K_0 and Wall's finiteness obstruction (Lecture I)

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Outline

- Introduce the projective class group $K_0(R)$.
- Discuss examples.
- State Swan's Theorem.
- Discuss its algebraic and topological significance (e.g., finiteness obstruction).

The projective class group

Definition (Projective R-module)

An R-module P is called projective if it satisfies one of the following equivalent conditions:

- P is a direct summand in a free R-module;
- The following lifting problem has always a solution

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & \nwarrow_{\overline{f}} & \uparrow{\scriptstyle f} & & \\ & & P & & \end{array}$$

- If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

- Over a field or, more generally, over a principal ideal domain every projective module is free.
- If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree.
  For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.
- Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Example (Representations of finite groups)

Let F be a field of characteristic p for p a prime number or 0. Let G be a finite group.
Then F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G.
It is a free FG-module only if G is trivial.

Definition (Projective class group $K_0(R)$)

Define the projective class group of an (associative) ring R (with unit)

$$K_0(R)$$

to be the following abelian group:

- Generators are isomorphism classes $[P]$ of finitely generated projective R-modules P;
- The relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Exercise

Show that $K_0(R)$ is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.

- A ring homomorphism $f\colon R \to S$ induces a homomorphism of abelian groups

$$f_*\colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P].$$

- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.

- The reduced projective class group $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

- $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.

- Compatibility with products

  The two projections from $R \times S$ to R and S induce an isomorphism

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

- Morita equivalence

  Let R be a ring and $M_n(R)$ be the ring of (n,n)-matrices over R. Then there is a natural isomorphism

$$K_0(R) \xrightarrow{\cong} K_0(M_n(R)).$$

Example (Principal ideal domains)

If R is a principal ideal domain and F is its quotient field, then we obtain mutually inverse isomorphisms

$$\begin{array}{rclcrcl} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & & n & \mapsto & [R^n]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example (Representation ring)

- Let G be a finite group and let F be a field of characteristic zero.
- Then the representation ring $R_F(G)$ is the same as $K_0(FG)$.
- $K_0(FG) \cong R_F(G)$ is the finitely generated free abelian group with the irreducible G-representations as basis.
- For instance $K_o(\mathbb{C}[\mathbb{Z}/n]) \cong \mathbb{Z}^n$.

Exercise

Compute $K_0(\mathbb{C}[S_3])$.

Example (Dedekind domains)

- Let R be a Dedekind domain, for instance the ring of integers in an algebraic number field.
- The ideal class group $C(R)$ is the abelian group of equivalence classes of ideals.
- Then we obtain an isomorphism

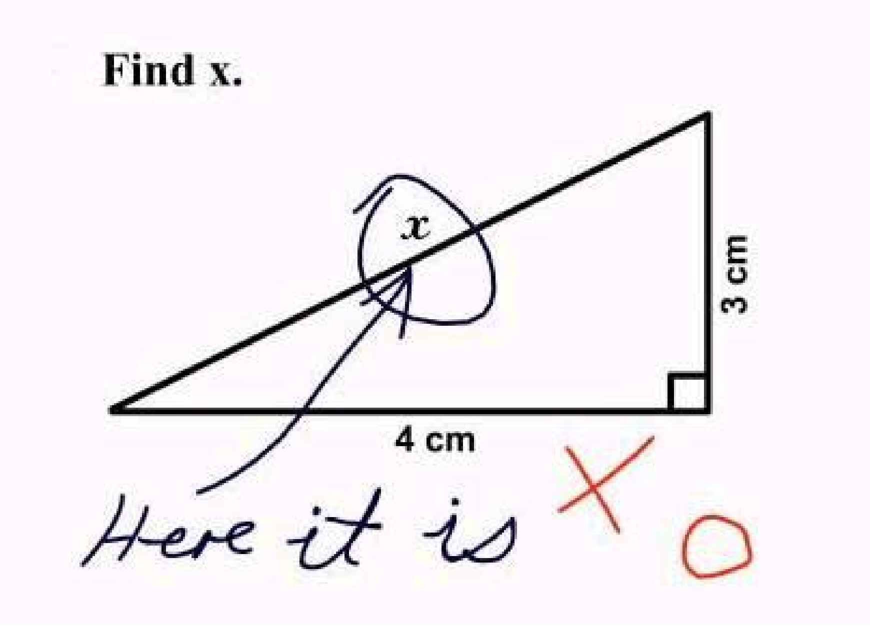
$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

- The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p.

