

Wolfgang Lück and Tibor Macko

Surgery Theory: Foundations

February 1, 2024

Springer Nature



Preface

Surgery theory was created in the sixties to solve classification problems for manifolds and since then has led to an enormous number of striking results. It has many interactions with other areas of mathematics, such as algebra, differential geometry, geometric group theory, algebraic K -theory, number theory, and the theory of operator algebras. Surgery theory also promises to be a major tool in geometry and topology in the future.

Surgery theory was initiated by Kervaire and Milnor in their paper [216] on the classification of homotopy spheres. Surgery theory for simply connected closed manifolds was developed systematically in Browder's book [55]. The book of Wall [414] established surgery theory for arbitrary fundamental groups. It also contains numerous new results on the classification of closed manifolds. The main tool in surgery theory is the surgery exact sequence due to Browder, Novikov, Sullivan, and Wall. It combines the homotopy theory of manifolds with the L -theory of quadratic forms over the group rings of their fundamental groups in order to obtain classification results about manifolds. The work of Kirby and Siebenmann [219] made it possible to do surgery also in the topological category. Quinn [337] gave a description of the surgery exact sequence as the long exact sequence of homotopy groups of a fibration and identified one of its maps as the so-called assembly map. Ranicki [344] developed a chain complex version of algebraic L -theory, answering a request by Wall [414, Chapter 17G], and later provided an algebraic description of the assembly map [348]. The Farrell–Jones Conjecture [150] about the assembly maps and its proofs for a large class of groups allow computations of L -groups of infinite groups and open the door to many applications of surgery theory for compact manifolds with infinite fundamental groups.

The goal of this book is to present an accurate, comprehensible, complete, and detailed introduction to surgery theory, which is useful for various groups of readers, such as experts in surgery theory, experienced mathematicians, who may not be experts in surgery, but just want to learn or use it, and also, of course, advanced undergraduate and graduate students. This is quite a challenge since surgery theory is sophisticated and complicated, requiring a large amount of previous knowledge, and since a lot of material has been accumulated, but not all details are well docu-

mented. We tried to find a reasonable compromise between the intention to present many details in full generality, to fix some bugs in the literature, to motivate the constructions, theorems, and proofs, and the desire to allow the reader to just browse through the book to get a first impression, or find a solution to a specific problem or an answer to a specific question, without necessarily going through all of the text.

Throughout the book we rely on the basics of the surgery theory developed in recent decades. None of the main theorems or concepts are new, but there are places where our approach to certain details is novel.

Bonn, February, 2024

Wolfgang Lück
Tibor Macko

Contents

1 Introduction	1
1.1 Some Classical Problems that Can Be Attacked by Surgery Theory	1
1.2 Overview of the Contents of this Book	3
1.3 Outlook	8
1.4 How to Use this Book	9
1.5 Prerequisites	11
1.6 Acknowledgement	12
2 The s-Cobordism Theorem	15
2.1 Introduction	15
2.2 Handlebody Decompositions	19
2.3 Handlebody Decompositions and CW -Structures	25
2.4 Reducing the Handlebody Decomposition	29
2.5 Handlebody Decompositions and Whitehead Groups	34
2.6 Notes	37
3 Whitehead Torsion	39
3.1 Introduction	39
3.2 Whitehead Groups	41
3.3 Algebraic Approach to Whitehead Torsion	44
3.4 Geometric Approach to Whitehead Torsion	52
3.5 Reidemeister Torsion and (Generalised) Lens Spaces	57
3.5.1 Lens Spaces	57
3.5.2 Generalised Lens Spaces	58
3.5.3 Homotopy Classification of Generalised Lens Spaces	59
3.5.4 Reidemeister Torsion	63
3.5.5 Simple Homotopy Classification of Generalised Lens Spaces	65
3.5.6 A Variant of Reidemeister Torsion	66
3.5.7 The Classification of Lens Spaces	69

3.5.8	The Homeomorphism Classification of Generalised Lens Spaces	72
3.6	The Spherical Space Form Problem	72
3.7	Notes	73
4	The Surgery Step and ξ-Bordism	75
4.1	Introduction	75
4.2	The CW-Version of the Surgery Step	76
4.3	Motivation for the Surgery Step	79
4.4	Motivation for the Bundle Data	86
4.5	Immersions and Embeddings	88
4.6	The Surgery Step	94
4.7	The Pontrjagin–Thom Construction	98
4.8	Notes	103
5	Poincaré Duality	105
5.1	Introduction	105
5.2	Local Coefficients	106
5.3	Passage to Group Rings	108
5.4	The Determinant Line Bundle	111
5.5	The Intrinsic Fundamental Class	114
5.6	Poincaré Duality	120
5.6.1	Rings with Involutions	120
5.6.2	Poincaré Complexes	121
5.6.3	The Signature	131
5.6.4	The Degree of a Map	138
5.7	Notes	139
6	The Spivak Normal Structure	141
6.1	Introduction	141
6.2	Spherical Fibrations	142
6.3	The Thom Isomorphism	153
6.4	The Normal Fibration of a Connected Finite CW-Complex	163
6.5	The Spivak Normal Structure and Its Main Properties	166
6.6	Existence of Poincaré Complexes without Vector Bundle Reduction	170
6.7	Characterisation of Poincaré Duality in Terms of the Normal Fibration	175
6.8	The Existence of the Spivak Normal Fibration	177
6.9	Spanier–Whitehead Duality	178
6.10	The Uniqueness of the Spivak Normal Fibration	181
6.11	Notes	191
7	Normal Maps and the Surgery Problem	193
7.1	Introduction	193
7.2	An Informal Preview	194
7.3	Rank k Normal Invariants	196

