The equivariant Lefschetz fixed point theorem for proper cocompact G-manifolds

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Abstract

Suppose one is given a discrete group G, a cocompact proper G-manifold M, and a G-self-map $f\colon M \to M$. Then we introduce the equivariant Lefschetz class of f, which is globally defined in terms of cellular chain complexes, and the local equivariant Lefschetz class of f, which is locally defined in terms of fixed point data. We prove the equivariant Lefschetz fixed point theorem, which says that these two classes agree. As a special case, we prove an equivariant Poincaré-Hopf Theorem, computing the universal equivariant Euler characteristic in terms of the zeros of an equivariant vector field, and also obtain an orbifold Lefschetz fixed point theorem. Finally, we prove a realization theorem for universal equivariant Euler characteristics.

Key words: equivariant Lefschetz class, equivariant Lefschetz fixed point theorem, proper cocompact G-manifold, equivariant vector field.

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0. Introduction

Let us recall the classical Lefschetz fixed point theorem. Let $f\colon M \to M$ be a smooth self-map of a compact smooth manifold M, such that $\mathrm{Fix}(f) \cap \partial M = \emptyset$ and for each $x \in \mathrm{Fix}(f)$, the determinant of the linear map $(\mathrm{id} - T_xf)\colon T_xM \to T_xM$ is different from zero. Denote by T_xM^c the one-point compactification of T_xM, which is homeomorphic to a sphere. Let $(\mathrm{id} - T_xf)^c\colon T_xM^c \to T_xM^c$ be the homeomorphism induced by the self-homeomorphism $(\mathrm{id} - T_xf)\colon T_xM \to T_xM$. Denote by $\deg((\mathrm{id} - T_xf)^c)$ its degree, which is 1 or -1, depending on whether $\det(\mathrm{id} - T_xf)$ is positive or negative. Let

$$L^{\mathbb{Z}[\{1\}]}(f) \;:=\; \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathbb{Q}}(H_p(f;\mathbb{Q})) \;=\; \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathbb{Z}}(C_p(f))$$

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be the classical Lefschetz number of f, where $H_p(f;\mathbb{Q})$ is the map on the singular homology with rational coefficients and $C_p(f)$ is the chain map on the cellular $\mathbb{Z}$-chain complex induced by f for some smooth triangulation of M. The Lefschetz fixed point theorem says that under the conditions above the fixed point set $\operatorname{Fix}(f) = \{x \in M \mid f(x) = x\}$ is finite and

$$L^{\mathbb{Z}[\{1\}]}(f) \quad = \quad \sum_{x \in \operatorname{Fix}(f)} \deg((\operatorname{id} - T_x f)^c). \tag{0.1}$$

For more information about it we refer for instance to [1].

The purpose of this paper is to generalize this to the following equivariant setting. Let G be a (not necessarily finite) discrete group G. A smooth G-manifold M is a smooth manifold with an action of G by diffeomorphisms. It is called *cocompact* if the quotient space $G\backslash M$ is compact. It is *proper* if the map $G \times M \to M \times M,\ (g,m) \mapsto (g \cdot m, m)$ is proper; when the action is cocompact, this happens if and only if all isotropy groups are finite. One can equip M with the structure of a proper finite G-CW-complex by an equivariant smooth triangulation [4]. The main result of this paper is

Theorem 0.2 (Equivariant Lefschetz fixed point theorem) *Let G be a discrete group. Let M be a cocompact proper G-manifold (possibly with boundary) and let $f\colon M \to M$ be a smooth G-map. Suppose that $\operatorname{Fix}(f) \cap \partial M = \emptyset$ and for each $x \in \operatorname{Fix}(f)$ the determinant of the linear map $\operatorname{id} - T_x f\colon T_x M \to T_x M$ is different from zero.*

Then $G\backslash \operatorname{Fix}(f)$ is finite, the equivariant Lefschetz class *of f (see Definition 3.6)*

$$\Lambda^G(f) \quad \in \quad U^G(M)$$

is defined in terms of cellular chain complexes, and the local equivariant Lefschetz class *of f (see Definition 4.6)*

$$\Lambda^G_{\mathrm{loc}}(f) \quad \in \quad U^G(M)$$

is defined. Also $\Lambda^G(f)$ and $\Lambda^G_{\mathrm{loc}}(f)$ depend only on the differentials $T_x f$ for $x \in \operatorname{Fix}(f)$, and

$$\Lambda^G(f) \ = \ \Lambda^G_{\mathrm{loc}}(f).$$

If G is trivial, Theorem 0.2 reduces to (0.1). We emphasize that we want to treat arbitrary discrete groups and take the component structure of the various fixed point sets into account.

In Section 1 we will define the orbifold Lefschetz number, which can also be viewed as an L^2-Lefschetz number, and prove the orbifold Lefschetz fixed point theorem 2.1 in Section 2. It is both a key ingredient in the proof of and a special case of the equivariant Lefschetz fixed point theorem 0.2.

In Section 3 we introduce the *equivariant Lefschetz class* $\Lambda^G(f)$, which is globally defined in terms of cellular chain complexes, and in Section 4 we introduce the *local equivariant Lefschetz class* $\Lambda^G_{\mathrm{loc}}(f)$, which is locally defined in terms of the differentials at the fixed points. These two are identified by the equivariant Lefschetz fixed point theorem 0.2, whose proof is completed in Section 5.

A classical result (the Poincaré-Hopf Theorem) says that the Euler characteristic of a compact smooth manifold can be computed by counting (with signs) the zeros of a vector field which is transverse to the zero-section and points outward at the boundary. This is a corollary of the classical Lefschetz fixed point theorem (0.1) via the associated flow. In Section 6 we will extend this result to the equivariant setting for proper cocompact G-manifolds by defining the universal equivariant Euler characteristic, defining the index of an equivariant vector field which is transverse to the zero-section and points outward at the boundary, and proving their equality in Theorem 6.6. As an illustration we explicitly compute the universal equivariant Euler characteristic and the local equivariant index of an equivariant vector field for the standard action of the infinite dihedral group on $\mathbb{R}$ in Example 6.9.

To prove Theorem 6.6 was one motivation for this paper, since it is a key ingredient in [10]. There a complete answer is given to the question of what information is carried by the element $\mathrm{Eul}^G(M) \in KO_0^G(M)$, the class defined by the equivariant Euler operator for a proper cocompact G-manifold M. Rosenberg [11] has already settled this question in the non-equivariant case by perturbing the Euler operator by a vector field and using the classical result that the Euler characteristic can be computed by counting the zeros of a vector field. The equivariant version of this strategy will be applied in [10], which requires having Theorem 6.6 available.

In Section 7 we discuss the problem whether there exists a proper smooth G-manifold M with prescribed sets $\pi_0(M^H)$ for $H \subseteq G$ such that $\chi^G(M)$ realizes a given element in $U^G(M)$. A necessary and sufficient condition for this is given in Theorem 7.6. The sufficiency part of the proof is based on a construction called *multiplicative induction* or *coinduction*. Again these results will have applications in [10].

The paper is organized as follows:

1. The orbifold Lefschetz number

In order to define the various Lefschetz classes and prove the various Lefschetz fixed point theorems for cocompact proper G-manifolds, we need some input about traces.

Let R be a commutative associative ring with unit, for instance $R = \mathbb{Z}$ or $R = \mathbb{Q}$. Let $u\colon P \to P$ be an endomorphism of a finitely generated projective RG-module. Choose a finitely generated projective RG-module Q and an isomorphism $v\colon P \oplus Q \xrightarrow{\cong} \bigoplus_{i\in I} RG$ for some finite index set I. We obtain an RG-endomorphism

$$v \circ (u \oplus 0) \circ v^{-1}\colon \bigoplus_{i\in I} RG \to \bigoplus_{i\in I} RG.$$

Let $A = (a_{i,j})_{i,j\in I}$ be the matrix associated to this map, i.e.,

$$v \circ (u \oplus 0) \circ v^{-1}(\{w_i \mid i \in I\}) \;=\; \left\{ \sum_{i\in J} w_i \cdot a_{i,j} \;\middle|\; j \in I \right\}.$$

Define

$$\operatorname{tr}_{RG}\colon RG \;\to\; R, \quad \sum_{g\in G} r_g \cdot g \;\mapsto\; r_1 \tag{1.1}$$

where r_1 is the coefficient of the unit element $1 \in G$. Define the *RG-trace of u* by

$$\operatorname{tr}_{RG}(u) \;:=\; \sum_{i\in I} \operatorname{tr}_{RG}(a_{ii}) \quad \in R. \tag{1.2}$$

We omit the easy and well-known proof that this definition is independent of the various choices such as Q and v and that the following Lemma 1.3 is true.

Lemma 1.3 (*a*) *Let $u\colon P \to Q$ and $v\colon Q \to P$ be RG-maps of finitely generated projective RG-modules. Then*

$$\operatorname{tr}_{RG}(v \circ u) = \operatorname{tr}_{RG}(u \circ v);$$

(*b*) *Let P_1 and P_2 be finitely generated projective RG-modules. Let*

$$\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} : P_1 \oplus P_2 \to P_1 \oplus P_2$$

be a RG-self-map. Then

$$\operatorname{tr}_{RG} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \;=\; \operatorname{tr}_{RG}(u_{1,1}) + \operatorname{tr}_{RG}(u_{2,2});$$

(*c*) *Let* $u_1, u_2 \colon P \to P$ *be* RG*-endomorphisms of a finitely generated projective* RG*-module and* $r_1, r_2 \in R$. *Then*

$$\operatorname{tr}_{RG}(r_1 \cdot u_1 + r_2 \cdot u_2) \;=\; r_1 \cdot \operatorname{tr}_{RG}(u_1) + r_2 \cdot \operatorname{tr}_{RG}(u_2);$$

(*d*) *Let* $\alpha \colon G \to K$ *be an inclusion of groups and* $u \colon P \to P$ *be an endomorphism of a finitely generated projective* RG*-module. Then induction with* α *yields an endomorphism* $\alpha_* u$ *of a finitely generated projective* RK*-module, and*

$$\operatorname{tr}_{RK}(\alpha_* u) \;=\; \operatorname{tr}_{RG}(u);$$

(*e*) *Let* $\alpha \colon H \to G$ *be an inclusion of groups with finite index* $[G : H]$ *and* $u \colon P \to P$ *be an endomorphism of a finitely generated projective* RG*-module. Then the restriction to* RH *with* α *yields an endomorphism* $\alpha^* u$ *of a finitely generated projective* RH*-module, and*

$$\operatorname{tr}_{RH}(\alpha^* u) \;=\; [G : H] \cdot \operatorname{tr}_{RG}(u);$$

(*f*) *Let* $H \subseteq G$ *be finite such that* $|H|$ *is invertible in* R. *Let* $u \colon R[G/H] \to R[G/H]$ *be a* RG*-map which sends* $1H$ *to* $\sum_{gH \in G/H} r_{gH} \cdot gH$. *Then* $R[G/H]$ *is a finitely generated projective* RG*-module and*

$$\begin{aligned} \operatorname{tr}_{RG}(u) &= |H|^{-1} \cdot r_{1H}; \\ \operatorname{tr}_{RG}(\operatorname{id}_{R[G/H]}) &= |H|^{-1}. \end{aligned}$$

Let G be a discrete group. A relative G-CW-complex (X, A) is *finite* if and only if X is obtained from A by attaching finitely many equivariant cells, or, equivalently, $G \backslash (X/A)$ is compact. A relative G-CW-complex (X, A) is *proper* if and only if the isotropy group G_x of each point $x \in X - A$ is finite (see for instance [7, Theorem 1.23 on page 18]). Let $(f, f_0) \colon (X, A) \to (X, A)$ be a cellular G-self-map of a finite proper relative G-CW-complex (X, A). Let R be a commutative ring such that for any $x \in X - A$ the order of its isotropy group G_x is invertible in R. Then the cellular RG-chain complex $C_*(X, A)$ is finite projective, i.e., each chain module is finitely generated projective and $C_p(X, A) = 0$ for $p \geq d$ for some integer d.

