Chern characters for the equivariant K-theory of proper G-CW-complexes

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Abstract. We first construct a classifying space for defining equivariant K-theory for proper actions of discrete groups. This is then applied to construct equivariant Chern characters with values in Bredon cohomology with coefficients in the representation ring functor $R(-)$ (tensored by the rationals). And this in turn is applied to prove some versions of the Atiyah-Segal completion theorem for real and complex K-theory in this setting.

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In an earlier paper [8], we showed that for any discrete group G, equivariant K-theory for finite proper G-CW-complexes can be defined using equivariant vector bundles. This was then used to prove a version of the Atiyah-Segal completion theorem in this situation. In this paper, we continue to restrict attention to actions of discrete groups, and begin by constructing an appropriate classifying space which allows us to define $K^*_G(X)$ for an arbitrary proper G-complex X. We then construct rational-valued equivariant Chern characters for such spaces, and use them to prove some more general versions of completion theorems.

In fact, we construct two different types of equivariant Chern character, both of which involve Bredon cohomology with coefficients in the system $\big(G/H \mapsto R(H)\big)$. The first,

$$\mathrm{ch}^*_X : K^*_G(X) \longrightarrow H^*_G(X;\mathbb{Q}\otimes R(-)),$$

is defined for arbitrary proper G-complexes. The second, a refinement of the first, is a homomorphism

$$\widetilde{\mathrm{ch}}^*_X : K^*_G(X) \longrightarrow \mathbb{Q}\otimes H^*_G(X;R(-)),$$

but defined only for finite dimensional proper G-complexes for which the isotropy subgroups on X have bounded order. When X is a *finite* proper G-complex (i.e., X/G is a finite CW-complex), then $H^*_G(X;R(-))$ is finitely generated, and these two target groups are isomorphic. The second Chern character is important when proving the completion theorems. The idea for defining equivariant Chern characters with values in Bredon cohomology $H^*_G(X;\mathbb{Q}\otimes R(-))$ was first due to Słomińska [12]. A complex-valued Chern character was constructed earlier by Baum and Connes [4], using very different methods.

The completion theorem of [8] is generalized in two ways. First, we prove it for real as well as complex K-theory. In addition, we prove it for families of subgroups in the sense of Jackowski [7]. This means that for each finite proper G-complex X and each family $\mathcal{F}$ of subgroups of G, $K^*_G(E_{\mathcal{F}}(G)\times X)$ is shown to be isomorphic to a certain completion of $K^*_G(X)$. In particular, when $\mathcal{F}=\{1\}$, then $E_{\mathcal{F}}(G)=EG$, and this becomes the usual completion theorem.

The classifying spaces for equivariant K-theory are constructed here using Segal's Γ-spaces. This seems to be the most convenient form of topological group completion in our situation. However, although Γ-spaces do produce spectra, as described in [10], the spectra they produce are connective, and hence not what is needed to define equivariant K-theory directly. So instead, we define $K^{-n}_G(-)$ and $KO^{-n}_G(-)$ for all $n\geq 0$ using classifying *spaces* constructed from a Γ-space, then prove Bott periodicity, and use that to define the groups in positive degrees. One could, of course, construct an equivariant spectrum (or an $\mathsf{Or}(G)$-spectrum

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in the sense of [6]) by combining our classifying space K_G with the Bott map $\Sigma^2 K_G \to K_G$; but the approach we use here seems the simplest way to do it.

By comparison, in [6], equivariant K-*homology* groups $K_*^G(X)$ were defined by using certain covariant functors $\mathbf{K}^{\mathrm{top}}$ from the orbit category $\mathsf{Or}(G)$ to spectra. This construction played an important role in [6] in reformulating the Baum-Connes conjecture. In general, one expects an equivariant homology theory to be classified by a covariant functor from the orbit category to spaces or spectra, and an equivariant cohomology theory to be classified by a contravariant functor. But in fact, when defining equivariant K-theory here, it turned out to be simplest to do so via a classifying G-space, rather than a classifying functor from $\mathsf{Or}(G)$ to spaces.

We would like in particular to thank Chuck Weibel for suggesting Segal's paper and the use of Γ-spaces, as a way to avoid certain problems we encountered when first trying to define the multiplicative structure on $K_G(X)$.

The paper is organized as follows. The classifying spaces for $K_G^{-n}(-)$ and $KO_G^{-n}(-)$ are constructed in Section 1; and the connection with G-vector bundles is described. Products are then constructed in Section 2, and are used to define Bott homomorphisms and ring structures on $K_G^*(X)$; and thus to complete the construction of equivariant K-theory as a multiplicative equivariant cohomology theory. Homomorphisms in equivariant K-theory involving changes of groups are then constructed in Section 3. Finally, the equivariant Chern characters are constructed in Section 5, and the completion theorems are formulated and proved in Section 6. Section 4 contains some technical results about rational characters.

1. A classifying space for equivariant K-theory

Our classifying space for equivariant K-theory for proper actions of an infinite discrete group is constructed using Γ-spaces in the sense of Segal. So we begin by summarizing the basic definitions in [10].

Let Γ be the category whose objects are finite sets, and where a morphism $\theta : S \to T$ sends each element $s \in S$ to a subset $\theta(s) \subseteq T$ such that $s \neq s'$ implies $\theta(s) \cap \theta(s') = \emptyset$. Equivalently, if $\mathcal{P}(S)$ denotes the set of subsets of S, one can regard a morphism in Γ as a map $\mathcal{P}(S) \to \mathcal{P}(T)$ which sends disjoint unions to disjoint unions. For all $n \geq 0$, $\mathbf{n}$ denotes the object $\{1, \ldots, n\}$. (In particular, $\mathbf{0}$ is the empty set.) There is an obvious functor from the simplicial category Δ to Γ, which sends each object $[n] = \{0, 1, \ldots, n\}$ in Δ to $\mathbf{n}$, and where a morphism in Δ — an order preserving map $\varphi : [m] \to [n]$ — is sent to the morphism $\theta_\varphi : \mathbf{m} \to \mathbf{n}$ in Γ which sends i to $\{j \,|\, \varphi(i-1) < j \leq \varphi(i)\}$.

A Γ-space is a functor $\underline{A} : \Gamma^{\mathrm{op}} \to \mathsf{Spaces}$ which satisfies the following two conditions:

(i) $\underline{A}(\mathbf{0})$ is a point; and

(ii) for each $n > 1$, the map $\underline{A}(\mathbf{n}) \longrightarrow \prod_{i=1}^n \underline{A}(\mathbf{1})$, induced by the inclusions $\kappa_i : \mathbf{1} \to \mathbf{n}$ ($\kappa_i(1) = \{i\}$), is a homotopy equivalence.

(In fact, Segal only requires that $\underline{A}(\mathbf{0})$ be contractible; but for our purposes it is simpler to assume it is always a point.) Note that each $A(S)$ has a basepoint: the image of $A(\mathbf{0})$ induced by the unique morphism $S \to \mathbf{0}$. We write $A = \underline{A}(\mathbf{1})$, thought of as the "underlying space" of the Γ-space $\underline{A}$. A Γ-space $\underline{A} : \Gamma^{\mathrm{op}} \to \mathsf{Spaces}$ can be regarded as a simplicial space via restriction to Δ, and $|\underline{A}|$ denotes its topological realization (nerve) as a simplicial space.

If $\underline{A}$ is a Γ-space, then $\underline{BA}$ denotes the Γ-space $\underline{BA}(S) = |\underline{A}(S \times -)|$; and this is iterated to define $\underline{B^n A}$ for all n. Thus, $B^n A = \underline{B^n A}(\mathbf{1})$ is the realization of the n-simplicial space which sends $(S_1, \ldots, S_n)$ to $\underline{A}(S_1 \times \cdots \times S_n)$. Since $\underline{A}(\mathbf{0})$ is a point, we can identify ΣA $(= \Sigma(\underline{A}(\mathbf{1})))$ as a subspace of $BA \cong |\underline{A}|$; and this induces by adjointness a map $A \to \Omega BA$. Upon iterating this, we get maps $\Sigma(B^n A) \to B^{n+1} A$ for all

n; and these make the sequence $A, BA, B^2A, \ldots$ into a spectrum. This is "almost" an Ω-spectrum, in that $B^nA \simeq \Omega B^{n+1}A$ for all $n \geq 1$ [10, Proposition 1.4].

Note that for any Γ-space $\underline{A}$, the underlying space $A = \underline{A}(\mathbf{1})$ is an H-space: multiplication is defined to be the composite of a homotopy inverse of the equivalence $\underline{A}(\mathbf{2}) \xrightarrow{\simeq} \underline{A}(\mathbf{1}) \times \underline{A}(\mathbf{1})$ with the map $\underline{A}(\mathbf{2}) \to \underline{A}(\mathbf{1})$ induced by $m_2 : \mathbf{1} \to \mathbf{2}$ ($m_2(1) = \{1, 2\}$). Then $A \simeq \Omega BA$ if $\pi_0(A)$ is a group; and ΩBA is the topological group completion of A otherwise. All of this is shown in [10, §1].

We work here with equivariant Γ-spaces; i.e., with functors $\underline{A} : \Gamma^{\mathrm{op}} \to G\text{-}\mathsf{Spaces}$ for which $\underline{A}(\mathbf{0})$ is a point, and for which $\big(\underline{A}(\mathbf{n})\big)^H \to \prod_{i=1}^n \big(\underline{A}(\mathbf{1})\big)^H$ is a homotopy equivalence for all $H \subseteq G$. In other words, restriction to fixed point sets of any $H \subseteq G$ defines a Γ-space $\underline{A}^H$; and the properties of equivariant Γ-spaces follow immediately from those of nonequivariant ones. For example, Segal's [10, Proposition 1.4] implies immediately that for any equivariant Γ-space $\underline{A}$, $B^nA \to \Omega B^{n+1}A$ is a weak equivalence for all $n \geq 1$ in the sense that it restricts to an equivalence $(B^nA)^H \simeq (\Omega B^{n+1}A)^H$ for all $H \subseteq G$. This motivates the following definitions.

If $\mathcal{F}$ is any family of subgroups of G, then a *weak $\mathcal{F}$-equivalence* of G-spaces is a G-map whose restriction to fixed point sets of any subgroup in $\mathcal{F}$ is a weak homotopy equivalence in the usual sense. The following lemma about maps to weak equivalences is well known; we note it here for later reference.

Lemma 1.1. *Fix a family $\mathcal{F}$ of subgroups of G, and let $f : Y \to Y'$ be any weak $\mathcal{F}$-equivalence. Then for any G-complex X all of whose isotropy subgroups are in $\mathcal{F}$, the map*

$$f_* : [X, Y]_G \xrightarrow{\;\cong\;} [X, Y']_G$$

is a bijection. More generally, if $A \subseteq X$ is any G-invariant subcomplex, and all isotropy subgroups of $X \smallsetminus A$ are in $\mathcal{F}$, then for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_0} & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\alpha} & Y' \end{array}$$

of G-maps, there is an extension of α_0 to a G-map $\widetilde{\alpha} : X \to Y$ such that $f \circ \widetilde{\alpha} \simeq \alpha$ (equivariantly homotopic), and $\widetilde{\alpha}$ is unique up to equivariant homotopy.

Proof. The idea is the following. Fix a G-orbit of cells $\big(G/H \times D^n \to X\big)$ in X whose boundary is in A. Then, since $Y^H \to (Y')^H$ is a weak homotopy equivalence, the map $eH \times D^n \to X^H \to (Y')^H$ can be lifted to Y^H (up to homotopy), and this extends equivariantly to a G-map $G/H \times D^n \to Y$. Upon continuing this procedure, we obtain a lifting of α to a G-map $\widetilde{\alpha} : X \to Y$ which extends α_0. This proves the existence of a lifting in the above square (and the surjectivity of f_* in the special case); and the uniqueness of the lifting follows upon applying the same procedure to the pair $X \times I \supseteq (X \times \{0, 1\}) \cup (A \times I)$. □

Now fix a discrete group G. Let $\mathcal{E}(G)$ be the category whose objects are the elements of G, and with exactly one morphism between each pair of objects. Let $\mathcal{B}(G)$ be the category with one object, and one morphism for each element of G. (Note that $|\mathcal{E}(G)| = EG$ and $|\mathcal{B}(G)| = BG$; hence the notation.) When necessary to be precise, g_a will denote the morphism $a \to ga$ in $\mathcal{E}(G)$. We let G act on $\mathcal{E}(G)$ via right multiplication: $x \in G$ acts on objects by sending a to ax and on morphisms by sending g_a to g_{ax}. Thus, for any $H \subseteq G$, the orbit category $\mathcal{E}(G)/H$ is the groupoid whose objects are the cosets in G/H, and with one morphism $g_{aH} : aH \to gaH$ for each $g \in G$: a category which is equivalent to $\mathcal{B}(H)$. Note in particular that $\mathcal{B}(G) \cong \mathcal{E}(G)/G$.

In order to deal simultaneously with real and complex K-theory, we let F denote one of the fields $\mathbb{C}$ or $\mathbb{R}$. Set $F^\infty = \bigcup_{n=1}^\infty F^n$: the space of all infinite sequences in F with finitely many nonzero terms. Let $F\text{-}\mathsf{mod}$ be the category whose objects are the finite dimensional vector subspaces of F^∞, and whose morphisms are F-linear *isomorphisms*. The set of objects of $F\text{-}\mathsf{mod}$ is given the discrete topology, and the space of morphisms between any two objects has the usual topology.

For any finite set S, an S-*partitioned* vector space is an object V of $F\text{-}\mathtt{mod}$, together with a direct sum decomposition $V=\bigoplus_{s\in S}V_s$. Let $F\langle S\rangle\text{-}\mathtt{mod}$ denote the category of S-partitioned vector spaces in $F\text{-}\mathtt{mod}$, where morphisms are isomorphisms which respect the decomposition. In particular, $F\langle\mathbf{0}\rangle\text{-}\mathtt{mod}$ has just one object $0\subseteq F^\infty$ and one morphism. A morphism $\theta:S\to T$ induces a functor $F\langle\theta\rangle$ from $F\langle T\rangle\text{-}\mathtt{mod}$ to $F\langle S\rangle\text{-}\mathtt{mod}$, by sending $V=\bigoplus_{t\in T}V_t$ to $W=\bigoplus_{s\in S}W_s$ where $W_s=\bigoplus_{t\in\theta(s)}V_t$.