7.4	Normal Maps	201
7.5	Notes	217
8	The Even-Dimensional Surgery Obstruction	219
8.1	Introduction	219
8.1.1	The Nature of the Surgery Obstruction	219
8.1.2	The Intrinsic Versions	221
8.1.3	The Simply Connected Case	221
8.1.4	The Surgery Problem Relative Boundary and Wall's Realisation Theorem	222
8.2	Normal Γ -Maps	223
8.3	Intersection and Self-intersection Pairings	227
8.3.1	Intersections of Immersions	228
8.3.2	Self-intersections of Immersions	232
8.4	Surgery Kernels and Forms	238
8.4.1	Surgery Kernels	239
8.4.2	Symmetric Forms and Surgery Kernels	245
8.4.3	Quadratic Forms and Surgery Kernels	252
8.5	Even-Dimensional L -Groups	258
8.5.1	The Definition of $L_{2k}(R)$ via Forms	258
8.5.2	The L -Groups of \mathbb{Z} in Dimensions $n = 4k$	262
8.5.3	The L -Groups of \mathbb{Z} in Dimensions $n = 4k + 2$	265
8.6	The Even-Dimensional Surgery Obstruction in the Universal Covering Case	270
8.6.1	Normal Maps of Pairs	270
8.6.2	Surgery Kernels for Maps of Pairs	271
8.6.3	Proof of Theorem 8.112	278
8.7	The Intrinsic L -Group and Surgery Obstruction in Even Dimensions	285
8.7.1	Conjugation Invariant Functors	286
8.7.2	Half-Conjugation Invariant Functors	288
8.7.3	The Definition of Intrinsic L -Groups in Even Dimensions	292
8.7.4	The Intrinsic Even-Dimensional Surgery Obstruction	293
8.7.5	Bordism Invariance in Even Dimensions for a Not Necessarily Cylindrical Target	299
8.7.6	The Simply Connected Case in Even Dimensions	300
8.7.7	The Role of the Bundle Data in the Highly Connected Case	305
8.8	The Even-Dimensional Surgery Obstruction Relative Boundary	308
8.8.1	Normal Maps of Pairs	308
8.8.2	Normal Bordism for Normal Maps of Pairs	309
8.8.3	The Intrinsic Even-Dimensional Surgery Obstruction Relative Boundary	311
8.8.4	Kernels for Maps of Triads	315
8.9	Realisation of Even-Dimensional Surgery Obstructions	317
8.10	Notes	325

9 The Odd-Dimensional Surgery Obstruction	327
9.1 Introduction	327
9.2 Odd-Dimensional L -Groups	329
9.2.1 The Definition of $L_{2k+1}(R)$ via Formations	330
9.2.2 Presentations of Formations	334
9.2.3 The Definition of $L_{2k+1}(R)$ via Automorphisms	339
9.2.4 The Odd-Dimensional L -Groups of \mathbb{Z}	344
9.3 The Odd-Dimensional Surgery Obstruction in the Universal Covering Case	360
9.3.1 The Kernel Formation	361
9.3.2 The Bordism Formation	370
9.3.3 Proof of Theorem 9.60	379
9.4 The Intrinsic Surgery Obstruction in Odd Dimensions	386
9.4.1 The Definition of Intrinsic L -Group in Odd Dimensions	386
9.4.2 The Intrinsic Odd-Dimensional Surgery Obstruction	386
9.4.3 Bordism Invariance in Odd Dimensions for Not Necessarily Cylindrical Target	387
9.4.4 The Simply Connected Case in Odd Dimensions	388
9.5 The Odd-Dimensional Surgery Obstruction Relative Boundary	389
9.6 Realisation of Odd-Dimensional Surgery Obstructions	389
9.7 Notes	392
10 Decorations and the Simple Surgery Obstruction	393
10.1 Introduction	393
10.2 Decorated L -Groups	393
10.3 Reidemeister U -Torsion	398
10.4 The Simple Surgery Obstruction	402
10.4.1 The Even-Dimensional Case	403
10.4.2 The Odd-Dimensional Case	404
10.4.3 Main Properties of the Simple Surgery Obstruction	404
10.5 Notes	406
11 The Geometric Surgery Exact Sequence	407
11.1 Introduction	407
11.2 The Geometric Structure Set	407
11.3 The Set of Normal Maps	410
11.4 The Surgery Obstruction Groups	411
11.5 The Geometric Surgery Exact Sequence	413
11.6 The Piecewise Linear and Topological Categories	415
11.7 Rigidity	419
11.8 Group Structures in the Surgery Exact Sequences	420
11.9 Some Information about G/O , G/PL and G/TOP	422
11.10 Functorial Properties of the Geometric Surgery Exact Sequence	424
11.11 Notes	427

12 Homotopy Spheres	431
12.1 Introduction	431
12.2 The Group of Homotopy Spheres	433
12.3 The Surgery Exact Sequence for Homotopy Spheres	435
12.4 The J -Homomorphism and Stably Framed Bordism	442
12.5 The Computation of bP_{n+1}	446
12.6 The Group Θ_n/bP_{n+1}	450
12.7 The Kervaire–Milnor Braid	450
12.8 Notes	453
13 The Geometric Surgery Obstruction Group and Surgery Obstruction	459
13.1 Introduction	459
13.2 Surgery on and Normal Bordisms for Pairs	460
13.2.1 Short Review of Normal Bordism for Manifolds Relative Boundary	460
13.2.2 Surgery on the Boundary	461
13.3 The π - π -Theorem	462
13.3.1 Algebraic Proof of the π - π -Theorem 13.4	464
13.3.2 Preliminaries for the Proof of the π - π -Theorem 13.4	464
13.3.3 Proof of the π - π -Theorem 13.4 in the Even-Dimensional Case	466
13.3.4 Proof of the π - π -Theorem 13.4 in the Odd-Dimensional Case	472
13.4 The Geometric Surgery Obstruction Group and Surgery Obstruction	481
13.4.1 The Construction of the Geometric Surgery Groups	481
13.4.2 Making the Reference Map a π_0 - and π_1 -Isomorphism	482
13.4.3 The Geometric Surgery Obstruction	489
13.4.4 Identifying the Geometric Surgery Obstruction Groups with the Algebraic L -Groups	490
13.5 The Geometric Rothenberg Sequence	491
13.6 The Geometric Shaneson Splitting	497
13.7 Notes	499
14 Chain Complexes	501
14.1 Introduction	501
14.2 Modules over Associative Rings	502
14.3 Modules over Rings with Involution	502
14.4 Some Basic Chain Complex Constructions	504
14.5 Further Chain Complex Constructions over Rings with Involution	507
14.6 Homotopy Theory of Chain Complexes	510
14.6.1 Chain Homotopy	510
14.6.2 Pushouts	512
14.6.3 Mapping Cylinders, Mapping Cones and Suspensions	513
14.6.4 Cofibrations and Homotopy Cofibrations	520
14.6.5 Homotopy Pushouts	524