Definition 1.4 *Define the* orbifold Lefschetz number *of* (f, f_0) *by*

$$L^{RG}(f, f_0) \;:=\; \sum_{p \geq 0} (-1)^p \cdot \operatorname{tr}_{RG}(C_p(f, f_0)) \quad \in R. \tag{1.5}$$

One easily proves using Lemma 1.3

Lemma 1.6 *Let* $(f, f_0) \colon (X, A) \to (X, A)$ *be a cellular* G*-self-map of a finite proper relative* G*-*CW*-complex such that* $|G_x|$ *is invertible in* R *for each* $x \in X - A$. *Then:*

(a) *The equivariant Lefschetz number $L^{RG}(f, f_0)$ depends only on the G-homotopy class of (f, f_0);*

(b) *Let $(g, g_0)\colon (X, A) \to (Y, B)$ and $(h, h_0)\colon (Y, B) \to (X, A)$ be cellular G-maps of finite proper relative G-CW-complexes such that $|G_x|$ is invertible in R for each $x \in X - A$ and $|G_y|$ is invertible in R for each $y \in Y - B$. Then*
$$L^{RG}(g \circ h, g_0 \circ h_0) \;=\; L^{RG}(h \circ g, h_0 \circ g_0);$$

(c) *Let $\alpha\colon G \to K$ be an inclusion of groups. Then induction with α yields a cellular K-self-map $\alpha_*(f, f_0)$ of a finite proper relative K-CW-complex, and*
$$L^{RK}(\alpha_*(f, f_0)) \;=\; L^{RG}(f, f_0);$$

(d) *Let $\alpha\colon H \to G$ be an inclusion of groups with finite index $[G : H]$. Then restriction with α yields a cellular H-self-map $\alpha^*(f, f_0)$ of a finite proper relative H-CW-complex, and*
$$L^{RH}(\alpha^* f) \;=\; [G : H] \cdot L^{RG}(f).$$

Remark 1.7 The rational number $L^{\mathbb{Q}G}(f, f_0)$ agrees with the L^2*-Lefschetz number* $L^{(2)}(f, f_0; \mathcal{N}(G))$ introduced in [9, Section 6.8]. It can be read off from the map induced by (f, f_0) on the L^2-homology of (X, X_0) by the analog of the usual formula, namely by
$$L^{\mathbb{Q}G}(f, f_0) \;=\; L^{(2)}(f, f_0; \mathcal{N}(G)) \;=\; \sum_{p \geq 0} (-1)^p \cdot \operatorname{tr}_{\mathcal{N}(G)} \left(H_p^{(2)}(f, f_0; \mathcal{N}(G)) \right),$$
where $\operatorname{tr}_{\mathcal{N}(G)}$ is the standard trace of the group von Neumann algebra $\mathcal{N}(G)$. A similar formula exists in terms of $H_n(X, X_0; \mathbb{Q})$ only under the very restrictive assumption, that each $\mathbb{Q}G$-module $H_p(X, X_0; \mathbb{Q})$ is finitely generated projective. If G is finite, then (X, X_0) is a finite relative CW-complex and
$$L^{\mathbb{Q}G}(f, f_0) \;=\; \frac{1}{|G|} \cdot L^{\mathbb{Z}[\{1\}]}(f, f_0).$$

The following description of $L^G(f)$ will be useful later. Let $I_p(X, A)$ be the set of path components of $X_p - X_{p-1}$. This is the same as the set of open cells of (X, A) regarded as relative CW-complex (after forgetting the group action). The group G acts on $I_p(X, A)$. For an open p-cell e let G_e be its isotropy group, $\overline{e}$ be its closure and $\partial e = e - \overline{e}$. Then $\overline{e}/\partial e$ is homeomorphic to S^p and there is a homeomorphism
$$h\colon \bigvee_{e' \in I_p(X,A)} \overline{e'}/\partial e' \xrightarrow{\cong} X_p / X_{p-1}.$$

For an open cell $e \in I_p(X, A)$ define the *incidence number*
$$\operatorname{inc}(f, e) \quad \in \quad \mathbb{Z} \tag{1.8}$$

to be the degree of the composition

$$\overline{e}/\partial e \xrightarrow{i_e} \bigvee_{e' \in I_p(X,A)} \overline{e'}/\partial e' \xrightarrow{h} X_p/X_{p-1}$$
$$\xrightarrow{f} X_p/X_{p-1} \xrightarrow{h^{-1}} \bigvee_{e' \in I_p(X,A)} \overline{e'}/\partial e' \xrightarrow{\mathrm{pr}_e} \overline{e}/\partial e,$$

where i_e is the obvious inclusion and pr_e is the obvious projection. Obviously $\mathrm{inc}(f,e) = \mathrm{inc}(f,ge)$ for $g \in G$. One easily checks using Lemma 1.3

Lemma 1.9 *Let (X,A) be a finite proper relative G-CW-complex. Consider a cellular G-map $(f,f_0)\colon (X,A) \to (X,A)$. Then*

$$L^{\mathbb{Q}G}(f,f_0) \;=\; \sum_{p \geq 0} (-1)^p \cdot \sum_{Ge \in G \backslash I_p(X,A)} |G_e|^{-1} \cdot \mathrm{inc}(f,e).$$

Proof : By definition,

$$L^{\mathbb{Q}G}(f,f_0) \;=\; \sum_{p \geq 0} (-1)^p \cdot \mathrm{tr}_{\mathbb{Q}G}(C_p(f,f_0)).$$

But $C_p(f,f_0)$ is the $\mathbb{Q}G$-endomorphism induced by (f,f_0) on $\mathbb{Q}I_p(X,A)$. This $\mathbb{Q}G$-module splits as a direct sum of submodules, one for each G-orbit Ge in $I_p(X,A)$. And $\mathrm{tr}_{\mathbb{Q}G}$ of the G-action on the submodule corresponding to Ge is $|G_e|^{-1} \cdot \mathrm{inc}(f,e)$. ■

2. The orbifold Lefschetz fixed point theorem

This section is devoted to the proof of:

Theorem 2.1 (The orbifold Lefschetz fixed point theorem) *Let M be a cocompact proper G-manifold (possibly with boundary) and let $f\colon M \to M$ be a smooth G-map. Suppose that $\mathrm{Fix}(M) \cap \partial M = \emptyset$ and for any $x \in Fix(f)$ the determinant of the map $(\mathrm{id}_{T_xM} - T_xf)$ is different from zero. Then $G\backslash \mathrm{Fix}(f)$ is finite, and*

$$L^{\mathbb{Q}G}(f) \;=\; \sum_{G\backslash \mathrm{Fix}(f)} |G_x|^{-1} \cdot \deg\left((\mathrm{id}_{T_xM} - T_xf))^c\right).$$

Theorem 2.1 above will be a key ingredient in the proof of the equivariant Lefschetz fixed point theorem 0.2. On the other hand Theorem 0.2 implies Theorem 2.1.

Let us first consider as an illustration the easy case, where G is finite. Then

$$L^{\mathbb{Q}G}(f) = |G|^{-1} \cdot L^{\mathbb{Q}[\{1\}]}(f) = |G|^{-1} \cdot L^{\mathbb{Z}[\{1\}]}(f)$$

by Lemma 1.6 (d) and $L^{\mathbb{Z}[\{1\}]}(f)$ is the (ordinary) Lefschetz number of the self-map $f\colon M \to M$ of the compact manifold M. The non-equivariant Lefschetz fixed point theorem says

$$L^{\mathbb{Z}\{1\}}(f) \;=\; \sum_{\mathrm{Fix}(f)} \deg\left((\mathrm{id}_{T_xM} - T_xf)^c\right).$$

Thus Theorem 2.1 follows for finite G. The proof in the case of an infinite group cannot be reduced to the non-equivariant case in such an easy way since M is not compact anymore. Instead we extend the proof in the non-equivariant case to the equivariant setting.

Proof of Theorem 2.1:

Since the argument is a bit complicated technically, even though no individual step is that hard, we start by giving the reader an outline of the proof. The idea is to construct a good G-invariant simplicial structure with respect to which we can compute the equivariant Lefschetz number. This structure will have the property that it will be clear that simplices away from $\mathrm{Fix}(f)$ do not contribute anything to the answer. Thus we will be reduced to a local calculation around the fixed points, where smoothness of f and the non-degeneracy condition on $(\mathrm{id}_{T_xM} - T_xf)$ will reduce the calculation to linear algebra.

Fix a G-invariant Riemannian metric on M. Choose $\epsilon_1 > 0$ such that for all $x \in M$ the exponential map is defined on $D_{\epsilon_1}T_xM = \{v \in T_xM \mid \|v\| \le \epsilon_1\}$, where $\|v\|$ for $v \in T_xM$ is the norm coming from the Riemannian metric. Such $\epsilon_1 > 0$ exists because $G\backslash M$ is compact. The image N_{x,ϵ_1} of the exponential map on $D_{\epsilon_1}T_xM$ is a G_x-submanifold of M and a compact neighborhood of x. The exponential map induces a G_x-diffeomorphism

$$\exp_{x,\epsilon_1}\colon D_{\epsilon_1}T_xM \xrightarrow{\cong} N_{x,\epsilon_1}$$

with $\exp_{x,\epsilon_1}(0) = x$ whose differential at 0 is the identity under the canonical identification $T_0D_{\epsilon_1}T_xM = T_xM$.

