“Stable Prime Decompositions of Four-Manifolds” by

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We would like to dedicate this paper to Bill Browder on the occasion of his sixtieth birthday.

Abstract: The main result of this paper is a four-dimensional stable version of Kneser's conjecture on the splitting of three-manifolds as connected sums. Namely, let M be a topological respectively smooth compact connected four-manifold (with orientation or *Spin*-structure). Suppose that $\pi_1(M)$ splits as $*_{i=1}^n \Gamma_i$ such that the image of $\pi_1(C)$ in $\pi_1(M)$ is subconjugated to some Γ_i for each component C of ∂M. Then M is stably homeomorphic respectively diffeomorphic (preserving the orientation or *Spin*-structure) to a connected sum $\sharp_{i=1}^n M_i$ with $\Gamma_i = \pi_1(M_i)$. Stably means that one allows additional connected sums with some copies of $S^2 \times S^2$ on both sides. We also prove a uniqueness statement. As a consequence we obtain the existence and uniqueness of the stable prime decomposition of compact connected four-manifolds (with orientation or *Spin*-structure). The main technical ingredients are the bordism approach to the stable classification of manifolds due to the first author and the Kurosh Subgroup Theorem.

Key words: Stable splitting of four-manifolds as connected sums, Kneser's conjecture, stable classification and bordism theory, Kurosh Subgroup Theorem

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Introduction

A compact connected orientable smooth three-manifold M has a so called prime decomposition. Namely, M is oriented diffeomorphic to a connected sum $\sharp_{i=1}^n M_i$ of oriented manifolds M_i which are prime, i.e. if M_i is diffeomorphic to $M_i' \sharp M_i''$, then M_i' or M_i'' is oriented diffeomorphic to S^3. The manifolds M_i are unique up to order and oriented diffeomorphism.

The corresponding result cannot hold for four-manifolds. For example $(S^2 \times S^2) \sharp \mathbb{C}P^2$ is diffeomorphic to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \sharp \mathbb{C}P^2$. Or, for a simply connected four-dimensional *Spin*-manifold M with non-trivial signature $M \sharp M^-$ is homeomorphic and often diffeomorphic to a connected sum of $(S^2 \times S^2)$'s. The problem here is, that the value of the signature for different pieces is not determined by the large manifold. A natural way to overcome these difficulties is to allow connected sum with an arbitrary *simply connected* closed four-manifold. Up to connected sum with simply connected closed manifolds, we prove the corresponding result in dimension four.

A *stable oriented diffeomorphism* from a four-manifold M to N is an orientation preserving diffeomorphism from $M \sharp k(S^2 \times S^2)$ to $N \sharp \overline{k}(S^2 \times S^2)$ for some non-negative integers k and $\overline{k}$. If the manifolds are equipped with a *Spin*-structure, we can in addition require that these structures are preserved. We call a connected compact orientable smooth four-manifold M *stably prime* if M_i stably oriented diffeomorphic to $M_i'\sharp M_i''$ implies that M_i' or M_i'' is simply connected and closed.

Theorem 0.1 (Stable Prime Decomposition) *Let M be a connected compact oriented smooth four-manifold. Then:*

1. *There are stably prime oriented four-manifolds $M_1, M_2, \ldots, M_n$ and a stable oriented diffeomorphism*
$$f : M \longrightarrow \sharp_{i=1}^n M_i.$$
2. *Let $f' : M \longrightarrow \sharp_{i=1}^{n'} M_i'$ be another stable oriented diffeomorphism for stably prime oriented four-manifolds $M_1', M_2', \ldots, M_{n'}'$. Suppose that none of the M_i's and M_i''s is simply connected and closed. Then $n = n'$ and $M_i \sharp S_i$ and $M'_{\sigma(i)}\sharp S_i'$ are stably oriented diffeomorphic for $i \in \{1, 2, \ldots, n\}$, appropriate simply connected closed oriented four-manifolds S_i and S_i' and a permutation σ.*

Closely related to prime decompositions is Kneser's conjecture. Let M be a compact connected three-manifold with incompressible boundary whose fundamental group admits a splitting $\alpha : \pi_1(M) \longrightarrow \Gamma_1 * \Gamma_2$. Kneser's conjecture whose proof can be found in [6, chapter 7] says that there are manifolds M_1 and M_2 with Γ_1 and Γ_2 as fundamental groups and a homeomorphism $M \longrightarrow M_1\sharp M_2$ inducing α on the fundamental groups. Kneser's conjecture fails even in the closed case in dimensions ≥ 5 by results of Cappell [2],[3]. Counterexamples of closed orientable four-manifolds which even do not split up to homotopy and examples of closed orientable four-manifolds which split topologically but not smoothly are constructed by the authors of this article in [9]. But again it holds stably. We restrict ourselves to oriented manifolds. For simplicity we state in the introduction only an easy to formulate special case of our more general results whose precise statements are given in section 1. A group π is *indecomposable* if π is non-trivial and $\pi \cong \Gamma_1 * \Gamma_2$ implies that Γ_1 or Γ_2 is trivial.

Theorem 0.2 (Stable Kneser Decomposition) *If M is a closed connected smooth oriented four-manifold with non-trivial fundamental group, then there are oriented smooth four-manifolds $M_1, M_2, \ldots, M_n$ with indecomposable $\pi_1(M_i)$ for $i \in \{1, 2, \ldots, n\}$, such that M and $\sharp_{i=1}^n M_i$ are stably oriented diffeomorphic.*

If we have two splittings $\sharp_{i=1}^n M_i$ and $\sharp_{i=1}^{n'} M_i'$ of M as above, then $n = n'$ and $M_i\sharp S_i$ and $M'_{\sigma(i)}\sharp S_i'$ are oriented diffeomorphic for $i \in \{1, 2, \ldots, n\}$, appropriate simply connected closed smooth manifolds S_i, S_i' and a permutation $\sigma \in \Sigma_n$. If M is a Spin-manifold and we equip M_i and M_i' with the Spin-structures induced from a stable diffeomorphism as in Theorem 0.1, we can take S_i and S_i' as Spin-manifolds and the diffeomorphisms Spin-structure preserving.
■

Since the stable diffeomorphism type of a simply connected closed smooth four-manifold is determined by the type of the intersection form (I = odd = non-*Spin* or II = even = *Spin*) and the signature, we can take for the manifolds S either $r(\mathbb{C}P^2) \sharp\, p(S^2 \times S^2)$ in the case I or $rK \sharp\, p(S^2 \times S^2)$ in the case II, where $\mathbb{C}P^2$ is the complex projective space of complex dimension two and K is the Kummer surface (note that by Rohlin's theorem the signature is divisible by $16 = -\operatorname{sign}(K)$).

Both results have topological versions. All manifolds are topological, "diffeomorphic" must be substituted by "homeomorphic" and in the last paragraph $r(\mathbb{C}P^2) \sharp\, p(S^2 \times S^2)$ respectively $rK \sharp\, p(S^2 \times S^2)$ must be substituted by $r(\mathbb{C}P^2) \sharp\, s(E_8) \sharp\, p(S^2 \times S^2)$ respectively $r(E_8) \sharp\, p(S^2 \times S^2)$ where E_8 is the simply connected closed topological four-manifold with E_8 as intersection form whose existence is proved by Freedman [5, Theorem 1.7].

We mention that this article is motivated by a paper of Hillman [7] which shows the existence of a stable splitting for a closed connected four-manifold M with fundamental group $\pi_1(M) = \Gamma_1 * \Gamma_2$. (Actually Hillman only proves that after adding $\mathbb{C}P^2$ or $\overline{\mathbb{C}}P^2$ one gets a splitting but his argument can be modified to give the original statement).

The paper is organized as follows :

The precise statements of our results, also for compact connected four-manifolds with boundary are given in section 1 and from this we deduce Theorems 0.1 and 0.2 using Kurosh's Subgroup Theorem. Section 2 summarizes the bordism approach to the stable classification due to the first author in a setup which is adequate for the purposes of this article and contains some preliminary results. One may skip section 2 and turn directly to the proofs of the main theorems in the following sections and get back to section 2 when necessary.