Let $\underline{\mathrm{Vec}}_G^F$ be the Γ-space defined by setting

$$\underline{\mathrm{Vec}}_G^F(S)\overset{\text{def}}{=}\bigl|\mathrm{func}(\mathcal{E}(G),F\langle S\rangle\text{-}\mathtt{mod})\bigr|$$

for each finite set S. Here, $\mathrm{func}(\mathcal{C},\mathcal{D})$ denotes the category of functors from $\mathcal{C}$ to $\mathcal{D}$. We give this the G-action induced by the action on $\mathcal{E}(G)$ described above. This is made into a functor on Γ via composition with the functors $F\langle\theta\rangle$.

By definition, $\underline{\mathrm{Vec}}_G^F(\mathbf{0})$ is a point. To see that $\underline{\mathrm{Vec}}_G^F$ is an equivariant Γ-space, it remains to show for each n and H that the map $\bigl(\underline{\mathrm{Vec}}_G^F(\mathbf{n})\bigr)^H\to\prod_{i=1}^n\bigl(\underline{\mathrm{Vec}}_G^F(\mathbf{1})\bigr)^H$ is a homotopy equivalence. The target is the nerve of the category of functors from $\mathcal{E}(G)/H$ to n-tuples of objects in $F\text{-}\mathtt{mod}$, while the source can be thought of as the nerve of the full subcategory of functors from $\mathcal{E}(G)/H$ to n-tuples of vector subspaces which are independant in F^∞. And these two categories are equivalent, since every object in the larger one is isomorphic to an object in the smaller (and the set of objects is discrete).

For all finite $H\subseteq G$, $\bigl(\mathrm{Vec}_G^F\bigr)^H$ is the disjoint union, taken over isomorphism classes of finite dimensional H-representations, of the classifying spaces of their automorphism groups. We will see later that Vec_G^F classifies G-vector bundles over proper G-complexes. So it is natural to define equivariant K-theory using the its group completion $KF_G\overset{\text{def}}{=}\Omega B\mathrm{Vec}_G^F$, regarded as a pointed G-space.

In the following definition, $[-,-]_G$ and $[-,-]_G^{\boldsymbol{\cdot}}$ denote sets of homotopy classes of G-maps, and of pointed G-maps, respectively.

Definition 1.2. *For each proper G-complex X, set*

$$K_G(X)=[X,K\mathbb{C}_G]_G\quad\textit{and}\quad KO_G(X)=[X,K\mathbb{R}_G]_G.$$

For each proper G-CW-pair (X,A) and each $n\ge 0$, set

$$K_G^{-n}(X,A)=[\Sigma^n(X/A),K\mathbb{C}_G]_G^{\boldsymbol{\cdot}}\quad\textit{and}\quad KO_G^{-n}(X,A)=[\Sigma^n(X/A),K\mathbb{R}_G]_G^{\boldsymbol{\cdot}}.$$

The usual cohomological properties of the $KF_G^{-n}(-)$ follow directly from the definition. Homotopy invariance and excision are immediate; and the exact sequence of a pair and the Mayer-Vietoris sequence of a pushout square are shown using Puppe sequences to hold in degrees ≤ 0. Note in particular the relations

$$\begin{aligned}KF_G^{-n}(X)&\cong\mathrm{Ker}\bigl[KF_G(S^n\times X)\longrightarrow KF_G(X)\bigr]\\KF_G^{-n}(X,A)&\cong\mathrm{Ker}\bigl[KF_G^{-n}(X\cup_A X)\longrightarrow KF_G^{-n}(X)\bigr],\end{aligned}\tag{1.3}$$

for any proper G-CW-pair (X,A) and any $n\ge 0$.

The following lemma will be needed in the next section. It is a special case of the fact that $\underline{\mathrm{Vec}}_G^F$ and KF_G (at least up to homotopy) are independent of our choice of category of F-vector spaces.

Lemma 1.4. *For any monomorphism $\alpha:F^\infty\to F^\infty$, the induced map $\alpha_*:\underline{\mathrm{Vec}}_G^F\to\underline{\mathrm{Vec}}_G^F$, defined by composition with $F\text{-}\mathtt{mod}\xrightarrow{\alpha(-)}F\text{-}\mathtt{mod}$, is G-homotopic to the identity. In particular, α_* induces the identity on $K_G(X)$.*

Proof. The functor $\bigl(V\mapsto\alpha(V)\bigr)$ is naturally isomorphic to the identity. □

In [8], we defined $\mathbb{K}_G(X)$, for any proper G-complex X, to be the Grothendieck group of the monoid of vector bundles over X. We next construct natural homomorphisms $\mathbb{K}_G(X)\to K_G(X)$, for all proper

G-complexes X, which are isomorphisms if X/G is a finite complex (this is the situation where the $\mathbb{K}_G^*(X)$ form an equivariant cohomology theory).

For each $n \geq 0$, let $F^n\text{-}\mathtt{mod} \subseteq F\text{-}\mathtt{mod}$ be the full subcategory of n-dimensional vector subspaces in F^∞. Let $F^n\text{-}\mathtt{frame}$ denote the category whose objects are the pairs (V, b), where V is an object of $F^n\text{-}\mathtt{mod}$ and b is an ordered basis of V; and whose morphisms are the isomorphisms which send ordered basis to ordered basis. The set of objects is given the topology of a disjoint union of copies of $GL_n(F)$ (one for each V in $F^n\text{-}\mathtt{mod}$). Note that there is a unique morphism between any pair of objects in $F^n\text{-}\mathtt{frame}$. Set

$$\mathrm{Vec}_G^{F,n} = \bigl|\mathrm{func}(\mathcal{E}(G), F^n\text{-}\mathtt{mod})\bigr| \qquad \text{and} \qquad \widetilde{\mathrm{Vec}_G^{F,n}} = \bigl|\mathrm{func}(\mathcal{E}(G), F^n\text{-}\mathtt{frame})\bigr|,$$

with the action of $G \times GL_n(F)$ on $\widetilde{\mathrm{Vec}_G^{F,n}}$ induced by the G-action on $\mathcal{E}(G)$ and the $GL_n(F)$-action on the set of ordered bases of each n-dimensional V. Let $\tau_n : \widetilde{\mathrm{Vec}_G^{F,n}} \to \mathrm{Vec}_G^{F,n}$ be the G-map induced by the forgetful functor $F^n\text{-}\mathtt{frame} \to F^n\text{-}\mathtt{mod}$. Then $GL_n(F)$ acts freely and properly on $\widetilde{\mathrm{Vec}_G^{F,n}}$. And τ_n induces a G-homeomorphism $\widetilde{\mathrm{Vec}_G^{F,n}}/GL_n(F) \cong \mathrm{Vec}_G^{F,n}$, since for any $\varphi : V \to V'$ in $F\text{-}\mathtt{mod}$, a lifting of V or V' to $F^n\text{-}\mathtt{frame}$ determines a unique lifting of the morphism.

Let $H \subseteq G \times GL_n(F)$ be any subgroup. If $H \cap (1 \times GL_n(F)) \neq 1$, then $(\widetilde{\mathrm{Vec}_G^{F,n}})^H = \emptyset$, since $GL_n(F)$ acts freely on $\widetilde{\mathrm{Vec}_G^{F,n}}$. So assume $H \cap (1 \times GL_n(F)) = 1$. Then H is the graph of some homomorphism $\varphi : H' \to GL_n(F)$ $(H' \subseteq G)$, and $(\widetilde{\mathrm{Vec}_G^{F,n}})^H$ is the nerve of the (nonempty) category of φ-equivariant functors $\mathcal{E}(G) \to F^n\text{-}\mathtt{frame}$, with a unique morphism between any pair of objects (since there is a unique morphism between any pair of objects in $F^n\text{-}\mathtt{frame}$). In particular, this shows that $(\widetilde{\mathrm{Vec}_G^{F,n}})^H$ is contractible.

Thus, $\widetilde{\mathrm{Vec}_G^{F,n}}$ is a universal space for those $(G \times GL_n(F))$-complexes upon which $GL_n(F)$ acts freely (cf. [8, §2]). The frame bundle of any n-dimensional G-F-vector bundle over a G-complex X is such a complex, and hence n-dimensional G-F-vector bundles over X are classified by maps to $\mathrm{Vec}_G^{F,n} = \widetilde{\mathrm{Vec}_G^{F,n}}/GL_n(F)$. It follows that

$$\mathrm{EVec}_G^{F,n} = \widetilde{\mathrm{Vec}_G^{F,n}} \times_{GL_n(F)} F^n \longrightarrow \mathrm{Vec}_G^{F,n}$$

is a universal n-dimensional G-F-vector bundle. And $[X, \mathrm{Vec}_G^{F,n}]_G \cong \mathbb{V}\mathrm{ect}_G^{F,n}(X)$: the set of isomorphism classes of n-dimensional G-F-vector bundles over X.

If E is any G-F-vector bundle over X, we let $[\![E]\!] \in KF_G(X) = [X, KF_G]_G$ be the composite of the classifying map $X \to \mathrm{Vec}_G^F$ for E with the group completion map $\mathrm{Vec}_G^F \to \Omega B\mathrm{Vec}_G^F = KF_G$. Any pair E, E' of vector bundles over X is induced by a G-map

$$X \longrightarrow \mathrm{Vec}_G^F \times \mathrm{Vec}_G^F = |\,\mathrm{func}(\mathcal{E}(G), F\text{-}\mathtt{mod} \times F\text{-}\mathtt{mod})| \simeq |\,\mathrm{func}(\mathcal{E}(G), F\langle\mathbf{2}\rangle\text{-}\mathtt{mod})|;$$

and upon composing with the forgetful functor $F\langle\mathbf{2}\rangle\text{-}\mathtt{mod} \to F\text{-}\mathtt{mod}$ we get the classifying map for $E \oplus E'$. The direct sum operation on $\mathbb{V}\mathrm{ect}_G^F(X)$ is thus induced by the H-space structure on Vec_G^F, and $[\![E \oplus E']\!] = [\![E]\!] + [\![E']\!]$ for all E, E'.

Proposition 1.5. *The assignment $\bigl([E] \mapsto [\![E]\!]\bigr)$ defines a homomorphism*

$$\gamma_X : \mathbb{K}F_G(X) \longrightarrow KF_G(X),$$

for any proper G-complex X. This extends to natural homomorphisms $\gamma_{X,A}^{-n} : \mathbb{K}F_G^{-n}(X, A) \to KF_G^{-n}(X, A)$, for all proper G-CW-pairs (X, A) and all $n \geq 0$; which are isomorphisms when restricted to the category of finite proper G-CW-pairs.

Proof. By the above remarks, $\bigl([E] \mapsto [\![E]\!]\bigr)$ defines a homomorphism of monoids from $\mathbb{V}\mathrm{ect}_G^F(X)$ to $KF_G(X)$, and hence a homomorphism of groups

$$\gamma_X : \mathbb{K}F_G(X) \longrightarrow KF_G(X).$$

Homomorphisms $\gamma_{X,A}^{-n}$ (for all proper G-CW-pairs (X, A)) are then constructed via the definitions

$$\mathbb{K}F_G^{-n}(X) \overset{\text{def}}{=} \mathrm{Ker}[\mathbb{K}F_G(S^n \times X) \to \mathbb{K}F_G(X)]$$

and $\mathbb{K}F_G^{-n}(X,A) \overset{\text{def}}{=} \operatorname{Ker}[\mathbb{K}F_G^{-n}(X\cup_A X) \to \mathbb{K}F_G^{-n}(X)]$ used in [8], together with the analogous relations (1.3) for $K_G^*(-)$. These homomorphisms clearly commute with boundary maps.

It remains to check that γ_X^{-n} is an isomorphism whenever X is a finite proper G-complex. Since $\mathbb{K}F_G(-)$ and $KF_G(-)$ are both cohomology theories in this situation, it suffices, using the Mayer-Vietoris sequences for pushout squares

$$\begin{tikzcd} G/H \times S^{m-1} \arrow[r] \arrow[d, "\varphi"] & G/H \times D^m \arrow[d] \\ X \arrow[r] & (G/H \times D^m) \cup_\varphi X, \end{tikzcd}$$

to do this when $X = G/H \times S^m$ for finite $H \subseteq G$ and any $m \geq 0$. Using (1.3) again, it suffices to show that $\gamma_X = \gamma_X^0$ is an isomorphism whenever $X = G/H \times Y$ for any finite complex Y with trivial G-action. By definition,

$$KF_G(G/H \times Y) = \bigl[G/H \times Y, KF_G\bigr]_G \cong [Y, (KF_G)^H];$$

while $\mathbb{K}F_G(G/H \times Y)$ is the Grothendieck group of the monoid

$$\mathbb{V}\mathrm{ect}_G^F(G/H \times Y) \cong \bigl[G/H \times Y, \mathrm{Vec}_G^F\bigr]_G \cong \bigl[Y, (\mathrm{Vec}_G^F)^H\bigr].$$

Since $\pi_0((\mathrm{Vec}_G^F)^H)$ is a free abelian monoid (the monoid of isomorphism classes of H-representations), [10, Proposition 4.1] applies to show that $[-, (KF_G)^H]$ is universal among representable functors from compact spaces to abelian groups which extend $\mathbb{V}\mathrm{ect}_G^F(G/H\times-) \cong \mathbb{V}\mathrm{ect}_H^F(-)$. And since $\mathbb{K}_H$ is representable as a functor on compact spaces with trivial action (H is finite), it is the universal functor, and so $[Y, (KF_G)^H] \cong \mathbb{K}_H(Y) \cong \mathbb{K}_G(G/H \times Y)$. □

2. Products and Bott periodicity

We now want to construct Bott periodicity isomorphisms, and define the multiplicative structures on $K_G^*(X)$ and $KO_G^*(X)$. Both of these require defining pairings of classifying spaces; thus pairings of Γ-spaces. A general procedure for doing this was described by Segal [10, §5], but a simpler construction is possible in our situation.

Fix an isomorphism $\mu : F^\infty \otimes F^\infty \to F^\infty$ ($F = \mathbb{C}$ or $\mathbb{R}$), induced by some bijection between the canonical bases. This induces a functor

$$\mu_* : F\langle S\rangle\text{-}\mathsf{mod} \times F\langle T\rangle\text{-}\mathsf{mod} \longrightarrow F\langle S{\times}T\rangle\text{-}\mathsf{mod},$$

and hence (for any discrete groups H and G)

$$\mu_* : \underline{\mathrm{Vec}}_H^F(S) \wedge \underline{\mathrm{Vec}}_G^F(T) \longrightarrow \underline{\mathrm{Vec}}_{H\times G}^F(S{\times}T). \tag{2.1}$$

This is an $(H{\times}G)$-equivariant map of bi-Γ-spaces, and after taking their nerves (and loop spaces) we get maps

$$\underset{= KF_H \wedge KF_G}{\Omega B\mathrm{Vec}_H^F \wedge \Omega B\mathrm{Vec}_G^F} \longrightarrow \Omega^2\bigl(B\mathrm{Vec}_H^F \wedge B\mathrm{Vec}_G^F\bigr) \xrightarrow{\Omega^2|\mu_*|} \Omega^2 B^2\mathrm{Vec}_{H\times G}^F \simeq \underset{= KF_{H\times G}}{\Omega B\mathrm{Vec}_{H\times G}^F}. \tag{2.2}$$

By Lemma 1.4, these maps are all independent (up to homotopy) of the choice of $\mu : F^\infty \otimes F^\infty \to F^\infty$.