Since $G\backslash M$ is compact, we can choose $\epsilon_2 > 0$ such that $f(N_{x,\epsilon_2}) \subseteq N_{x,\epsilon_1}$ and $T_xf(D_{\epsilon_2}T_xM) \subseteq D_{\epsilon_1}T_xM$ holds for all $x \in \mathrm{Fix}(f)$. Notice that $\exp_{x,\epsilon_1}$ restricted to $D_{\epsilon_2}T_xM$ is $\exp_{x,\epsilon_2}$. We want to change f up to G-homotopy without changing $\mathrm{Fix}(f)$ such that $\exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}$ and T_xf agree on $D_{\epsilon_3}T_xM$ for some positive number $\epsilon_3 > 0$ and all $x \in \mathrm{Fix}(f)$. Consider $x \in \mathrm{Fix}(f)$. Notice that $\exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}$ sends 0 to 0 and has T_xf as differential at 0 under the canonical identification $T_0D_{\epsilon_1}T_xM = T_xM$. By Taylor's theorem we can find a constant $C_1 > 0$ such that with respect to the norm on T_xM induced by the Riemannian metric on M

$$||\exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}(v) - T_xf(v)|| \;\le\; C_1 \cdot ||v||^2 \quad \text{for } v \in D_{\epsilon_2}T_xM. \tag{2.2}$$

Since $\det(\mathrm{id} - T_xf) \neq 0$, we can find a constant $C_2 > 0$ such that

$$||T_xf(v) - v|| \;\ge\; C_2 \cdot ||v|| \qquad \text{for } v \in T_xM. \tag{2.3}$$

Choose a smooth function $\phi\colon [0,\epsilon_2] \to [0,1]$ with the properties that $\phi(t) = 1$ for $t \le \min\{C_2/3C_1, \epsilon_2/3\}$ and $\phi(t) = 0$ for $t \ge \min\{C_2/2C_1, \epsilon_2/2\}$. Define

$$h\colon D_{\epsilon_2}T_xM \times [0,1] \ \to \ D_{\epsilon_1}T_xM$$

by

$$h(y,t) \ := \ \big(1 - t\phi(||v||)\big) \cdot \exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}(v) + t\phi(||v||) \cdot T_xf(v).$$

Obviously h is a G_x-homotopy from $h_0 = \exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}$ to a G_x-map h_1. The homotopy h is stationary outside $D_{\min\{C_2/2C_1,\epsilon_2/2\}}T_xM$ and h_1 agrees with T_xf on $D_{\epsilon_3}T_xM$ if we put $\epsilon_3 = \min\{C_2/3C_1, \epsilon_2/3\}$. Each map h_t has on $D_{\min\{C_2/2C_1,\epsilon_2/2\}}T_xM$ only one fixed point, namely 0. This follows from the following estimate based on (2.2) and (2.3) for $v \in D_{\min\{C_2/2C_1,\epsilon_2/2\}}T_xM$:

$$\begin{aligned}
&||h_t(v) - v|| \\
&\ge \ ||T_xf(v) - v|| - \\
&\qquad\qquad ||(1 - t\phi(||v||)) \cdot \exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}(v) - (1 - t\phi(||v||)) \cdot T_xf(v)|| \\
&= \ ||T_xf(v) - v|| - (1 - t\phi(||v||)) \cdot || \exp_{x,\epsilon_1}^{-1} \circ f \circ \exp_{x,\epsilon_2}(v) - T_xf(v)|| \\
&\ge \ C_2 \cdot ||v|| - (1 - t\phi(||v||)) \cdot C_1 \cdot ||v||^2 \\
&\ge \ (C_2 - (1 - t\phi(||v||)) \cdot C_1 \cdot ||v||) \cdot ||v|| \\
&\ge \ C_2 \cdot ||v||/2.
\end{aligned}$$

In particular we see that the only fixed point of f on $N_{\min\{C_2/2C_1,\epsilon_2/2\},x}$ is x. After possibly decreasing ϵ_1 we can assume without loss of generality that $N_{\epsilon_1,x} \cap N_{\epsilon_1,y} = \emptyset$ for $x, y \in \mathrm{Fix}(f), x \ne y$. Since M is cocompact, $G\backslash \mathrm{Fix}(f)$ is finite.

Since $h_t = h_0$ has no fixed points outside $D_{\min\{C_2/2C_1,\epsilon_2/2\}}T_xM$, each map h_t has only one fixed point, namely 0. Since the G_x-homotopy h is stationary outside $D_{\min\{C_2/2C_1,\epsilon_2/2\}}T_xM$, it extends to a G-homotopy from f to a G-map f' such that $\mathrm{Fix}(f) = \mathrm{Fix}(f')$ and

$$\exp_{x,\epsilon_1}^{-1} \circ f' \circ \exp_{x,\epsilon_2}(v) = T_xf'(v) = T_xf(v)$$

holds for each $x \in \mathrm{Fix}(f)$ and each $v \in D_{\epsilon_3}T_xM$. In the sequel we will identify $D_{\epsilon_1}T_xM$ with the compact neighborhood $N_{\epsilon_1,x}$ of x by $\exp_{x,\epsilon_1}$ for $x \in \mathrm{Fix}(f)$. Since $L^{\mathbb{Q}G}(f)$ depends only on the G-homotopy class of f, we can assume in the sequel that f agrees with $T_xf\colon D_{\epsilon_3}T_xM \to D_{\epsilon_1}T_xM$ on $D_{\epsilon_3}T_xM$ for each $x \in \mathrm{Fix}(f)$.

Next we analyze the G_x-linear map $T_xf\colon T_xM \to T_xM$ for $x \in \mathrm{Fix}(f)$. We can decompose the orthogonal G_x-representation T_xM as

$$T_xM \ = \ \bigoplus_{i=1}^{n} V_i^{m_i}$$

for pairwise non-isomorphic irreducible G_x-representations V_1, V_2, ... , V_n and positive integers m_1, m_2, ..., m_n. The G_x-linear automorphism T_xf splits as

$\oplus_{i=1}^n f_i$ for G_x-linear automorphisms $f_i\colon V_i^{m_i} \to V_i^{m_i}$. Let $D_i = \operatorname{End}_{\mathbb{R}G_x}(V_i)$ be the skew-field of G_x-linear endomorphisms of V_i. It is either the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the skew-field of quaternions $\mathbb{H}$. There is a canonical isomorphism of normed vector spaces

$$\operatorname{End}_{\mathbb{R}G_i}(V_i^{m_i}) \cong M_{m_i}(D_i).$$

Since the open subspace $GL_{m_i}(D_i) \subseteq M_{m_i}(D_i)$ is connected for $D_i = \mathbb{C}$ and $D_i = \mathbb{H}$ and the sign of the determinant induces a bijection $\pi_0(GL_{m_i}(\mathbb{R})) \to \{\pm 1\}$, we can connect $f_i \in \operatorname{Aut}_{G_x}(V_i)$ by a (continuous) path to either $\operatorname{id}\colon V_i^{m_i} \to V_i^{m_i}$ or to $-\operatorname{id}_{V_i} \oplus \operatorname{id}_{V_i^{m_i-1}}\colon V_i^{m_i} \to V_i^{m_i}$. This implies that we can find a decomposition

$$T_xM = V_x \oplus W_x$$

of the orthogonal G_x-representation T_xM into orthogonal G_x-subrepresentations and a (continuous) path $w_t\colon T_xM \to T_xM$ of linear G_x-maps from T_xf to $2\cdot \operatorname{id}_{V_x} \oplus 0_{W_x}$ such that $\operatorname{id} - w_t$ is an isomorphism for all $t \in [0,1]$. Since w_t is continuous on the compact set $[0,1]$, there is a constant $C_3 \geq 1$ such that for each $v \in T_xM$ and each $t \in [0,1]$,

$$||w_t(v)|| \quad \leq \quad C_3 \cdot ||v||.$$

Choose a smooth function $\psi\colon [0, \epsilon_3/C_3] \to [0,1]$ such that $\psi(t) = 1$ for $t \leq \epsilon_3/3C_3$ and $\psi(t) = 0$ for $t \geq 2\epsilon_3/3C_3$. Define a G_x-homotopy

$$u\colon D_{\epsilon_3/C_3}T_xM \times [0,1] \;\to\; D_{\epsilon_3}T_xM, \qquad (y,t) \;\mapsto\; w_{t\cdot\psi(||v||)}(v).$$

This is a G_x-homotopy from $f|_{D_{\epsilon_3}T_xM} = T_xf|_{D_{\epsilon_3}T_xM} = w_0|_{D_{\epsilon_3}T_xM}$ to a linear G_x-map u_1. The map u_1 and the map $2\operatorname{id}_{V_x} \oplus 0_{W_x}\colon T_xM \to T_xM$ agree on $D_{\epsilon_3/3C_3}T_xM$. For each $t \in [0,1]$ the map $u_t\colon D_{\epsilon_3/C_3}T_xM \to D_{\epsilon_3}T_xM$ has only one fixed point, namely 0, since this is true for w_t for each $t \in [0,1]$ by construction. The G_x-homotopy u is stationary outside $D_{2\epsilon/3C_3}T_xM$. Hence it can be extended to a G-homotopy $U\colon M \times [0,1] \to M$ which is stationary outside $D_{2\epsilon/3C_3}T_xM$. Since

$$\begin{aligned} L^{\mathbb{Q}G}(f) &= L^{\mathbb{Q}G}(U_1); \\ \deg((\operatorname{id} - T_xf)^c) &= \deg((\operatorname{id} - T_xU_1)^c), \end{aligned}$$

we can assume without loss of generality that f looks on $D_{\epsilon_3/C_3}T_xM$ like

$$2\operatorname{id}_{V_x} \oplus 0_{W_x}\colon T_xM = V_x \oplus W_x \to T_xM = V_x \oplus W_x$$

for each $x \in \operatorname{Fix}(f)$. By scaling the metric with a constant, we can arrange that we can take $\epsilon/3C_3 = 1/2$ and $\epsilon_1 = 2$, in other words, we can identify D_2T_xM with a neighborhood of x in M and f is given on $D_{1/2}T_xM$ by $T_xf = 2\cdot\operatorname{id}_{V_x} \oplus 0_{W_x}$.

Let d be the metric on M coming from the Riemannian metric. Choose an integer $\delta > 0$ such that the inequality $d(y, f(y)) \geq \delta$ holds for each $y \in M$, which does not lie in $D_{1/2}T_xM$ for each $x \in \mathrm{Fix}(f)$. Consider $x \in \mathrm{Fix}(f)$. Choose G_x-equivariant triangulations on the unit spheres SV_x and SW_x such that the diameter of each simplex measured with respect to the metric d is smaller than $\delta/8$. Equip $[0,1]$ with the CW-structure whose 0-skeleton is $\{\frac{i}{2n} \mid i = 0,1,2,\ldots,2n\}$ for some positive integer n which will be specified later. Equip D_1V_x with the G_x-CW-structure which is induced from the product G_x-CW-structure on $SV_x \times [0,1]$ by the quotient map

$$SV_x \times [0,1] \to DV_x,\ (y,t) \mapsto t \cdot y.$$

This is not yet the structure of a simplicial G_x-complex since the cells look like cones over simplices or products of simplices. The cones over simplices are again simplices and will not be changed. There is a standard way of subdividing a product of simplices to get a simplicial structure again. We use the resulting simplicial G_x-structure on D_1V_x. It is actually a G_x-equivariant triangulation. Define analogously a G_x-simplicial structure on D_1W.

Notice that $D_{1/2}V_x \subseteq D_1V_x$ inherits a G_x-CW-simplicial substructure. We will also use a second G_x-simplicial structure on $D_{1/2}V_x$, which will be denoted by $D_{1/2}V'_x$. It is induced by the product G_x-CW-structure on $SV_x \times [0,1]$ above together with the the quotient map

$$SV_x \times [0,1] \to D_{1/2}V_x,\ (y,t) \mapsto t/2 \cdot y.$$

The G_x-simplicial -structure on $D_{1/2}V'_x$ is finer than the one on $D_{1/2}V_x$ but agrees with the one on $D_{1/2}V_x$ on the boundary. The map $2\,\mathrm{id}\colon V_x \to V_x$ induces an isomorphism of G_x-simplicial complexes $2\,\mathrm{id}\colon D_{1/2}V'_x \xrightarrow{\cong} D_{1/2}V_x$, but it does not induce a simplicial map $2\,\mathrm{id}\colon D_{1/2}V_x \to D_1V_x$. The latter map is at least cellular with respect to the G_x-CW-structures induced from the G_x-simplicial structures since the p-skeleton of $D_{1/2}V_x$ is contained in the p-skeleton of $D_{1/2}V'_x$.

