On the Borel Conjecture and related topics

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March 2008

Outline and goal

- Present a list of prominent conjectures such as the one due to Bass, Borel, Farrell-Jones, Kaplansky and Novikov.
- Discuss the relations among these conjectures.
- State our main theorem which is joint work with Bartels. It says that these conjectures are true for an interesting class of groups including word-hyperbolic groups and CAT(0)-groups.
- Discuss consequences and open cases.
- Make a few comments about the proof.

Some prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Projective class groups)

Let R be a regular ring. Suppose that G is torsionfree. Then:

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective;*
- *If R is a principal ideal domain, then* $\widetilde{K}_0(RG) = 0$.

- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.,e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
- Let G be a finitely presented group. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\operatorname{Wh}(G)$ *vanishes.*

- Fix $n \geq 6$. The vanishing of $\operatorname{Wh}(G)$ is equivalent to the following geometric statement: Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial.

Conjecture (Moody's Induction Conjecture)

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G*

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

is bijective;

- *Let F be a field of characteristic p for a prime number p. Then the map*

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

is bijective.

- If G is torsionfree, the Induction Conjecture says that everything comes from the trivial subgroup and we rediscover some of the previous conjectures.

- The various versions of the Bass Conjecture fit into this context as well.
- Roughly speaking, the Bass Conjecture extends basic facts of the representation theory of finite groups to the projective class group of infinite groups.

Conjecture (L^2-torsion)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of a closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering.
- If M is hyperbolic and has odd dimension, its L^2-torsion is up to a non-zero dimension constant its volume.
- The conjecture above allows to extend the notion of volume to word hyperbolic groups whose L^2-Betti numbers all vanish.
- It also gives interesting invariants for group automorphisms.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0,f_0) = \operatorname{sign}_x(M_1,f_1).$$

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic.

- This is the topological version of Mostow rigidity. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.
- Examples due to Farrell-Jones (1989) show that the Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.

- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture. Namely, in the Borel Conjecture the fundamental group can be complicated but there are no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.
- A systematic study of topologically rigid manifolds is presented in a paper by Kreck-Lück (2006), where a kind of interpolation between the Poincaré Conjecture and the Borel Conjecture is studied.
- There is also an existence part of the Borel Conjecture. Namely, if X is an aspherical finite Poincaré complex, then X is homotopy equivalent to an ANR-homology manifold. One may also hope that X is homotopy equivalent to a closed manifold. But then one runs into Quinn's resolutions obstruction which seem to be a completely different story (see Byrant-Ferry-Mio-Weinberger (1995)). The question is whether it vanishes for closed aspherical manifolds.

Conjecture (K-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG.
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.

Lemma

Let R be a regular ring and let G be a torsionfree group such that K-theoretic Farrell-Jones Conjecture holds. Then

- $K_n(RG) = 0$ *for* $n \leq -1$*;*
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*
- *The Whitehead group* $\mathrm{Wh}(G)$ *is trivial.*

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG;\mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG;\mathbf{K}_R) \cong K_0(RG).$$

- A similar argument works for $\mathrm{Wh}(G) = 0$.

Conjecture (L-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG;\mathbf{L}_R^{\langle-\infty\rangle}\big) \to L_n^{\langle-\infty\rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$
$$\mathcal{S}^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Browder, Novikov, Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\text{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L}^{\langle -\infty\rangle})$ is bijective for $k \geq n+1$ and injective for $k = n$ if both the K-theoretic and L-theoretic Farrell-Jones Conjectures hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.

The general formulation of the Farrell-Jones Conjecture

Conjecture (Farrell-Jones Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in an additive G-category $\mathcal{A}$ for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_{\mathcal{A}}) \to H_n^G(pt, \mathbf{K}_{\mathcal{A}}) = K_n(\mathcal{A} * G)$$

is bijective for all $n \in \mathbb{Z}$.