14.6.6	Homotopy Cocartesian Squares	525
14.6.7	Pullbacks	527
14.6.8	Homotopy Fibres	528
14.6.9	Relating the Homotopy Cofibre and the Homotopy Fibre	531
14.6.10	Homotopy Pullbacks	538
14.6.11	Homotopy Colimits of Sequences	542
14.6.12	Homotopy Limits of Inverse Systems of Maps	551
14.6.13	Chain Complexes of Projective Modules	551
14.7	Notes	554
15	Algebraic Surgery	555
15.1	Introduction	555
15.2	Overview	557
15.2.1	Statement of the Main Results	558
15.2.2	Some Basic Notions and Explanations	559
15.2.3	Outline of the Proofs	565
15.3	Structured Chain Complexes	566
15.3.1	Basic Definitions	567
15.3.2	The Symmetric Construction	577
15.3.3	Products	581
15.3.4	Homotopy Invariance	583
15.3.5	The Suspension Map	585
15.3.6	The Quadratic Construction	595
15.3.7	Algebraic Poincaré Complexes	618
15.3.8	Equivariant S -Duality and Umkehr Maps	619
15.3.9	Wu Classes	623
15.3.10	Homotopy $\mathbb{Z}[\mathbb{Z}/2]$ -Chain Maps	626
15.4	Forms and Formations	632
15.4.1	Homotopy Theory of Highly Connected Structured Chain Complexes	633
15.4.2	Revision of Forms and Formations	636
15.4.3	Complexes versus Forms and Formations	638
15.5	L -Groups in Terms of Chain Complexes	649
15.5.1	Pairs	649
15.5.2	Cobordisms	666
15.5.3	The Algebraic Thom Construction	683
15.5.4	The Algebraic Boundary	693
15.5.5	Algebraic Surgery	708
15.6	The Identification of the L -Groups	724
15.7	The Identification of the Surgery Obstructions	740
15.7.1	Euler Classes	742
15.7.2	Self-Intersections	744
15.7.3	Putting it Together	745
15.8	Simple L -groups and Simple Surgery Obstruction in Terms of Chain Complexes	751

15.9 Applications of Algebraic Surgery	753
15.9.1 Product formulas	753
15.9.2 The Algebraic Surgery Transfers for Fibrations	754
15.9.3 The Algebraic Rothenberg Sequence	755
15.9.4 The Algebraic Shaneson Splitting	757
15.9.5 The Algebraic Surgery Exact Sequence	758
15.9.6 The Total Surgery Obstruction	763
15.10 Notes	763
16 Brief Survey of Computations of L-Groups	765
16.1 Introduction	765
16.2 More Decorated L -Groups	766
16.3 Finite Groups	767
16.4 Torsionfree Groups	771
16.5 The Farrell–Jones Conjecture	773
16.5.1 The K -theoretic Farrell–Jones Conjecture with Coefficients in Additive G -Categories	774
16.5.2 The L -theoretic Farrell–Jones Conjecture with Coefficients in Additive G -Categories with Involution	776
16.5.3 Reducing the Family	777
16.5.4 The Full Farrell–Jones Conjecture	779
16.6 Notes	781
17 The Homotopy Type of G/TOP, G/PL and G/O	783
17.1 Introduction	783
17.2 Localisation	784
17.3 G/TOP	785
17.4 G/PL	789
17.5 G/O	791
17.6 H -Space Structures on G/TOP and G/PL	791
17.7 Splitting Invariants	795
17.8 Milnor Manifolds and Kervaire Manifolds	800
17.9 Notes	802
18 Computations of Topological Structure Sets of some Prominent Closed Manifolds	803
18.1 Introduction	803
18.2 Fake Spaces	804
18.3 Products of Spheres	807
18.4 Complex Projective Spaces	809
18.5 Quaternionic Projective Spaces	810
18.6 The Join and the Transfer	811
18.7 The ρ -Invariant	813
18.8 Real Projective Spaces	818
18.9 Lens Spaces	822

18.10 Aspherical Manifolds	828
18.11 Some Torus Bundles over Lens Spaces	829
18.12 Notes	831
19 Topological Rigidity	833
19.1 Introduction	833
19.2 Topological Rigidity and the Surgery Exact Sequence	833
19.3 Aspherical Closed Manifolds	838
19.3.1 Homotopy Classification of Aspherical CW-Complexes	838
19.3.2 Low Dimensions	839
19.3.3 Non-Positive Curvature	839
19.3.4 Torsionfree Discrete Subgroups of Almost Connected Lie Groups	840
19.3.5 Hyperbolisation	840
19.3.6 Exotic Aspherical Closed Manifolds	841
19.3.7 Non-Aspherical Closed Manifolds	843
19.4 The Borel Conjecture	843
19.5 Non-Aspherical Topologically Rigid Closed Manifolds	844
19.5.1 Spheres	845
19.5.2 3-Manifolds	845
19.5.3 Products of Two Spheres	845
19.5.4 Homology Spheres	846
19.5.5 Connected Sums	846
19.5.6 Homology Isomorphisms and Topological Rigidity	846
19.6 Smooth Rigidity	847
19.7 Notes	847
20 Modified Surgery	849
20.1 Introduction	849
20.2 Stable Diffeomorphisms	850
20.3 Bordism Groups Associated to Spaces over BO	851
20.4 Modified Surgery	857
20.5 Normal k -Type	859
20.5.1 Embedding Spaces and Gauss Maps up to Contractible Choice	859
20.5.2 The Moore–Postnikov Factorisation	860
20.5.3 Normal k -Types	861
20.6 Classification up to Stable Diffeomorphisms	865
20.7 Tangential Approach	868
20.7.1 Passing to the Tangent Bundle	868
20.7.2 Connected Sum	869
20.8 Applications	870
20.8.1 Complete Intersections	870
20.8.2 Homogeneous Spaces	871
20.8.3 4-Manifolds	872

Contents	xv
20.9 Notes	873
21 Solutions of the Exercises	875
References	921
Notation	939
Index	945

Chapter 1

Introduction

1.1 Some Classical Problems that Can Be Attacked by Surgery Theory

In this section we give a list of concrete, classical, and prominent problems that have been (partially) solved by surgery theory. The list shall illustrate the high potential of surgery theory and the (partial) solutions of these problems will constitute the contents of this book.

