Hyperbolic groups with spheres as boundary and a stable version of the Cannon Conjecture

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September 2017

Preview of the main result

Conjecture (Gromov)

Let G be a torsionfree hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

The Conjecture is true for $n \geq 6$.

We also deal with the questions:

- Is there a stable solution to the conjecture in low dimensions?
- When is a Poincaré duality group the fundamental group of an aspherical closed manifold?

Hyperbolic spaces and hyperbolic groups

Definition (Hyperbolic space)

A δ-hyperbolic space X is a geodesic space whose geodesic triangles are all δ-thin.

A geodesic space is called hyperbolic if it is δ-hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic as a metric space.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$.

Definition (Boundary of a hyperbolic space)

Let X be a hyperbolic space. Define its **boundary** ∂X to be the set of equivalence classes of geodesic rays. Put

$$\overline{X} := X \amalg \partial X.$$

- Two geodesic rays $c_1, c_2 \colon [0, \infty) \to X$ are called **equivalent** if there exists $C > 0$ satisfying $d_X\big(c_1(t), c_2(t)\big) \leq C$ for $t \in [0, \infty)$.

Lemma

There is a topology on $\overline{X}$ with the properties:

- $\overline{X}$ *is compact and metrizable;*
- *The subspace topology $X \subseteq \overline{X}$ is the given one;*
- X *is open and dense in* $\overline{X}$.

- Let M be a simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then M is hyperbolic as a metric space and $\partial M = S^{\dim(M)-1}$.

Definition (Quasi-isometry)

A map $f\colon X \to Y$ of metric spaces is called a quasi-isometry if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality

  $$\lambda^{-1} \cdot d_X\big(x_1, x_2\big) - C \le d_Y\big(f(x_1), f(x_2)\big) \le \lambda \cdot d_X(x_1, x_2) + C$$

  holds for all $x_1, x_2 \in X$;
- For every y in Y there exists $x \in X$ with $d_Y\big(f(x), y\big) < C$.

Lemma (Švarc-Milnor Lemma)

Let X be a geodesic space. Suppose that the finitely generated group G acts properly, cocompactly and isometrically on X. Choose a base point $x \in X$. Then the map

$$f\colon G \to X, \quad g \mapsto gx$$

is a quasi-isometry.

Lemma (Quasi-isometry invariance of the Cayley graph)

The quasi-isometry type of the Cayley graph of a finitely generated group is independent of the choice of a finite set of generators.

Lemma (Quasi-isometry invariance of being hyperbolic)

The property "hyperbolic" is a quasi-isometry invariant of geodesic spaces.

Lemma (Quasi-isometry invariance of the boundary)

A quasi-isometry $f \colon X_1 \to X_2$ of hyperbolic spaces induces a homeomorphism

$$\partial X_1 \xrightarrow{\cong} \partial X_2.$$

Definition (Hyperbolic group)

A finitely generated group is called hyperbolic if its Cayley graph is hyperbolic.

Definition (Boundary of a hyperbolic group)

Define the boundary ∂G of a hyperbolic group to be the boundary of its Cayley graph.

Basic properties of hyperbolic groups

- A group G is hyperbolic if and only if it acts properly, cocompactly and isometrically on a hyperbolic space. In this case $\partial G = \partial X$.
- Let M be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- If G is virtually torsionfree and hyperbolic, then $\mathrm{vcd}(G) = \dim(\partial G) + 1$.
- If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to S^n.
- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.

Gromov's Conjecture in low dimensions

Theorem (Casson-Jungreis, Freden, Gabai)

A hyperbolic group has S^1 as boundary if and only if it is a Fuchsian group.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

- In dimension four the only hyperbolic groups which are known to be good in the sense of Freedman are virtually cyclic.
- Possibly our results hold also in dimension 5.

ANR-homology manifolds

Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a closed ANR-homology manifold.

There is also the notion of a compact ANR-homology manifold with boundary.

- Every closed topological manifold is a closed ANR-homology manifold.
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

Definition (Disjoint Disk Property (DDP))

A homology ANR-manifold M has the disjoint disk property (DDP), if for any $\epsilon > 0$ and maps $f, g \colon D^2 \to M$, there are maps $f', g' \colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$

- A topological manifold of dimension ≥ 5 is a closed ANR-homology manifold, which has the DDP by transversality.

Poincaré duality groups

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

Theorem (Poincaré duality groups and ANR-homology manifolds Bartels-Lück-Weinberger)

Let G be a torsionfree group. Suppose that it satisfies the K- and L-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

The proof of the result above relies on

- Surgery theory as developed by Browder, Novikov, Sullivan, Wall for smooth manifolds and its extension to topological manifolds using the work of Kirby-Siebenmann.
- The algebraic surgery theory of Ranicki.
- The surgery theory for ANR-manifolds due to Bryant-Ferry-Mio-Weinberger and basic ideas of Quinn.
- The proof of the Farrell-Jones Conjecture for K- and L-theory for hyperbolic groups by Bartels-Lück.