We equip $D_1V_x \times D_1W_x$, $D_{1/2}V_x \times D_1W_x$ and $D_{1/2}V_x \times D_1W_x$ with the product G_x-simplicial structure. Again this requires subdividing products of simplices (except for products of a simplex with a vertex).

Recall that we have identified D_2T_xM with its image under the exponential map. Choose a complete set of representatives $\{x_1, x_2, \ldots, x_k\}$ for the G-orbits in $\mathrm{Fix}(f)$. By the construction above we get a G-triangulation on the G-submanifold $\coprod_{i=1}^k G \times_{G_{x_i}} D_1V_{x_i} \times D_1W_{x_i}$ of M such that the diameter of each simplex is smaller than $\delta/4$ if we choose the integer n above small enough. It can be extended to a G-triangulation K of M such that each simplex has a diameter less than $\delta/4$. Let K' be the refinement of K which agrees with K outside $D_{1/2}V_{x_i} \times D_1W$ and is $D_{1/2}V'_{x_i} \times D_1W$ on the subspace $D_{1/2}V_{x_i} \times D_1W$. Then $f\colon K' \to K$ is a G-map which is simplicial on $\coprod_{i=1}^k G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1W_{x_i}T_xM)$. The construction in the proof of the (non-equivariant) simplicial approximation theorem yields a subdivision K'' of K' such that K'' and K' agree on $\coprod_{i=1}^k G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1W_{x_i})$ and a G-homotopy $h\colon M \times [0,1] \to M$ from $h_0 = f$ to a simplicial map $h_1\colon K'' \to K$

such that h is stationary on $\coprod_{i=1}^{k} G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1 W_{x_i})$ and the track of the homotopy for each point in M lies within a simplex of K. Recall that any simplex of K has diameter at most $\delta/4$ and $d(y, f(y)) \geq \delta$ holds for $y \in M$ which does not lie in $\coprod_{i=1}^{k} G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1 W_{x_i})$. Hence for any simplex $e \in K''$ outside $\coprod_{i=1}^{k} G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1 W_{x_i})$ we have $h_1(e) \cap e = \emptyset$. The G-map $h_1 \colon K'' \to K''$ is not simplicial anymore but at least cellular with respect to the G-CW-structure on M coming from K''. This comes from the fact that each skeleton of K'' is larger than the one of K'.

Next we compute $\operatorname{inc}(h_1, e)$ for cells e in M with respect to the G-CW-structure induced by K''. Obviously $\operatorname{inc}(h_1, e) = 0$ if e does not belong to $\coprod_{i=1}^{k} G \times_{G_{x_i}} (D_{1/2}V'_{x_i} \times D_1 W_{x_i})$ since for such cells e we have $h_1(e) \cap e = \emptyset$. If e belongs to $D_{1/2}V'_{x_i} \times D_1 W_{x_i}$ its image under $h_1 = f = 2\operatorname{id}_{V_{x_i}} \oplus 0_{W_{x_i}}$ does not meet the interior of e unless it is the zero simplex sitting at $(0,0) \in V_{x_i} \oplus W_{x_i}$ or a simplex of the shape $\{t \cdot x \mid t \in [0, 1/4n], x \in e\} \times \{0\}$ for some simplex in $e \in SV_{x_i}$. Hence among the cells in $D_{1/2}V'_{x_i} \times D_1 W_{x_i}$ only the zero simplex sitting at $(0,0) \in V_{x_i} \oplus W_{x_i}$ and the simplex of the shape $\{t \cdot x \mid t \in [0, 1/4n], x \in e\} \times \{0\}$ for some simplex in $e \in SV_{x_i}$ can have non-zero incidence numbers $\operatorname{inc}(f, e)$ and one easily checks that these incidence numbers are all equal to 1. Hence, using Lemma 1.9 and the equality $\operatorname{inc}(f, e) = \operatorname{inc}(f, ge)$, we get:

$$\begin{aligned}
L^{\mathbb{Q}G}(f) &= L^{\mathbb{Q}G}(h_1) \\
&= \sum_{p \geq 0} (-1)^p \cdot \sum_{Ge \in G \backslash I_p(K'')} |G_e|^{-1} \cdot \operatorname{inc}(f, e) \\
&= \sum_{p \geq 0} (-1)^p \cdot \sum_{i=1}^{k} \sum_{\substack{G_{x_i} e \in \\ G_{x_i} \backslash I_p(D_{1/2}V'_{x_i} \times D_1 W)}} |(G_{x_i})_e|^{-1} \cdot \operatorname{inc}(f, e) \\
&= \sum_{p \geq 0} (-1)^p \cdot \sum_{i=1}^{k} |G_{x_i}|^{-1} \sum_{\substack{G_{x_i} e \in \\ G_{x_i} \backslash I_p(D_{1/2}V'_{x_i} \times D_1 W)}} |G_{x_i}/(G_{x_i})_e| \cdot \operatorname{inc}(f, e) \\
&= \sum_{p \geq 0} (-1)^p \cdot \sum_{i=1}^{k} |G_{x_i}|^{-1} \sum_{e \in I_p(D_{1/2}V'_{x_i} \times D_1 W)} \operatorname{inc}(f, e),
\end{aligned}$$

and

$$\begin{aligned}
L^{\mathbb{Q}G}(f) &= \sum_{i=1}^{k} |G_{x_i}|^{-1} \sum_{p\geq 0}(-1)^p \cdot \sum_{e\in I_p(D_{1/2}V'_{x_i}\times D_1 W)} \operatorname{inc}(f,e) \\
&= \sum_{i=1}^{k} |G_{x_i}|^{-1} \left(1+\sum_{p\geq 1}(-1)^p \cdot |I_{p-1}(SV_{x_i})|\right) \\
&= \sum_{i=1}^{k} |G_{x_i}|^{-1} \left(1-\chi(SV_{x_i})\right) \\
&= \sum_{i=1}^{k} |G_{x_i}|^{-1} (-1)^{\dim(V_{x_i})} \\
&= \sum_{i=1}^{k} |G_{x_i}|^{-1} \frac{\det(\operatorname{id}-T_{x_i}f)}{|\det(\operatorname{id}-T_{x_i}f)|} \\
&= \sum_{i=1}^{k} |G_{x_i}|^{-1} \deg((\operatorname{id}-T_{x_i}f)^c) \\
&= \sum_{Gx\in G\backslash \operatorname{Fix}(f)} |G_x|^{-1} \cdot \deg((\operatorname{id}-T_{x_i}f)^c).
\end{aligned}$$

This finishes the proof of Theorem 2.1. ■

3. The equivariant Lefschetz classes

In this section we define the equivariant Lefschetz class appearing in the equivariant Lefschetz fixed point Theorem 0.2. We will use the following notation in the sequel.

Notation 3.1 *Let G be a discrete group and $H \subseteq G$ be a subgroup. Let $NH = \{g \in G \mid gHg^{-1} = H\}$ be its* normalizer *and let $WH := NH/H$ be its* Weyl group.

Denote by consub(G) *the set of conjugacy classes (H) of subgroups $H \subseteq G$.*

Let X be a G-CW-complex. Put

$$\begin{aligned}
X^H &:= \{x \in X \mid H \subseteq G_x\}; \\
X^{>H} &:= \{x \in X \mid H \subsetneq G_x\},
\end{aligned}$$

where G_x is the isotropy group of x under the G-action.

Let $x\colon G/H \to X$ be a G-map. Let $X^H(x)$ be the component of X^H containing $x(1H)$. Put

$$X^{>H}(x) = X^H(x) \cap X^{>H}.$$

Let WH_x be the isotropy group of $X^H(x) \in \pi_0(X^H)$ under the WH-action.

Next we define the group $U^G(X)$, where the equivariant Lefschetz class will take its values.

Let $\Pi_0(G,X)$ be the *component category* of the G-space X in the sense of tom Dieck [2, I.10.3]. Objects are G-maps $x\colon G/H \to X$. A morphism σ from $x\colon G/H \to X$ to $y\colon G/K \to X$ is a G-map $\sigma\colon G/H \to G/K$ such that $y \circ \sigma$ and x are G-homotopic. A G-map $f\colon X \to Y$ induces a functor $\Pi_0(G,f)\colon \Pi_0(G,X) \to \Pi_0(G,Y)$ by composition with f. Denote by $\operatorname{Is}\Pi_0(G,X)$ the set of isomorphism classes $[x]$ of objects $x\colon G/H \to X$ in $\Pi_0(G,X)$. Define

$$U^G(X) \quad := \quad \mathbb{Z}[\operatorname{Is}\Pi_0(G,X)], \tag{3.2}$$

where for a set S we denote by $\mathbb{Z}[S]$ the free abelian group with basis S. Thus we obtain a covariant functor from the category of G-spaces to the category of abelian groups. Obviously $U^G(f) = U^G(g)$ if $f,g\colon X \to Y$ are G-homotopic.

There is a natural bijection

$$\operatorname{Is}\Pi_0(G,X) \quad \xrightarrow{\cong} \quad \coprod_{(H)\in \operatorname{consub}(G)} WH\backslash\pi_0(X^H), \tag{3.3}$$

which sends $x\colon G/H \to X$ to the orbit under the WH-action on $\pi_0(X^H)$ of the component $X^H(x)$ of X^H which contains the point $x(1H)$. It induces a natural isomorphism

$$U^G(X) \quad \xrightarrow{\cong} \quad \bigoplus_{(H)\in \operatorname{consub}(G)} \mathbb{Z}[WH\backslash\pi_0(X^H)]. \tag{3.4}$$

Let $\alpha\colon G \to K$ be a group homomorphism and X be a G-CW-complex. We obtain from α a functor

$$\alpha_*\colon \Pi_0(G,X) \to \Pi_0(K,\alpha_*X)$$

which sends an object $x\colon G/H \to X$ to the object $\alpha_*(x)\colon K/\alpha(H) = \alpha_*(G/H) \to \alpha_*X$ and similarly for morphisms. Thus we obtain an *induction homomorphism* of abelian groups

$$\alpha_*\colon U^G(X) \quad \to \quad U^K(\alpha_*X). \tag{3.5}$$

Next we define the equivariant Lefschetz class. Let X be a finite proper G-CW-complex. Let $f\colon X \to X$ be a cellular G-map such that for each subgroup $K \subseteq G$ the map $\pi_0(f^K)\colon \pi_0(X^K) \to \pi_0(X^K)$ is the identity. For any G-map $x\colon G/H \to X$ it induces a map

$$(f^H(x), f^{>H}(x))\colon (X^H(x), X^{>H}(x)) \to (X^H(x), X^{>H}(x))$$

of pairs of finite proper WH_x-CW-complexes. Then

$$L^{\mathbb{Z}WH_x}(f^H(x), f^{>H}(x)) \in \mathbb{Z}$$

is defined (see (1.5)) since the isotropy group under the WH_x-action of any point in $X^H(x) - X^{>H}(x)$ is trivial.