The following two problems represent the prototype of surgery problems, which, however, cannot be solved in full generality.

Problem 1.1 (Recognising manifolds) Let X be a connected finite CW -complex. Under which conditions is X homotopy equivalent to a closed manifold that is topological, PL (= piecewise linear), or smooth?

Problem 1.2 (Classifying manifolds) Let M and N be closed manifolds that are both topological, PL, or smooth. Under which conditions can one decide whether they are homeomorphic, PL homeomorphic, or diffeomorphic respectively. What are possible obstructions and under which conditions are they sufficient?

Exact formulations of both of these problems may vary. We sometimes consider modifications and use slightly different descriptions. Problem 1.1 may also be called an “existence problem” because we are asking whether there exists a manifold in the homotopy type of X . The variation of Problem 1.2 where we assume to begin with that M and N are homotopy equivalent may be called a “uniqueness problem” since we are asking how unique the manifolds in a given homotopy type are.

The next conjecture, a special case of Problem 1.2 in the topological category, is known to be true for all $n \geq 1$. Its proof uses surgery theoretic methods, except in dimension 3 where the proof relies on Ricci flow.

Conjecture 1.3 ((Generalised) Poincaré Conjecture) If M is a closed topological manifold homotopy equivalent to the standard n -sphere S^n , then M is homeomorphic to S^n .

The following problem, a special case of Problem [1.2](#) in the smooth category, triggered the development of surgery theory and will be discussed in Chapter [12](#)

Problem 1.4 (Homotopy spheres) Classify all oriented homotopy spheres, that means closed oriented smooth manifolds homotopy equivalent to the standard n -sphere, up to orientation preserving diffeomorphism.

Another problem, which is completely solved, is the following.

Problem 1.5 (Fake complex projective spaces) Classify all fake complex projective spaces, that means topological manifolds homotopy equivalent to the standard n -dimensional complex projective space, up to homeomorphism.

One can ask the corresponding question for other prominent manifolds, for example for fake real projective spaces or fake lens spaces. There are solutions in many interesting cases, as we will see in Chapter [18](#).

The next conjecture about aspherical manifolds, that means connected manifolds whose universal covering is contractible, will be treated in Chapter [19](#). It is known to be true if the fundamental group is contained in a large class of groups, which encompasses hyperbolic groups, CAT(0)-groups, solvable groups, and lattices in almost connected Lie groups, but is open in general. It is the topological version of Mostow rigidity.

Conjecture 1.6 (Borel Conjecture) Let M and N be closed aspherical topological manifolds. Then:

- (i) The fundamental groups $\pi_1(M)$ and $\pi_1(N)$ are isomorphic if and only if M and N are homeomorphic;
- (ii) Any homotopy equivalence $f: M \rightarrow N$ is homotopic to a homeomorphism;
- (iii) Any map $f: M \rightarrow N$ inducing an isomorphism between the fundamental groups is homotopic to a homeomorphism.

The next problem, which is essentially solved, triggered surgery theory for non-simply connected manifolds, see Section [3.6](#). It is a kind of generalisation of the Space Form Problem asking which finite groups occur as fundamental groups of closed Riemannian manifolds with constant positive sectional curvature.

Problem 1.7 (Spherical Space Form Problem) Which finite groups can act freely and topologically or smoothly respectively on a standard sphere, or, equivalently, occur as fundamental groups of closed manifolds whose universal covering is homeomorphic or diffeomorphic respectively to a standard sphere.

1.2 Overview of the Contents of this Book

Chapters 2 to 11 contain the core of surgery theory that leads to a general method for solving Problems 1.1 and 1.2, while Chapter 12 illustrates the method on the most prominent example of homotopy spheres. The following Chapters 13 to 17 contain additional theoretical tools that are needed to effectively solve Problems 1.1 and 1.2 in other cases, in particular in the topological category. Chapters 18 and 19 illustrate how all this is applied to the concrete examples from the list in the previous section. Finally, Chapter 20 is about an alternative approach called modified surgery.

Although it may not be obvious at first glance, Problems 1.1 and 1.2 are closely linked. The general slogan is that Problem 1.2 is a relative version of Problem 1.1. The general approach to both of these problems is to split them into several steps and to treat the steps separately. Both of these ideas are explained with more details at the appropriate places in the following brief survey of the individual chapters. In the book itself, each chapter has its own more detailed introduction.

In the overview below we will often only treat the smooth category. Nearly all notions and statements carry over to the PL and the topological category.

Chapter 2: The s -Cobordism Theorem

We state and prove the s -Cobordism Theorem 2.1. Roughly speaking, it says that a compact smooth cobordism W from the closed smooth manifold M_0 to the closed smooth manifold M_1 , for which the inclusion $M_i \rightarrow W$ is a simple homotopy equivalence for $i = 0, 1$ and $\dim(M_0) \geq 5$ holds, is diffeomorphic relative M_0 to the cylinder $M_0 \times [0, 1]$. This implies that M_0 and M_1 are diffeomorphic. Hence the s -Cobordism Theorem is highly relevant for the solution of Problem 1.2, namely, it is a cornerstone in the Surgery Program, see Remark 2.9 designed to solve Problem 1.2 by splitting it into three steps. Roughly, first find a (simple) homotopy equivalence, then construct a cobordism compatible with the (simple) homotopy equivalence, and finally improve this cobordism to an s -cobordism. If we get a positive answer in all three steps, then by the s -Cobordism Theorem we obtain a diffeomorphism.