Lemma 2.3. *For any discrete groups H and G, any H-space X, and any G-space Y, the following square commutes:*

$$\begin{tikzcd} \mathbb{K}F_H(X) \otimes \mathbb{K}F_G(Y) \arrow[r, "\gamma_X \times \gamma_Y"] \arrow[d, "\otimes"] & KF_H(X) \otimes KF_G(Y) \arrow[d, "\mu_*"] \\ \mathbb{K}F_{H\times G}(X \times Y) \arrow[r, "\gamma_{X\times Y}"] & KF_{H\times G}(X \times Y) \end{tikzcd}$$

where μ_* is the homomorphism induced by (2.2).

Proof. The pullback of the universal bundle $\mathrm{EVec}^F_{H\times G}$, via the pairing $\mathrm{Vec}^F_H \wedge \mathrm{Vec}^F_G \to \mathrm{Vec}^F_{H\times G}$ of (2.1), is isomorphic to the tensor product of the universal bundles EVec^F_H and EVec^F_G. This is clear if we identify $\mathrm{EVec}^F_G \cong \left|\mathrm{func}(\mathcal{E}(G), F\text{-}\mathtt{Bdl})\right|$ (and similarly for the other two bundles), where F-$\mathtt{Bdl}$ is the category of pairs (V,x) for V in F-$\mathtt{mod}$ and $x\in V$. $\square$

We now consider case where $H=1$, and hence where $KF_H = \mathbb{Z}\times BU$ or $\mathbb{Z}\times BO$. The product map (2.2), after composition with the Bott elements in $\pi_2(BU)$ or $\pi_8(BO)$, induces Bott maps

$$\beta^{\mathbb{C}}_* : \Sigma^2 K_G \longrightarrow K_G \qquad \text{and} \qquad \beta^{\mathbb{R}}_* : \Sigma^8 KO_G \longrightarrow KO_G. \tag{2.4}$$

Proposition 2.5. *For any proper CW-pair (X,A), the Bott homomorphisms*

$$b^{\mathbb{C}}_{X,A} : K^{-n}_G(X,A) \longrightarrow K^{-n-2}_G(X,A) \quad \text{and} \quad b^{\mathbb{R}}_{X,A} : KO^{-n}_G(X,A) \longrightarrow KO^{-n-8}_G(X,A)$$

are isomorphisms; and commute with the homomorphisms

$$\gamma^{-n}_{X,A} : \mathbb{K}F^{-n}_G(X,A) \to KF^{-n}_G(X,A).$$

Proof. The last statement follows immediately from Lemma 2.3.

By Lemma 1.1, it suffices to prove that the adjoint maps

$$K_G \longrightarrow \Omega^2 K_G \qquad \text{and} \qquad KO_G \longrightarrow \Omega^8 KO_G$$

to the pairings in (2.4) are weak homotopy equivalences after restricting to fixed point sets of finite subgroups of G. In other words, it suffices to prove that $b^{\mathbb{C}}_X : K_G(X) \to K^{-2}_G(X)$ and $b^{\mathbb{R}}_X : KO_G(X) \to KO^{-8}_G(X)$ are isomorphisms when $X = G/H \times S^n$ for any $n\geq 0$ and any finite $H\subseteq G$. And this follows since the Bott maps for $\mathbb{K}_G$ and $\mathbb{KO}_G$ are isomorphisms [8, Theorems 3.12 & 3.15], since $\mathbb{K}F^{-n}_G(X) \cong KF^{-n}_G(X)$ (Proposition 1.5), and since these isomorphisms commute with the Bott maps. $\square$

The $K^{-n}_G(X)$ and $KO^{-n}_G(X)$ can now be extended to (additive) equivariant cohomology theories in the usual way. But before stating this explicitly, we first consider the ring structure on $K_G(X)$. This is defined to be the composite

$$[X, KF_G]_G \times [X, KF_G]_G \longrightarrow [X, KF_{G\times G}]_G \xrightarrow{\Delta^*} [X, KF_G]_G,$$

where the first map is induced by the pairing in (2.2), and the second by restriction to the diagonal subcategory $\mathcal{E}(G) \subseteq \mathcal{E}(G\times G)$.

Before we can prove the ring properties of this multiplication, we must look more closely at the homotopy equivalence $\Omega B\mathrm{Vec}^F_G \xrightarrow{\simeq} \Omega^2 B^2 \mathrm{Vec}^F_G$ which appears in the definition of the product. In fact, there is more than one natural map from $\Omega^n B^n \mathrm{Vec}^F_G$ to $\Omega^{n+1} B^{n+1}\mathrm{Vec}^F_G$. For each $n\geq 0$ and each $k=0,\dots,n$, let $\iota^k_n : \Omega^n B^n \mathrm{Vec}^F_G \to \Omega^{n+1}B^{n+1}\mathrm{Vec}^F_G$ denote the map induced as $\Omega^n(f)$, where f is adjoint to the map $\Sigma B^n \mathrm{Vec}^F_G \to B^{n+1}\mathrm{Vec}^F_G$, induced by identifying $B^n\mathrm{Vec}^F_G(S_1,\dots,S_n)$ with $B^{n+1}\mathrm{Vec}^F_G(\dots,S_{k-1},\mathbf{1},S_k,\dots)$.

By a weak G-equivalence $f : X\to Y$ is meant a map of G-spaces which restricts to a weak equivalence $f^H : X^H \to Y^H$ for all $H\subseteq G$; i.e., a weak $\mathcal{F}$-equivalence in the notation of Lemma 1.1 when $\mathcal{F}$ is the family of all subgroups of G. Since we are interested equivariant Γ-spaces only as targets of maps from G-complexes, it suffices by Lemma 1.1 to work in a category where weak G-equivalences are inverted.

Lemma 2.6. *Let $\underline{A}$ be any G-equivariant Γ-space. Then for any $n\geq 1$, the maps $\iota^k_n : \Omega^n B^n A \to \Omega^{n+1}B^{n+1}A$ (for $0\leq k\leq n$) are all equal in the homotopy category of G-spaces where weak G-equivalences are inverted.*

Proof. For any $\sigma\in\Sigma_n$, let $\sigma_* : \Omega^n B^n A \to \Omega^n B^n A$ be the map induced by permuting the coordinates of $B^n A$ as an n-simplicial set, and by switching the order of looping. Then any two of the ι^k_{n-1} differ by composition with some appropriate σ_*, and so it suffices to show that the σ_* are all homotopic to the identity.

Consider the following commutative diagram

$$\begin{tikzcd}
\Omega BA \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}"'] & \Omega^{n+1}B^{n+1}A \arrow[d, "(1\times\sigma)_*"'] & \Omega^n B^n A \arrow[l, "\iota_n^n"'] \arrow[d, "\sigma_*"'] \\
\Omega BA \arrow[r, "\varphi"] & \Omega^{n+1}B^{n+1}A & \Omega^n B^n A, \arrow[l, "\iota_n^n"']
\end{tikzcd}$$

for any $\sigma \in \Sigma_n \subseteq \Sigma_{n+1}$, where $\varphi = \iota_n^0 \circ \cdots \circ \iota_1^0$ is induced by identifying $\underline{A}(S)$ with $\underline{A}(S, \mathbf{1}, \ldots, \mathbf{1})$. The diagram commutes, and all maps in it are weak G-equivalences by [10, Proposition 1.4]. So $(1 \times \sigma)_*$ and σ_* are both homotopic to the identity after inverting weak G-equivalences. $\square$

We are now ready to show:

Theorem 2.7. *For any discrete group and any proper G-complex X, the pairings μ_X define a structure of graded ring on $K_G^*(X)$ and on $KO_G^*(X)$, which make $K_G^*(-)$ and $KO_G^*(-)$ into multiplicative cohomology theories. The Bott isomorphisms*

$$b_X^{\mathbb{C}} : K_G^{-n}(X) \to K_G^{-n-2}(X) \qquad \text{and} \qquad b_X^{\mathbb{R}} : KO_G^{-n}(X) \to KO_G^{-n-8}(X)$$

are $K_G(X)$- or $KO_G(X)$-linear. And the canonical homomorphisms

$$\gamma_X^{\mathbb{C}} : \mathbb{K}_G^*(X) \to K_G^*(X) \qquad \text{and} \qquad \gamma_X^{\mathbb{R}} : \mathbb{KO}_G^*(X) \to KO_G^*(X)$$

are homomorphisms of rings.

Proof. As usual, set $F = \mathbb{C}$ or $\mathbb{R}$. We first check that μ_X makes $KF_G(X)$ into a commutative ring — i.e., that it is associative and commutative and has a unit.

To see that there is a unit, let $[F^1] \in \mathrm{Vec}_G^F$ denote the vertex for the constant functor $\mathcal{E}(G) \mapsto F^1 \in F\langle\mathbf{1}\rangle\text{-}\mathtt{mod}$, and set $[F^1]_\Omega = \iota_0^0([F^1]) \in \Omega B\mathrm{Vec}_G^F$. The following diagram commutes:

$$\begin{tikzcd}
\Omega B\mathrm{Vec}_G^F \arrow[r, "{[F^1]\wedge -}"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Vec}_G^F \wedge \Omega B\mathrm{Vec}_G^F \arrow[r, "\mu_*"] \arrow[d, "\iota_0^0\wedge\mathrm{Id}"', "\simeq"] & \Omega B\mathrm{Vec}_G^F \arrow[d, "\iota_1^0"', "\simeq"] \\
\Omega B\mathrm{Vec}_G^F \arrow[r, "{[F^1]_\Omega\wedge -}"] & \Omega B\mathrm{Vec}_G^F \wedge \Omega B\mathrm{Vec}_G^F \arrow[r, "\mu_*"] & \Omega^2 B^2\mathrm{Vec}_G^F;
\end{tikzcd}$$

and the composite in the top row is homotopic to the identity by Lemma 1.4. So the element $1 \in KF_G(X)$, represented by the constant map $X \mapsto [F^1]_\Omega \in KF_G$, is an identity for multiplication in $KF_G(X)$.

The commutativity of $KF_G(X)$ follows from Lemma 2.6 (the uniqueness of the map $\Omega BA \to \Omega^2 B^2 A$ after inverting weak G-equivalences); together with the fact that the pairing

$$\mu_* : B\mathrm{Vec}_G^F \wedge B\mathrm{Vec}_G^F \longrightarrow B^2\mathrm{Vec}_G^F$$

commutes up to homotopy using Lemma 1.4. And associativity follows since the triple products are induced by maps

$$\big(\Omega B\mathrm{Vec}_G^F\big)^{\wedge 3} \longrightarrow \Omega^3\big((B\mathrm{Vec}_G^F)^{\wedge 3}\big) \underset{\Omega^3|\mu_*\circ(\mathrm{Id}\wedge\mu_*)|}{\overset{\Omega^3|\mu_*\circ(\mu_*\wedge\mathrm{Id})|}{\rightrightarrows}} \Omega^3 B^3\mathrm{Vec}_G^F \xleftarrow{\simeq} \Omega B\mathrm{Vec}_G^F;$$

where the two maps in the middle are homotopic by Lemma 1.4, and the last could be any of the three possible maps by Lemma 2.6.

The extension of the product to negative gradings is straightforward, via the identifications of (1.3). For any $n, m \geq 0$, the composite

$$\begin{aligned} KF_G(S^n \times X) \otimes KF_G(S^m \times X) &\xrightarrow{\mathrm{proj}^*} KF_G(S^n \times S^m \times X) \otimes KF_G(S^n \times S^m \times X) \\ &\xrightarrow{\mu} KF_G(S^n \times S^m \times X) \end{aligned}$$

restricts to a product map $KF_G^{-n}(X) \otimes KF_G^{-m}(X) \to KF_G^{-n-m}(X)$. To see that the product has image in $KF_G^{-n-m}(X)$, just note that

$$\begin{aligned} KF_G^{-n-m}(X) &\cong \operatorname{Ker}\bigl[KF_G(S^{n+m} \times X) \longrightarrow KF_G(X)\bigr] \\ &= \operatorname{Ker}\bigl[KF_G(S^n \times S^m \times X) \longrightarrow KF_G(S^n \times X) \oplus KF_G(S^m \times X)\bigr]. \end{aligned}$$

This product is clearly associative, and graded commutative (where the change in sign comes from the degree of the switching map $S^{n+m} \to S^{m+n}$).

We next check that this product commutes with the Bott maps in the obvious way, so that it can be extended to $K_G^i(X)$ for all i. This means showing that the two maps

$$KF(S^n) \otimes KF_G(X) \otimes KF_G(X) \rightrightarrows KF_G(S^n \times X)$$

induced by the products constructed above are equal. And this follows from the same argument as that used to prove associativity of the internal product on $KF_G(X)$.

Finally, $\gamma : \mathbb{K}F_G^*(X) \to KF_G^*(X)$ is a ring homomorphism by Lemma 2.3. □

3. Induction, restriction, and inflation

In this section we explain how the natural maps defined on $\mathbb{K}_G(X)$ and $\mathbb{KO}_G(X)$ by induction and restriction carry over to $K_G(X)$ and $KO_G(X)$. Namely, we want to construct for any pair $H \subseteq G$ of discrete groups, any $F = \mathbb{C}$ or $\mathbb{R}$, any G-complex X, and any H-complex Y, natural induction and restriction maps

$$\operatorname{Ind}_H^G : KF_H^*(Y) \xrightarrow{\cong} KF_G^*(G \times_H Y) \qquad \text{and} \qquad \operatorname{Res}_H^G : KF_G^*(X) \longrightarrow KF_H^*(X|_H).$$

Furthermore, when $H \lhd G$ is a normal subgroup, we construct an inflation homomorphism

$$\operatorname{Infl}_{G/H}^G : KF_{G/H}^*(X/H) \longrightarrow KF_G^*(X),$$

which is an isomorphism whenever H acts freely on X. These maps correspond under the natural homomorphism $\mathbb{K}F_G^*(X) \to KF_G^*(X)$ to the obvious homomorphisms induced by induction, restriction, and pullback of vector bundles. They are all induced using the following maps between classifying spaces for equivariant K-theory.