Definition 3.6 *We define the* equivariant Lefschetz class *of* f

$$\Lambda^G(f) \quad \in \quad U^G(X)$$

by assigning to $[x\colon G/H \to X] \in \operatorname{Is}\Pi_0(G,X)$ *the integer*

$$L^{\mathbb{Z}WH_x}\left(f^H(x), f^{>H}(x)\colon (X^H(x), X^{>H}(x)) \to (X^H(x), X^{>H}(x))\right),$$

if $f^H\colon X^H \to X^H$ *maps* $X^H(x)$ *to itself, and zero otherwise.*

Since $X^{>H}(x) \neq X^H(x)$ and therefore $L^{\mathbb{Z}WH_x}(f^H(x), f^{>H}(x)) \neq 0$ holds only for finitely many elements $[x]$ in $\operatorname{Is}\Pi_0(G,X)$, Definition 3.6 makes sense. Notice for the sequel that $f^H(X^H(x)) \cap X^H(x) \neq \emptyset$ implies $f^H(X^H(x)) \subseteq X^H(x)$. The elementary proof that Lemma 1.6 implies the following lemma is left to the reader.

Lemma 3.7 *Let* X *be a finite proper* G*-CW-complex. Let* $f\colon X \to X$ *be a cellular* G*-map. Then*

(*a*) *The equivariant Lefschetz class* $\Lambda^G(f)$ *depends only on the cellular* G*-homotopy class of* f*;*

(*b*) *If* $f'\colon Y \to Y$ *is a cellular* G*-self-map of a finite* G*-CW-complex and* $h\colon X \to Y$ *is a cellular* G*-homotopy equivalence satisfying* $h \circ f \simeq_G f' \circ h$*, then* $U^G(h)\colon U^G(X) \xrightarrow{\cong} U^G(Y)$ *is bijective and sends* $\Lambda^G(f)$ *to* $\Lambda^G(f')$*;*

(*c*) *Let* $\alpha\colon G \to K$ *be an inclusion of groups. Denote by* $\alpha_* f$ *the cellular* K*-self-map obtained by induction with* α*. Then*

$$\Lambda^K(\alpha_* f) \;=\; \alpha_* \Lambda^G(f).$$

By the equivariant cellular approximation theorem (see for instance [7, Theorem 2.1 on page 32]) any G-map of G-CW-complexes is G-homotopic to a cellular G-map and two cellular G-maps which are G-homotopic are actually cellularly G-homotopic. Hence we can drop the assumption cellular in the sequel because of G-homotopy invariance of the equivariant Lefschetz class (see Lemma 3.7 (a)).

4. The local equivariant Lefschetz class

In this section we introduce the local equivariant Lefschetz class in terms of fixed point data. Before we can define it, we recall the classical notion of the equivariant Lefschetz class with values in the Burnside ring for a finite group.

Let K be a finite group. The abelian group $U^K(\{*\})$ is canonically isomorphic to the abelian group which underlies the Burnside ring $A(K)$. Recall that the *Burnside ring* is the Grothendieck ring of finite K-sets with the additive structure coming from disjoint union and the multiplicative structure coming from the Cartesian product.

Let X be a finite K-CW-complex. Define the *equivariant Lefschetz class with values in the Burnside ring* of f

$$\Lambda_0^K(f) \quad \in \quad A(K) = U^K(\{*\}) \tag{4.1}$$

by

$$\Lambda_0^K(f) \ := \sum_{(H) \in \operatorname{consub}(K)} L^{\mathbb{Z}WH}(f^H, f^{>H}) \cdot [K/H].$$

(Here and elsewhere the subscript $_0$ shall indicate that the corresponding invariant takes values in the Burnside ring and the component structure of the various fixed point sets is not taken into account.) Denote by

$$\operatorname{ch}_0^K \colon A(K) \quad \to \prod_{(H) \in \operatorname{consub}(K)} \mathbb{Z} \tag{4.2}$$

the *character map* which sends the class of a finite set S to the collection $\{|S^H| \mid (H) \in \operatorname{consub}(K)\}$ given by the orders of the various H-fixed point sets. The character map is a ring homomorphism, and it is injective (see Lemma 5.3). The equivariant Lefschetz class $\Lambda_0^K(f)$ is characterized by the property (see for instance [5, Theorem 2.19 on page 504]), [6, Lemma 3.3 on page 138])

$$\operatorname{ch}_0^K(\Lambda_0^K(f)) \quad = \quad \{L^{\mathbb{Z}[\{1\}]}(f^H) \mid (H) \in \operatorname{consub}(K)\}. \tag{4.3}$$

If $\operatorname{pr} \colon X \to \{*\}$ is the projection, then $U^K(\operatorname{pr}) \colon U^K(X) \to U^K(\{*\}) = A(K)$ sends $\Lambda^K(f)$ (see Definition 3.6) to $\Lambda_0^K(f)$ defined in (4.1).

Let V be a (finite-dimensional) K-representation and let $f \colon V^c \to V^c$ be a K-self-map of the one-point-compactification V^c. Define its *equivariant degree*

$$\operatorname{Deg}_0^K(f) \quad \in \quad A(K) = U^K(\{*\}) \tag{4.4}$$

by

$$\operatorname{Deg}_0^K(f) \ := \ (\Lambda_0^K(f) - 1) \cdot (\Lambda_0^K(\operatorname{id}_{V^c}) - 1).$$

Since the character map (4.2) is an injective ring homomorphism, we conclude from (4.3) above that $\operatorname{Deg}_0^K(f)$ is uniquely characterized by the equality

$$\operatorname{ch}_0^K(\operatorname{Deg}_0^K(f)) \quad = \quad \{\deg(f^H) \mid (H) \in \operatorname{consub}(K)\}, \tag{4.5}$$

where $\deg(f^H)$ is the degree of the self-map $f^H \colon (V^c)^H \to (V^c)^H$ of the connected closed orientable manifold $(V^c)^H$, if $\dim((V^c)^H) \geq 1$, and $\deg(f^H)$ is defined to be 1, if $\dim((V^c)^H) = 0$. The equivariant degree of (4.4) induces an isomorphism from the K-equivariant stable cohomotopy of a point to the Burnside ring $A(K)$ [3, Theorem 7.6.7 on page 190], [13].

Let M be a cocompact proper G-manifold (possibly with boundary). Let $f \colon M \to M$ be a smooth G-map. Denote by $\operatorname{Fix}(f) = \{x \in X \mid f(x) = x\}$ the *set of fixed points* of f. Suppose that for any $x \in \operatorname{Fix}(f)$ the determinant of the linear map $\mathrm{id} - T_xf\colon T_xM \to T_xM$ is different from zero. (One can always find a representative in the G-homotopy class of f which satisfies this assumption.) Then $G\backslash \operatorname{Fix}(f)$ is finite. Consider an element $x \in \operatorname{Fix}(f)$. Let $\alpha_x\colon G_x \to G$ be the inclusion. We obtain from $(\alpha_x)_*$ (see (3.5)) and $U^G(x)$ for $x \in \operatorname{Fix}(f)$ interpreted as a G-map $x\colon G/G_x \to X$ a homomorphism

$$U^{G_x}(\{*\}) \xrightarrow{(\alpha_x)_*} U^G(G/G_x) \xrightarrow{U^G(x)} U^G(X).$$

Thus we can assign to $x \in \operatorname{Fix}(f)$ the element $U^G(x)\circ(\alpha_x)_*(\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c))$, where $\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c)$ is the equivariant degree (see (4.4)) of the map induced on the one-point-compactifications by the isomorphism $(\mathrm{id} - T_xf)\colon T_xM \to T_xM$. One easily checks that this element depends only on the G-orbit of $x \in \operatorname{Fix}(X)$.

Definition 4.6 *We can define the* local equivariant Lefschetz class *by*

$$\Lambda^G_{\mathrm{loc}}(f) := \sum_{Gx \in G\backslash \operatorname{Fix}(f)} U^G(x) \circ (\alpha_x)_* \Big(\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c))\Big) \in U^G(M).$$

Now have defined all the ingredients appearing in the Equivariant Lefschetz fixed point theorem 0.2. Before we give its proof, we discuss the following example

Example 4.7 Let G be a discrete group and M be a cocompact proper G-manifold (possibly with boundary). Suppose that the isotropy group G_x of each point $x \in M$ has odd order. This holds automatically if G itself is a finite group of odd order. Let $f\colon M \to M$ be a smooth G-map. Suppose that $\operatorname{Fix}(f) \cap \partial M = \emptyset$ and for each $x \in \operatorname{Fix}(f)$ the determinant of the linear map $\mathrm{id} - T_xf\colon T_xM \to T_xM$ is different from zero. If H is a finite group of odd order, then the multiplicative group of units $A(H)^*$ of the Burnside ring is known to be $\{\pm 1\}$ [3, Proposition 1.5.1]. The element $\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c) \in A(G_x) = U^{G_x}(\{*\})$ satisfies $\Big(\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c)\Big)^2 = 1$ since this holds for its image under the injective ring homomorphism $\mathrm{ch}_0^{G_x}\colon A(G_x) \to \prod_{(H)\in \mathrm{consub}(G_x)} \mathbb{Z}$, whose coefficient at $(H) \in \mathrm{consub}(G_x)$ is $\deg((\mathrm{id} - T_xf)^c)^H) \in \{\pm 1\}$ (see (4.5)). Hence $\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c)$ belongs to $A(G_x)^* = \{\pm 1\}$. This implies that

$$\mathrm{Deg}_0^{G_x}((\mathrm{id} - T_xf)^c) = \frac{\det(\mathrm{id} - T_xf\colon T_xM \to T_xM)}{|\det(\mathrm{id} - T_xf\colon T_xM \to T_xM)|} \cdot [G_x/G_x].$$

Hence the definition of the local equivariant Lefschetz class reduces to

$$\Lambda^G_{\mathrm{loc}}(f) := \sum_{Gx \in G\backslash \operatorname{Fix}(f)} \frac{\det(\mathrm{id} - T_xf\colon T_xM \to T_xM)}{|\det(\mathrm{id} - T_xf\colon T_xM \to T_xM)|} \cdot [x\colon G/G_x \to M].$$

where $x\colon G/G_x \to M$ sends $g \cdot G_x$ to gx.

Remark 4.8 Equivariant Lefschetz classes for compact Lie groups were studied in [5]. In the non-equivariant setting, universal Lefschetz classes with values in certain K-groups were defined and analyzed in [8]. It seems to be possible to combine the K-theoretic invariants there with the equivariant versions presented here to obtain a universal equivariant Lefschetz class.

5. The proof of the equivariant Lefschetz fixed point theorem

This section is devoted to the proof of the equivariant Lefschetz fixed point Theorem 0.2.