The s -Cobordism Theorem 2.1 (in the topological category) implies the (Generalised) Poincaré Conjecture 1.3 in dimensions ≥ 5 .

Chapter 3: Whitehead Torsion

We give a systematic treatment of Whitehead torsion, which is the obstruction for a homotopy equivalence of finite CW -complexes to be a simple homotopy equivalence. This is relevant since it appears in the s -Cobordism Theorem 2.1. We also explain the classification of lens spaces by their Reidemeister torsion in Section 3.5. This yields a solution of Problem 1.2 in a very specific case where it can exceptionally be achieved without surgery theory from later chapters.

Chapter 4: The Surgery Step and ξ -Bordism

This chapter contains the first step towards a solution of Problem 1.1. Namely, we solve the following Problem 4.2: Given a map $f: M \rightarrow X$ from a closed n -dimensional smooth manifold M to a CW -complex X of finite type, can we modify

it, without changing the target, to a map $f': M' \rightarrow X$ with a closed n -dimensional smooth manifold as source such that f' is k -connected where $n = 2k$ or $n = 2k + 1$? The basic idea is to modify the procedure of making a map of finite CW -complexes highly connected by attaching cells to a finite sequence of so-called surgery steps, so that the source still remains a closed smooth manifold, a procedure commonly called “surgery below middle dimension”. We will see in the process that the presence of certain bundle data is desirable in order to make this work. These bundle data will be formalised in the technical notion of a normal map (also called a surgery problem) in Chapter [7](#).

Chapter 5: Poincaré Duality

We explain the notion of a finite Poincaré complex. This is relevant for Problem [1.1](#) since any finite CW -complex that is homotopy equivalent to a closed manifold is a finite Poincaré complex.

Chapter 6: The Spivak Normal Structure

Recall that a closed smooth manifold has a normal bundle, which is unique up to stable isomorphism. Its underlying sphere bundle is a spherical fibration unique up to stable fibre homotopy equivalence. In this chapter we show that any finite Poincaré complex possesses a Spivak normal structure, which is a spherical fibration coming with a certain collapse map and is unique up to stable fibre homotopy equivalence. Since this structure is homotopy invariant, we discover another obstruction for a finite Poincaré complex X to be homotopy equivalent to a closed manifold. Namely, its Spivak normal structure must have a vector bundle reduction, that means it must come from a vector bundle since the Spivak normal structure of a closed smooth manifold comes from its normal vector bundle by the Pontrjagin–Thom construction.

Chapter 7: Normal Maps and the Surgery Problem

We define the notion of a normal map of degree one motivated by the previous sections. Roughly speaking, a normal map of degree one is a map $f: M \rightarrow X$ from a closed smooth manifold M to a finite Poincaré complex X of degree one, which comes with bundle data, namely, a bundle map from the normal bundle of M to a vector bundle reduction ξ of the Spivak normal structure on X . The surgery problem, see Problem [7.40](#) now asks whether we can modify it by surgery to a normal map whose underlying map $f': M' \rightarrow X$ is a homotopy equivalence. We also show that the set of smooth normal bordism classes of smooth normal maps with the target a fixed finite Poincaré complex X (also called the set of smooth normal invariants) can be identified with the set of homotopy classes of maps from X to a certain space G/O , see Theorem [7.34](#). Analogous statements hold in the PL category and the topological category, see Theorem [11.24](#).

Summarising the development of the chapters so far, we see that the solution of Problem [1.1](#) is split into three steps as formulated in the Surgery Program for recognising manifolds [7.47](#). Roughly speaking, the first step is to check the necessary homotopical and homological condition on X , that means it must be a finite Poincaré complex. The second step is to find a normal map of degree one from

some closed smooth manifold M to X , which exists if the Spivak normal fibration has a vector bundle reduction. In the third step one tries to improve the map to a homotopy equivalence. Surgery below middle dimension from Chapter 4 yields first improvements towards the third step.

Now the slogan that Problem 1.2 is the relative version of Problem 1.1 becomes more apparent, see Remark 7.46. Moreover, the steps in Surgery Program 2.9 and in Surgery Program for recognising manifolds 7.47 correspond to each other as explained in the discussion after Remark 7.47.

Chapter 8: The Even-Dimensional Surgery Obstruction

Not every surgery problem can be solved; there are so-called surgery obstructions. In this chapter we construct the even-dimensional L -groups and the surgery obstructions taking values in them and show that in dimension ≥ 5 the vanishing of the surgery obstruction is equivalent to the existence of a solution of the surgery problem. The L -groups are defined in terms of quadratic forms over the group ring of the fundamental group of X . If the dimension n is divisible by four and X is simply connected, then the surgery obstruction group is \mathbb{Z} and the surgery obstruction is the difference of the signatures of M and X . In this particular case the surgery obstruction is independent of the bundle data, but that is not true in general. If the dimension n is even but not divisible by four and X is simply connected, the surgery obstruction group is $\mathbb{Z}/2$ and the surgery obstruction is the so-called Arf invariant, which definitely depends on the bundle data. This chapter yields the final step in the solution of Problem 1.1 in the even-dimensional case.

Chapter 9: The Odd-Dimensional Surgery Obstruction

We construct the odd-dimensional L -groups and the surgery obstructions taking values in them and show that in dimension ≥ 5 the vanishing of the surgery obstruction is equivalent to the existence of a solution of the surgery problem. The L -groups are defined in terms of automorphisms of quadratic forms, or, equivalently, in terms of formations over the group ring of the fundamental group of X . If X is simply connected and odd-dimensional, the surgery obstruction groups vanish and there are no surgery obstructions. This chapter yields the final step in the solution of Problem 1.1 in the odd-dimensional case.

Chapter 10: Decorations and the Simple Surgery Obstruction

We develop the simple version of the surgery obstruction groups and the surgery obstructions. The difference to the previous constructions is that we want the underlying map $f: M \rightarrow X$ to be a simple homotopy equivalence, while before we were only aiming at a homotopy equivalence. This is relevant in view of the s -Cobordism Theorem 2.1. In the definition of the surgery obstruction groups we now take finitely generated free modules coming with a basis as the underlying modules of quadratic forms and then consider the Whitehead torsion of the various isomorphisms appearing in the previous constructions.