Lemma 3.1. *Let $f : G' \to G$ be any homomorphism of discrete groups. Then composition with the induced functor $\mathcal{E}(f) : \mathcal{E}(G') \to \mathcal{E}(G)$ induces an G'-equivariant map $f^* : \underline{\mathrm{Vec}}_G^F \to \underline{\mathrm{Vec}}_{G'}^F$ of Γ-spaces, and hence a G'-equivariant map $f^* : KF_G \to KF_{G'}$ of classifying spaces. And for any subgroup $L \subseteq G'$ such that $L \cap \operatorname{Ker}(f) = 1$, f^* restricts to a homotopy equivalence $(KF_G)^{f(L)} \simeq (KF_{G'})^L$.*

Proof. This is immediate, except for the last statement. And if $L \subseteq G'$ is such that $L \cap \operatorname{Ker}(f) = 1$, then $L \cong f(L)$, the categories $\mathcal{E}(G')/L$ and $\mathcal{E}(G)/f(L)$ are both equivalent to the category $\mathcal{B}(L)$ with one object and endomorphism group L; and thus $(\underline{\mathrm{Vec}}_G^F)^{f(L)}(S) = |\operatorname{func}(\mathcal{E}(G)/f(L), F\langle S\rangle\text{-}\mathtt{mod})|$ is homotopy equivalent to $(\underline{\mathrm{Vec}}_{G'}^F)^L(S) = |\operatorname{func}(\mathcal{E}(G')/L, F\langle S\rangle\text{-}\mathtt{mod})|$ for each S in Γ. □

We first consider the restriction and induction homomorphisms.

Proposition 3.2. *Fix $F = \mathbb{C}$ or $\mathbb{R}$, and let $H \subseteq G$ be any pair of discrete groups. Let $i^* : KF_G \to KF_H$ be the map of Lemma 3.1.*

(a) *For any proper G-CW-pair (X, A), i^* induces a homomorphism of rings*

$$\operatorname{Res}_H^G : KF_G^*(X, A) \longrightarrow KF_H^*(X, A).$$

(b) *For any proper H-CW-pair (Y,B), i^* induces an isomorphism*

$$\mathrm{Ind}_H^G : KF_H^*(Y,B) \xrightarrow{\ \cong\ } KF_G^*(G\times_H Y, G\times_H B),$$

which is natural in (Y,B), and also natural with respect to inclusions of subgroups.

The restriction and induction maps both commute with the maps between $\mathbb{K}F_G(-)$ and $\mathbb{K}F_H(-)$ induced by induction and restriction of equivariant vector bundles.

Proof. It suffices to prove this when $A=\emptyset=B$ and $*=0$. The fact that $i^*: KF_G \to KF_H$ commutes with the Bott homomorphisms and the products follows directly from the definitions. So part (a) is clear.

The inverse of the homomorphism in (b) is defined to be the composite

$$[G\times_H Y, KF_G]_G \cong [Y, KF_G]_H \xrightarrow{\ i^*\circ -\ } [Y, KF_H]_H.$$

And since i^* restricts to a homotopy equivalence $(KF_G)^L \to (KF_H)^L$ for each finite $L\subseteq H$ (Lemma 3.1), this map is an isomorphism by Lemma 1.1.

The last statement is clear from the construction and the definition of $\gamma : \mathbb{K}F_G(-) \to KF_G(-)$. □

We next consider the inflation homomorphism.

Proposition 3.3. *Fix $F=\mathbb{C}$ or $\mathbb{R}$. Let G be any discrete group, and let $N \lhd G$ be a normal subgroup. Then for each proper G-CW-pair (X,A), there is an inflation map*

$$\mathrm{Infl}_{G/N}^G : KF_{G/N}^*(X/N, A/N) \longrightarrow KF_G^*(X,A),$$

which is natural in (X,A), which is a homomorphism of rings (if $A=\emptyset$), and which commutes with the homomorphism $\mathbb{K}F_{G/N}(X/N,A/N) \to \mathbb{K}F_G(X,A)$ induced considering G/N-vector bundles as G-vector bundles. And if N acts freely on X, then $\mathrm{Infl}_{G/N}^G$ is an isomorphism.

Proof. Let $f: G\to G/N$ denote the natural homomorphism, and let $f^*: KF_{G/N} \to KF_G$ be the induced map of Lemma 3.1. Define $\mathrm{Infl}_{G/N}^G$ to be the composite

$$[X/N, KF_{G/N}]_{G/N} \cong [X, KF_{G/N}]_G \xrightarrow{\ f^*\circ -\ } [X, KF_G]_G.$$

If N acts freely on X, then for each isotropy subgroup L of X, $L\cap N=1$, so $(f^*)^L : (KF_{G/N})^L \to (KF_G)^L$ is a homotopy equivalence by Lemma 3.1, and the inflation map is an isomorphism by Lemma 1.1. The other statements are clear. □

Another type of natural map will be needed when constructing the equivariant Chern character. Fix a discrete group G and a finite normal subgroup $N \lhd G$, and let $\mathrm{Irr}(N)$ be the set of isomorphism classes of irreducible *complex* N-representations. Let X be any proper G/N-complex. For any $V\in \mathrm{Irr}(N)$ and any G-vector bundle $E\to X$, let $\mathrm{Hom}_N(V,E)$ denote the vector bundle over X whose fiber over $x\in X$ is $\mathrm{Hom}_N(V,E_x)$ (each fiber of E is an N-representation). If $H\subseteq G$ is any subgroup which centralizes N, then we can regard $\mathrm{Hom}_N(V,E)$ as an H-vector bundle by setting $(hf)(x) = h{\cdot}f(x)$ for any $h\in H$ and any $f\in \mathrm{Hom}_N(V,E)$. We thus get a homomorphism

$$\Psi : \mathbb{K}_G(X) \longrightarrow \mathbb{K}_H(X)\otimes R(N),$$

where $\Psi([E]) = \sum_{V\in \mathrm{Irr}(N)} [\mathrm{Hom}_N(V,E)]\otimes[V]$. We need a similar homomorphism defined on $K_G^*(X)$.

Proposition 3.4. *Let G be a discrete group, let $N\lhd G$ be any finite normal subgroup, and let $H\subseteq G$ be any subgroup such that $[H,N]=1$. Then for any proper G/N-complex X, there is a homomorphism of rings*

$$\Psi = \Psi_{G;N,H}^X : K_G^*(X) \longrightarrow K_H^*(X)\otimes R(N),$$

which is natural in X and natural with respect to the degree-shifting maps $K_G^*(X) \to K_G^{*+n}(S^n\times X)$, and which has the following properties:

(a) *For any (complex) G-vector bundle $E \to X$,*

$$\Psi([\![E]\!]) = \sum_{V\in\mathrm{Irr}(N)} [\![\mathrm{Hom}_N(V,E)]\!] \otimes [V].$$

(b) *For any $G' \subseteq G$, $N' \subseteq N\cap G'$, and $H' \subseteq H\cap G'$, the following diagram commutes:*

$$\begin{tikzcd} K^*_G(X) \arrow[r, "\Psi^X_{G;N,H}"] \arrow[d, "\mathrm{Res}^G_{G'}"'] & K^*_H(X)\otimes R(N) \arrow[d, "\mathrm{Res}^H_{H'} \otimes \mathrm{Res}^N_{N'}"] \\ K^*_{G'}(X) \arrow[r, "\Psi^X_{G';N',H'}"] & K^*_{H'}(X)\otimes R(N'). \end{tikzcd}$$

Proof. Fix G, H, and N. For any irreducible N-representation V and any surjective homomorphism $p : \mathbb{C}[N] \twoheadrightarrow V$, composition with p defines a monomorphism

$$\mathrm{Hom}_N(V,W) \xrightarrow{-\circ p} \mathrm{Hom}_N(\mathbb{C}[N],W) = W$$

for any N-representation W; and thus allows us to identify $\mathrm{Hom}_N(V,W)$ as a subspace of W. In particular, there is a functor

$$p^* : \mathrm{func}(\mathsf{Or}(G)/N, \mathbb{C}\langle S\rangle\text{-}\mathsf{mod}) \longrightarrow \mathrm{func}(\mathsf{Or}(H), \mathbb{C}\langle S\rangle\text{-}\mathsf{mod})$$

which sends any α to the functor $h \mapsto \mathrm{Hom}_N(V,\alpha(hN)) \subseteq \alpha(hN)$. If $p' : \mathbb{C}[N] \twoheadrightarrow V'$ is another surjection of N-representations, where $V \cong V'$, then any isomorphism $V \xrightarrow{\cong} V'$ defines a natural isomorphism between p^* and $(p')^*$. We thus get a map of Γ-spaces

$$\psi_p : \underline{\mathrm{Vec}}^{\mathbb{C}}_G \longrightarrow \underline{\mathrm{Vec}}^{\mathbb{C}}_H$$

which is unique (independant of the projection p) up to H-equivariant homotopy. So this induces homomorphisms $\psi_V : K^{-n}_G(X) \to K^{-n}_H(X)$, for all proper G/N-complex X (and all $n \geq 0$), which depend only on V and not on p. The ψ_V clearly commute with the Bott maps, and thus extend to homomorphisms $\psi_V : K^*_G(X) \to K^*_H(X)$. So we can define Ψ by setting $\Psi(x) = \sum_{V\in\mathrm{Irr}(N)} \psi_V(x) \otimes [V]$. Point (a) is immediate; as is naturality in X and naturality for restriction to $G' \subseteq G$ or $H' \subseteq H$. Naturality with respect to the degree-shifting maps holds by construction.

We next show that Ψ is natural in N; i.e., that point (b) holds when $G' = G$ and $H' = H$. Let ψ_V be the homomorphisms defined above, for each irreducible N-representation V; and let $\psi'_W : K^*_G(X) \to K^*_H(X)$ be the corresponding homomorphism for each irreducible N'-representation W. For each $V \in \mathrm{Irr}(N)$ and each $W \in \mathrm{Irr}(N')$, set

$$n^V_W = \dim_{\mathbb{C}}\big(\mathrm{Hom}_{N'}(W,V)\big) = \dim_{\mathbb{C}}\big(\mathrm{Hom}_N(\mathrm{Ind}^N_{N'}(W),V)\big).$$

Thus, n^V_W is the multiplicity of W in the decomposition of $V|_{N'}$, as well as the multiplicity of V in the decomposition of $\mathrm{Ind}^N_{N'}(W)$. So for any $x \in K^*_G(X)$,

$$(\mathrm{Id}\otimes\mathrm{Res}^N_{N'})(\Psi_{G;N,H}(x)) = \sum_{V\in\mathrm{Irr}(N)} \psi_V(x)\otimes[V|_{N'}] = \sum_{W\in\mathrm{Irr}(N')}\Bigg(\sum_{V\in\mathrm{Irr}(N)} n^V_W\cdot\psi_V(x)\Bigg)\otimes[W];$$

and we will be done upon showing that $\psi'_W = \sum_V n^V_W\cdot\psi_V$ for each $W \in \mathrm{Irr}(N')$. Fix a surjection $p_0 : \mathbb{C}[N'] \twoheadrightarrow W$, and a decomposition $\mathrm{Ind}^N_{N'}(W) = \sum_{i=1}^k V_i$ (where the V_i are irreducible and $k = \sum_V n^V_W$). For each $1 \leq i \leq k$, let $p_i : \mathbb{C}[N] \twoheadrightarrow V_i$ be the composite of $\mathrm{Ind}^N_{N'}(p_0)$ followed by projection to V_i. Then

$$\psi_{p_0} = \bigoplus_{i=1}^k \psi_{p_i} : \big(\underline{\mathrm{Vec}}^{\mathbb{C}}_G\big)^N \longrightarrow \underline{\mathrm{Vec}}^{\mathbb{C}}_H$$

as maps of Γ-spaces, and so $\psi_W \simeq \sum_{i=1}^k \psi_{V_i}$ as maps $K^*_G(X) \to K^*_H(X)$.

It remains to show that Ψ is a homomorphism of rings. Since it is natural in N, and since $R(N)$ is detected by characters, it suffices to prove this when N is cyclic. For any $x, y \in K_G(X)$,

$$\Psi(x)\cdot\Psi(y) = \sum_{V,W\in\mathrm{Irr}(N)} \big(\psi_V(x)\cdot\psi_W(y)\big) \otimes [V\otimes W] \quad \text{and} \quad \Psi(xy) = \sum_{U\in\mathrm{Irr}(N)} \psi_U(xy)\otimes[U].$$

And thus $\Psi(x)\cdot\Psi(y) = \Psi(xy)$ since

$$\psi_U\circ\mu_* = \bigoplus_{\substack{V,W\in\mathrm{Irr}(G)\\ V\otimes W\cong U}} \mu_*\circ(\psi_V\wedge\psi_W) : \big(\underline{\mathrm{Vec}}_G^{\mathbb{C}}\big)^N \wedge \big(\underline{\mathrm{Vec}}_G^{\mathbb{C}}\big)^N \longrightarrow \underline{\mathrm{Vec}}_H^{\mathbb{C}},$$

as maps of Γ-spaces, for each $U\in\mathrm{Irr}(N)$. □

4. Characters and class functions

Throughout this section, G will be a *finite* group. We prove here some results showing that certain class functions are characters; results which will be needed in the next two sections.

For any field K of characteristic zero, a *K-character* of G means a class function $G \to K$ which is the character of some (virtual) K-representation of G. Two elements $g, h\in G$ are called *K-conjugate* if g is conjugate to h^a for some a prime to $n = |g| = |h|$ such that $(\zeta\mapsto\zeta^a)\in \mathrm{Gal}(K\zeta/K)$, where $\zeta = \exp(2\pi i/n)$. For example, g and h are $\mathbb{Q}$-conjugate if $\langle g\rangle$ and $\langle h\rangle$ are conjugate as subgroups, and are $\mathbb{R}$-conjugate if g is conjugate to h or h^{-1}.