First we define the *character map* for a proper G-CW-complex X:

$$\operatorname{ch}^G(X)\colon U^G(X) \quad \to \bigoplus_{\operatorname{Is}\Pi_0(G,X)} \mathbb{Q}. \tag{5.1}$$

We have to define for an isomorphism class $[x]$ of objects $x\colon G/H \to X$ in $\Pi_0(G,X)$ the component $\operatorname{ch}^G(X)([x])_{[y]}$ of $\operatorname{ch}^G(X)([x])$ which belongs to an isomorphism class $[y]$ of objects $y\colon G/K \to X$ in $\Pi_0(G,X)$, and check that $\chi^G(X)([x])_{[y]}$ is different from zero for at most finitely many $[y]$. Denote by $\operatorname{mor}(y,x)$ the set of morphisms from y to x in $\Pi_0(G,X)$. We have the left operation

$$\operatorname{Aut}(y) \times \operatorname{mor}(y,x) \to \operatorname{mor}(y,x), \qquad (\sigma,\tau) \mapsto \tau \circ \sigma^{-1}.$$

There is an isomorphism of groups

$$WK_y \xrightarrow{\cong} \operatorname{Aut}(y)$$

which sends $gK \in WK_y$ to the automorphism of y given by the G-map

$$R_{g^{-1}}\colon G/K \to G/K, \qquad g'K \mapsto g'g^{-1}K.$$

Thus $\operatorname{mor}(y,x)$ becomes a left WK_y-set.

The WK_y-set $\operatorname{mor}(y,x)$ can be rewritten as

$$\operatorname{mor}(y,x) \;=\; \{g \in G/H^K \mid g \cdot x(1H) \in X^K(y)\},$$

where the left operation of WK_y on $\{g \in G/H^K \mid g \cdot x(1H) \in Y^K(y)\}$ comes from the canonical left action of G on G/H. Since H is finite and hence contains only finitely many subgroups, the set $WK\backslash(G/H^K)$ is finite for each $K \subseteq G$ and is non-empty for only finitely many conjugacy classes (K) of subgroups $K \subseteq G$. This shows that $\operatorname{mor}(y,x) \neq \emptyset$ for at most finitely many isomorphism classes $[y]$ of objects $y \in \Pi_0(G,X)$ and that the WK_y-set $\operatorname{mor}(y,x)$ decomposes into finitely many WK_y orbits with finite isotropy groups for each object $y \in \Pi_0(G, X)$. We define

$$\mathrm{ch}^G(X)([x])_{[y]} \quad := \sum_{\substack{WK_y \cdot \sigma \in \\ WK_y \backslash \mathrm{mor}(y,x)}} |(WK_y)_\sigma|^{-1}, \tag{5.2}$$

where $(WK_y)_\sigma$ is the isotropy group of $\sigma \in \mathrm{mor}(y, x)$ under the WK_y-action. Notice that $\mathrm{ch}^G(X)([x])_{[y]}$ is the same as $\dim_{\mathbb{Q}WK_y}(\mathbb{Q}(\mathrm{mor}(y, x)))$, if one defines $\dim_{\mathbb{Q}WK_y}(P)$ of a finitely generated $\mathbb{Q}WK_y$-module P by $\mathrm{tr}_{\mathbb{Q}WK_y}(\mathrm{id}_P)$ (see Lemma 1.3 (f)). This agrees with the more general notion of the von Neumann dimension of the finitely generated Hilbert $\mathcal{N}(WK_y)$-module $l^2(\mathrm{mor}(y, x))$.

The character map $\mathrm{ch}^G(X)$ of (5.1) should not be confused with the isomorphism appearing in (3.4). If G is finite and $X = \{*\}$, then character map $\mathrm{ch}^G(X)$ of (5.1) and the character map ch_0^G of (4.2) are related under the identifications $U^G(\{*\}) = A(G)$ and $\mathrm{Is}\,\Pi_0(G, \{*\}) = \mathrm{consub}(G)$ by

$$(\mathrm{ch}_0^G)_{(H)} \;=\; |WH| \cdot \mathrm{ch}^G(\{*\})_{(H)}$$

for $(H) \in \mathrm{consub}(G)$.

Lemma 5.3 *The map* ch^G *of* (5.1) *is injective.*

Proof : Consider $u \in U^G(X)$ with $\mathrm{ch}^G(X)(u) = 0$. We can write u as a finite sum

$$u = \sum_{i=1}^{n} m_i \cdot [x_i \colon G/H_i \to X]$$

for some integer $n \geq 1$ and integers m_i such that $[x_i] = [x_j]$ implies $i = j$ and such that H_i is subconjugate to H_j only if $i \geq j$ or $(H_i) = (H_j)$. We have to show that $u = 0$. It suffices to prove

$$\mathrm{ch}^G(X)([x_i])_{[x_1]} \;=\; \begin{cases} 0 & \text{if } i > 1 \\ 1 & \text{if } i = 1 \end{cases}$$

because then $m_1 = 0$ follows and one can show inductively that $m_i = 0$ for $i = 1, 2, \ldots, n$. Suppose that $\mathrm{ch}^G(X)([x_i])_{[x_1]} \neq 0$. Then $\mathrm{mor}(x_1, x_i)$ is non-empty. This implies that $\mathrm{im}(x_i) \cap X^{H_1}$ is non-empty and hence that H_1 is subconjugate to H_i. From the way we have enumerated the H_i's we conclude $(H_i) = (H_1)$. Since $\mathrm{im}(x_i) \cap X^{H_1}(x_1)$ is non-empty, we get $[x_i] = [x_1]$ and hence $i = 1$. Since by definition $\mathrm{ch}^G(X)([x_1])_{[x_1]} \;=\; 1$, the claim follows. ■

Lemma 5.4 *Let $f \colon X \to X$ be a G-self-map of a finite proper G-CW-complex X. Let $[y]$ be an isomorphism class of objects $y \colon G/K \to X$ in $\Pi_0(G, X)$. Then*

$$\mathrm{ch}^G(X)(\Lambda^G(f))_{[y]} \;=\; L^{\mathbb{Q}WK_y}\left(f|_{X^K(y)} \colon X^K(y) \to X^K(y)\right),$$

if $f^K(X^K(y)) \subseteq X^K(y)$ and

$$\mathrm{ch}^G(X)(\Lambda^G(f))_{[y]} \;=\; 0$$

otherwise.

Proof : We first consider the case $f^K(X^K(y)) \subseteq X^K(y)$. Let X_p be the p-th skeleton of X. Then we can write X_p as a G-pushout

$$\begin{array}{ccc} \coprod_{i=1}^{n_p} G/H_i \times S^{p-1} & \xrightarrow{\coprod_{i=1}^{n_p} q_{p,i}} & X_{p-1} \\ \downarrow & & \downarrow \\ \coprod_{i=1}^{n_p} G/H_i \times D^p & \xrightarrow[\coprod_{i=1}^{n_p} Q_{p,i}]{} & X_p \end{array}$$

for an integer $n_p \geq 0$ and finite subgroups $H_i \subseteq G$. For each $i \in I_p$ let $x_{p,i} \colon G/H_i \to X$ be the G-map obtained by restricting the characteristic map $Q_{p,i}$ to $G/H_i \times \{0\}$. For $i = 1, 2, \ldots, p$ define

$$\operatorname{inc}(f,p,i) \quad = \quad \operatorname{inc}(f,e_i) \tag{5.5}$$

where e_i is the open cell $Q_{p,i}(gH_i \times (D^p - S^{p-1}))$ for any choice of $gH_i \in G/H_i$ and $\operatorname{inc}(f, e_i)$ is the incidence number defined in (1.8). Since f is G-equivariant, the choice of $gH_i \in G/H_i$ does not matter.

Now the G-CW-structure on X induces a WK_y-CW-structure on $X^K(y)$ whose p-skeleton $X^K(y)_p$ is $X^K(y) \cap X_p$. Let $x \colon G/H \to X$ be an object in $\Pi_0(G, X)$. Recall that $\operatorname{mor}(y, x)$ is the set of morphisms from y to x in $\Pi_0(G, X)$ and carries a canonical WK_y-operation. One easily checks that there is a WK_y-pushout diagram

$$\begin{array}{ccc} \coprod_{i=1}^{n_p} \operatorname{mor}(y, x_{p,i}) \times S^{p-1} & \xrightarrow{\coprod_{i=1}^{n_p} q^K_{p,i}(y)} & X^K(y)_{p-1} \\ \downarrow & & \downarrow \\ \coprod_{i=1}^{n_p} \operatorname{mor}(y, x_{p,i}) \times D^p & \xrightarrow[\coprod_{i=1}^{n_p} Q^K_{p,i}(y)]{} & X^K(y)_p\,, \end{array}$$

where the maps $q^K_{p,i}(y)$ and $Q^K_{p,i}(y)$ are induced by the maps $q_{p,i}$ and $Q_{p,i}$ in the obvious way. We conclude from Lemma 1.9

$$\begin{aligned} & L^{\mathbb{Q}[WK_y]}(f^K(y)) \\ & \quad = \sum_{p \geq 0} (-1)^p \cdot \sum_{i=1}^{n_p} \sum_{\substack{WK_y \cdot \sigma \in \\ WK_y \backslash \operatorname{mor}(y, x_{p,i})}} |(WK_y)_\sigma|^{-1} \cdot \operatorname{inc}(f,p,i), \end{aligned} \tag{5.6}$$

where the incidence number $\operatorname{inc}(f,p,i)$ has been defined in (5.5).

Analogously we get for any object $x \colon G/H \to X$ a WH_x-pushout diagram

$$\begin{array}{ccc} \coprod_{\substack{i=1,2\ldots,n_p \\ [x_{p,i}]=[x]}} \operatorname{Aut}(x,x) \times S^{p-1} & \xrightarrow{\coprod_{i=1}^{n_p} q^H_{p,i}(x)} & X^H(x)_{p-1} \cup X^{>H}(x) \\ \downarrow & & \downarrow \\ \coprod_{\substack{i=1,2\ldots,n_p \\ [x_{p,i}]=[x]}} \operatorname{Aut}(x,x) \times D^p & \xrightarrow[\coprod_{i=1}^{n_p} Q^H_{p,i}(x)]{} & X^H(x)_p \cup X^{>H}(x) \end{array}$$

describing how the p-skeleton of the relative WH_x-CW-complex $(X^H(x), X^{>H}(x))$ is obtained from its $(p-1)$-skeleton. We conclude from Lemma 1.9

$$L^{\mathbb{Z}[WH_x]}\left(f^H(x), f^{>H}(x))\colon (X^H(x), X^{>H}(x)) \to (X^H(x), X^{>H}(x))\right) = \sum_{p\geq 0}(-1)^p \cdot \sum_{\substack{i=1,2,\ldots,n_p\\ [x_{p,i}]=[x]}} \operatorname{inc}(f,p,i), \quad (5.7)$$

where $\operatorname{inc}(f,p,i)$ is the number introduced in (5.5). We get from the Definition 3.6 of $\Lambda^G(f)$ and (5.7)

$$\operatorname{ch}^G(X)(\Lambda^G(f))_{[y]} \quad = \quad \sum_{p\geq 0}(-1)^p \cdot \sum_{i=1}^{n_p} \operatorname{inc}(f,p,i) \cdot \operatorname{ch}^G(X)([x_{p,i}])_{[y]}. \quad (5.8)$$

Now Lemma 5.4 follows from the definition (5.2) of $\operatorname{ch}^G(X)([x_{p,i}])_{[y]}$ and equations (5.6) and (5.8) in the case $f^K(X^K(y)) \subseteq X^K(y)$.