Chapter 11: The Geometric Surgery Exact Sequence

We introduce the surgery exact sequence in Theorems [11.22](#) and [11.25](#) see also Remark [11.23](#). It is the realisation of the Surgery Program [2.9](#) and thus yields a general method for solving Problem [1.2](#). The surgery exact sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater than or equal to five.

Its simple topological version for an n -dimensional closed topological manifold N from Theorem [11.25](#) aims at the computation of the simple structure set $\mathcal{S}^{\text{TOP},s}(N)$. Elements in $\mathcal{S}^{\text{TOP},s}(N)$ are represented by simple homotopy equivalences $f: M \rightarrow N$ with a closed topological manifold M as source and N as target. Two such elements $f: M \rightarrow N$ and $f': M' \rightarrow N$ represent the same element if there is a homeomorphism $h: M \rightarrow M'$ such that $f' \circ h$ and f are homotopic. The surgery exact sequence is an exact sequence of abelian groups of the shape

$$\begin{aligned} \mathcal{N}^{\text{TOP}}(N \times [0, 1], N \times \{0, 1\}) &\rightarrow L_{n+1}^s(\mathbb{Z}\pi, w) \rightarrow \mathcal{S}^{\text{TOP},s}(N) \\ &\rightarrow \mathcal{N}^{\text{TOP}}(N) \rightarrow L_n^s(\mathbb{Z}\pi, w) \end{aligned}$$

where $L_n^s(\mathbb{Z}\pi, w)$ is the simple algebraic L -group of the integral group ring of the fundamental group π of N with the orientation homomorphism w , the normal invariants $\mathcal{N}^{\text{TOP}}(N)$ are given by the surgery problems with target N , and the first and the fourth map are given by taking surgery obstructions. Here one needs to require either $n \geq 5$ or that $n = 4$ and the fundamental group is good in the sense of Freedman, see [\[157\]](#) [\[158\]](#), and Remark [8.30](#).

Note that a closed topological manifold N has the property that any simple homotopy equivalence $M \rightarrow N$ from a closed topological manifold to N is homotopic to a homeomorphism if and only if the structure set $\mathcal{S}^{\text{TOP},s}(N)$ consists of precisely one element, namely the one given by id_N .

The surgery exact sequence in the smooth category from Theorem [11.22](#) is in general not an exact sequence of abelian groups, only of pointed sets, see Section [11.8](#).

Chapter 12: Homotopy Spheres

This chapter is devoted to the classification of oriented homotopy spheres up to oriented diffeomorphism, where a homotopy sphere is a smooth closed manifold that is homotopy equivalent to S^n . This boils down to calculating the structure set $\mathcal{S}(S^n)$ in the smooth category. The input is the geometric surgery exact sequence in the smooth category from Chapter [11](#), calculations of the L -groups in the simply connected case from Chapters [8](#) and [9](#), and homotopy theoretic results about the so-called J -homomorphisms, which shed light on the normal invariants. The maps in the sequence are determined by studying the signature and the Arf invariant of surgery problems.

Information gained by these calculations yields results about various classifying spaces, which are organised in the so-called Kervaire–Milnor braid, see Section [12.7](#).

Chapter 13: The Geometric Surgery Obstruction Group and Surgery Obstruction

This chapter is equivalent to the famous Chapter 9 in Wall's book [414]. We give a geometric approach to the L -groups and the surgery obstruction based on bordism theory. This is convenient in some situations where the necessary algebra is hard to analyse or not even available, such as controlled or equivariant surgery.

Chapter 14: Chain Complexes

This chapter has two goals. Firstly, we summarise the sign conventions that we use in the subsequent chapter about algebraic surgery and in fact throughout the book. These have been well thought through, and we hope that they will become standard. Unfortunately, in the literature many different sign conventions are used. The second goal of this chapter is to review some basic homotopy theory of chain complexes. Both of these topics provide background for the next chapter.

Chapter 15: Algebraic Surgery

We introduce a chain complex version of the L -groups and of the surgery obstructions and identify them with the L -groups and surgery obstructions from Chapters 8 and 9 see Theorem 15.3 and Theorem 15.4. So the chapter contains a presentation of Ranicki's theory of algebraic surgery where forms and formations are uniformly generalised to algebraic Poincaré chain complexes and their algebraic cobordism theory. One drawback of Ranicki's presentation in the original sources is that he very often describes certain constructions, such as algebraic surgery, only by writing down formulas without giving any structural insight. In our exposition we try to give certain general chain complex constructions that shall motivate the outcome and lead finally to explicit formulas. Moreover, we always use our sign conventions whereas Ranicki uses different sign conventions in different papers.

Chapter 16: Brief Survey of Computations of L -Groups

We give a brief survey of computations of L -group of group rings $\mathbb{Z}\pi$. For finite π the calculations were mostly done in the previous century and involve using representation theory and number theory. For infinite π nowadays the main tool is the Farrell–Jones Conjecture, which will be extensively treated in the book in preparation [261].

Chapter 17: The Homotopy Type of G/TOP , G/PL and G/O

We review how to determine the homotopy types of the classifying spaces G/PL and G/TOP , see Section 11.9 and Theorem 17.6. This leads to the computation of the set of normal invariants in the topological category in terms of singular cohomology after localising at 2 and in terms of KO -theory after inverting 2, see (11.41) and Theorem 11.24.

Chapter 18: Computations of Topological Structure Sets of some Prominent Closed Manifolds

We discuss how surgery theory and in particular the surgery exact sequence lead to computations of topological structure sets. We treat products of spheres, complex and real projective spaces, lens spaces, and tori. This leads to the classification of closed topological manifolds that are homotopy equivalent to these spaces up to homeomorphism. In particular we completely solve Problem [1.5](#). We have treated this problem for the standard sphere in the smooth category already in Chapter [12](#) about homotopy spheres.