1. Kneser Splittings for Manifolds with Boundary

All manifolds are assumed to be compact. We will formulate and prove our results for smooth manifolds. With the same modifications as explained in the introduction the analogous results hold for topological manifolds. The proofs are also identically the same replacing everywhere the smooth objects by the corresponding topological ones.

We will use the following convention on fundamental groups. Let $f : X \longrightarrow Y$ be a map of path-connected spaces. If we write $\pi_1(X)$, we mean $\pi_1(X, x)$ after some choice of base point $x \in X$. The homomorphism $\pi_1(f) : \pi_1(X) \longrightarrow \pi_1(Y)$ is the composition of

$\pi_1(f,x) : \pi_1(X,x) \longrightarrow \pi_1(Y,f(x))$ and the isomorphism $c(w) : \pi_1(Y,f(x)) \longrightarrow \pi_1(Y,y)$ given by conjugation with a path w joining $f(x)$ and y. Notice that $\pi_1(f)$ is only well-defined up to inner automorphisms of $\pi_1(Y)$. Given a connected sum $\sharp_{i=1}^n M_i$ we will use the following composition of isomorphisms as identification

$$*_{i=1}^n \pi_1(M_i) \xrightarrow{j^{-1}} *_{i=1}^n \pi_1(M_i - \mathrm{int}(D^4)) \xrightarrow{k} \pi_1(\sharp_{i=1}^n M_i)$$

where j and k are induced by the inclusions in the obvious way. Notice again that this identification is only well-defined up to inner automorphisms. We call a component C of ∂M *π_1-null* if the inclusion induces the trivial map $\pi_1(C) \longrightarrow \pi_1(M)$.

Theorem 1.3 (Existence of Stable Splitting) *Let M be an oriented connected four-manifold with non-trivial fundamental group. Let*

$$\alpha : \pi_1(M) \longrightarrow *_{i=1}^n \Gamma_i$$

be a group isomorphism such that each Γ_i is non-trivial. Suppose for any component C of ∂M that the image of the composition $\alpha \circ \pi_1(j) : \pi_1(C) \longrightarrow *_{i=1}^n \Gamma_i$ is subconjugated to one of the Γ_i's for $j : C \longrightarrow M$ the inclusion.

Then there are oriented connected four-manifolds $M_1, M_2, \ldots, M_n$ with identifications $\pi_1(M_i) \longrightarrow \Gamma_i$ and oriented simply connected four-manifolds $N_1, N_2, \ldots, N_p$ and a stable oriented diffeomorphism

$$f : M \longrightarrow \sharp_{i=1}^n M_i \ \sharp\ \sharp_{j=1}^p N_j$$

such that the composition

$$\pi_1(M) \xrightarrow{\pi_1(f)} \pi_1(\sharp_{i=1}^n M_i \ \sharp\ \sharp_{j=1}^p N_j) \longrightarrow *_{i=1}^n \pi_1(M_i) \longrightarrow *_{i=1}^n \Gamma_i$$

agrees with α up to inner automorphisms, no boundary component of the M_i's is π_1-null and each N_i has a connected non-empty boundary. ■

Theorem 1.4 (Uniqueness of Stable Splitting) *Let $M_1, M_2, \ldots, M_n$ and $M_1', M_2', \ldots, M_n'$ be oriented connected four-manifolds with non-trivial fundamental groups $\Gamma_i = \pi_1(M_i)$ and $\Gamma_i' = \pi_1(M_i')$ such that no boundary component of them is π_1-null. Let $N_1, N_2, \ldots, N_p$ and $N_1', N_2', \ldots, N_q'$ be oriented simply connected four-manifolds whose boundaries are connected and non-empty. Let*

$$f : \sharp_{i=1}^n M_i \ \sharp\ \sharp_{j=1}^p N_j \longrightarrow \sharp_{i=1}^n M_i' \ \sharp\ \sharp_{j=1}^q N_j'$$

be a stable diffeomorphism, which is either oriented or Spin-structure preserving, if the underlying manifolds are Spin. Denote the homomorphism induced by f on the fundamental groups by

$$f_* : *_{i=1}^n \Gamma_i \longrightarrow *_{i=1}^n \Gamma_i'.$$

Suppose for $i \in \{1, 2, \ldots, n\}$ that $\mathrm{pr}_i' \circ f_* \circ j_i$ is an isomorphism where $\mathrm{pr}_i' : *_{i=1}^n \Gamma_i' \longrightarrow \Gamma_i'$ is the canonical projection and $j_i : \Gamma_i \longrightarrow *_{i=1}^n \Gamma_i$ is the canonical inclusion. Then:

For $i \in \{1, 2, \ldots, n\}$ we have $f(\partial M_i) = \partial M_i'$ and there are simply connected oriented closed four-manifolds S_i and S_i' and oriented diffeomorphisms

$$f_i : M_i \sharp S_i \longrightarrow M_i' \sharp S_i'$$

which extend $f|_{\partial M_i} : \partial M_i \longrightarrow \partial M_i'$ and induce up to inner automorphism $\mathrm{pr}_i' \circ f_* \circ j_i$ on the fundamental groups. Moreover, we have $p = q$ and there is an appropriate permutation σ such that $f(\partial N_j) = \partial N_{\sigma(j)}'$ and there are oriented simply connected closed four-manifolds T_j and T_j' and oriented diffeomorphisms

$$g_j : N_j \sharp T_j \longrightarrow N_{\sigma(j)}' \sharp T_j'$$

which extend $f|_{\partial N_j} : \partial N_j \longrightarrow \partial N_j'$. If the manifolds are Spin-manifolds we can choose the manifolds S_i, S_i', T_j and T_j' as Spin-manifolds and f_i and g_i Spin-structure preserving. ■

We finish this section by deriving Theorems 0.2 and 0.1 from these results and the following version of Kurosh's Subgroup Theorem (see [4, Theorem 8,chapter 7 on page 175]).

Theorem 1.5 (Kurosh Subgroup Theorem) *Let H be a subgroup of the free product $G = *_{i\in I} G_i$. There is a suitable chosen set of representatives $g \in \overline{g}$ for the double cosets $\overline{g} \in H\backslash G/G_i$ for each $i \in I$ and a free subgroup $F \subset G$ satisfying*

$$H \;=\; F \;*\; *_{i\in I}\left(*_{\overline{g}\in H\backslash G/G_i}\, gG_ig^{-1} \cap H\right).$$

Let π be a non-trivial finitely generated group. A *Kurosh splitting* is an isomorphism

$$\alpha : \pi \longrightarrow *_{i=0}^{n} \Gamma_i$$

such that Γ_0 is free and Γ_j is indecomposabel and not infinite cyclic for $j = 1, 2, \ldots, n$. Recall that Γ_i is called indecomposable if Γ_i is non-trivial and $\Gamma_i \cong \Gamma_i' * \Gamma_i''$ implies that Γ_i' or Γ_i'' is trivial. The existence and the following uniqueness statement for a second Kurosh splitting $\alpha' : \pi \longrightarrow *_{i=0}^{n'} \Gamma_i'$ follow from Kurosh Subgroup Theorem 1.5. If $j_i : \Gamma_i \longrightarrow *_{i=0}^{n} \Gamma_i$ and $\mathrm{pr}_i' : *_{i=0}^{n} \Gamma_i' \longrightarrow \Gamma_i'$ are the inclusion and projection, then $n = n'$ and there is a permutation σ such that $\mathrm{pr}_{\sigma(i)}' \circ \alpha' \circ \alpha^{-1} \circ j_i$ is an isomorphism for $i \in \{0, 1, \ldots, n\}$. Theorem 0.2 now follows from the above Theorems.

Before we prove Theorem 0.1, we characterize the property "stably prime" in terms of the fundamental group data.

Lemma 1.6 *A connected compact orientable four-manifold M is stably prime if and only if it satisfies the following conditions.*

1. *There is no isomorphism* $\alpha : \pi_1(M) \longrightarrow \Gamma_1 * \Gamma_2$ *for non-trivial groups* Γ_1 *and* Γ_2 *such that for each component* C *of the boundary the composition of* α *and the map induced by the inclusion* $\pi_1(C) \longrightarrow \pi_1(M)$ *has an image which is subconjugated to* Γ_1 *or* Γ_2.