Solutions to the exercises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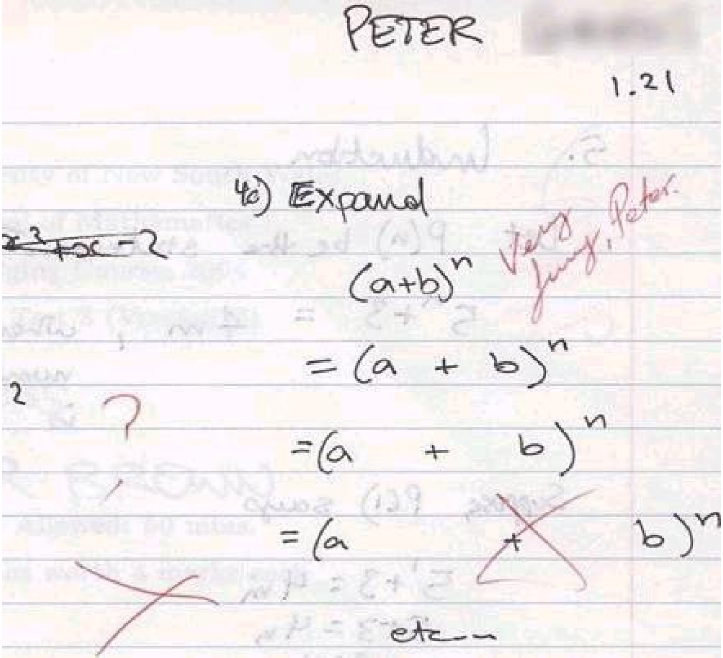
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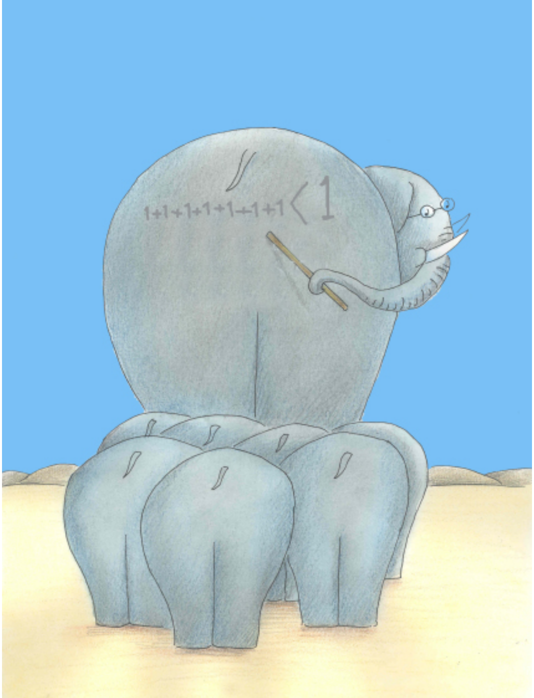
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- Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X.

- This is the zero-th term of a generalized cohomology theory $K^*(X)$, called topological K-theory, which is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = \{0\}$.

- Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Exercise

Show that the $C(S^2)$-module of sections of the tangent bundle TS^2 is finitely generated projective and and even stably finitely generated free, but not finitely generated free.

Theorem (Swan (1962))

There is an isomorphism

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Wall's finiteness obstruction

Definition (Finitely dominated)

A CW-complex X is called finitely dominated if there exists a finite (= compact) CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

Problem

Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

Definition (Wall's finiteness obstruction)

A finitely dominated CW-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its finiteness obstruction as follows:

- Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$.
- Let $C_*(\widetilde{X})$ be the cellular chain complex, which is a free $\mathbb{Z}\pi$-chain complex.
- Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$.
- Define
$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

- Let $f_*\colon C_* \to D_*$ be a R-chain homotopy equivalence of finite projective R-chain complexes. We want to show that

$$\sum_n (-1)^n \cdot [C_n] = \sum_n (-1)^n \cdot [D_n].$$

- Define the mapping cone $\operatorname{cone}(f_*)$ of f_* to be the chain complex whose n-th differential is

$$\operatorname{cone}(f_*)_n := C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}} \operatorname{cone}(f_*)_{n-1} := C_{n-2} \oplus D_{n-1}$$

- It is contractible if and only if f_* is a R-chain homotopy equivalence.

- Let E_* be any contractible R-chain complex.
- Let γ and δ be two chain contractions.
- Define R-homomorphisms

$$\begin{aligned} (e_* + \gamma_*)_{\mathrm{odd}} : E_{\mathrm{odd}} &\to E_{\mathrm{ev}}; \\ (e_* + \delta_*)_{\mathrm{ev}} : E_{\mathrm{ev}} &\to E_{\mathrm{odd}}. \end{aligned}$$

- Put

$$\begin{aligned} \mu_n &:= (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n; \\ \nu_n &:= (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n. \end{aligned}$$

- One easily checks that

$$(\mathrm{id} + \mu_*)_{\mathrm{odd}},$$
$$(\mathrm{id} + \nu_*)_{\mathrm{ev}}$$

and both compositions

$$(e_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (e_* + \delta_*)_{\mathrm{ev}}$$
$$(e_* + \delta_*)_{\mathrm{ev}} \circ (\mathrm{id} + \nu_*)_{\mathrm{ev}} \circ (e_* + \gamma_*)_{\mathrm{odd}}$$

are given by upper triangular matrices whose diagonal entries are identity maps.

- In particular these four maps are isomorphisms.
- This implies that $(e_* + \gamma_*)_{\mathrm{odd}} \colon E_{\mathrm{odd}} \to E_{\mathrm{ev}}$ is an isomorphism.

- Hence $\sum_n (-1)^n \cdot [E_n] = 0$ in $K_0(R)$.
- If we apply this to $E_* = \operatorname{cone}(f_*)$, we get in $K_0(R)$

$$\sum_n (-1)^n \cdot [C_{n-1} \oplus D_n] = \sum_n (-1)^n \cdot \big([C_{n-1}] + [D_n]\big) = 0.$$

- This implies in $K_0(R)$

$$\sum_n (-1)^n \cdot [C_n] = \sum_n (-1)^n \cdot [D_n].$$

Theorem (Wall (1965))

A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ *vanishes.*

Exercise

Show that a finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex.

- Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Theorem (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$)

The following statements are equivalent for a finitely presented group G:

- *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex;*
- $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$.

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsion free G)

If G is torsion free, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

Cliffhanger

Question

What is $K_1(R)$?

To be continued

Stay tuned

Next talk: Tuesday 9:15