Now suppose that $f^K(X^K(y)) \cap X^K(y) = \emptyset$. For any object $x\colon G/H \to X$ with $\operatorname{mor}(y,x) \neq \emptyset$ we conclude $f^H(X^H(x)) \cap X^H(x) = \emptyset$ and hence $\Lambda^G(f)$ assigns to $[x]$ zero by definition. We conclude $\operatorname{ch}^G(X)(\Lambda^G(f))_{[y]} = 0$ from the definition of ch^G. This finishes the proof of Lemma 5.4. ∎

Lemma 5.9 *Let M be a cocompact smooth proper G-manifold. Let $f\colon M \to M$ be a smooth G-map. Suppose that for any $x \in Fix(f)$ the determinant* $\det(\operatorname{id}_{T_xf} - T_xf)$ *is different from zero. Let $y\colon G/K \to M$ be an object in $\Pi_0(G,M)$. Denote by $(WK_y)_x$ the isotropy group for $x \in \operatorname{Fix}(f|_{M^K(y)})$ under the WK_y-action on $M^K(y)$.*

Then $G\backslash\operatorname{Fix}(f)$ is finite and we get

$$\operatorname{ch}^G(M)_{[y]}(\Lambda^G_{\mathrm{loc}}(f)) = \sum_{\substack{WK_y\cdot x\in\\ WK_y\backslash\operatorname{Fix}(f|_{M^K(y)})}} |(WK_y)_x|^{-1} \cdot \deg\left(\left(\operatorname{id}_{T_xM^K(y)} - T_x(f|_{M^K(y)})\right)^c\right).$$

Proof : We have already shown in Theorem 2.1 that $G\backslash\operatorname{Fix}(f)$ is finite.

Consider $x \in M$. Let G_x be its isotropy group under the G-action and denote by $x\colon G/G_x \to M$ the G-map $g \mapsto gx$. Let $\alpha_x\colon G_x \to G$ be the inclusion. Given a morphism $\sigma\colon y \to x$, let $(WK_y)_\sigma$ be its isotropy group under the WK_y-action and let (K_σ) be the element in $\operatorname{consub}(G_x)$ given by $g^{-1}Kg$ for any element $g \in G$ for which $\sigma\colon G/K \to G/G_x$ sends $g'K$ to $g'gG_x$. Notice that (K_σ) depends only on $WK_y \cdot \sigma \in WK_y\backslash\operatorname{mor}(y,x)$. We first show for the composition

$$U^{G_x}(\{*\}) \xrightarrow{(\alpha_x)_*} U^G(G/G_x) \xrightarrow{U^G(x)} U^G(M) \xrightarrow{\operatorname{ch}^G(M)_{[y]}} \mathbb{Q}$$

that for each $u \in U^{G_x}(\{*\})$

$$\begin{aligned} & \operatorname{ch}^G(M)_{[y]} \circ U^G(x) \circ (\alpha_x)_*(u) \\ & \qquad = \sum_{\substack{WK_y\cdot\sigma\in \\ WK_y\backslash \operatorname{mor}(y,x)}} |(WK_y)_\sigma|^{-1} \cdot \operatorname{ch}_0^{G_x}(u)_{(K_\sigma)}. \end{aligned} \tag{5.10}$$

It suffices to check this for a basis element $u = [G_x/L] \in U^G(\{*\}) = A(G_x)$ for some subgroup $L \subseteq G_x$. Let $\operatorname{pr}\colon G/L \to G/G_x$ be the projection. We get from the definitions

$$\begin{aligned} & \operatorname{ch}^G(M)_{[y]} \circ U^G(x) \circ (\alpha_x)_*([G/L]) \\ & \qquad = \operatorname{ch}^G(M)_{[y]}([x \circ \operatorname{pr}\colon G/L \to M]) \\ & \qquad = \sum_{\substack{WK_y\cdot\tau\in \\ WK_y\backslash \operatorname{mor}(y,x\circ\operatorname{pr})}} |(WK_y)_\tau|^{-1}. \end{aligned} \tag{5.11}$$

We get a WK_y-map

$$q\colon \operatorname{mor}(y, x\circ\operatorname{pr}) \to \operatorname{mor}(y,x), \qquad \tau \mapsto \operatorname{pr}\circ\tau.$$

We can write

$$\operatorname{mor}(y, x\circ\operatorname{pr}) \;=\; \coprod_{\substack{WK_y\cdot\sigma\in \\ WK_y\backslash \operatorname{mor}(y,x)}} WK_y \times_{(WK_y)_\sigma} q^{-1}(\sigma).$$

The $(WK_y)_\sigma$-set $q^{-1}(\sigma)$ is a finite disjoint union of orbits

$$q^{-1}(\sigma) \;=\; \coprod_{i\in I(\sigma)} (WK_y)_\sigma/A_i.$$

This implies

$$\operatorname{mor}(y, x\circ\operatorname{pr}) \;=\; \coprod_{\substack{WK_y\cdot\sigma\in \\ WK_y\backslash \operatorname{mor}(y,x)}} \coprod_{i\in I(\sigma)} WK_y/A_i$$

and hence

$$\sum_{\substack{WK_y\cdot\tau\in \\ WK_y\backslash \operatorname{mor}(y,x\circ\operatorname{pr})}} |(WK_y)_\tau|^{-1} \;=\; \sum_{\substack{WK_y\cdot\sigma\in \\ WK_y\backslash \operatorname{mor}(y,x)}} \sum_{i\in I(\sigma)} |A_i|^{-1}.$$

Putting this into (5.11) yields

$$\begin{aligned} & \operatorname{ch}^G(M)_{[y]} \circ U^G(x) \circ (\alpha_x)_*([G/L]) \\ & \qquad = \sum_{\substack{WK_y\cdot\sigma\in \\ WK_y\backslash \operatorname{mor}(y,x)}} \sum_{i\in I(\sigma)} |A_i|^{-1}. \end{aligned} \tag{5.12}$$

Obviously

$$|q^{-1}(\sigma)| = |(WK_y)_\sigma| \cdot \sum_{i \in I(\sigma)} |A_i|^{-1}. \tag{5.13}$$

If $\sigma\colon G/K \to G/G_x$ sends $g'K$ to $g'gG_x$ for appropriate $g \in G$ with $g^{-1}Kg \subseteq G_x$, then there is a bijection

$$G_x/L^{g^{-1}Kg} \xrightarrow{\cong} q^{-1}(\sigma)$$

which maps $g'L$ to the morphism $R_{gg'}\colon G/K \to G/L,\ \ g''K \to g''gg'L$. This implies

$$|q^{-1}(\sigma)| \quad = \quad |G_x/L^{g^{-1}Kg}| \;=\; \mathrm{ch}_0^{G_x}([G_x/L])_{(K_\sigma)}. \tag{5.14}$$

We conclude from the equations (5.12), (5.13) and (5.14) above that (5.10) follows for $u = [G_x/L]$ and hence for all $u \in U^{G_x}(\{*\}) = A(G_x)$.

Now we are ready to prove Lemma 5.9. We get from the Definition 4.6 of $\Lambda^G_{\mathrm{loc}}(f)$ and from (5.10)

$$\begin{aligned}\mathrm{ch}^G(M)_{[y]}(\Lambda^G_{\mathrm{loc}}(v)) \;=\; & \sum_{Gx \in G\backslash \mathrm{Fix}(f)} \ \sum_{\substack{WK_y\cdot\sigma \in \\ WK_y\backslash \mathrm{mor}(y,x)}} \\ & |(WK_y)_\sigma|^{-1} \cdot \mathrm{ch}_0^{G_x}\left(\mathrm{Deg}_0^{G_x}\left((\mathrm{id} - T_xf)^c\right)\right)_{(K_\sigma)}. \end{aligned} \tag{5.15}$$

We conclude from (4.5)

$$\mathrm{ch}_0^{G_x}\left(\mathrm{Deg}_0^{G_x}\left((\mathrm{id} - T_xf)^c\right)\right)_{(K_\sigma)} \quad = \quad \deg\left(\left(\mathrm{id}_{T_xM^{K_\sigma}} - T_xf^{K_\sigma}\right)^c\right). \tag{5.16}$$

If $\sigma\colon G/K \to G/G_x$ is of the form $g'K \mapsto g'gG_x$ for appropriate $g \in G$ with $g^{-1}Kg \subseteq G_x$, then

$$\deg\left(\left(\mathrm{id}_{T_xM^{K_\sigma}} - T_xf^{K_\sigma}\right)^c\right) \;=\; \deg\left(\left(\mathrm{id}_{T_{gx}M^{K}(y)} - T_{gx}(f|_{M^K(y)})\right)^c\right) \tag{5.17}$$

since f is G-equivariant. There is a bijection of WK_y-sets

$$\coprod_{Gx \in G\backslash \mathrm{Fix}(f)} \mathrm{mor}(y,x) \xrightarrow{\cong} \mathrm{Fix}(f|_{M^K(y)})$$

which sends $\sigma \in \mathrm{mor}(y,x)$ to $\sigma(1K)\cdot x$. Now Lemma 5.9 follows from (5.15), (5.16) and (5.17). ■

Because of Lemma 5.3, Lemma 5.4 and Lemma 5.9 the equivariant Lefschetz fixed point Theorem 0.2 follows from the orbifold Lefschetz fixed point Theorem 2.1. ■

6. Euler characteristic and index of a vector field in the equivariant setting

Definition 6.1 *Let X be a finite G-CW-complex X. We define the* universal equivariant Euler characteristic *of X*

$$\chi^G(X) \quad \in \quad U^G(X)$$

by assigning to $[x\colon G/H \to X] \in \operatorname{Is}\Pi_0(G,X)$ the (ordinary) Euler characteristic of the pair of finite CW-complexes $(WH_x\backslash X^H(x), WH_x\backslash X^{>H}(x))$. If X is proper, we define its orbifold Euler characteristic

$$\chi^{\mathbb{Q}G}(X) \quad := \quad \sum_{p\geq 0}(-1)^p \sum_{G\cdot e\in G\backslash I_p(X)} |G_e|^{-1} \qquad \in \mathbb{Q},$$

where $I_p(X)$ is the set of the open cells of the CW-complex X (after forgetting the group action) and G_e is the isotropy group of e under the G-action on $I_p(X)$.

One easily checks that $\chi^G(X)$ agrees with the equivariant Lefschetz class $\Lambda^G(\mathrm{id}_X)$ (see Definition 3.6). It can also be expressed by counting equivariant cells. If $x\colon G/H \to X$ is an object in $\Pi_0(G,X)$ and $\sharp_p([x])$ is the number of equivariant p-dimensional cells of orbit type G/H which meet the component $X^H(x)$, then

$$\chi^G(X)([x]) \quad = \quad \sum_{p\geq 0}(-1)^p\cdot \sharp_p([x]). \tag{6.2}$$

The universal equivariant Euler characteristic is the universal additive invariant for finite G-CW-complexes in the sense of [7, Theorem 6.7].