2. *If* M *has a* π_1*-null boundary component, then* M *is simply connected and* ∂M *is non-empty and connected.*

Proof : If one of the conditions above is violated, Theorem 1.3 gives a splitting of M into $M_1 \sharp M_2$ such that neither M_1 nor M_2 is simply connected and closed. Conversely, given such a splitting, one sees immediately, that at least one of the conditions above is not fullfilled. ■

In particular a connected closed orientable four-manifold is stably prime if and only if $\pi_1(M)$ is trivial or indecomposable. Now, we prove Theorem 0.1.

Proof : 1.) The existence of f follows from the following inductive process. If M is stably prime, the process stops. If M is not stably prime, choose a stable oriented diffeomorphism $M \longrightarrow M_1 \sharp M_2$ such that neither M_1 nor M_2 is simply connected and closed. Now apply this process to both M_1 and M_2. It remains to show that this process stops after a finite number of steps. This follows from Lemma 1.6, the Grushko-Neumann Theorem [10, Theorem 1.8 and Corollary 1.9 on page 178] which implies that the rank of a group, i.e. the minimal number of generators, is additive under free products and the simple fact that M has only finitely many π_1-null boundary components.

2.) Consider a stable oriented diffeomorphism

$$f : (\sharp_{i=1}^{l} L_i) \sharp (\sharp_{i=1}^{n} M_i) \sharp (\sharp_{i=1}^{p} N_i) \longrightarrow (\sharp_{i=1}^{l'} L_i') \sharp (\sharp_{i=1}^{n'} M_i') \sharp (\sharp_{i=1}^{p'} N_i')$$

such that each L_i, L_i', M_i, M_i', N_i and N_i' is stably prime, each L_i is closed and has infinite cyclic fundamental group, none of the M_i's and M_i''s is simply connected or has both infinite cyclic fundamental group and empty boundary, and each N_i and N_i' is simply connected and has a non-empty boundary. Notice that any finite connected sum of stably prime connected four-manifolds can be written in this way if none of the summands is simply connected and closed. We conclude from Lemma 1.6 that none of the M_i's and M_i''s has a π_1-null boundary component and that the boundary of each N_i and N_i' is non-empty and connected. We abbreviate in the sequel $\Gamma_i = \pi_1(M_i)$ and $\Gamma_i' = \pi_1(M_i')$ and introduce the finitely generated free groups $\Gamma_0 = *_{i=1}^{l} \pi_1(L_i)$ and $\Gamma_0' = *_{i=1}^{l'} \pi_1(L_i)$. The map induced on the fundamental groups by f is denoted by

$$f_* : *_{i=0}^{n} \Gamma_i \longrightarrow *_{j=0}^{n'} \Gamma_i'.$$

Fix an index $i \in \{1, 2, \ldots, n\}$. We apply Kurosh Subgroup Theorem 1.5 to $f_*(\Gamma_i) \subset *_{j=0}^{n} \Gamma_j'$ and obtain

$$f_*(\Gamma_i) = F * *_{j=0}^{n'} \left(*_{\overline{g} \in f_*(\Gamma_i) \backslash *_{j=0}^{n'} \Gamma_j' / \Gamma_j'} \; g \Gamma_j' g^{-1} \cap f_*(\Gamma_i) \right).$$

Let C be a boundary component of M_i. There is an index $j \in \{1, 2, \ldots, n'\}$ such that $f(C) \subset M_j'$. If $j_* : \pi_1(C) \longrightarrow \Gamma_i$ is the map induced by the inclusion, then $f_*(j_*(\pi_1(C)))$ is subconjugated to Γ'_j. Hence there is $g_0 \in *_{i=0}^{n'}\Gamma'_i$ such that $f_*(j_*(\pi_1(C))) \subset g_0\Gamma'_j g_0^{-1} \cap f_*(\Gamma_i)$ holds. We conclude from Kurosh Subgroup Theorem 1.5 that $f_*(j_*(\pi_1(C)))$ is subconjugated to $g\Gamma'_j g^{-1} \cap f_*(\Gamma_i)$ for appropriate $\overline{g} \in f_*(\Gamma_i)\backslash *_{j=0}^{n'} \Gamma'_j/\Gamma'_j$. Recall that M_i is stably prime, has no π_1-null boundary component and it is not true that M_i has both infinite cyclic fundamental group and empty boundary. We derive from Lemma 1.6 applied to the isomorphism induced by f_*

$$\pi_1(M_i) = \Gamma_i \longrightarrow f_*(\Gamma_i) \; = \; F \; * \; *_{j=0}^{n'} \left(*_{\overline{g} \in f_*(\Gamma_i)\backslash *_{j=0}^{n'}\Gamma'_j/\Gamma'_j} \, g\Gamma'_j g^{-1} \cap f_*(\Gamma_i) \right).$$

that there is a unique index $\sigma(i) \in \{1, 2, \ldots, n\}$ and $\overline{g} \in f_*(\Gamma_i)\backslash *_{j=0}^{n'} \Gamma'_j/\Gamma'_j$ satisfying

$$f_*(\Gamma_i) \; = \; g\Gamma'_{\sigma(i)} g^{-1}.$$

We get a map $\sigma : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n'\}$. Completely analogously one defines a map $\sigma' : \{1, 2, \ldots, n'\} \longrightarrow \{1, 2, \ldots, n\}$ such that for each $j \in \{1, 2, \ldots, n'\}$ there is $g \in *_{i=0}^{n}\Gamma_i$ satisfying

$$f_*^{-1}(\Gamma'_j) \; = \; g\Gamma_{\sigma'(j)} g^{-1}.$$

Let $\mathrm{pr}'_j : *_{j=0}^{n}\Gamma'_i \longrightarrow \Gamma'_j$ be the canonical projection and $j_i : \Gamma_i \longrightarrow *_{i=0}^{n}\Gamma_i$ be the canonical inclusion. We conclude for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 2, \ldots, n'\}$ that the composition $\mathrm{pr}'_j \circ f_* \circ j_i$ is an isomorphism if $j = \sigma(i)$ and trivial otherwise. Hence $\sigma' \circ \sigma = \mathrm{id}$ and following diagram commutes for appropriate $\overline{f_*}$

$$\begin{array}{ccc} *_{i=0}^{n}\Gamma_i & \xrightarrow{f_*} & *_{i=0}^{n}\Gamma'_i \\ \mathrm{pr}_0 \downarrow & & \downarrow \mathrm{pr}'_0 \\ \Gamma_0 & \xrightarrow[\overline{f_*}]{} & \Gamma'_0 \end{array}$$

The same argument applied to f_*^{-1} shows that $\sigma \circ \sigma' = \mathrm{id}$ and that $\overline{f_*}$ has an inverse. Hence σ and σ' are inverse to one another, $n = n'$ and the composition $\mathrm{pr}'_{\sigma(i)} \circ f_* \circ j_i$ is an isomorphism for $i \in \{0, 1, 2, \ldots, n\}$ if we put $\sigma(0) = 0$.

From Theorem 1.4 we conclude that $M_i \sharp S_i$ and $M_{\sigma(i)} \sharp S'_i$ are stably oriented diffeomorphic for each $i \in \{1, 2, \ldots, n\}$, appropriate simply connected closed oriented four-manifolds S_i and S'_i and that $p = p'$ and $N_i \sharp T_i$ and $N'_{\tau(i)} \sharp T'_i$ are stably oriented diffeomorphic for each $i \in \{1, 2, \ldots, p\}$, appropriate simply connected closed oriented four-manifolds T_i and T'_i and permutation τ. Since Γ_0 and Γ'_0 are isomorphic, we get $l = l'$. Each L_i and L'_i is stably isomorphic to $S^1 \times S^3$ after adding simply connected closed oriented four-manifolds for $i \in \{1, 2, \ldots, l\}$ by Theorem 2.1 and Lemma 2.3 since L_i and L'_i are closed and have infinite cyclic fundamental groups. This finishes the proof of Theorem 0.1.