Chapter 19: Topological Rigidity

A closed topological manifold N is called topologically rigid if any homotopy equivalence $M \rightarrow N$ with a closed topological manifold M as source and N as target is homotopic to a homeomorphism. In this chapter we want to study the question of which closed topological manifolds are topologically rigid. Section [19.4](#) is devoted to the Borel Conjecture predicting that any aspherical closed manifold is topologically rigid. Examples of non-aspherical closed manifolds that are topologically rigid are discussed in Section [19.5](#).

In Section [19.6](#) we briefly discuss the rarity of smooth rigidity in high dimensions.

Chapter 20: Modified Surgery

In this chapter we digress from the main line of the book, which treats the classification of manifolds with a given homotopy type via classical surgery, and discuss aspects of the use of surgery theory to classify manifolds with less homotopy theoretic input. Specifically we discuss variations of the Surgery Program, see Remark [2.9](#) which were pioneered by Kreck [\[225\]](#) and are often called modified surgery. Modified surgery might not have the general structural impact as surgery has on the classification of manifolds, on prominent conjectures such as those of Borel or Novikov, or on index theory and C^* -algebras, but leads in some special but very interesting cases to better and beautiful results, for instance for complete intersections, homogeneous spaces, and 4-manifolds.

1.3 Outlook

Here is a (not necessarily complete) list (in alphabetical order) of topics that we were not able to treat in this book in detail or at all, but which are very interesting. Some of them could be part of sequels to this book (not necessarily written by the authors of this book). For some items we include references where these topics have already been addressed and where further references can be found.

- Algebraic surgery in the setting of ∞ -categories and the relation of algebraic surgery to hermitian K -theory, see [\[69, 70, 71, 271\]](#);
- ANR-homology manifolds and the Quinn obstruction, see [\[67, 338, 339\]](#);

- Applications of surgery theory to knot theory, see [245, 351];
- Applications to questions from differential geometry, in particular to the existence of Riemannian metrics with positive scalar curvature, see [362];
- Automorphism groups of closed manifolds, see [430];
- Computations of the L -groups of group rings of finite groups, see [177];
- Controlled surgery theory [328, 330];
- Equivariant surgery theory [76, 136, 137, 138, 139, 262, 263, 335];
- Finite H -spaces, see [6, 34];
- Full presentation of the proof of the Spherical Space Form Problem, see [120, 277];
- Mapping surgery to analysis, see [184, 185, 186, 440];
- Parametrised surgery, see [163, 162, 195];
- Poincaré surgery, Poincaré embeddings, and LS -groups, see [179], [221], [414, Chapter 11];
- Stratified surgery theory; see [424];
- Surgery in dimension 4, see [37, 159];
- Surgery in the topological category, see [219];
- The Novikov Conjecture, see [153, 154, 226];
- The total surgery obstruction, see [234, 343, 348];
- UNil-terms and splitting obstructions, see [77, 78, 79, 104].

1.4 How to Use this Book

As mentioned before, the potential readers may vary from established experts on surgery theory to advanced students without any previous knowledge about surgery theory. Obviously the various groups of readers have rather different expectations and needs. On the one hand we want to give correct and complete definitions, theorems and proofs, but we also want to allow the reader to browse through the text and get a first impression or a global picture. This leads of course to some tension that we tried to solve as explained below.

A typical example is the notion of a normal map and normal bordism. The definition is quite lengthy, see for instance Definition [7.13] and Definition [7.15]. This is actually necessary, as none of the items occurring there can be dropped when one wants to set up the theory and give accurate proofs in full generality. But when one is working or thinking about a problem or wants to get a first impression, one should work with an extract of these definitions as explained for instance in Section [7.2]. It can also be useful to make simplifying assumptions, for instance, that all manifolds are orientable, or, equivalently, that the orientation homomorphism $w: \pi \rightarrow \{\pm 1\}$ is trivial, or even that every manifold is simply connected. Then one can ignore the local coefficient systems and work with ordinary homology, and one does not have to deal with group rings but only with the ring \mathbb{Z} of integers. In daily life one may get as far as to say that a normal map of degree one is a map $f: M \rightarrow X$ of degree one with connected orientable source and target covered by bundle data, without really memorising what the bundle data are.

Another example of how one can use the book in different ways concerns Chapter 2 on the s -Cobordism Theorem. The minimal approach is to go through the introduction and ignore the rest; that suffices completely to go on with the book. Or one may want to get the full proof and therefore go through all the material of Chapter 2 and at least parts of Chapter 3. This is explained in the Guide of Chapter 2.

The rather long Chapter 15 on algebraic surgery can also be read in rather different ways. One may ignore all the motivations and structural explanations, just concentrate on the formulas, completely leave out the proofs, and just browse through the definitions and main theorems. Or one may want to understand all the details and get an insight into why the definitions and proofs are set up as they stand, and therefore read everything. Again here a reader should first go through the introduction and the guide at its end (and then through the Overview given in Section 15.2), before she or he decides which parts of the chapter she or he wants to read in which reading modus.

One can apply surgery theory in the smooth, PL or topological category. In other words one may consider smooth compact manifolds and try to classify them up to diffeomorphism, compact PL manifolds and try to classify them up to PL homeomorphism, or compact topological manifolds and try to classify them up to homeomorphism. When we explain some technical constructions, such as the surgery step, the bundle data, and so on, we will work in the smooth category since there all the notions such as tangent bundle, normal bundle, transversality and so on are well documented. All this carries over to the PL category and the topological category. We will not go into the sophisticated details of how this can be done since it would go beyond the scope of this book. For the topological category the seminal work of Kirby–Siebenmann [219] is needed. A good reference for the PL category is Rourke–Sanderson [367]. So basic tools such as the surgery exact sequence do exist in all three categories.

The classification results do of course depend on whether we are working in the smooth, PL, or topological category. It turns out that the nicest results occur in the topological category. The reader will have to accept the fact that we develop surgery theory in detail only in the smooth category, but will also apply it to the topological category without further explanations.

Very often we will make the assumption that the dimensions of the manifolds under consideration are greater than or equal to 5. The problem is that the so-called Whitney trick applies only under this dimension assumption. The problem with the Whitney trick can be solved and hence surgery can also be carried out in dimension 4, provided that we work in the topological category and the fundamental group π is good in the sense of Freedman, see Remark 8.30. All of the results presented in this book with the dimension condition ≥ 5 extend to dimension 4 in the topological category if the fundamental group is good. The reader has to live with the fact that we do not explain what is behind these ideas of Freedman, but refer for instance to [37, 159].

