones Conjecture). *The assembly maps induced by the projection $E_{\mathcal{VCYC}}(G) \to G/G$*

$$\begin{aligned} \text{asmb}\colon \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}) &\to K_n(RG); \\ \text{asmb}\colon \mathcal{H}_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}^{-\infty}) &\to L_n^{-\infty}(RG), \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$.

5.3. Completion Theorem

Let G be a discrete group. For a proper finite G-CW-complex let $K_G^*(X)$ be its equivariant K-theory defined in terms of equivariant finite dimensional complex vector bundles over X. Let $I \subseteq K_G^0(\underline{E}G)$ be the augmentation ideal, i.e. the kernel of the map $K^0(\underline{E}G) \to \mathbb{Z}$ sending the class of an equivariant complex vector bundle to its complex dimension. Let $K_G^*(\underline{E}G)\widehat{_I}$ be the I-adic completion of $K_G^*(\underline{E}G)$ and let $K^*(BG)$ be the topological K-theory of BG.

Theorem 29 (Completion Theorem for discrete groups). *Let G be a discrete group such that there exists a finite model for $\underline{E}G$. Then there is a canonical isomorphism*

$$K^*(BG) \xrightarrow{\cong} K_G^*(\underline{E}G)\widehat{_I}.$$

5.4. Classifying Spaces for Equivariant Bundles

The equivariant K-theory for finite proper G-CW-complexes appearing above can be extended to arbitrary proper G-CW-complexes (including the multiplicative structure) using Γ-spaces in the sense of Segal and involving classifying spaces for equivariant vector bundles. These classifying spaces for equivariant vector bundles are again classifying spaces of certain Lie groups and certain families

5.5. Equivariant Homology and Cohomology

Classifying spaces for families play a role in computations of equivariant homology and cohomology for compact Lie groups such as equivariant bordism. Rational computations of equivariant (co-)-homology groups are possible in general using Chern characters for discrete groups and proper G-CW-complexes

6. Finiteness Conditions

The questions whether there exists finite models, models of finite type or models or finite-dimensional models for $\underline{E}G$ or wthat is the minimal value of $\dim(\underline{E}G)$ is quite interesting and an obvious extention of the same question for BG.

Remark 30 (Algebraic criterion). Let G be discrete. In the classical case one can read off the possible dimension of BG from the homological algebra of $\mathbb{Z}G$, in particular in terms of the cohomological dimension of the trivial $\mathbb{Z}G$-modul $\mathbb{Z}$. There are analogous results for $E_{\mathcal{F}}(G)$ if one considers modules over the orbit category $\mathrm{Or}(G)$, in particular the constant contravariant $\mathbb{Z}\mathrm{Or}(G)$-module $\mathbb{Z}_{\mathcal{F}}$ whose value is $\mathbb{Z}$ on G/H for $H \in \mathcal{F}$ and $\{0\}$ on G/H for $H \notin \mathcal{F}$. This gives in principle a complete answer in algebraic terms but is often hard to apply in concrete situations.

6.1. Some conditions for finite-dimensional models

As an illustration we give a small selection of results on this topic to due to Connolly, Dunwoody, Kropholler, Kozniewsky, L., Leary, Meintrup, Mislin and Nucinkis and others.

Theorem 31 (Discrete subgroups of Lie groups). *Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup K. Let $G \subseteq L$ be a discrete subgroup of L.*

Then L/K with the left G-action is a model for $\underline{E}G$.

Suppose additionally that G contains a torsionfree subgroup $\Delta \subseteq G$ of finite index. Then we have

$$\operatorname{vcd}(G) \leq \dim(L/K)$$

and equality holds if and only if $G\backslash L$ is compact.

Theorem 32 (A criterion for 1-dimensional models). *Let G be a discrete group. Then there exists a 1-dimensional model for $\underline{E}G$ if and only the cohomological dimension of G over the rationals $\mathbb{Q}$ is less or equal to one.*

Theorem 33. Virtual cohomological dimension and $\dim(\underline{E}G)$ *Let G be a discrete group which contains a torsionfree subgroup of finite index and has* **virtual cohomological dimension**$\operatorname{vcd}(G) \leq d$. *Let $l \geq 0$ be an integer such that the length $l(H)$ of any finite subgroup $H \subset G$ is bounded by l.*

Then we have $\operatorname{vcd}(G) \leq \dim(\underline{E}G)$ for any model for $\underline{E}G$ and there exists a model for $\underline{E}G$ of dimension $\max\{3, d\} + l$.