- $E_{\mathcal{VC}\text{yc}}(G)$ is the classifying space of the family of virtually cyclic subgroups.
- $H_*^G(-; \mathbf{K}_{\mathcal{A}})$ is the G-homology theory satisfying for every $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}_{\mathcal{A}}) = K_n(\mathcal{A} * H).$$

- If one takes for $\mathcal{A}$ the category of finitely generated projective R-modules, then $K_n(\mathcal{A} * G)$ becomes $K_n(RG)$.
- The formulation with additive categories allows the presence of G-actions on the coefficient ring R and the twisting of the involution by an orientation homomorphism and gives automatically certain inheritance properties.
- We think of it as an advanced induction theorem (such as Artin's or Brower's induction theorem for representations of finite groups).

Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture for any additive G-category $\mathcal{A}$, then all the conjectures mentioned above follow for G.

The status of the Farrell-Jones Conjecture

Theorem (Main Theorem Bartels-Lück(2008))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures hold with coefficients in any additive G-category (with involution) has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If we demand on the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all $\operatorname{CAT}(0)$-groups.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- On examples is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.
- Bartels-Echterhoff-Lück(2007) show that the Bost Conjecture with coefficients in C^*-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0\big(\mathcal{A} \rtimes_{l^1} G\big) \to K_0\big(\mathcal{A} \rtimes_{C_r^*} G\big)$$

is not bijective for all G-C^*-algebras $\mathcal{A}$.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\text{Out}(F_n)$;
  - Thompson groups.
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems and have some exotic properties which may cause the failure of the Farrell-Jones Conjecture.
- One needs a property which can be used to detect a non-trivial element which is not in the image of the assembly map or is in its kernel.

Computational aspects

Theorem (The algebraic K-theory of torsionfree hyperbolic groups)

Let G be a torsionfree hyperbolic group and let R be a ring (with involution). Then we get an isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG);$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG);$$

Theorem (L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

K-theory versus L-theory

- So far the K-theory case has been easier to handle.
- The reason is that at some point a transfer argument comes in. After applying the transfers the element gets controlled on the total space level and then is pushed down to the base space.
- The transfer $p^!$ for a fiber bundle $F \colon E \to B$ has in K-theory the property that $p^! \circ p_*$ is multiplication with the Euler characteristic. In most situations F is contractible and hence obviously $p^! \circ p_*$ is the identity what is needed for the proof.
- In the L-theory case $p^! \circ p_*$ is multiplication with the signature. If the fiber is a sphere, then $p^! \circ p_*$ is zero.
- One needs a construction which makes out of a finite CW-complex with Euler characteristic 1 a finite Poincare complex with signature 1 or a chain complex or module analogue.

- Such a construction is given by the multiplicative hyperbolic form.
- Given a finitely projective R-module P over the commutative ring R, define a symmetric bilinear R-form $H_\otimes(P)$ by

$$\big(P \otimes P^*\big) \times \big(P \otimes P^*\big) \to R, \quad (p \otimes \alpha, q \otimes \beta) \mapsto \alpha(q) \cdot \beta(p).$$

  If one replaces $\otimes$ by $\oplus$ and $\cdot$ by $+$, this becomes the standard hyperbolic form.
- The multiplicative hyperbolic form induces a ring homomorphism

$$K_0(R) \to L^0(R), \quad [P] \mapsto [H_\otimes(P)].$$

- It is an isomorphism for $R = \mathbb{Z}$.

Comments on the proof

Here are the basic steps of the proof of the main Theorem.

Step 1: Interpret the assembly map as a forget control map. Then the task is to give a way of gaining control.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}$ holds if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- Z-set-condition
  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;
- Long thin coverings
  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists a $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^\beta \times \{x\} \subset U$. Here g^β denotes the β-ball around g in G with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above. This is often done by constructing a certain flow space and use the flow to let a given not yet perfect covering flow into a good one. The construction of the flow space for CAT(0)-space is one of the main ingredients.