The book contains a number of exercises. They come in two flavours. A few of them contain additional information or a computation that may be needed later. Most of them are not needed for the exposition of the book, but give some illuminating

insight. Moreover, the reader may test whether she or he has understood the text, or improve her or his understanding by trying to solve the exercises. Note that hints to the solutions of the exercises are given in Chapter [21](#)

Readers wishing to find a specific topic are advised to first look at the Overview of the Contents of this Book in Section [1.2](#) in order to find the right chapter and then that chapter's introduction. Each introduction to a chapter concludes with a guide, which may help the reader to figure out how to access the contents of that chapter. The extensive index at the end of the book can also be used to find the right spot for a specific topic. The index contains an item Theorem, under which all theorems with their names appearing in the book are listed, and analogously there is an item Conjecture.

We have successfully used parts of this book for seminars, reading courses, and advanced courses for students.

The reader may also consult other monographs on surgery theory such as [55](#), [93](#), [252](#), [352](#), [414](#). Further surveys article or more information can be found for instance in [72](#), [73](#), [145](#), [146](#), [153](#), [154](#), [219](#), [226](#), [348](#), [424](#), [425](#).

1.5 Prerequisites

We require that the reader is familiar with the basics of the following concepts and notions. Readers can learn these topics from the suggested references, but there are many more books and monographs available.

- *CW*-complexes, Cellular Approximation Theorem, Whitehead Theorem, see [45](#), [178](#), [399](#);
- Covering theory, universal coverings, see [45](#), [178](#), [399](#);
- Homology, cohomology, cup and cap-product, signature, characteristic classes [45](#), [178](#), [198](#), [308](#), [399](#);
- Homotopy groups, fibrations, cofibrations, Hurewicz Theorem, see [45](#), [178](#), [399](#);
- Topological and smooth manifolds, see [45](#), [49](#), [224](#), [237](#), [241](#);
- Vector bundles, normal and tangent bundle of a smooth manifold, see [45](#), [49](#), [189](#), [224](#), [308](#);
- Classifying spaces for groups and for vector bundles, fibre bundles, and fibrations, see [198](#), [279](#), [289](#), [308](#);
- Transversality, regular values, immersions, and submersions, see [45](#), [49](#), [189](#), [224](#);
- Group rings, modules and chain complexes over a non-commutative ring, see [59](#), [238](#), [327](#), [421](#);
- Bordism of manifolds, bordism ring, see [189](#), [308](#).

1.6 Acknowledgement

The origin of this book dates back to 2001 when the Summer school on Topology of high-dimensional manifolds was held at ICTP in Trieste. Wolfgang Lück gave a series of lectures and the resulting notes [252] were published as *A basic introduction to surgery theory* in the Proceedings of the school.

However, the idea that the notes could be expanded into a book that would provide a more comprehensive tour of the foundations of the subject persisted. In 2011 Wolfgang Lück approached Diarmuid Crowley and Tibor Macko with an offer to contribute to this endeavour. At the time all three were based in Bonn, and the two younger mathematicians joined the project.

After his moves to Aberdeen and later to Melbourne, Diarmuid Crowley left the team working on the book. The authors are grateful to him for his contributions.

Diarmuid Crowley was scientifically involved with the material covered in Chapters 4 to 9 and 11 and 12, and specifically with versions of these chapters prior to the introduction of the intrinsic fundamental class. His involvement was most significant in Chapters 6, 8 and 9. He also contributed material which persists in Chapters 4, 11 and 12. Specific contributions of his that we would like to mention are:

- Material on the construction of Poincaré complexes without vector bundle reduction in Section 6.6 and the proof of the uniqueness of the Spivak normal fibration covered in Sections 6.9 and 6.10.
- Material on the groups $L_{2k}(\mathbb{Z})$ in Sections 8.5.2 and 8.5.3, the role of the bundle data in the surgery obstruction for highly connected normal maps in Section 8.7.7 and the discussion of surgery kernels for maps of pairs and triads in Sections 8.6.2, 8.8.2 and 8.8.4.
- Chapter 9, in particular the proof that $L_{4k+1}(\mathbb{Z}) = 0$ in Section 9.2.4.
- Material on the spaces G/O, G/PL and G/TOP in Section 11.9 and on the algebraic properties of surgery exact sequences in Section 11.8.

His insights, through our many conversations over time, have contributed to our understanding of the subject and many of its subtleties.

The authors also want to thank the participants of the one year course *Introduction to Surgery Theory*, which took place during the Covid pandemic via ZOOM in 2020 and 2021. There were many very fruitful discussions during the lectures and the tutorials, which helped to improve the exposition a lot. Special thanks go to Frieder Jäckel, Dominik Kirstein, and Christian Kremer.

There are many more mathematicians who made very useful comments about the book, including Spiros Adams-Florou, Serhii Dylida, Ian Hambleton, Fabian Hebestreit, Samuel Kalužný, Daniel Kasprowski, Matthias Kreck, Markus Land, Mark Powell, Andrew Ranicki, Ajay Raj, Arunima Ray, Julia Semikina, Wolfgang Steimle, Peter Teichner, Simona Veselá, Shmuel Weinberger, Christoph Winges, and the (unknown) referees. We are grateful to Philipp Kühn for helping us with the pictures.

Finally the first author wants to thank in particular his wife Sibylle for her patience.

This book project was funded by the Leibniz-Award of the first author granted by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), the ERC Advanced Grant “KL2MG-interactions” (no. 662400) of the first author granted by the European Research Council, and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – GZ 2047/1, Projekt-ID 390685813, Hausdorff Center for Mathematics at Bonn.

The second author was supported by the following grants: “Topology of high-dimensional manifolds” in the scheme “Returns”, VEGA 1/0101/17 and VEGA 1/0596/21 of the Ministry of Education of Slovakia, and APVV-16-0053 of the Slovak Research and Development Agency.