Proposition 4.1. *Fix a finite extension K of $\mathbb{Q}$, and let $A\subseteq K$ be its ring of integers. Let $f : G\to A$ be any function which is constant on K-conjugacy classes. Then $|G|\cdot f$ is an A-linear combination of K-characters of G.*

Proof. Set $n = |G|$, for short. Let $V_1,\ldots,V_k$ be the distinct irreducible $K[G]$-representations, let χ_i be the character of V_i, set $D_i = \mathrm{End}_{K[G]}(V_i)$ (a division algebra over K), and set $d_i = \dim_K(D_i)$. Then by [11, Theorem 25, Cor. 2],

$$|G|\cdot f = \sum_{i=1}^k r_i\chi_i \qquad \text{where} \qquad r_i = \frac{1}{d_i}\sum_{g\in G} f(g)\chi_i(g^{-1});$$

and we must show that $r_i\in A$ for all i. This means showing, for each $i = 1,\ldots,k$, and each $g\in G$ with K-conjugacy class $\mathrm{conj}_K(g)$, that $|\mathrm{conj}_K(g)|\cdot\chi_i(g)\in d_iA$.

Fix i and g; and set $C = \langle g\rangle$, $m = |g| = |C|$, and $\zeta = \exp(2\pi i/m)$. Then $\mathrm{Gal}(K(\zeta)/K)$ acts freely on the set $\mathrm{conj}_K(g)$: the element $(\zeta\mapsto\zeta^a)$ acts by sending h to h^a. So $[K(\zeta){:}K]\big||\mathrm{conj}_K(g)|$.

Let $V_i|_C = W_1^{a_1}\oplus\cdots\oplus W_t^{a_t}$ be the decomposition as a sum of irreducible $K[C]$-modules. For each j, $K_j \overset{\text{def}}{=} \mathrm{End}_{K[C]}(W_j)$ is the field generated by K and the r-th roots of unity for some $r|m$ ($m = |C|$), and $\dim_{K_j}(W_j) = 1$. So $\dim_K(W_j)\big|[K(\zeta){:}K]$. Also, $d_i\big|\dim_K(W_j^{a_j})$, since $W_j^{a_j}$ is a D_i-module; and thus $d_i\big|a_j\cdot|\mathrm{conj}_K(g)|$. So if we set $\xi_j = \chi_{W_j}(g)\in A$, then

$$|\mathrm{conj}_K(g)|\cdot\chi_i(g) = |\mathrm{conj}_K(g)|\cdot\sum_{j=1}^t a_j\xi_j \in d_iA,$$

and this finishes the proof. □

For each prime p and each element $g\in G$, there are unique elements g_r of order prime to p and g_u of p-power order, such that $g = g_rg_u = g_ug_r$. As in [11, §10.1], we refer to g_r as the p'-component of g. We say that a class function $f : G\to\mathbb{C}$ is *p-constant* if $f(g) = f(g_r)$ for each $g\in G$. Equivalently, f is p-constant if and only if $f(g) = f(g')$ for all $g, g'\in G$ such that $[g, g'] = 1$ and $g^{-1}g'$ has p-power order.

Lemma 4.2. *Fix a finite group G, a prime p, and a field K of characteristic zero. Then a p-constant class function $\varphi : G \to K$ is a K-character of G if and only if $\varphi|_H$ is a K-character of H for all subgroups $H \subseteq G$ of order prime to p.*

Proof. Recall first that G is called K-elementary if for some prime q, $G = C_m \rtimes Q$, where C_m is cyclic of order m, $q\nmid m$, Q is a q-group, and the conjugation action of Q on $K[C_m]$ leaves invariant each of its field components. By [11, §12.6, Prop. 36], a K-valued class function of G is a K-character if and only if its restriction to any K-elementary subgroup of G is a K-character. Thus, it suffices to prove the lemma when G is K-elementary.

Assume first that G is q-K-elementary for some prime $q \neq p$. Fix a subgroup $H \subseteq G$ of p-power index and order prime to p, and let $\alpha : G \twoheadrightarrow H$ be the surjection with $\alpha|_H = \mathrm{Id}$. Set $p^a = |\operatorname{Ker}(\alpha)|$. Then

$$\operatorname{Aut}(\operatorname{Ker}(\alpha)) \cong (\mathbb{Z}/p^a)^* \cong (1 + p\mathbb{Z}/p^a) \times (\mathbb{Z}/p)^*,$$

where the first factor is a p-group. Hence for any $g \in H$ and $x \in \operatorname{Ker}(\alpha)$, either $[g, x] = 1$ and hence $g = (gx)_r$; or $gxg^{-1} = x^i$ for some $i \not\equiv 1 \pmod p$ and hence g is conjugate to gx. In either case, $\varphi(gx) = \varphi(g)$. Thus, $\varphi = (\varphi|_H)\circ\alpha$, and this is a K-character of G since $\varphi|_H$ is by assumption a K-character of H.

Now assume G is p-K-elementary. Write $G = C_m \rtimes P$, where $p\nmid m$ and P is a p-group. Let S be the set of primes which divide m. For each $I \subseteq S$, let $C_I \subseteq C_m$ be the product of the Sylow p-subgroups for $p \in I$, set $G_I = C_I \rtimes P$, and let $\alpha_I : G \twoheadrightarrow G_I$ be the homomorphism which is the identity on G_I.

For each $I \subseteq S$, we can consider $K[C_I]$ as a G-representation via the conjugation action of P; and each C_I-irreducible summand of $K[C_I]$ is P-invariant and hence G-invariant. Thus, each irreducible $K[C_I]$-representation can be extended to a $K[G_I]$-representation upon which $P \cap C_G(C_I)$ acts trivially. Hence, since $\varphi|_{C_I}$ is a K-character of C_I; there is a K-character χ_I of G_I such that $\chi_I(gx) = \chi_I(x) = \varphi(x)$ for all $x \in C_I$ and $g \in P$ such that $[g, C_I] = 1$.

Now set

$$\chi = \sum_{J\subseteq I\subseteq S} (-1)^{|I \smallsetminus J|}(\chi_I\circ\alpha_J),$$

a K-character of G. We claim that $\varphi = \chi$. Since both are class functions, it suffices to show that $\varphi(gx) = \chi(gx)$ for all commuting $g \in P$ and $x \in C_m = C_S$. Fix such g and x, and let $X \subseteq S$ be the set of all primes $p\big||x|$. Then $[g, C_X] = 1$, and so

$$\begin{aligned}
\chi(gx) &= \sum_{J\subseteq I\subseteq S} (-1)^{|I\smallsetminus J|}\chi_I(\alpha_J(gx)) = \sum_{J\subseteq I\subseteq S} (-1)^{|I\smallsetminus J|}\chi_I(g{\cdot}\alpha_J(x)) \\
&= \sum_{J\subseteq I\subseteq X} (-1)^{|I\smallsetminus J|}\varphi(\alpha_J(x)) + \sum_{J\subseteq I\not\subseteq X} (-1)^{|I\smallsetminus J|}\chi_I(g{\cdot}\alpha_J(x)) \\
&= \varphi(x) = \varphi(gx).
\end{aligned}$$

Note, in the second line, that all terms in the second sum cancel since $\alpha_J(x) = \alpha_{J'}(x)$ if $J = J' \cap X$, and all terms in the first sum cancel except that where $J = I = X$. □

When $A = \mathbb{Z}$ and $K = \mathbb{Q}$, Proposition 4.1 and Lemma 4.2 combine to give:

Corollary 4.3. *Fix a finite group G and a prime p. Let $f : G \to \mathbb{Z}$ be any function which is p-constant, and constant on $\mathbb{Q}$-conjugacy classes in G. Set $|G| = m{\cdot}p^r$ where $p\nmid m$. Then $m{\cdot}f$ is a $\mathbb{Q}$-character of G.*

5. The equivariant Chern character

We construct here two different equivariant Chern characters, both defined on the equivariant complex K-theory of proper G-complexes. The first is defined for arbitrary X (with proper G-action), and sends

$K_G^*(X)$ to the Bredon cohomology group $H_G^*(X;\mathbb{Q}\otimes_{\mathbb{Z}}R(-))$. The second is defined only when X is finite dimensional and has bounded isotropy, and takes values in $\mathbb{Q}\otimes_{\mathbb{Z}}H_G^*(X;R(-))$.

We first fix our notation for dealing with Bredon cohomology [5]. Let $\mathsf{Or}(G)$ denote the orbit category: the category whose objects are the orbits G/H for $H\subseteq G$, and where $\mathrm{Mor}_{\mathsf{Or}(G)}(G/H,G/K)$ is the set of G-maps. A coefficient system for Bredon cohomology is a functor $F:\mathsf{Or}(G)^{\mathrm{op}}\to\mathsf{Ab}$. For any such functor F and any G-complex X, the Bredon cohomology $H_G^*(X;F)$ is the cohomology of a certain cochain complex $C_G^*(X;F)$, where $C_G^n(X;F)$ is the direct product over all orbits of n-cells of type G/H of the groups $F(G/H)$. This can be expressed functorially as a group of morphisms of functors on $\mathsf{Or}(G)$:

$$C_G^n(X,F)=\mathrm{Hom}_{\mathsf{Or}(G)}\bigl(\underline{\mathrm{C}}_n(X),F\bigr),$$

where $\underline{\mathrm{C}}_n(X):\mathsf{Or}(G)^{\mathrm{op}}\to\mathsf{Ab}$ is the functor $\underline{\mathrm{C}}_n(X)(G/H)=C_n(X^H)$.

Clearly, the coefficient system F need only be defined on the subcategory of orbit types which occur in the G-complex X. In particular, since we work here only with proper actions, we restrict attention to the full subcategory $\mathsf{Or}_f(G)$ of orbits G/H for *finite* $H\subseteq G$. Let $R(-)$ denote the functor on $\mathsf{Or}_f(G)$ which sends G/H to $R(H)$: a functor on the orbit category via the identification $R(H)\cong K_G^0(G/H)$. More precisely, a morphism $G/H\to G/K$ in $\mathsf{Or}_f(G)$, where $gH\mapsto gaK$ for some $a\in G$ with $a^{-1}Ha\subseteq K$, is sent to the homomorphism $R(K)\to R(H)$ induced by restriction and conjugation by a.

Since $R(-)$ is a functor from the orbit category to rings, there is a pairing

$$C_G^*(X;R(-))\otimes C_G^*(X;R(-))\longrightarrow C_G^*(X\times X;R(-))$$

for any proper X, and hence a similar pairing in cohomology. Via restriction to the diagonal subspace $X\subseteq X\times X$ this defines a ring structure on $H_G^*(X;R(-))$.

The equivariant Chern character will be constructed here by first reinterpreting $H_G^*(X;\mathbb{Q}\otimes R(-))$ as a certain group of homomorphisms of functors, and then directly constructing a map from $K_G(X)$ to such homomorphisms. This will be done with the help of another category, $\mathsf{Sub}_f(G)$, which is closely related to $\mathsf{Or}_f(G)$. The objects of $\mathsf{Sub}_f(G)$ are the finite subgroups of G, and

$$\mathrm{Mor}_{\mathsf{Sub}_f(G)}(H,K)\subseteq\mathrm{Hom}(H,K)/\mathrm{Inn}(K)$$

is the subset consisting of those monomorphisms induced by conjugation and inclusion in G. There is a functor $\mathsf{Or}_f(G)\to\mathsf{Sub}_f(G)$ which sends an orbit G/H to the subgroup H, and which sends a morphism $\bigl(xH\mapsto xaK\bigr)$ in $\mathsf{Or}_f(G)$ to the homomorphism $\bigl(x\mapsto a^{-1}xa\bigr)$ from H to K. Via this functor, we can think of $\mathsf{Sub}_f(G)$ as a quotient category of $\mathsf{Or}_f(G)$.

Let $\underline{\mathrm{C}}_*^{\mathrm{qt}}(X),\underline{\mathrm{H}}_*^{\mathrm{qt}}(X):\mathsf{Sub}_f(G)^{\mathrm{op}}\longrightarrow\mathsf{Ab}$ be the functors

$$\underline{\mathrm{C}}_*^{\mathrm{qt}}(X)(H)=C_*(X^H/C_G(H))\qquad\text{and}\qquad\underline{\mathrm{H}}_*^{\mathrm{qt}}(X)(H)=H_*(X^H/C_G(H)).\tag{5.1}$$

For any functor $F:\mathsf{Sub}_f(G)^{\mathrm{op}}\to\mathsf{Ab}$, regarded also as a functor on $\mathsf{Or}_f(G)^{\mathrm{op}}$,

$$\mathrm{Hom}_{\mathsf{Or}_f(G)}(\underline{\mathrm{C}}_*(X),F)\cong\mathrm{Hom}_{\mathsf{Sub}_f(G)}(\underline{\mathrm{C}}_*^{\mathrm{qt}}(X),F),\tag{5.2}$$

since

$$\mathrm{Hom}_{C_G(H)}(C_*(X^H),F(H))\cong\mathrm{Hom}(C_*(X^H/C_G(H)),F(H))$$

for each H (and $C_G(H)$ is the group of automorphisms of G/H in $\mathsf{Or}_f(G)$ sent to the identity in $\mathsf{Sub}_f(G)$). In particular, (5.2) will be applied when $F=R(-)$, regarded as a functor on $\mathsf{Sub}_f(G)$ as well as on $\mathsf{Or}_f(G)$.

As noted above, for any coefficient system F, the cochain complex $C_G^*(X;F)$ can be identified as a group of homomorphisms of functors on $\mathsf{Or}(G)$. The following lemma says that the Bredon cohomology groups $H_G^*(X;\mathbb{Q}\otimes R(-))$ have a similar description, but using functors on $\mathsf{Sub}_f(G)^{\mathrm{op}}$.