2. Stable Classification and Bordism Theory

In this section we explain the necessary details of the bordism approach to the stable classification of manifolds due to the first author and prove some preliminary lemmas. Recall that all manifolds are assumed to be compact and we restrict ourselves to smooth manifolds.

We begin with organizing the bookkeeping of the fundamental group data. We consider pairs (π, w_2) which consist of a finitely presented group π and an element w_2 in $H^2(\pi;\mathbb{Z}/2) \coprod \{\infty\}$. We call two such pairs (π, w_2) and (π', w_2') *equivalent* if there is an isomorphism $f : \pi \longrightarrow \pi'$ with the properties that either $w_2 = \infty$ and $w_2' = \infty$ or $w_2 \in H^2(\pi;\mathbb{Z}/2)$, $w_2' \in H^2(\pi';\mathbb{Z}/2)$ and $f^*(w_2') = w_2$ holds. A *type* T is an equivalence class $[\pi, w_2]$ of such pairs.

An oriented manifold determines a type $T(M)$, called the *normal* 1-*type*, for which a representative is given as follows. Put $\pi = \pi_1(M)$. Let $g : M \longrightarrow K(\pi, 1)$ be a classifying map of the universal covering and denote by $w_k(M) \in H^k(M;\mathbb{Z}/2)$ the k-th Stiefel-Whitney class of the normal bundle of M. If $w_2(\widetilde{M}) \neq 0$ holds for the universal covering $\widetilde{M}$, then put $w_2 = \infty$. Otherwise let w_2 be the unique element satisfying $g^*(w_2) = w_2(M)$. The unique existence follows from the exact sequence coming from the Serre spectral sequence of the fibration $\widetilde{M} \longrightarrow M \longrightarrow K(\pi, 1)$

$$0 \longrightarrow H^2(K(\pi,1);\mathbb{Z}/2) \xrightarrow{g^*} H^2(M;\mathbb{Z}/2) \longrightarrow H^2(\widetilde{M};\mathbb{Z}/2).$$

Two homotopy equivalent manifolds have the same normal 1-type.

Before we introduce the relevant bordism groups, we recall how to convert a continous map $u : X \longrightarrow K$ into a fibration $u' : P(u) \longrightarrow K$. Define

$$P(u) = \{(x, w) \mid w(0) = u(x)\} \subset X \times \mathrm{map}(I, K)$$

and $u'(x, w) = w(1)$. Define the map $u'' : P(u) \longrightarrow X$ by sending (x, w) to x and define the homotopy $\psi : u \circ u'' \simeq u'$ by sending $((x, w), t)$ to $w(t)$. The triple $(P(u), u'', \psi)$ has the universal property that for any space Z together with maps $f' : Z \to K$ and $f'' : Z \longrightarrow X$ and homotopy $\phi : u \circ f'' \simeq f'$ there is precisely one map $g : Z \longrightarrow P(u)$ such that

$$f'' = u'' \circ g, \quad f' = u' \circ g \quad \text{and} \quad \phi = \psi \circ (g \times \mathrm{id}).$$

Namely, define $g(z) = (f''(z), \psi_z)$ for ψ_z the path sending t to $\psi(z, t)$. There is a map $i : X \longrightarrow P(u)$ sending x to $(x, c_{u(x)})$ where $c_{u(x)}$ is the constant path in K at $u(x)$. It is a homotopy inverse of u'' and its composition with u' is u.

A type T determines a fibration $\mathcal{B}(\mathcal{T})$ over BSO or over $BSpin$, if $w_2 = 0$, as follows. Let $[\pi, w_2]$ be a representative of T. If $w_2 = \infty$ define it as the trivial fibration

$$\mathcal{B}(\mathcal{T}) = \mathcal{BSO} \times \mathcal{K}(\pi, \infty) \to \mathcal{BSO}$$

over BSO. If $w_2 = 0$ we define it as the trivial fibration

$$\mathcal{B}(\mathcal{T}) = \mathcal{BS}_{\surd}\rangle\backslash \times \mathcal{K}(\pi, \infty) \to \mathcal{BS}_{\surd}\rangle\backslash$$

over $BSpin$. If $w_2 \neq 0,\ \infty$ represent w_2 by a map $u : K(\pi, 1) \to K(\mathbb{Z}/2, 2)$ with corresponding fibration $P(u)$ over $K(\mathbb{Z}/2, 2)$. Represent the universal Stiefel Whitney class by a map $q : BSO \to K(\mathbb{Z}/2, 2)$. Then define our fibration by the pullback

$$\mathcal{B}(\mathcal{T}) = \amalg^*(\mathcal{P}(\sqcap)) \to \mathcal{BSO}.$$

These fibrations are up to fibre homotopy equivalence uniquely determined by T. In all three cases there are projection maps to $K(\pi,1)$ denoted by $p_{K(\pi,1)}$.

Suppose that M has normal 1-type $T(M)$ and let $g : M \to K(\pi,1)$ be a map satisfying $g^* w_2 = w_2(\nu(M))$ if $w_2(M) = w_2(\nu(M)) \neq \infty$. Then the normal Gauss map $\nu : M \to BSO$ or $\nu : M \to BSpin$, if M is has a *Spin*-structure, admits a lift ρ over $\mathcal{B}(\mathcal{T})$ as follows. If $w_2 = \infty$ it is given by the normal Gauss map together with g. If $w_2 \neq \infty$ it is given by the normal Gauss map together with g and with a homotopy between the composition of the two maps to $K(\mathbb{Z}/2,2)$. We call such a lift ρ a *normal structure* of M in $\mathcal{B}(\mathcal{T}(\mathcal{M}))$ compatible with g and the orientation resp. *Spin*-structure. If a normal structure ρ is a 2-equivalence, it is called a *normal 1-smoothing*. Notice that a normal structure ρ is a normal 1-smoothing if and only if the underlying map g induces an isomorphism on π_1.

Given a fibration $\mathcal{B} \to \mathcal{BSO}$ or $\mathcal{B} \to \mathcal{BS}_{\surd}\rangle\backslash$ we denote the bordism group of n-dimensional closed oriented or *Spin*-manifolds together with a lift of the normal Gauss map over $\mathcal{B}$ by

$$\Omega_n(\mathcal{B}).$$

If ρ is a normal 1-smoothing of M in $\mathcal{B}(\mathcal{T}(\mathcal{M}))$, then the pair (M,ρ) determines an element in $\Omega_4(\mathcal{B}(\mathcal{T}(\mathcal{M})))$.

Now we can formulate the main result of the bordism approach to the stable classification of connected four-manifolds due to the first author [8].

Theorem 2.1 (Stable Classification of Four-Manifolds by Bordism Theory) *Let M_1 and M_2 be connected four-manifolds with orientation respectively Spin-structure and $\partial f : \partial M_1 \longrightarrow \partial M_2$ be a diffeomorphism which preserves the induced orientation respectively Spin-structure. Suppose that the normal 1-type of M_1 and M_2 is equal to T and denote by $\mathcal{B}(\mathcal{T})$ any representation of the associated fibration. Let $g_i : M_i \to K(\pi,1)$ be classifying maps of the universal covering respecting w_2 such that $g_2|_{\partial M_2} \circ \partial f = g_1|_{\partial M_1}$.*

1. *There exists a stable oriented (Spin-structure preserving, if M_i are Spin) diffeomorphism*

$$f : M_1 \longrightarrow M_2$$

   extending ∂f such that the maps $g_2 \circ f$ and g_1 to $K(\pi,1)$ are homotopic if and only if there are normal 1-smoothings ρ_i of M_i compatible with g_i and the orientations resp. Spin-structures, such that $\rho_2|_{\partial M_2} \circ \partial f$ and $\rho_1|_{\partial M_1}$ agree and

$$[M_1^- \cup_{\partial f} M_2, \rho_1^- \cup_{\partial f} \rho_2] = 0 \qquad \in \Omega_4(\mathcal{B}(\mathcal{T}))$$

2. *Given a manifold with boundary together with a lift of the normal Gauss-map to $\mathcal{B}(\mathcal{T})$, it is bordant relative boundary to a normal 1-smoothing.*

The strategy for proving the main Theorems is to analyse how the bordism group decomposes if the fundamental group splits as a free product. For this the following categorial considerations are useful.