Theorem (Bestvina-Mess)

A torsionfree hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary and S^{n-1} have the same Čech cohomology.

Corollary

Let G be a torsionfree word-hyperbolic group. Let $n \geq 6$. Then the following statements are equivalent:

1. *The boundary ∂G has the integral Čech cohomology of S^{n-1};*
2. *G is a Poincaré duality group of dimension n;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
4. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5. Then M is a topological manifold if and only if M has the DDP and $\iota(M) = 1$.*

Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace "closed ANR-homology manifold" by "closed topological manifold" in the theorem above.
- In general the Quinn obstruction is not a homotopy invariant but it is a homotopy invariant for aspherical closed ANR-homology manifolds, provided that the Farrell-Jones Conjecture holds.
- However, some experts expect the answer no.
- I am not an expert and hope that the answer is yes.

Theorem (Quasi-isometry invariance of Quinn's resolution obstruction Bartels-Lück-Weinberger)

Let G_1 and G_2 be torsionfree hyperbolic groups.

- *Let G_1 and G_2 be quasi-isometric. Then G_1 is a Poincaré duality group of dimension n if and only if G_2 is;*
- *Let M_i be an aspherical closed ANR-homology manifold with $\pi_1(M_i) \cong G_i$ for $i = 1, 2$. If ∂G_1 and ∂G_2 are homeomorphic, then the Quinn obstructions of M_1 and M_2 agree;*
- *Let G_1 and G_2 be quasi-isometric. Then there exists an aspherical closed topological manifold M_1 with $\pi_1(M_1) = G_1$ if and only if there exists an aspherical closed topological manifold M_2 with $\pi_1(M_2) = G_2$.*

Hyperbolic groups with spheres as boundary

Theorem (Hyperbolic groups with spheres as boundary Bartels-Lück-Weinberger)

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

1. *The boundary ∂G is homeomorphic to S^{n-1};*
2. *There is a closed aspherical topological manifold M together with an isomorphism $u_M \colon \pi_1(M) \xrightarrow{\cong} G$ such that its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

If the first statement is true, the manifold appearing above is unique up to homeomorphism (taking u_M into account).

Exotic Examples

By hyperbolization techniques due to Charney, Davis, Januskiewicz one can find the following examples:

Examples (Exotic universal coverings)

Given $n \geq 5$, there are aspherical closed topological manifolds M of dimension n with hyperbolic fundamental group $G = \pi_1(M)$ satisfying:

- The universal covering $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$ and ∂G is not homeomorphic to S^{n-1}.
- M is smooth and $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ but ∂G is not S^{n-1}.

Example (No smooth structures)

For every $k \geq 2$ there exists a torsionfree hyperbolic group G with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold M with $\pi_1(M) \cong G$. In particular G is not the fundamental group of a closed smooth Riemannian manifold with $\sec(M) < 0$.

Theorem (Davis-Fowler-Lafont)

For every $n \geq 6$ there exists an aspherical closed topological manifold with hyperbolic fundamental group which is not triangulable.

Theorem (Bartels-Lück)

For every $n \geq 5$ closed aspherical topological manifolds with hyperbolic fundamental groups are topologically rigid.

Corollary

For any $n \geq 6$ there exists a hyperbolic group which is the fundamental group of an aspherical topological manifold but not the fundamental group of an aspherical triangulable topological manifold.

A stable version of the Cannon Conjecture in the torsionfree case

Conjecture (Cannon's Conjecture in the torsionfree case)

A torsionfree hyperbolic group G has S^2 as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

Theorem (Bestvina-Mess)

Let G be an infinite hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold M. Then M is hyperbolic and G satisfies the Cannon's Conjecture.

Theorem (Ferry-Lück-Weinberger (in preparation))

Let G be a torsionfree hyperbolic group with S^{k-1} as boundary and $l \geq 0$ be an integer with $k + l \geq 6$.

- *Then there is a closed aspherical $(k+l)$-dimensional manifold M with an isomorphism $u_M \colon \pi_1(M) \xrightarrow{\cong} G \times \mathbb{Z}^l$.*
- *If N is another closed aspherical manifold with an isomorphism $u_N \colon \pi_1(N) \xrightarrow{\cong} G \times \mathbb{Z}^l$, then there is a homeomorphism $f \colon M \to N$ with $\pi_1(f) = u_N^{-1} \circ u_M$.*

Corollary

Let G be a torsionfree hyperbolic group with S^2 as boundary. Let $l \geq 3$ be a natural number. Choose a closed aspherical manifold M together with an isomorphism $u_M \colon \pi_1(M) \cong G \times \mathbb{Z}^l$.

Then the following assertions are equivalent:

- *The Cannon Conjecture for G is true;*
- *There exists a homeomorphism $f \colon N \times T^l \xrightarrow{\cong} M$ for some closed manifold N.*