Lemma 5.3. *Fix a discrete group G and a proper G-complex X. Then (5.2) induces an isomorphism of rings*

$$\Phi_X:H_G^*(X;\mathbb{Q}\otimes R(-))\xrightarrow{\ \cong\ }\mathrm{Hom}_{\mathsf{Sub}_f(G)}\bigl(\underline{H}_*^{\mathrm{qt}}(X),\mathbb{Q}\otimes R(-)\bigr).$$

Proof. Since

$$C^*_G(X;\mathbb{Q}\otimes R(-)) \cong \mathrm{Hom}_{\mathsf{Or}_f(G)}(\underline{C}_*(X),\mathbb{Q}\otimes R(-)) \cong \mathrm{Hom}_{\mathsf{Sub}_f(G)}(\underline{C}^{\mathrm{qt}}_*(X),\mathbb{Q}\otimes R(-)),$$

this will follow immediately once we show that $\mathbb{Q}\otimes R(-)$ is injective as a functor $\mathsf{Sub}_f(G)^{\mathrm{op}} \to \mathsf{Ab}$. It suffices to prove this after tensoring with $\mathbb{C}$; i.e., it suffices to prove that $\mathrm{Cl}(-)$ (complex valued class functions) is injective. And this holds since for any $F : \mathsf{Sub}_f(G)^{\mathrm{op}} \to \mathsf{Ab}$,

$$\mathrm{Hom}_{\mathsf{Sub}_f(G)}(F,\mathrm{Cl}(-)) \cong \prod_g \mathrm{Hom}_{\mathsf{Sub}_f(G)}(F,\mathrm{Cl}_g(-)) \cong \prod_g \mathrm{Hom}(F(\langle g\rangle),\mathbb{C});$$

where both products are taken over any set of conjugacy class representatives for elements of finite order in G, and where $\mathrm{Cl}_g(H)$ denotes the space of class functions on H which vanish on all elements not G-conjugate to g. □

We are now ready to define the Chern character

$$\mathrm{ch}^*_X : K^*_G(X) \longrightarrow H^*_G(X;\mathbb{Q}\otimes R(-))$$

for any proper G-complex X. Here and in the following theorem, we regard $K^*_G(-)$ as being $\mathbb{Z}/2$-graded; so that ch^*_X sends $K^0_G(X)$ to $H^{\mathrm{ev}}_G(X;\mathbb{Q}\otimes R(-))$ and sends $K^1_G(X)$ to $H^{\mathrm{odd}}_G(X;\mathbb{Q}\otimes R(-))$. By Lemma 5.3, it suffices to define homomorphisms

$$\mathrm{ch}^H_X : K^*_G(X) \longrightarrow \mathrm{Hom}\big(H_*(X^H/C_G(H))\,,\,\mathbb{Q}\otimes R(H)\big),$$

for each finite subgroup $H \subseteq G$, which are natural in H in the obvious way. We define ch^H_X to be the following composite:

$$\begin{aligned}
K^*_G(X) \xrightarrow{\mathrm{Res}} K^*_{N_G(H)}(X^H) \xrightarrow{\Psi} K^*_{C_G(H)}(X^H)\otimes R(H) \xrightarrow{(\mathrm{proj})^*} K^*_{C_G(H)}(EG{\times}X^H)\otimes R(H)\\
\xrightarrow[\cong]{\mathrm{Infl}^{-1}} K^*(EG{\times}_{C_G(H)}X^H)\otimes R(H) \xrightarrow{\mathrm{ch}\otimes\mathrm{Id}} H^*\big(EG{\times}_{C_G(H)}X^H;\mathbb{Q}\big)\otimes R(H)\\
\xleftarrow[\cong]{(\mathrm{proj})^*} H^*\big(X^H/C_G(H);\mathbb{Q}\big)\otimes R(H) \cong \mathrm{Hom}\big(H_*(X^H/C_G(H)),\mathbb{Q}\otimes R(H)\big).
\end{aligned} \tag{5.4}$$

Here, Ψ is the homomorphism defined in Proposition 3.4, ch denotes the ordinary Chern character, and $(\mathrm{proj})^*$ in the bottom line is an isomorphism since all fibers of the projection from $EG\times_{C_G(H)}X^H$ to $X^H/C_G(H)$ are $\mathbb{Q}$-acyclic (classifying spaces of finite groups). By the naturality properties of Ψ shown in Proposition 3.4, $\prod_H \mathrm{ch}^H_X$ takes values in $\mathrm{Hom}_{\mathsf{Sub}_f(G)}\big(\underline{H}^{\mathrm{qt}}_*(X),\mathbb{Q}\otimes R(-)\big)$, and hence (via Lemma 5.3) defines an equivariant Chern character

$$\mathrm{ch}^*_X : K^*_G(X) \longrightarrow H^*_G(X;\mathbb{Q}\otimes R(-)) \cong \mathrm{Hom}_{\mathsf{Sub}_f(G)}\big(\underline{H}^{\mathrm{qt}}_*(X),\mathbb{Q}\otimes R(-)\big).$$

All of the maps in (5.4) are homomorphisms of rings, and hence ch^*_X is also a homomorphism of rings. Also, the ch^*_X commute with degree-changing maps $K^*_G(X) \to K^{*+m}(S^m{\times}X)$ (i.e., product with the fundamental class of S^m) and similarly in cohomology, since all maps in (5.4) do so. They are thus natural with respect to boundary maps in Mayer-Vietoris sequences.

Theorem 5.5. *For any finite proper G-complex X, the Chern character ch^*_X extends to an isomorphism of rings*

$$\mathbb{Q}\otimes\mathrm{ch}^*_X : \mathbb{Q}\otimes K^*_G(X) \xrightarrow{\;\cong\;} H^*_G(X;\mathbb{Q}\otimes R(-)).$$

Proof. For any finite subgroup $H \subseteq G$,

$$K^0_G(G/H) \cong R(H) \cong H^0_G(G/H;R(-)), \quad \text{and} \quad K^1_G(G/H) = 0 = H^{\neq 0}_G(G/H;R(-)).$$

From the definition in (5.4) (and since the non-equivariant Chern character $K(\mathrm{pt}) \to H^0(\mathrm{pt})$ is the identity map), we see that $\mathbb{Q}\otimes\mathrm{ch}^*_{G/H}$ is the identity map under the above identifications. The Chern characters for $G/H{\times}D^n$ and $G/H{\times}S^{n-1}$) are thus isomorphisms for all n. The theorem now follows by induction on the number of orbits of cells in X, together with the Mayer-Vietoris sequences for pushouts $X = X' \cup_\varphi (G/H{\times}D^n)$ (and the 5-lemma). □

Theorem 5.5 means that the $\mathbb{Q}$-localization of the classifying space K_G splits as a product of equivariant Eilenberg-Maclane spaces. Hence for any proper G-complex X, there is an isomorphism $K^*_G(X;\mathbb{Q}) \xrightarrow{\cong} H^*_G(X;\mathbb{Q}\otimes R(-))$, where the first group is defined via the localized spectrum (and is *not* in general isomorphic to $\mathbb{Q}\otimes K^*_G(X,A)$).

The coefficient system $\mathbb{Q}\otimes R(-)$, and hence its cohomology, splits in a natural way as a product indexed over cyclic subgroups of G of finite order. For any cyclic group S of order $n<\infty$, we let $\mathbb{Z}[\zeta_S]\subseteq\mathbb{Q}(\zeta_S)$ denote the cyclotomic ring and field generated by the n-th roots of unity; but regarded as quotient rings of the group rings $\mathbb{Z}[S^*]\subseteq\mathbb{Q}[S^*]$ ($S^*=\operatorname{Hom}(S,\mathbb{C}^*)$). In other words, we fix an identification of the n-th roots of unity in $\mathbb{Q}(\zeta_S)$ with the irreducible characters of S. The kernel of the homomorphism $R(S)\cong\mathbb{Z}[S^*]\twoheadrightarrow\mathbb{Z}[\zeta_S]$ is precisely the ideal of elements whose characters vanish on all generators of S.

Lemma 5.6. *Fix a discrete group G, and let $\mathcal{S}(G)$ be a set of conjugacy class representatives for the cyclic subgroups $S\subseteq G$ of finite order. Then for any proper G-complex X, there is an isomorphism of rings*

$$H^*_G(X;\mathbb{Q}\otimes R(-))\cong\prod_{S\in\mathcal{S}(G)}\bigl(H^*(X^S/C_G(S);\mathbb{Q}(\zeta_S))\bigr)^{N(S)},$$

where $N(S)$ acts via the conjugation action on $\mathbb{Q}(\zeta_S)$ and translation on $X^S/C_G(S)$. If, furthermore, the isotropy subgroups on X have bounded order, then the homomorphism of rings

$$\begin{aligned} H^*_G(X;R(-)) \longrightarrow \prod_{S\in\mathcal{S}(G)} H\Bigl(\bigl(C^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])\bigr)^{N(S)}\Bigr)& \\ \longrightarrow \prod_{S\in\mathcal{S}(G)}\bigl(H^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])\bigr)^{N(S)},& \quad (1)\end{aligned}$$

induced by restriction to cyclic subgroups and by the projections $R(S)\twoheadrightarrow\mathbb{Z}[\zeta_S]$, has kernel and cokernel of finite exponent.

Proof. By (5.2),

$$C^*_G(X;R(-))\cong\operatorname{Hom}_{\mathsf{Or}_f(G)}(\underline{C}_*(X),R(-))\cong\operatorname{Hom}_{\mathsf{Sub}_f(G)}(\underline{C}^{\mathrm{qt}}_*(X),R(-)).$$

For each $S\in\mathcal{S}(G)$, let $\chi_S\in\operatorname{Cl}(G)$ be the idempotent class function: $\chi_S(g)=1$ if $\langle g\rangle$ is conjugate to S, and $\chi_S(g)=0$ otherwise. By Proposition 4.1, for each finite subgroup $H\subseteq G$, $(\chi_S)|_H$ is the character of an idempotent $e^H_S\in\mathbb{Q}\otimes R(H)$. Set $\mathbb{Q}R_S(H)=e^H_S\cdot(\mathbb{Q}\otimes R(H))$, and let $R_S(H)\subseteq\mathbb{Q}R_S(H)$ be the image of $R(H)$ under the projection. This defines a splitting $\mathbb{Q}\otimes R(-)=\prod_{S\in\mathcal{S}(G)}\mathbb{Q}R_S(-)$ of the coefficient system. For each S and H,

$$\mathbb{Q}R_S(S)=\mathbb{Q}(\zeta_S)\quad\text{and so}\quad \mathbb{Q}R_S(H)\cong\operatorname{map}_{N(S)}\Bigl(\operatorname{Mor}_{\mathsf{Sub}_f(G)}(S,H)\,,\,\mathbb{Q}(\zeta_S)\Bigr).$$

It follows that

$$C^*_G(X;\mathbb{Q}R_S(-))\cong\operatorname{Hom}_{\mathsf{Sub}_f(G)}(\underline{C}^{\mathrm{qt}}_*(X),\mathbb{Q}R_S(-))\cong\operatorname{Hom}_{\mathbb{Q}[N(S)]}\bigl(C_*(X^S/C_G(S)),\mathbb{Q}(\zeta_S)\bigr);$$

and hence $H^*_G(X;\mathbb{Q}R_S(-))\cong\bigl(H^*(X^S/C_G(S));\mathbb{Q}(\zeta_S)\bigr)^{N(S)}$.

Now assume there is a bound on the orders of isotropy subgroups on X, and let m be the least common multiple of the $|G_x|$. By Proposition 4.1 again, $me^H_S\in R(H)$ for each $S\in\mathcal{S}(G)$ and each isotropy subgroup H. So there are homomorphisms of functors

$$R(-)\;\underset{j}{\overset{i}{\rightleftarrows}}\;\prod_{S\in\mathcal{S}(G)}R_S(-),$$

where i is induced by the projections $R(H)\twoheadrightarrow R_S(H)$ and j by the homomorphisms $R_S(H)\xrightarrow{me^H_S\cdot}R(H)$ (regarding $R_S(H)$ as a quotient of $R(H)$); and $i\circ j$ and $j\circ i$ are both multiplication by m. For each S, the monomorphism

$$C^*_G(X;R_S(-))\cong\operatorname{Hom}_{\mathbb{Z}[N(S)]}\bigl(C_*(X^S/C_G(S)),\mathbb{Z}[\zeta_S]\bigr)\longrightarrow C^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])$$

is split by the norm map for the action of $N(S)/C_G(S)$, and hence the kernel and cokernel of the induced homomorphism

$$H^*_G(X;R_S(-)) \longrightarrow \big(H^*(X^S/C_G(S);\mathbb{Z}[\zeta_S])\big)^{N(S)}$$

have exponent dividing $\varphi(m)$ (since $|N(S)/C_G(S)|\big|\,|\operatorname{Aut}(S)|\big|\varphi(m)$). The composite in (1) thus has kernel and cokernel of exponent $m{\cdot}\varphi(m)$. □

By the first part of Proposition 5.6, the equivariant Chern character can be regarded as a homomorphism

$$\mathrm{ch}^*_X : K^*_G(X) \longrightarrow \prod_{S\in\mathcal{S}(G)} \big(H^*(X^S/C_G(S);\mathbb{Q}(\zeta_S))\big)^{N(S)},$$

where $\mathcal{S}(G)$ is as above. This is by construction a product of ring homomorphisms.

We now apply the splitting of Lemma 5.6 to construct a second version of the equivariant rational Chern character: one which takes values in $\mathbb{Q}\otimes H^*_G(X;R(-))$ rather than in $H^*_G(X;\mathbb{Q}\otimes R(-))$. The following lemma handles the nonequivariant case.

Lemma 5.7. *There is a homomorphism $n!\mathrm{ch} : K^*(X) \to H^{\le 2n}(X;\mathbb{Z})$, natural on the category of CW-complexes, whose composite to $H^*(X;\mathbb{Q})$ is $n!$ times the usual Chern character truncated in degrees greater than $2n$. Furthermore, $n!\mathrm{ch}$ is natural with respect to suspension isomorphisms $K^*(X) \cong \widetilde{K}^{*+m}(\Sigma^m(X_+))$, and is multiplicative in the sense that $\big(n!\mathrm{ch}(x)\big){\cdot}\big(n!\mathrm{ch}(y)\big) = n!{\cdot}\big(n!\mathrm{ch}(xy)\big)$ for all $x,y\in K(X)$ (in both cases after restricting to the appropriate degrees).*

Proof. Define $n!\mathrm{ch} : K^0(X) \to H^{\mathrm{ev},\le 2n}(X;\mathbb{Z})$ to be the following polynomial in the Chern classes:

$$n!{\cdot}\sum_{i=1}^{n}\Big(1 + x_i + \frac{x_i^2}{2!} + \cdots + \frac{x_i^n}{n!}\Big) \in \mathbb{Z}[c_1,\dots,c_n] = \mathbb{Z}[x_1,x_2,\dots,x_n]^{\Sigma_n}.$$

Here, as usual, c_k is the k-th elementary symmetric polynomial in the x_i. This is extended to $K^{-1}(X) \cong \widetilde{K}(\Sigma(X_+))$ in the obvious way. The relations all follow from the usual relations between Chern classes in the rings $H^*(BU(m))$. □

We are now ready to construct the integral Chern character. What this really means is that under certain restrictions on X, some multiple of the rational Chern character ch^*_X of Theorem 5.5 can be lifted to the integral Bredon cohomology group $H^*_G(X;R(-))$.