Denote $K = K(\mathbb{Z}/2,2)$ or $K = *$. Define a category $\mathcal{C}$ as follows. An object (X,u) is a map $u : X \longrightarrow K$ and a morphism $(f,\phi) : (X,u) \longrightarrow (Y,v)$ consists of a map $f : X \longrightarrow Y$ together with a homotopy $\phi : v \circ f \simeq u$. The composition $(g,\psi) \circ (f,\phi)$ is defined by $(g \circ f, (\psi \circ (f \times \mathrm{id})) * \phi)$ where $*$ denotes the composition of homotopies. If the homotopy ψ is the constant homotopy, we abbreviate (f,ψ) by f. Two morphisms (f_0,ψ_0) and (f_1,ψ_1) from (X,u) to (Y,v) are called homotopic if they can be connected by a continuous one parameter family of morphisms (f_t,ϕ_t). The following elementary facts will frequently be used in the sequel.

Lemma 2.2 *Let $(f,\phi) : (X,u) \longrightarrow (Y,v)$ be a morphism. If $g : X \longrightarrow Y$ is a map homotopic to f, then there is a homotopy $\psi : v \circ g \simeq u$ such that the morphisms (f,ϕ) and (g,ψ) are homotopic. If $f : X \longrightarrow Y$ is a homotopy equivalence, then there is a morphism $(g,\psi) : (Y,v) \longrightarrow (X,u)$ such that both compositions of (f,ϕ) and (g,ψ) are homotopic to the identity morphism.* ∎

Let $q : B \to K$ be a fixed map. Given an object (X,u), we have the pullback

$$\begin{array}{ccc} \mathcal{B}(\sqcap) & \xrightarrow{\bar{q}} & P(u) \\ {\scriptstyle \bar{u}}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \xrightarrow[q]{} & K \end{array}$$

For a manifold M and appropriate choices of q and u, the fibration $\mathcal{B}(\sqcap)$ over B corresponds to a normal 1-type as described above. More precisely, if $w_2 = \infty$, let $B = BSO$, $K = *$ and $X = K(\pi_1(M),1)$. For $w_2 = 0$ choose $B = BSpin$ instead of BSO. For $w_2 \neq 0$, ∞ choose $B = BSO$, $K = K(\mathbb{Z}/2,2)$, $X = K(\pi_1(M),1)$ and q and u maps representing the second Stiefel Whitney classes. Then in all three cases $\mathcal{B}(\mathcal{T}(\mathcal{M})) = \mathcal{B}(\sqcap)$.

For the special purpose of this paper the formulation of $\mathcal{B}(\sqcap)$ has the advantage that it separates the categorial input, namely the fundamental group data encoded in $K(\pi,1)$ from the other data like orientation and *Spin*-structure.

Define

$$\Omega_n(X,u) \;=\; \Omega_n(\mathcal{B}(\sqcap))$$

A morphism $(f,\phi) : (X,u) \longrightarrow (Y,v)$ defines by the universal property of the construction $P(-)$ a fiber map $P(f,\phi) : P(u) \longrightarrow P(v)$ where fiber map means $P(f,\phi) \circ v' = u'$. If we apply Ω_n to it, we obtain a homomorphism denoted by

$$\Omega_n(f,\phi) : \Omega_n(X,u) \longrightarrow \Omega_n(Y,v).$$

Clearly this is a functor on $\mathcal{C}$. Moreover, it is a generalized homology theory in the sense that it has the following properties. It is homotopy invariant, i.e., homotopic morphisms induce the same homomorphism. There is a Mayer-Vietoris sequence in the following sense. Consider the following pushout

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X \end{array}$$

with i_1 a cofibration. Put $j_0 = j_2 \circ i_2 = j_1 \circ i_1$. Let (X,u) be an object. We obtain objects (X_k, u_k) by $u_k = u \circ j_k$ and morphisms $j_k : (X_k, u_k) \longrightarrow (X,u)$ for $k = 0,1,2$. Recall that we omit constant homotopies in our notation for morphisms. Now there is a long exact Mayer-Vietoris sequence

$$\ldots \xrightarrow{\delta} \Omega_n(X_0,u_0) \xrightarrow{(\Omega_n(i_1),\Omega_n(i_2))} \Omega_n(X_1,u_1) \oplus \Omega_n(X_2,u_1) \xrightarrow{\Omega_n(j_1)-\Omega_n(j_2)} \Omega_n(X,u)$$
$$\xrightarrow{\delta} \Omega_{n-1}(X_0,u_0) \longrightarrow \quad \ldots$$

Namely, we obtain a pushout with a cofibration as horizontal upper arrow

$$\begin{array}{ccc} \overline{P(u_0)} & \xrightarrow{\overline{P(i_1)}} & \overline{P(u_1)} \\ {\scriptstyle \overline{P(i_2)}}\downarrow & & \downarrow{\scriptstyle \overline{P(j_1)}} \\ \overline{P(u_2)} & \xrightarrow[\overline{P(j_2)}]{} & \overline{P(u_0)} \end{array}$$

Notice that the bordism group $\Omega_n(\mathcal{B})$ can be identified with the bordism group of the stable vector bundle over $\mathcal{B}$ which is the pullback of the universal bundle over BSO respectively $BSpin$ and thus the existence of the Mayer-Vietoris sequence follows by standard arguments, namely the Pontrjagin-Thom construction and the fact that stable homotopy is a generalized homology theory [1, Kapitel II].

Let (X,u) be an object with path-connected X. Denote by $*$ the space consisting of one point. Consider an object $(*,v)$ and a morphism $(j,\mu) : (*,v) \longrightarrow (X,u)$. Define

$$\widetilde{\Omega}_n(X,u) \ := \ \operatorname{cok}\left(\Omega_n(j,\mu) : \Omega_n(*,v) \longrightarrow \Omega_n(X,u)\right)$$

We want to show that the definition of $\widetilde{\Omega}_n(X,u)$ is independent of the choice of v and (j,μ) and that a morphism $(f,\phi) : (X,u) \longrightarrow (Y,v)$ induces a homomorphism making the following diagram commute for pr the canonical projection.

$$\begin{array}{ccc} \Omega_n(X,u) & \xrightarrow{\text{pr}} & \widetilde{\Omega}_n(X,u) \\ {\scriptstyle \Omega_n(f,\phi)}\downarrow & & \downarrow{\scriptstyle \widetilde{\Omega}_n(f,\phi)} \\ \Omega_n(Y,v) & \xrightarrow[\text{pr}]{} & \widetilde{\Omega}_n(Y,v) \end{array}$$

This follows from the following fact and Lemma 2.2. If $(j', \mu') : (*, v') \longrightarrow (Y, v)$ is a morphism and (j, μ) and (f, ϕ) are as above, then there is a morphism $(\mathrm{id}, \psi) : (*, v) \longrightarrow (*, v')$ such that $(f, \phi) \circ (j, \mu)$ and $(j', \mu') \circ (\mathrm{id}, \psi)$ are homotopic morphisms.

The reduced group $\widetilde{\Omega}_n(X, u)$ is relevant for our Uniqueness Theorem 1.4 since there we classify stably up to connected sum with a simply connected oriented resp. *Spin*-manifold and by Theorem 2.1 this is decided in the reduced bordism group.

Next we make some computations for this generalized homology theory. Recall that sign denotes the signature. The group $\Omega_n(*, v)$ is either equal to Ω_n^{SO} if $B = BSO$, or to Ω_n^{Spin} if $B = BSpin$.