Example 34 (Virtually poly-cyclic groups). Let the group Δ be **virtually poly-cyclic**, i.e. Δ contains a subgroup Δ' of finite index for which there is a finite sequence $\{1\} = \Delta'_0 \subseteq \Delta'_1 \subseteq \ldots \subseteq \Delta'_n = \Delta'$ of subgroups such that Δ'_{i-1} is normal in Δ'_i with cyclic quotient Δ'_i/Δ'_{i-1} for $i = 1, 2, \ldots, n$. Denote by r the number of elements $i \in \{1, 2, \ldots, n\}$ with $\Delta'_i/\Delta'_{i-1} \cong \mathbb{Z}$. The number r is called the **Hirsch rank**. The group Δ contains a torsionfree subgroup of finite index. We call Δ' **poly-$\mathbb{Z}$** if $r = n$, i.e. all quotients Δ'_i/Δ'_{i-1} are infinite cyclic. Then

1. $r = \text{vcd}(\Delta)$;

2. $r = \max\{i \mid H_i(\Delta'; \mathbb{Z}/2) \neq 0\}$ for one (and hence all) poly-$\mathbb{Z}$ subgroup $\Delta' \subset \Delta$ of finite index;

3. There exists a finite r-dimensional model for $\underline{E}\Delta$ and for any model $\underline{E}\Delta$ we have $\dim(\underline{E}\Delta) \geq r$.

6.2. Reduction to discrete groups

The **discretization** G_d of a topological group G is the same group but now with the discrete topology.

Theorem 35 (Passage from topological groups to totally disconnected groups). *Let G be a locally compact second countable Hausdorff group. Put $\overline{G} := G/G^0$. Then there is a G-CW-model for $\underline{E}G$ that is d-dimensional or finite or of finite type respectively if and only if $\underline{E}\overline{G}$ has a $\overline{G}$-CW-model that is d-dimensional or finite or of finite type respectively.*

Theorem 36 (Passage from totally disconnected groups to discrete groups). *Let G be a locally compact totally disconnected Hausdorff group and let $\mathcal{F}$ be a family of subgroups of G. Then there is a G-CW-model for $E_{\mathcal{F}}(G)$ that is d-dimensional or finite or of finite type respectively if and only if there is a G_d-CW-model for $E_{\mathcal{F}}(G_d)$ that is d-dimensional or finite or of finite type respectively.*

7. Counterexamples

The following problem is stated by Brown It created a lot of activities and many of the results stated above were motivated by it.

Problem 37. *For which discrete groups* G*, which contain a torsionfree subgroup of finite index and has virtual cohomological dimension* $\leq d$*, does there exist a* d*-dimensional* G*-*CW*-model for* $\underline{E}G$*?*

Leary and Nucinkis have constructed many very interesting examples of discrete groups some of which are listed below. Their main technical input is an equivariant version of the constructions due to Bestvina and Brady. These examples show that the answer to the Problems 37 and to other problems appearing in the literature is **not** positive in general. A group G is **of type VF** if it contains a subgroup $H \subseteq G$ of finite index for which there is a finite model for BH.

1. For any positive integer d there exist a group G of type VF which has virtually cohomological dimension $\leq 3d$, but for which any model for $\underline{E}G$ has dimension $\geq 4d$;

2. There exists a group G with a finite cyclic subgroup $H \subseteq G$ such that G is of type VF but the centralizer $C_G H$ of H in G is not of type FP_∞;

3. There exists a group G of type VF which contains infinitely many conjugacy classes of finite subgroups;

4. There exists an extension $1 \to \Delta \to G \to \pi \to 1$ such that $\underline{E}\Delta$ and $\underline{E}G$ have finite G-CW-models but there is no G-CW-model for $\underline{E}\pi$ of finite type.

8. The Orbit Space of $\underline{E}G$

We will see that in many computations of the group (co-)homology, of the algebraic K- and L-theory of the group ring or the topological K-theory of the reduced C^*-algebra of a discrete group G a key problem is to determine the homotopy type of the quotient space $G\backslash\underline{E}G$ of $\underline{E}G$. The following result shows that this is a difficult problem in general and can only be solved in special cases where som extra geometric input is available. It was proved by Leary and Nucinkis based on ideas due to Baumslag-Dyer-Heller and Kan and Thurston.

Theorem 38 (The homotopy type of $G\backslash\underline{E}G$). *Let X be a connected CW-complex. Then there exists a group G such that $G\backslash\underline{E}G$ is homotopy equivalent to X.*