The Farrell-Jones Conjecture (Lecture II)

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Outline

- We briefly explain homology theories and how they arise from spectra.
- We state the Farrell-Jones-Conjecture and the Baum-Connes Conjecture for torsionfree groups.
- We discuss applications of these conjectures such as the Kaplansky Conjecture, Novikov Conjecture and the Borel Conjecture.
- We explain that the formulations for torsionfree groups cannot extend to arbitrary groups and state the general versions.
- We give a report about the status of the Farrell-Jones Conjecture.

Homology theory

Definition (Homology theory)

A homology theory $\mathcal{H}_*$ is a covariant functor from the category of CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

  If (X,A) is a CW-pair and $f\colon A \to B$ is a cellular map , then

$$\mathcal{H}_n(X,A) \xrightarrow{\cong} \mathcal{H}_n(X \cup_f B, B).$$

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Definition (continued)

- Disjoint union axiom

$$\bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right).$$

- If the CW-complex X is the union of two subcomplexes X_1 and X_2 and we put $X_0 = X_1 \cap X_2$, then there is a long exact Mayer-Vietoris sequence

$$\cdots \to \mathcal{H}_{n+1}(X_0) \to \mathcal{H}_{n+1}(X_1) \oplus \mathcal{H}_{n+1}(X_2) \to \mathcal{H}_{n+1}(X)$$
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Theorem (Homology theories and spectra)

Let $\mathbf{E}$ *be a spectrum. Then we obtain a homology theory* $H_*(-;\mathbf{E})$ *by*

$$H_n(X,A;\mathbf{E}) := \pi_n\left((X \cup_A \operatorname{cone}(A)) \wedge \mathbf{E}\right).$$

It satisfies

$$H_n(pt;\mathbf{E}) = \pi_n(\mathbf{E}).$$

Any homology theory arises in this way.

- The following conjectures are motivated by computations which reveal a homological flavour of K and L-theory of group rings.

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The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- BG is the classifying space of the group G.
- $K_n(BG)$ is the topological K-homology of BG.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G which is the closure in the norm topology of $\mathbb{C}G$ considered as subalgebra of $\mathcal{B}(l^2(G))$.
- There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G;\mathbb{R})).$$

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The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of R;
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$ for $n \in \mathbb{Z}$.

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The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

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- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
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Consequences of the Isomorphism Conjectures for torsionfree groups

- Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the K-theoretic and L-theoretic respectively Farrell-Jones Conjecture for the coefficient ring R.
- Let $\mathcal{BC}$ be the class of groups which satisfy the Baum-Connes Conjecture.

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Lemma

Supose that R is a regular ring, G is torsionfree and $G \in \mathcal{FJ}_K(R)$*. Then*

- $K_n(RG) = 0$ *for* $n \leq -1$*;*
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

Lemma

Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$*. Then the Whitehead group* $\mathrm{Wh}(G)$ *is trivial.*

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

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In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
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Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let F be a field and let G be a torsionfree group with $G \in \mathcal{FJ}_K(F)$. Then 0 and 1 are the only idempotents in FG.

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Proof.

- Let p be an idempotent in FG. We want to show $p \in \{0, 1\}$.
- Denote by $\epsilon\colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module.
  Since $G \in \mathcal{FJ}_K(F)$, we can conclude that

  $$i_*\colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is surjective.
  Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

  $$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

Proof (continued).

- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain

$$FG^m \cong i_*(P) \oplus FG^n.$$

- Inserting this in the first equation yields

$$(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.$$

- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$.
- This implies $(p)^k = 0$ and hence $p = 0$.

□

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f \colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\mathrm{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g \colon M_0 \to M_1$ and homotopy equivalence $f_i \colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\mathrm{sign}_x(M_0, f_0) = \mathrm{sign}_x(M_1, f_1).$$

Theorem (Baum-Connes Conjecture and the Farrell-Jones Conjecture imply the Novikov Conjecture)

The Novikov Conjecture is true if the assembly map appearing in the Baum-Connes Conjecture or in the L-theoretic Farrell-Jones Conjecture are rationally injective.

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- The Novikov Conjecture predicts for a homotopy equivalence $f\colon M \to N$ of closed aspherical manifolds

$$f_*(\mathcal{L}(M)) = \mathcal{L}(N).$$

- This is surprising since this is not true in general and in many case one could detect that two specific closed homotopy equivalent manifolds cannot be diffeomorphic by the failure of this equality to be true.
- A deep theorem of Novikov (1965) predicts that $f_*(\mathcal{L}(M)) = \mathcal{L}(N)$ holds for a homeomorphism of closed manifolds.
- Hence an explanation why the Novikov Conjecture may be true for closed aspherical manifolds is due to the next conjecture.

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Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.
  A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).
- There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-L. (2005)).

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If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

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What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
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Classifying spaces for families

Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of G is a set of (closed) subgroups of G which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lll} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

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Let $\mathcal{F}$ be a family of subgroups of G. A model for the classifying G-CW-complex for the family $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

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Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

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is bijective for all $n \in \mathbb{Z}$.

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Status of the Farrell-Jones Conjectures

- There are more general versions of the Farrell-Jones Conjecture, where one allows twisted coefficients which can actually be additive G- categories. In the sequel we refer to this general version.

Theorem (Main Theorem (Bartels-Echterhoff-Farrell-Lück-Reich-Rüping-Wegner (2008-2012)))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds. It has the following properties:

- *Hyperbolic groups belong to $\mathcal{FJ}$;*
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Theorem (continued)

- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- $\mathrm{CAT}(0)$*-groups belong to $\mathcal{FJ}$;*
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Open problems

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