Lemma 2.3 *1. The following table gives generators and explicite isomorphisms for the various bordism groups:*

$\Omega_3^{SO} = 0$	
$\Omega_3^{Spin} = 0$	
$\mathrm{sign} : \ \Omega_4^{SO} \xrightarrow{\cong} \mathbb{Z}$	$\mathbb{C}P^2$
$\mathrm{sign} : \ \Omega_4^{Spin} \xrightarrow{\cong} 16 \cdot \mathbb{Z}$	K

2. $\widetilde{\Omega}_4(K(F, 1), u)) = 0$ *for F a finitely generated free group and both cases $B = BSO$ or $B = BSpin$.*

Proof : 1.) is standard. 2.) follows from the Mayer-Vietoris sequence applied to a wedge of S^1's and to the pushout which decribes S^1 as the identification of the two end points of $[0, 1]$ to one point. ■

3. Proof of the Existence of a Stable Kneser Splitting

In this section we prove Theorem 1.3. We recall that the normal 1-type of a manifold M determines a fibration $\mathcal{B}(\mathcal{T}(\mathcal{M}))$. With the notation of the last section, if M has $w_2 = \infty$, let $B = BSO$, $K = *$ and $X = K(\pi_1(M), 1)$. For $w_2 = 0$ choose $B = BSpin$ instead of BSO. For $w_2 \neq 0, \infty$ choose $B = BSO$, $K = K(\mathbb{Z}/2, 2)$, $X = K(\pi_1(M), 1)$ and q and u maps representing the second Stiefel Whitney classes. Then in all three cases $\mathcal{B}(\mathcal{T}(\mathcal{M})) = \mathcal{B}(\sqcap)$.

Firstly we show that we can assume without loss of generality that no boundary component C of M is π_1-null, i.e. the inclusion induces the trivial map $\pi_1(C) \longrightarrow \pi_1(M)$. Let C_1, C_2, …,C_m be the π_1-null boundary components of M. Since $\Omega_3(*, v)$ is trivial by Lemma 2.3, there is a nullbordism N_i for each C_i with respect to $(*, v)$. By 0- and 1-dimensional surgery on the interior of N_i we can achieve that N_i is simply connected. Define

$$\widehat{M} = M \ \cup_{C_1} N_1^- \cup_{C_2} \ldots \cup_{C_m} N_m^-$$

By Theorem 2.1 there is a stable oriented diffeomorphism $f : M \longrightarrow \widehat{M} \sharp N_1 \sharp \ldots \sharp N_m$ which induces on the fundamental groups the isomorphism induced by the inclusion of M in $\widehat{M}$. No boundary component of $\widehat{M}$ is π_1-null. Obviously it suffices to prove the claim for $\widehat{M}$.

If C is a component of ∂M, there is by assumption an index $i \in \{1, 2, \ldots, n\}$ such that the image of $\alpha \circ \pi_1(j)$ for $j : C \longrightarrow M$ the inclusion is subconjugated to Γ_i. Since we also assume that this image is non-trivial, this index is unique. For $i \in \{1, 2, \ldots, n\}$ let $\partial_i M$ be the union of those components C of ∂M for which this index is i. Since the inclusion $\partial M \longrightarrow M$ is a cofibration, we can construct maps $g : M \longrightarrow \vee_{i=1}^n K(\Gamma_i, 1)$ and $\partial_i g : \partial_i M \longrightarrow K(\Gamma_i, 1)$ for $i \in \{1, 2, \ldots, n\}$ such that the restriction of g to $\partial_i M$ is the composition of g_i with the canonical inclusion $j_i : K(\Gamma_i, 1) \longrightarrow \vee_{i=1}^n K(\Gamma_i, 1)$ and g induces α on the fundamental groups. Choose pointed maps $u_i : K(\Gamma_i, 1) \longrightarrow K$ such that the composition $u = (\vee_{i=1}^n u_i) \circ g : M \longrightarrow K$ corresponds to the Stiefel-Whitney classes of M in the case where K is not a point, but $K(\mathbb{Z}/2, 2)$. Let ρ be normal 1-smoothings of M in $\mathcal{B}(\sqcap)$ compatible with g and the orientation resp. *Spin*-structure. Denote the restriction of ρ to $\partial_i M$ by $\partial_i \rho$. By construction the homomorphism

$$\oplus_{i=1}^n \Omega_3(j_i) : \oplus_{i=1}^n \Omega_3(K(\Gamma_i, 1), u_i) \longrightarrow \Omega_3(\vee_{i=1}^n K(\Gamma_i, 1), \vee_{i=1}^n u_i)$$

sends $([\partial_i M, \partial_i \rho] \mid i = 1, 2, \ldots, n)$ to the element $[\partial M, \rho|_{\partial M}]$ which is zero since (M, ρ) is a nullbordism for its representative. This homomorphism is injective by a Mayer-Vietoris argument and Lemma 2.3. Hence we can find nullbordisms (V_i, σ_i) for $(\partial_i M, \partial_i \rho)$ with respect to $(K(\Gamma_i, 1), u_i)$ for $i \in \{1, 2, \ldots, n\}$. By the same argument as above the homomorphism

$$\oplus_{i=1}^n \Omega_4(j_i) : \oplus_{i=1}^n \Omega_4(K(\Gamma_i, 1), u_i) \longrightarrow \Omega_4(\vee_{i=1}^n K(\Gamma_i, 1), \vee_{i=1}^n u_i)$$

is surjective. Let $([W_i, \tau_i] \mid i = 1, 2, \ldots, n)$ be a preimage of $-\left[M^- \cup_{\partial M} \coprod_{i=1}^n V_i, \rho^- \cup \coprod_{i=1}^n \sigma_i\right]$ and then we get

$$\left[M^- \cup_{\partial M} \coprod_{i=1}^n (V_i \coprod W_i), \rho^- \cup \coprod_{i=1}^n (\sigma_i \coprod \tau_i)\right] = 0 \qquad \in \Omega_4(\vee_{i=1}^n K(\Gamma_i, 1), \vee_{i=1}^n u_i).$$

By Theorem 2.1 $(V_i \coprod W_i), (\sigma_i \coprod \tau_i)$ is bordant relative boundary to a normal 1-smoothing (M_i, ρ_i). We have

$$[M \cup_{\partial M} \sharp_{i=1}^n M_i, \rho \cup \sharp_{i=1}^n (j_i \circ \rho_i)] = 0 \qquad \in \Omega_4(\vee_{i=1}^n K(\Gamma_i, 1), \vee_{i=1}^n u_i).$$

By Theorem 2.1 there is a stable oriented diffeomorphism

$$f : M \longrightarrow \sharp_{i=1}^n M_i$$

such that the composition $\sharp_{i=1}^n (j_i \circ g_i) \circ f$ is homotopic to $g : M \longrightarrow \vee_{i=1}^n K(\Gamma_i, 1)$. This finishes the proof of Theorem 1.3. ■

4. Proof of the Uniqueness Result

This section is devoted to the proof of Theorem 1.4. We firstly show that we can assume without loss of generality that none of the manifolds N_j respectively N'_j are present. By counting the π_1-null components we conclude $p = q$. After possibly renumbering the N'_j's, we can assume that f maps ∂N_j to $\partial N'_j$ for all $j \in \{1, 2, \ldots, q\}$. Since each N_j and N'_j is simply connected, the desired oriented diffeomorphism $g_j : N_j \sharp T_j \longrightarrow N'_j \sharp T'_j$ exists by Theorem 2.1. Again by Theorem 2.1 there is a stable orientation respectively *Spin*-structure preserving diffeomorphism

$$\sharp_{i=1}^n M_i \longrightarrow \sharp_{i=1}^n M_i \sharp \sharp_{j=1}^p N_j \cup_{\partial N_j} N_j^-$$

inducing on the fundamental group the obvious isomorphism and the claim follows.