Proposition 5.8. *Let G be a discrete group, and let X be a finite dimensional proper G-complex whose isotropy subgroups have bounded order. Then there is a homomorphism*

$$\widetilde{\mathrm{ch}}^*_X : K^*_G(X) \longrightarrow \mathbb{Q}\otimes H^*_G(X;R(-)),$$

natural in such X, whose composite to $H^*_G(X;\mathbb{Q}\otimes R(-))$ is the map ch^*_X of Theorem 5.5. Furthermore, $\widetilde{\mathrm{ch}}^*_X$ induces an isomorphism of rings $\mathbb{Q}\otimes K^*_G(X) \xrightarrow{\cong} \mathbb{Q}\otimes H^*_G(X;R(-))$. And for any finite subgroup $K\subseteq G$, $\widetilde{\mathrm{ch}}^0_{G/K}$ is the identity map under the identifications $K_G(G/K)\cong R(G/K)\cong H^0_G(G/K;R(-))$.

Proof. Fix X, and choose any integer $n\ge \dim(X)/2$. Set $m = \mathrm{lcm}\big\{|G_x|\,\big|\,x\in X\big\}$ and $N = n!{\cdot}m^{4n}$. For each $S\in G$ of finite order, let $\widetilde{\mathrm{ch}}^S_X$ be the following composite:

$$\begin{aligned}
K^*_G(X) \xrightarrow{\mathrm{Res}} K^*_{N_G(S)}(X^S) \xrightarrow{\ \Psi\ } K^*_{C_G(S)}(X^S)\otimes R(S) \xrightarrow{(\mathrm{proj})^*} K^*_{C_G(S)}(EG\times X^S)\otimes R(S)\\
\xrightarrow[\cong]{\mathrm{Infl}^{-1}} K^*(EG\times_{C_G(S)}X^S)\otimes R(S) \xrightarrow{n!\mathrm{ch}} H^{\le 2n}\big(EG\times_{C_G(S)}X^S\big)\otimes R(S)\\
\xrightarrow{m^{4n}(\mathrm{proj}^*)^{-1}} H^*\big(X^S/C_G(S)\big)\otimes R(S) \longrightarrow H^*(X^S/C_G(S);\mathbb{Z}[\zeta_S]).
\end{aligned}$$

Here, Ψ is the homomorphism of Lemma 3.4, and Infl is the inflation isomorphism of Proposition 3.3. The first map in the bottom row is well defined since $H^*(X^S/C(S);\mathbb{Z}[\zeta_S]) \xrightarrow{\mathrm{proj}^*} H^*(EG\times_{C(S)}X^S;\mathbb{Z}[\zeta_S])$ has kernel and cokernel of exponent m^{2n} (this follows from the spectral sequence for the projection, all of whose fibers are of the form BG_x for $x \in X$). The last map is induced by the projection $R(S) \twoheadrightarrow \mathbb{Z}[\zeta_S]$. All of these maps are homomorphisms of rings (up to the obvious integer multiples).

Now let $\mathcal{S}(G)$ be any set of conjugacy class representatives for cyclic subgroups $S \subseteq G$ of finite order. Define $\widetilde{\mathrm{ch}}^*_X$ to be the composite

$$\widetilde{\mathrm{ch}}^*_X : K^*_G(X) \xrightarrow{\frac{1}{N}\prod \widetilde{\mathrm{ch}}^S_X} \mathbb{Q} \otimes \Big(\prod_{S \in \mathcal{S}(G)} \big(H^*(X^S/C_G(S); \mathbb{Z}[\zeta_S])\big)^{N(S)} \Big) \cong \mathbb{Q} \otimes H^*_G(X; R(-)),$$

where the isomorphism is that of Lemma 5.6. The naturality of $\widetilde{\mathrm{ch}}^*_X$, its independence of the choice of n, and its relation with ch^*_X, are immediate from the construction. Also, $\widetilde{\mathrm{ch}}^*_X$ is natural with respect to the degree-changing maps $K^*(X) \to K^{*+m}(S^m \times X)$ (and similarly in cohomology). In particular, this means that it commutes with all maps in Mayer-Vietoris sequences.

It remains to prove that $\widetilde{\mathrm{ch}}^*_X$ induces an isomorphism on $\mathbb{Q} \otimes K^*_G(X)$. This is done by induction on $\dim(X)$, using the obvious Mayer-Vietoris sequences. So it suffices to show it for (possibly infinite) disjoint unions $\coprod_{i \in I} G/H_i$ of orbits. Both groups are zero in odd degrees. And in even degrees,

$$\begin{array}{ccc} \mathbb{Q} \otimes K_G\big(\coprod_{i \in I} G/H_i\big) & \xrightarrow{\widetilde{\mathrm{ch}}^0_X} & \mathbb{Q} \otimes H^{\mathrm{ev}}_G\big(\coprod_{i \in I} G/H_i; R(-)\big) \\ \cong \mathbb{Q} \otimes \big(\prod_i R(H_i)\big) & & \cong \mathbb{Q} \otimes \big(\prod_i R(H_i)\big) \end{array}$$

is the identity map under these identifications. □

6. Completion theorems

Throughout this section, G is a discrete group. We want to prove completion theorems for finite proper G-complexes: theorems which show that $K^*_G(E \times X)$, when E is a "universal space" of a certain type, is isomorphic to a certain completion of $K^*_G(X)$. The key step will be to construct elements of $K^*_G(X)$ whose restrictions to orbits in X are sufficiently "interesting". And this requires a better understanding of the "edge homomorphism" for $K^*_G(X)$.

For any finite dimensional proper G-complex X, the skeletal filtration of $K^*_G(X)$ induces a spectral sequence

$$E_2^{p,2*} \cong H^p_G(X; R(-)) \Longrightarrow K^*_G(X).$$

If X also has bounded isotropy, the Chern character $\widetilde{\mathrm{ch}}_X$ of Proposition 5.8 is an isomorphism (after tensoring with $\mathbb{Q}$) from the limit of this spectral sequence to its E_2-term. It follows that the spectral sequence collapses rationally; i.e., that the images of all differentials in the spectral sequence consist of torsion elements.

Of particular interest is the edge homomorphism of the spectral sequence. This is a homomorphism

$$\epsilon_X : K^*_G(X) \longrightarrow H^0_G(X; R(-)),$$

which is induced by restriction to the 0-skeleton of X under the identification

$$H^0_G(X; R(-)) = \mathrm{Ker}\big[K_G(X^{(0)}) \longrightarrow K^1_G(X^{(1)}, X^{(0)})\big] = \mathrm{Im}\big[K_G(X^{(1)}) \longrightarrow K_G(X^{(0)})\big].$$

Alternatively, $H^0_G(X; R(-))$ can be thought of as the inverse limit, taken over all isotropy subgroups H of X and all connected components of X^H, of the representation rings $R(H)$; and the edge homomorphism sends an element of $K^*_G(X)$ to the collection of its restrictions to elements of $K^*_G(Gx) \cong R(G_x)$ at all points $x \in X$.

As an application of the integral Chern character of Proposition 5.8, we get:

Proposition 6.1. *Let X be any finite dimensional proper G-complex whose isotropy subgroups have bounded order. Then for any $\xi \in H^0_G(X;R(-))$, there is $k>0$ such that $k\cdot\xi$ and ξ^k lie in the image of the edge homomorphism*

$$\epsilon_X : K_G(X) \longrightarrow H^0_G(X;R(-)).$$

Similarly, for any $\xi \in H^0_G(X;RO(-))$, there is $k>0$ such that $k\cdot\xi$ and ξ^k lie in the image of the edge homomorphism

$$\epsilon_X : KO_G(X) \longrightarrow H^0_G(X;RO(-)).$$

Proof. The usual homomorphisms between $R(-)$ and $RO(-)$, and between $K^*_G(-)$ and $KO^*_G(-)$, induced by $(\mathbb{C}\otimes_{\mathbb{R}})$ and by forgetting the complex structure, show that up to 2-torsion, $KO^*_G(X)$ and $H^0_G(X;RO(-))$ are the fixed point sets under complex conjugation of the groups $K^*_G(X)$ and $H^0_G(X;R(-))$, respectively. So the edge homomorphism in the orthogonal case is also surjective modulo torsion. The rest of the argument is identical in the real and complex cases; we restrict to the complex case for simplicity.

By Proposition 5.8, the integral Chern character for $X^{(0)}$ is the identity under the usual identifications $K_G(G/K) \cong R(K) \cong H^0_G(G/K;R(-))$ for an orbit G/K (K finite). So by the naturality of $\widetilde{\mathrm{ch}}_X$, the composite

$$\mathbb{Q}\otimes K_G(X) \xrightarrow[\cong]{\widetilde{\mathrm{ch}}^0_X} \mathbb{Q}\otimes H^{\mathrm{ev}}_G(X;R(-)) \twoheadrightarrow \mathbb{Q}\otimes H^0_G(X;R(-)) \subseteq \mathbb{Q}\otimes K_G(X^{(0)}) is just the map induced by restriction onto$$

X$^{(0)}$. So rationally, the edge homomorphism is just the projection of the integral Chern character $\widetilde{\mathrm{ch}}$ onto $H^0_G(X;R(-))$, and is in particular surjective. And hence, for any $\xi \in H^0_G(X;R(-))$, there is some $k>0$ such that $k\cdot\xi \in \epsilon_X(K_G(X))$.

It remains to show that $\xi^k \in \mathrm{Im}(\epsilon_X)$ for some k. If we knew that the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,2*} \cong H^p_G(X;R(-)) \Longrightarrow K^*_G(X)$$

were multiplicative (i.e., that the differentials were derivations), then the result would follow directly. As we have seen, all differentials in the spectral sequence have finite order. Hence, for each $r \geq 2$ and each $\eta \in E_r^{0,2*}$, there is some $k>0$ such that

$$k\cdot d_r(\eta)=0, \qquad \text{and hence} \qquad d_r(\eta^k) = k\cdot d_r(\eta)\eta^{k-1} = 0.$$

Upon iteration, this shows that for any $\xi \in H^0_G(X;R(-)) = E_2^{0,0}$, there exists $k>0$ such that $k\cdot\xi$ and ξ^k both survive to $E_\infty^{0,0}$; and hence lie in the image of the edge homomorphism.

Rather than prove the multiplicativity of the spectral sequence, we give the following more direct argument. Identify

$$\xi \in H^0_G(X;R(-)) = \mathrm{Im}\big[K_G(X^{(1)}) \longrightarrow K_G(X^{(0)})\big].$$

Assume, for some $r \geq 2$, that ξ lies in the image of $K_G(X^{(r-1)})$; we prove that some power of ξ lies in the image of $K_G(X^{(r)})$.

Fix $\widetilde{\xi} \in K_G(X^{(r-1)})$ such that $\mathrm{res}_{X^{(0)}}(\widetilde{\xi}) = \xi$. Since $r \geq 2$,

$$\mathrm{Im}\big[H^0_G(X^{(r-1)};R(-)) \longrightarrow H^0_G(X^{(0)};R(-))\big] = \mathrm{Im}\big[H^0_G(X^{(r)};R(-)) \longrightarrow H^0_G(X^{(0)};R(-))\big].$$

Hence, since the Chern character is rationally an isomorphism, there exists k such that $k\cdot\xi$ lies in the image of $K_G(X^{(r)})$, or equivalently such that

$$\begin{aligned} k\cdot\widetilde{\xi} &\in \mathrm{Ker}\Big[K_G(X^{(r-1)}) \longrightarrow K_G(X^{(0)}) \xrightarrow{d} K^1_G(X^{(r)},X^{(0)})\Big] \\ &= \mathrm{Ker}\Big[K_G(X^{(r-1)}) \xrightarrow{d} K^1_G(X^{(r)},X^{(r-1)}) \longrightarrow K^1_G(X^{(r)},X^{(0)})\Big]. \end{aligned} \tag{1}$$

In Lemma 6.2 below, we will show that there is a $K_G(X^{(r-1)})$-module structure on the relative group $K^1_G(X^{(r)},X^{(r-1)})$ which makes the boundary map $d : K_G(X^{(r-1)}) \to K^1_G(X^{(r)},X^{(r-1)})$ into a derivation. Then $d(\widetilde{\xi}^k) = k\cdot\widetilde{\xi}^{k-1}\cdot d(\widetilde{\xi})$, so $\widetilde{\xi}^k$ lies in the kernel in (1), and hence $\xi^k = \mathrm{res}_{X^{(0)}}(\widetilde{\xi}^k)$ lies in the image of $K_G(X^{(r)})$. $\square$

It remains to prove:

Lemma 6.2. *Let X be any proper G-complex. Then, for any $r \geq 2$, one can put a $K_G(X^{(r-1)})$-module structure on $K^1_G(X^{(r)}, X^{(r-1)})$ in such a way that for any $\alpha, \beta \in K_G(X^{(r-1)})$,*

$$d(\alpha\beta) = \alpha{\cdot}d\beta + \beta{\cdot}d\alpha \in K^1_G(X^{(r)}, X^{(r-1)}).$$

Proof. We can assume $X = X^{(r)}$. Write $Y = X^{(r-1)}$, for short. Fix a map $\Delta : X \to Z \overset{\text{def}}{=} X \times Y \cup Y \times X$ which is homotopic to the diagonal, and such that $\Delta|_Y$ is equal to the diagonal map. Since Z contains the $r+1$-skeleton of $X \times X$, Δ is unique up to homotopy (rel Y). In particular, if $T : Z \to Z$ is the map which switches coordinates, then $T{\circ}\Delta \simeq \Delta$ (rel Y).