Suppose for a moment that $\sharp_{i=1}^n M_i$ and $\sharp_{i=1}^n M'_i$ are *Spin*. We want to show that M_i and M'_i are diffeomorphic modulo connected sum with appropriate simply connected four-manifolds with *Spin*-structure. Notice that for all M_i and M'_i the normal 1-type has $w_2 = 0$. Thus we have to show that M_i and M'_i have same normal 1-type and admit normal 1-smoothings in $\mathcal{B}_\rangle$ which induce the right map on π_1 as stated in Theorem 1.4 and are compatible with $f|_{\partial M_i}$ such that M_i and M'_i are bordant rel. boundary (identified via $f|_{\partial M_i}$) in the reduced bordism group corresponding to the normal 1-type, which here is $\widetilde{\Omega}_4^{Spin}(K(\Gamma'_i, 1))$. If $\sharp_{i=1}^n M_i$ and $\sharp_{i=1}^n M'_i$ are just oriented manifolds we are allowed to modify M_i and M'_i by connected sum with any oriented simply connected four-manifold and after adding copies of $\mathbb{C}P^2$ we can assume that the normal 1-type for all M_i and M'_i has $w_2 = \infty$. Then we have to show that M_i and M'_i have same normal 1-type and admit normal 1-smoothings in $\mathcal{B}_\rangle$ which induce the right map on π_1 as stated in Theorem 1.4 and are compatible with $f|_{\partial M_i}$ such that M_i and M'_i are bordant relative boundary (identified via $f|_{\partial M_i}$) in the reduced bordism group corresponding to the normal 1-type, which here is $\widetilde{\Omega}_4(K(\Gamma'_i, 1)) = \widetilde{\Omega}_4^{SO}(K(\Gamma'_i, 1))$. In the following proof the argument is identically the same in the *Spin*-case and in the oriented case and thus we restrict ourselves for simplicity to the oriented case.

We first show $f(\partial M_i) = f(\partial M'_i)$. Let C be a component of ∂M_i for $i \in \{0, 1, \ldots, n\}$. Since $\mathrm{pr}'_i \circ f_* \circ j_i$ is an isomorphism and C is not π_1-null in M_i, the image of the composition of pr'_i and the homomorphism $\pi_1(f(C)) \longrightarrow \pi_1(\sharp_{i=1}^n M'_i) = *_{i=1}^n \Gamma'_i$ induced by the inclusion is non-trivial. This implies $f(C) \subset M'_i$.

Let W' be obtained from $\coprod_{i=1}^n M'_i \times [0, 1]$ by attaching 1-handles to $\coprod_{i=1}^n M'_i \times \{1\}$ such that

$$\partial W' = \coprod_{i=1}^n (M'_i)^- \cup_{\coprod_{i=1}^n \partial M'_i} \sharp_{i=1}^n M'_i$$

where we identify M'_i with $M'_i \cup_{\coprod_{i=1}^n \partial M'_i \times \{0\}} \coprod_{i=1}^n \partial M'_i \times [0, 1]$. Define analogously W for the M_i's. Let $V = W^- \cup_f W'$ be obtained by glueing W and W' together along f. Choose a map $h'_i : M'_i \longrightarrow K(\Gamma'_i, 1)$ inducing the identity on the fundamental groups and mapping the embedded disk where the 1-handles are attached to the base point. Let $h : W' \longrightarrow \vee_{i=1}^n K(\Gamma'_i, 1)$ be the map which is on $M_i \times [0, 1]$ the composition of the projection $M_i \times [0, 1] \longrightarrow M_i$, h_i and the canonical inclusion of $K(\Gamma'_i, 1)$ into $\vee_{i=1}^n K(\Gamma_i, 1)$ and on the one-handles the constant map. Since the inclusion of W into V is 3-connected we can extend this map to a map

$h : V \longrightarrow \vee_{i=1}^n K(\Gamma_i, 1)$. Notice for the sequel that the restriction of this map to M_i' composed with the projection $\mathrm{pr}_k' : \vee_{i=1}^n K(\Gamma_i', 1) \longrightarrow K(\Gamma_k, 1)$ is the constant map for $k \neq i$. Restricting a normal structure of V compatible with h to $M_i \cup_{f|_{\partial M_i}} M_i'$ yields a normal structure ρ_i for $M_i \cup_{f|_{\partial M_i}} M_i'$ with respect to $\vee_{i=1}^n K(\Gamma_i', 1)$. In the sequel we abbreviate $M_i^- \cup_{f|_{\partial M_i}} M_i'$ by $M_i \cup M_i'$. We conclude

Lemma 4.1 *We have*

$$\sum_{i=1}^n [M_i \cup M_i', \rho_i] \;=\; 0 \qquad \in \widetilde{\Omega}_4(\vee_{i=1}^n K(\Gamma_i', 1)). \qquad \blacksquare$$

The projection $\mathrm{pr}_j : \vee_{i=1}^n K(\Gamma_i', 1) \longrightarrow K(\Gamma_j, 1)$ induces a homomorphisms

$$\widetilde{\Omega}_4(\mathrm{pr}_j) : \widetilde{\Omega}_4(\vee_{i=1}^n K(\Gamma_i', 1)) \longrightarrow \widetilde{\Omega}_4(K(\Gamma_j', 1)).$$

Notice that

$$\oplus_{i=1}^n \widetilde{\Omega}_4(j_i) : \oplus_{i=1}^n \widetilde{\Omega}_4(K(\Gamma_i', 1)) \longrightarrow \widetilde{\Omega}_4(\vee_{i=1}^n K(\Gamma_i', 1))$$

and

$$\oplus_{i=1}^n \widetilde{\Omega}_4(\mathrm{pr}_i) : \widetilde{\Omega}_4(\vee_{i=1}^n K(\Gamma_i', 1)) \longrightarrow \oplus_{i=1}^n \widetilde{\Omega}_4(K(\Gamma_i', 1))$$

are isomorphisms, inverse to one another, by a Mayer-Vietoris argument and Lemma 2.3.

Lemma 4.2 *For $i, k \in \{1, 2, \ldots, n\}$ with $i \neq k$ we get*

$$\widetilde{\Omega}_4(\mathrm{pr}_k') \, [M_i \cup M_i', \rho_i] = 0 \qquad \in \widetilde{\Omega}_4(K(\Gamma_k', 1)). \qquad \blacksquare$$

Notice that Theorem 1.4 follows from Lemma 4.1 and Lemma 4.2 because they imply together with the pair of inverse isomorphisms above

$$[M_i \cup M_i', \mathrm{pr}_i' \circ \rho_i] = 0 \qquad \in \widetilde{\Omega}_4(K(\Gamma_i', 1)).$$

for $i \in \{1, 2, \ldots, n\}$ and then one can apply Theorem 2.1. So it remains to prove Lemma 4.2.

Let C_1, C_2, $\ldots$,C_m be the components of ∂M_i. Let $k_j : \pi_1(C_j) \longrightarrow \Gamma_i$ be the homomorphism induced by the inclusion. Similarly, define $k_j' : \pi_1(f(C_j)) \longrightarrow \Gamma_i'$. If $g : G \longrightarrow H$ is a group homomorphism, denote by $H /\!\!/ g$ the pushout of groups of $* \longleftarrow G \xrightarrow{g} H$. This is the same as the quotient of H by the normal subgroup generated by the image of g.

Lemma 4.3 *Suppose that ∂M_i is non-empty. Then there is an isomorphism*

$$\alpha : \left(\Gamma_i /\!\!/ *_{j=1}^m k_j\right) \; * F \longrightarrow \pi_1(M_i / \partial M_i)$$

for F a finitely generated free group of rank $m-1$ and a map

$$\beta : \Gamma_i /\!\!/ *_{j=1}^m k_j \longrightarrow \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1 \leq l \leq n, l \neq i} \Gamma'_l$$

such that the composition of β with the projection

$$\overline{\mathrm{pr}'_i} : \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1 \leq l \leq n, l \neq i} \Gamma'_l \longrightarrow \Gamma'_i /\!\!/ *_{j=1}^m k'_j$$

is an isomorphism and the following diagram commutes up to inner automorphisms of Γ'_k for $j : \Gamma_i /\!\!/ *_{j=1}^m k_j \longrightarrow \Gamma_i /\!\!/ *_{j=1}^m k_j \ * F$ the canonical inclusion and $\overline{g_i}$ induced by $\mathrm{pr}'_k \circ g_i$

$$\begin{array}{ccc}
\Gamma_i /\!\!/ *_{j=1}^m k_j & \xrightarrow{\alpha \circ j} & \pi_1(M_i/\partial M_i) \\
\Big\downarrow \beta & & \Big\downarrow \pi_1(\overline{g_i}) \\
\left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1 \leq l \leq n, l \neq i} \Gamma'_l & \xrightarrow[\mathrm{pr}'_k]{} & \Gamma'_k
\end{array}$$

Before we prove Lemma 4.3, we explain how Lemma 4.2 and hence Theorem 1.4 follow from it. We only treat the more difficult case where ∂M_i is non-empty, the other case is similiar and does not use Lemma 4.3.

The underlying map of the normal structure ρ_i is $g_i \cup g'_i : M_i \cup M'_i \longrightarrow \vee_{i=1}^n K(\Gamma'_i, 1)$. The composition $\mathrm{pr}'_k \circ g'_i$ is the constant map to the base point. Therefore, we obtain a factorization of $\mathrm{pr}'_k \circ (g_i \cup g'_i) : M_i \cup M'_i \longrightarrow K(\Gamma'_k, 1)$ as the composition of the projection $q : M_i \cup M'_i \longrightarrow M_i/\partial M_i$ and the map $\overline{g_i} : M_i/\partial M_i \longrightarrow K(\Gamma'_k, 1)$ induced by $\mathrm{pr}'_k \circ g_i$. The element $\widetilde{\Omega}_4(\mathrm{pr}'_k)\,[M_i \cup M'_i, \rho_i]$ in $\widetilde{\Omega}_4(K(\Gamma'_k, 1))$ lies in the image of the composition $\widetilde{\Omega}_4(\overline{g_i}) \circ \widetilde{\Omega}_4(q)$. The map $\overline{g_i}$ induces a map $K(\overline{g_i}, 1) : K(\pi_1(M_i/\partial M_i), 1) \longrightarrow K(\Gamma'_k, 1)$ which induces a homomorphism

$$\widetilde{\Omega}_4(K(\overline{g_i}, 1)) : \widetilde{\Omega}_4(K(\pi_1(M_i/\partial M_i), 1)) \longrightarrow \widetilde{\Omega}_4(K(\Gamma'_k, 1))$$

As $\overline{g_i}$ factorizes over $K(\overline{g_i}, 1)$, it suffices to show that $\widetilde{\Omega}_4(K(\overline{g_i}, 1))$ is trivial. If $j : G \longrightarrow G * F$ is the inclusion for F a finitely generated free group, then the homomorphism

$$\widetilde{\Omega}_4(K(j, 1)) : \widetilde{\Omega}_4(K(G, 1)) \longrightarrow \widetilde{\Omega}_4(K(G * F, 1))$$

is an isomorphism by a Mayer-Vietoris argument and Lemma 2.3. Hence it suffices to show because of Lemma 4.3 that the homomorphism

$$\widetilde{\Omega}_4(K(\mathrm{pr}'_k \circ \beta, 1)) : \widetilde{\Omega}_4(K(\Gamma_i /\!\!/ *_{j=1}^m k_j, 1)) \longrightarrow \widetilde{\Omega}_4(K(\Gamma'_k, 1))$$

is trivial. From the Kurosh Subgroup Theorem 1.5 there are subgroups A_p and a free subgroup F' of $\left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1 \leq l \leq n, l \neq i} \Gamma'_l$ such that each A_p is subconjugated to $\Gamma'_i /\!\!/ *_{j=1}^m k'_j$ or some of the Γ_l's and the image of β is given by

$$\mathrm{im}(\beta) \ = \ F' \ * \ *_{p=1}^q A_p.$$

Since the composition of β with $\overline{\mathrm{pr}'_i} : \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1\le l\le n, l\ne i} \Gamma'_l \longrightarrow \Gamma'_i /\!\!/ *_{j=1}^m k'_j$ is injective by Lemma 4.3 each A_p is subconjugated to $\Gamma'_i /\!\!/ *_{j=1}^m k'_j$. The inclusion $\iota : *_{p=1}^q A_p \longrightarrow \mathrm{im}(\beta)$ induces by the argument above an isomorphism

$$\widetilde{\Omega}_4(K(\iota,1)) : \widetilde{\Omega}_4(\vee_{p=1}^q K(A_p,1)) \longrightarrow \widetilde{\Omega}_4(\mathrm{im}(\beta))$$

Since $\widetilde{\Omega}_4(K(\mathrm{pr}'_k \circ \beta, 1))$ factorizes through $\widetilde{\Omega}_4(\mathrm{im}(\beta))$ and the composition of ι with the projection $\mathrm{pr}'_k : \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1\le l\le n, l\ne i} \Gamma'_l \longrightarrow \Gamma'_k$ is trivial, $\widetilde{\Omega}_4(K(\mathrm{pr} \circ \beta, 1))$ is trivial. This finishes the proof that Lemma 4.3 implies Lemma 4.2 and hence Theorem 1.4.

It remains to prove Lemma 4.3. The map $\alpha : \left(\Gamma_i /\!\!/ *_{j=1}^m k_j\right) \ * F \longrightarrow \pi_1(M_i/\partial M_i)$ is given on $\Gamma_i /\!\!/ *_{j=1}^m k_j$ by the map which is induced by the projection $M_i \longrightarrow M_i/\partial M_i$. The free group F has $m-1$ generators and α sends the i-th generator to the class in $\pi_1(M_i/\partial M_i)$ represented by some path in M_i joining C_1 and C_{j+1}. Notice that $M_i/\partial M_i$ is up to homotopy the same as attaching to each boundary component C_j the cone over C_j and then attaching $m-1$ one-cells such that the j-th one-cell joins the top of the cone of C_1 and C_{j+1} for $1 \le j \le m-1$. One easily checks using Seifert-van Kampen Theorem that α is an isomorphism.

For appropriate choices of elements w_j in $*_{i=1}^n \Gamma'_i$ the following diagram commutes

$$\begin{array}{ccc} *_{j=1}^m \pi_1(C_j) & \xrightarrow{*_{j=1}^m \pi_1(f|_{C_j})} & *_{j=1}^m \pi_1(f(C_j)) \\ {\scriptstyle *_{j=1}^m k_j}\downarrow & & \downarrow{\scriptstyle *_{j=1}^m c(w_j)\circ k'_j} \\ \Gamma_i & \xrightarrow[f_*\circ j_i]{} & *_{i=1}^n \Gamma'_i \end{array}$$

where $c(w_j)$ denotes conjugation with w_j. The map β is the homomorphism making the following diagram commute

$$\begin{array}{ccc} \Gamma_i & \xrightarrow{f_*\circ j_i} & *_{i=1}^n \Gamma'_i \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \Gamma_i /\!\!/ *_{j=1}^m k_j & \xrightarrow[\beta]{} & \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1\le l\le n, l\ne i} \Gamma'_l \end{array}$$

Since $*_{j=1}^m \pi_1(f|_{C_j})$ is an isomorphism and $\mathrm{pr}'_i \circ f_* \circ j_i$ is an isomorphism by assumption, the map $\mathrm{pr}'_i \circ f_* \circ j_i$ induces an isomorphism β_1 making the following diagram commute

$$\begin{array}{ccc} \Gamma_i & \xrightarrow{\mathrm{pr}'_i\circ f_*\circ j_i} & \Gamma'_i \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \Gamma_i /\!\!/ *_{j=1}^m k_j & \xrightarrow[\beta_1]{} & \Gamma'_i /\!\!/ *_{j=1}^m c(\mathrm{pr}'_i(w_j)) \circ k'_j \end{array}$$

The identity on Γ'_i induces an isomorphism

$$\beta_2 : \Gamma'_i /\!\!/ *_{j=1}^m c(\mathrm{pr}'_i(w_j)) \circ k'_j \longrightarrow \Gamma'_i /\!\!/ *_{j=1}^m k'_j$$

The composition of β with the projection $\overline{\mathrm{pr}'_i} : \left(\Gamma'_i /\!\!/ *_{j=1}^m k'_j\right) \ * \ *_{1\le l\le n, l\neq i} \Gamma'_l \longrightarrow \Gamma'_i /\!\!/ *_{j=1}^m k'_j$ is the composition of the isomorphisms β_1 and β_2 and hence bijective. The diagram commutes up to inner automorphisms of Γ'_k by construction. This finishes the proof of Lemma 4.3 and hence of Theorem 1.4. ■

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