Now, for $\alpha \in K_G(Y)$ and $x \in K^1_G(X, Y)$, let $\alpha{\cdot}x \in K^1_G(X, Y)$ be the image of $\alpha \times x$ under the following composite

$$\alpha \times x \in K^1_G(Y \times X, Y \times Y) \cong K^1_G(Z, X \times Y) \xrightarrow{\text{incl}^*} K^1_G(Z, Y \times Y) \xrightarrow{\Delta^*} K^1_G(X, Y).$$

Here, the external product $\alpha \times x$ is induced by the pairing $K_G \wedge K_G \to K_{G\times G} \to K_G$ of (2.2); or equivalently is defined to be the internal product of $\text{proj}_1^*(\alpha) \in K_G(Y \times X)$ and $\text{proj}_2^*(x) \in K^1_G(Y \times X, Y \times Y)$. We can thus consider $K_G(X, Y)$ as a $K_G(Y)$-module. In particular, the relation $(\alpha\beta){\cdot}x = \alpha{\cdot}(\beta{\cdot}x)$ follows since the two composites $(\Delta \times \text{Id}_X){\circ}\Delta$ and $(\text{Id}_X \times\Delta){\circ}\Delta$ are homotopic as maps from X to

$$(X{\times}Y{\times}Y) \cup (Y{\times}X{\times}Y) \cup (Y{\times}Y{\times}X).$$

Now consider the following commutative diagram:

$$\begin{array}{ccccc}
K_G(Y \times Y) & \xrightarrow{d} & K^1_G(Z, Y \times Y) & \xrightarrow{\cong} & K^1_G\big((X, Y) \times Y\big) \oplus K^1_G\big(Y \times (X, Y)\big) \\
\Big\downarrow{\scriptstyle \Delta^*} & & \Big\downarrow{\scriptstyle \Delta^*} & & \Big\uparrow{\scriptstyle \cong} \\
K_G(Y) & \xrightarrow{d} & K^1_G(X, Y) & \longleftarrow & K^1_G\big(Z, Y \times X\big) \oplus K^1_G\big(Z, X \times Y\big)
\end{array}$$

where the isomorphisms hold by excision. For any $\alpha, \beta \in K_G(Y)$, the external product $\alpha \times \beta \in K_G(Y \times Y)$ is sent, by the maps in the top row, to the pair $(d\alpha \times \beta, \alpha \times d\beta)$. This follows from the linearity of the differential (which holds in any multiplicative cohomology theory). And since $T{\circ}\Delta \simeq \Delta$, as noted above, we have

$$d(\alpha\beta) = \Delta^*(d(\alpha \times \beta)) = \beta{\cdot}d\alpha + \alpha{\cdot}d\beta. \quad \square$$

As an immediate consequence of Proposition 6.1, we now get:

Corollary 6.3. *Assume that G is discrete. Fix any family $\mathcal{F}$ of finite subgroups of G of bounded order, and let*

$$\mathbf{V} = \big(V_H\big) \in \varprojlim_{H\in\mathcal{F}} R(H) \qquad \text{or} \qquad \mathbf{V}' = \big(V'_H\big) \in \varprojlim_{H\in\mathcal{F}} RO(H)$$

be any system of compatible (virtual) representations. Then for any finite dimensional proper G-complex X all of whose isotropy subgroups lie in $\mathcal{F}$, there is an integer $k > 0$, and elements $\alpha, \beta \in K_G(X)$ (or $\alpha', \beta' \in KO_G(X)$), such that $\alpha|_x = k{\cdot}V_{G_x}$ and $\beta|_x = (V_{G_x})^k$ (or $\alpha'|_x = k{\cdot}V'_{G_x}$ and $\beta'|_x = (V'_{G_x})^k$) for all $x \in X$.

Proof. Let ξ be the image of $\mathbf{V}$ under the ring homomorphism

$$\varprojlim_{H\in\mathcal{F}} R(H) \longrightarrow H^0_G(X; R(-))$$

(or similarly in the orthogonal case); and apply Proposition 6.1. $\square$

Corollary 6.3 can be thought of as a generalization of [8, Theorem 2.7]. It was that result which was the key to proving the completion theorem in [8], and Corollary 6.3 plays a similar role in proving the more general completion theorems here.

In what follows, a *family* of subgroups of a discrete group G will always mean a set of subgroups closed under conjugation and closed under taking subgroups.

Lemma 6.4. *Let X be a proper n-dimensional G-complex. Set*

$$I = \mathrm{Ker}[K_G^*(X) \xrightarrow{\text{res}} K_G^*(X^{(0)})].$$

Then $I^{n+1} = 0$.

Proof. Fix any elements $x \in I^n$ and $y \in I$. By induction, we can assume that x vanishes in $K_G^*(X^{n-1})$, and hence that it lifts to an element $x' \in K_G^*(X, X^{(n-1)})$. Recall that $K_G^*(X, X^{(n-1)})$ is a $K_G^*(X)$-module, and the map $K_G^*(X, X^{(n-1)}) \to K_G^*(X)$ is $K_G^*(X)$-linear. But $I{\cdot}K_G^*(X, X^{(n-1)}) = 0$, since I vanishes on orbits; so $yx' = 0$, and hence $yx = 0$ in $K_G^*(X)$. □

As in earlier sections, in order to handle the complex and real cases simultaneously, we set $F = \mathbb{C}$ or $\mathbb{R}$, and write $KF_G^*(-)$ and $RF(-)$ for the equivariant K-theory and representation rings over F.

Fix any finite proper G-complex X, and let $f : X \to L$ be any map to a finite dimensional proper G-complex L whose isotropy subgroups have bounded order. Let $\mathcal{F}$ be any family of finite subgroups of G. Regard $KF_G^*(X)$ as a module over the ring $KF_G(L)$. Set

$$I = I_{\mathcal{F},L} = \mathrm{Ker}\Big[KF_G(L) \xrightarrow{\text{res}} \prod_{H\in\mathcal{F}} KF_H(L^{(0)})\Big].$$

For any $n \geq 0$, the composite

$$I^n{\cdot}KF_G^*(X) \subseteq KF_G^*(X) \xrightarrow{\text{proj}^*} KF_G^*(E_{\mathcal{F}}(G) \times X) \xrightarrow{\text{res}} KF_G^*((E_{\mathcal{F}}(G) \times X)^{(n-1)})$$

is zero, since the image is contained in $IKF_G^*((E_{\mathcal{F}}(G){\times}_G X)^{(n-1)})^n = 0$ which vanishes by Lemma 6.4. This thus defines a homomorphism of pro-groups

$$\lambda_{\mathcal{F}}^{X,f} : \big\{KF_G^*(X)/I^n{\cdot}K_G^*(X)\big\}_{n\geq 1} \longrightarrow \big\{KF_G^*\big((E_{\mathcal{F}}(G) \times X)^{(n-1)}\big)\big\}_{n\geq 1}.$$

Theorem 6.5. *Fix $F = \mathbb{C}$ or $\mathbb{R}$. Let G be a discrete group, and let $\mathcal{F}$ be a family of subgroups of G closed under conjugation and under subgroups. Fix a finite proper G-complex X, a finite dimensional proper G-complex Z whose isotropy subgroups have bounded order, and a G-map $f : X \to Z$. Regard $KF_G^*(X)$ as a module over $KF_G(Z)$, and set*

$$I = I_{\mathcal{F},Z}^F = \mathrm{Ker}\Big[KF_G(Z) \xrightarrow{\text{res}} \prod_{H\in\mathcal{F}} KF_H(Z^{(0)})\Big].$$

Then

$$\lambda_{\mathcal{F}}^{X,f} : \big\{KF_G^*(X)/I^n{\cdot}KF_G^*(X)\big\}_{n\geq 1} \longrightarrow \big\{KF_G^*\big((E_{\mathcal{F}}(G) \times X)^{(n-1)}\big)\big\}_{n\geq 1}$$

is an isomorphism of pro-groups. Also, the inverse system $\big\{KF_G^*\big((E_{\mathcal{F}}(G) \times X)^{(n)}\big)\big\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. In particular,

$$\varprojlim{}^1 KF_G^*\big((E_{\mathcal{F}}(G) \times X)^{(n)}\big) = 0;$$

and $\lambda_{\mathcal{F}}^{X,f}$ induces an isomorphism

$$KF_G^*(X)\widehat{{}_I} \xrightarrow{\cong} KF_G^*(E_{\mathcal{F}}(G) \times X) \cong \varprojlim KF_G^*\big((E_{\mathcal{F}}(G) \times X)^{(n)}\big).$$

Proof. Assume that $\lambda_{\mathcal{F}}^{X,f}$ is an isomorphism. Then the system $\big\{KF_G^*\big((E_{\mathcal{F}}(G){\times}X)^{(n)}\big)\big\}_{n\geq 1}$ satisfies the Mittag-Leffler condition because $\big\{KF_G^*(X)/I^n\big\}$ does. In particular,

$$\varprojlim{}^1 KF_G^*\big((E_{\mathcal{F}}(G){\times}X)^{(n)}\big) = 0, \text{ and so } KF_G^*(E_{\mathcal{F}}(G){\times}X) \cong \varprojlim KF_G^*\big((E_{\mathcal{F}}(G){\times}X)^{(n)}\big)$$

(cf. [3, Proposition 4.1]).

It remains to show that $\lambda_{\mathcal{F}}^{X,f}$ is an isomorphism.

Step 1 Assume first that $X = G/H$, for some finite subgroup $H \subseteq G$. Let $\mathcal{F}|H$ be the family of subgroups of H contained in $\mathcal{F}$, and consider the following commutative diagram:

$$\begin{array}{ccccc} KF_G(Z) & \xrightarrow{f^*} & KF_G^*(G/H) & \xrightarrow{\mathrm{pr}_1} & KF_G^*(E_{\mathcal{F}}(G)\times G/H) \\ & & \Big\downarrow{\scriptstyle \mathrm{ev}_{eH}}\,\cong & & \Big\downarrow{\scriptstyle q}\,\cong \\ RF(H) & \xrightarrow[\cong]{} & KF_H^*(\mathrm{pt}) & \xrightarrow{\mathrm{pr}_2} & KF_H^*(E_{\mathcal{F}|H}(H)). \end{array}$$

Here, pr_2 induces an isomorphism of pro-groups

$$\bigl\{KF_H^*(*)/I_{\mathcal{F}}(H)^n{\cdot}KF_H^*(*)\bigr\}_{n\ge1} \longrightarrow \bigl\{KF^*\bigl((BH)^{(n-1)}\bigr)\bigr\}_{n\ge1}$$

by the theorem of Jackowski [7, Theorem 5.1], where

$$I_{\mathcal{F}}(H) = \mathrm{Ker}\Bigl[RF(H) \longrightarrow \prod_{L\in\mathcal{F}|H} RF(L)\Bigr] \supseteq I' \overset{\text{def}}{=} \mathrm{ev}_{f(eH)}(I).$$

(The theorem in [7] is stated only for complex K-theory, but as noted afterwards, the proof applies equally well to the real case.) We want to show that pr_1 induces an isomorphism of pro-groups

$$\bigl\{KF_G^*(G/H)/I^n{\cdot}KF_G^*(G/H)\bigr\}_{n\ge1} \longrightarrow \bigl\{KF_G^*\bigl((E_{\mathcal{F}}(G)\times G/H)^{(n-1)}\bigr)\bigr\}_{n\ge1}.$$

So we must show that for some k, $I_{\mathcal{F}}(H)^k \subseteq I'$.

This means showing that the ideal $I_{\mathcal{F}}(H)/I'$ is nilpotent; or equivalently (since $R(H)$ is noetherian) that it is contained in all prime ideals of $R(H)/I'$ (cf. [2, Proposition 1.8]). In other words, we must show that every prime ideal of $R(H)$ which contains I' also contains $I_{\mathcal{F}}(H)$. Fix any prime ideal $\mathfrak{P} \subseteq R(H)$ which does *not* contain $I_{\mathcal{F}}(H)$. Set $\zeta = \exp(2\pi i/|H|)$ and $A = \mathbb{Z}[\zeta]$. By a theorem of Atiyah [1, Lemma 6.2], there is a prime ideal $\mathfrak{p} \subseteq A$ and an element $s \in H$ such that

$$\mathfrak{P} = \{v \in R(G) \,|\, \chi_v(s) \in \mathfrak{p}\}.$$

(This is stated in [1] only in the complex case, but the same arguement applies to prime ideals in the real representation ring.) Also, s is not an element of any $L \in \mathcal{F}$, since $\mathfrak{P} \not\supseteq I_{\mathcal{F}}(H)$. Set $p = \mathrm{char}(A/\mathfrak{p})$ (possibly $p = 0$).

For any $g \in G$ of finite order, we let g_r represent its p-regular component: the unique $g_r \in \langle g\rangle$ such that $p\nmid|g_r|$ and $|(g_r)^{-1}g|$ is a power of p ($g_r = g$ if $p = 0$). By [1, Lemma 6.3], we can replace s by s_r without changing the ideal $\mathfrak{P}$; and can thus assume that $p\nmid|s|$.

Let m' be the least common multiple of the orders of isotropy subgroups in Z, and let m be the largest divisor of m' prime to p ($m = m'$ if $p = 0$). Define $\varphi : \mathrm{tors}(G) \to \mathbb{Z}$ by setting $\varphi(g) = 0$ if $g_r \in L$ for some $L \in \mathcal{F}$, and $\varphi(g) = m$ otherwise. By Corollary 4.3, $\varphi|_L$ is a rational character of L for each $L \in \mathrm{Isotr}(Z)$. So by Corollary 6.3, there is $k > 0$ and an element $\xi \in K_G(Z)$ whose restriction to any orbit has character the restriction of φ^k. In other words, $\xi \in I = I^F_{\mathcal{F},Z}$, and so $\varphi^k|_H$ is the character of an element $v \in I'$. But then $\chi_v(s) = \varphi(s)^k \notin \mathfrak{p}$, so $v \notin \mathfrak{P}$, and thus $\mathfrak{P} \not\supseteq I'$.

Step 2 By Step 1, the theorem holds when $\dim(X) = 0$. So we now assume that $\dim(X) = m > 0$. Assume $X = Y \cup_\varphi \bigl(G/H\times D^m\bigr)$, for some attaching map $\varphi : G/H\times S^{m-1} \to Y$. We can assume inductively that the theorem holds for Y, $G/H\times S^{m-1}$, and $G/H\times D^m \simeq G/H$.

All terms in the Mayer-Vietoris sequence

$$\longrightarrow KF_G^*(X) \longrightarrow KF_G^*(Y) \oplus KF_G^*(G/H\times D^m) \longrightarrow KF_G^*(G/H\times S^{m-1}) \longrightarrow$$

are $KF_G(X)$-modules and all homomorphisms are $KF_G(X)$-linear; and the $KF_G(Z)$-module structure on each term is induced from the $KF_G(X)$-module structure. So if we let $I' \subseteq KF_G(X)$ be the ideal generated by the image of I; then dividing out by $(I')^n$ is the same as dividing out by I^n for all terms. In addition,

$KF_G(X)$ is noetherian (in fact, a finitely generated abelian group), and so this Mayer-Vietoris sequence induces an exact sequence of pro-groups

$$\longrightarrow \bigl\{KF_G^*(X)/I^n\bigr\}_{n\geq 1} \longrightarrow \bigl\{KF_G^*(Y)/I^n \oplus KF_G^*(G/H\times D^m)/I^n\bigr\}_{n\geq 1} \longrightarrow \bigl\{KF_G^*(G/H\times S^{m-1})/I^n\bigr\}_{n\geq 1} \longrightarrow$$

by [8, Lemma 4.1]. There is a similar Mayer-Vietoris exact sequence of the pro-groups

$$\bigl\{KF_G^*\bigl((E_{\mathcal{F}}(G)\times -)^{(n-1)}\bigr)\bigr\}_{n\geq 1};$$

and the theorem now follows from the 5-lemma for pro-groups together with the induction assumptions. □

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