The orbifold Euler characteristic $\chi^{\mathbb{Q}G}(X)$ can be identified with $L^{\mathbb{Q}G}(\mathrm{id}_X)$ (see Definition 1.4) or with the more general notion of the L^2-Euler characteristic $\chi^{(2)}(X;\mathcal{N}(G))$. In analogy with $L^{(2)}(f,f_0;\mathcal{N}(G))$ (see Remark 1.7), one can compute $\chi^{(2)}(X;\mathcal{N}(G))$ in terms of L^2-homology

$$\chi^{(2)}(X;\mathcal{N}(G)) \;=\; \sum_{p\geq 0}(-1)^p\cdot \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(X;\mathcal{N}(G))\right),$$

where $\dim_{\mathcal{N}(G)}$ denotes the von Neumann dimension (see for instance [9, Section 6.6]).

We conclude from Lemma 5.4:

Lemma 6.3 *Let X be a finite proper G-CW-complex X. Let $[y]$ be an isomorphism class of objects $y\colon G/K \to X$ in $\pi_0(G,X)$. Then*

$$\mathrm{ch}^G(X)_{[y]}(\chi^G(X)) \;=\; \chi^{\mathbb{Q}WK_y}(X^K(y)).$$

Next we want to express the universal equivariant Euler characteristic of a cocompact proper G-manifold M, possibly with boundary, in terms of the zeros of an equivariant vector field.

Consider an *equivariant vector field* Ξ on M, i.e., a G-equivariant section of the tangent bundle TM of M. Suppose that Ξ is *transverse to the zero-section* $i\colon M \to TM$, i.e., if $\Xi(x) = 0$, then $T_{\Xi(x)}TM$ is the sum of the subspaces given by the images of $T_x\Xi\colon T_xM \to T_{\Xi(x)}(TM)$ and $T_xi\colon T_xM \to T_{\Xi(x)}(TM)$. Any equivariant vector field on M can be changed by an arbitrary small perturbation into one which is transverse to the zero-section. We want to assign to such a Ξ its *equivariant index* as follows.

Since Ξ is transverse to the zero-section, the set $\mathrm{Zero}(\Xi)$ of points $x \in M$ with $\Xi(x) = 0$ is discrete. Hence $G\backslash \mathrm{Zero}(\Xi)$ is finite, since G acts properly on M and $G\backslash M$ is compact by assumption. Fix $x \in \mathrm{Zero}(\Xi)$. The zero-section $i\colon M \to TM$ and the inclusion $j_x\colon T_xM \to TM$ induce an isomorphism of G_x-representations

$$T_xi \oplus T_xj_x\colon T_xM \oplus T_xM \xrightarrow{\cong} T_{i(x)}(TM)$$

if we identify $T_{i(x)}(T_xM) = T_xM$ in the obvious way. If pr_k denotes the projection onto the k-th factor for $k = 1, 2$ we obtain a linear G_x-equivariant isomorphism

$$d_x\Xi\colon \quad T_xM \xrightarrow{T_x\Xi} T_{\Xi(x)}(TM) \xrightarrow{(T_xi \oplus T_xj_x)^{-1}} T_xM \oplus T_xM \xrightarrow{\mathrm{pr}_2} T_xM. \tag{6.4}$$

Notice that we obtain the identity if we replace pr_2 by pr_1 in the expression (6.4) above. The G_x-map $d_x\Xi$ induces a G_x-map $(d_x\Xi)^c\colon T_xM^c \to T_xM^c$ on the one-point compactification. Define analogously to the *local equivariant Lefschetz class* (see Definition 4.6) the *equivariant index* of Ξ

$$i^G(\Xi) \quad \in \quad U^G(M) \tag{6.5}$$

by

$$i^G(\Xi) \;:=\; \sum_{Gx \in G\backslash \mathrm{Zero}(\Xi)} U^G(x) \circ (\alpha_x) \left(\mathrm{Deg}_0^{G_x}((d_x\Xi)^c)\right).$$

We say that Ξ points *outward at the boundary* if for each $x \in \partial M$ the tangent vector $\Xi(x) \in T_xM$ does not lie in the subspace $T_x\partial M$ and is contained in the half space T_xM^+ of tangent vectors $u \in T_xM$ for which there is a path $w\colon [-1, 0] \to M$ with $w(0) = x$ and $w'(0) = u$. If $\partial M = \emptyset$, this condition is always satisfied.

Theorem 6.6 (Equivariant Euler characteristic and vector fields) *Let M be a cocompact proper smooth G-manifold. Let Ξ be a G-equivariant vector field which is transverse to the zero-section and points outward at the boundary. Then we get in $U^G(M)$*

$$\chi^G(M) \;=\; i^G(\Xi).$$

Proof : Let $\Phi\colon M \times (-\infty, 0] \to M$ be the flow associated to the vector field Ξ. It is defined on $M \times (-\infty, 0]$ since Ξ is equivariant and points outward and M is cocompact. Moreover, each map $\Phi_{-\epsilon}\colon M \to M$ is a G-diffeomorphism and G-homotopic to id_M for $\epsilon > 0$. This implies

$$\Lambda^G(\Phi_{-\epsilon}) \;=\; \Lambda^G(\mathrm{id}_M) \;=\; \chi^G(M).$$

Because of the equivariant Lefschetz fixed point Theorem 0.2 it remains to prove for some $\epsilon > 0$

$$i^G(\Xi) \;=\; \Lambda^G_{\mathrm{loc}}(\Phi_{-\epsilon}).$$

If we choose $\epsilon > 0$ small enough, the diffeomorphism $\phi_{-\epsilon}\colon M \to M$ will have as set of fixed points $\mathrm{Fix}(\phi_{-\epsilon})$ precisely $\mathrm{Zero}(\Xi)$. It suffices to prove for $x \in \mathrm{Zero}(\Xi) = \mathrm{Fix}(\Phi_{-\epsilon})$

$$\mathrm{Deg}_0^{G_x}((d_x\Xi)^c) \;=\; \mathrm{Deg}_0^{G_x}((\mathrm{id} - T_x\Phi_{-\epsilon})^c).$$

Recall that the character map $\mathrm{ch}_0^{G_x}\colon A(G_x) \to \prod_{(H)\in\mathrm{consub}(G_x)} \mathbb{Z}$ of (4.2) is injective. We conclude from (4.5) that it suffices to prove for any subgroup $H \subseteq G_x$ the equality of degrees of self-maps of the closed orientable manifold $((T_xM)^c)^H = ((T_xM)^H)^c$

$$\deg\left(((d_x\Xi)^c)^H\right) \quad = \quad \deg\left(((\mathrm{id} - T_x\Phi_{-\epsilon})^c)^H\right). \tag{6.7}$$

It suffices to treat the case $H = \{1\}$ and $\dim(M) \geq 1$ — the other cases are completely analogous or follow directly from the definitions. Since (6.7) is of local nature, we may assume $M = \mathbb{R}^n$ and $x = 0$. In the sequel we use the standard identification $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Then the vector field Ξ becomes a smooth map $\Xi\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\Xi(0) = 0$ and $d_0\Xi$ becomes the differential $T_0\Xi$. Let Φ be the flow associated to Ξ. Choose $\epsilon \geq 0$ and an open neighborhood $U \subseteq \mathbb{R}^n$ of 0 such that Φ is defined on $U \times [-\epsilon, 0]$. By Taylor's theorem we can find a smooth map $\eta\colon U \times [-\epsilon, 0] \to \mathbb{R}^n$ such that for $t \in [-\epsilon, 0]$ and $u \in U$,

$$\phi_t(u) = u + t \cdot \Xi(u) + t^2 \cdot \eta_t(u)\ .$$

This implies

$$T_0\phi_t = \mathrm{id} + t \cdot (T_0\Xi + t \cdot T_0\eta_t)\,.$$

Since $[-\epsilon, 0]$ is compact, we can find a constant C independent of t such that the operator norm of $T_0\eta_t$ satisfies $\|T_0\eta_t\| < C$ for $t \in [-\epsilon, 0]$. The differential $T_0\Xi$ is an isomorphism by assumption. Hence $T_0\Xi + t \cdot T_0\eta_t$ is invertible for $t \in [-D, 0]$ if we put $D := \min\{\epsilon, C^{-1} \cdot ||(T_0\Xi)^{-1}||^{-1}\}$. Hence we get for $t \in [-D, 0]$ that $\mathrm{id} - T_0\Phi_t$ is invertible and

$$\frac{\det(\mathrm{id} - T_0\Phi_t)}{|\det(\mathrm{id} - T_0\Phi_t)|} \;=\; \frac{\det(T_0\Xi + t \cdot T_0\eta_t)}{|\det(T_0\Xi + t \cdot T_0\eta_t)|} \;=\; \frac{\det(T_0\Xi)}{|\det(T_0\Xi)|}.$$

Hence (6.7) follows if we take $\epsilon > 0$ small enough. This finishes the proof that Theorem 6.6 follows from Theorem 0.2. ■

Remark 6.8 Let M be a proper cocompact G-manifold without boundary. If M possesses a nowhere-vanishing equivariant vector field, then $\chi^G(M) = 0$ by the Theorem 6.6. The converse is true if M satisfies the weak gap hypothesis that $\dim(M^{>G_x}(x)) \leq \dim(M^{G_x}(x)) - 2$ holds for each $x \in G_x$. The proof is done by induction over the orbit bundles and the induction step is reduced to the non-equivariant case. The weak gap condition ensures that $M^{G_x}(x) - M^{>G_x}(x)$ is connected for $x \in M$. It is satisfied if all isotropy groups of M have odd order. For finite groups G more information about this question can be found in [12, Remark 2.5 (iii) on page 32].

Example 6.9 Let D be the infinite dihedral group $D = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$. We use the presentation $D = \langle s, t \mid s^2 = 1, s^{-1}ts = t^{-1} \rangle$ in the sequel. The subgroups $H_0 = \langle t \rangle$ and $H_1 = \langle ts \rangle$ have order two and $\{\{1\}, H_0, H_1\}$ is a complete system of representatives for the conjugacy classes of finite subgroups of D. The infinite dihedral group D acts on $\mathbb{R}$ by $s \cdot r = -r$ and $t \cdot r = r + 1$ for $r \in \mathbb{R}$. The interval $[0, 1/2]$ is a fundamental domain for the D-action. There is a D-CW-structure on $\mathbb{R}$ with $\{n \mid n \in \mathbb{Z}\} \coprod \{n + 1/2 \mid n \in \mathbb{Z}\}$ as zero-skeleton. Let $x_i \colon D/H_i \to \mathbb{R}$ be the D-map sending $1H_i$ to 0 for $i = 0$ and to $1/2$ for $i = 1$. Let $y \colon D \to \mathbb{R}$ be the D-map sending 1 to 0. Then $\mathbb{R}$ has two equivariant 0-cells, which have x_0 and x_1 as their characteristic maps, and one equivariant 1-cell of orbit type $D/\{1\}$.

We get $\operatorname{Is}\Pi_0(D; \mathbb{R}) = \{[x_0], [x_1], [y]\}$. Recall that $U^D(\mathbb{R})$ is the free $\mathbb{Z}$-module with basis $\operatorname{Is}\Pi_0(D; \mathbb{R})$. Hence we write

$$U^D(\mathbb{R}) = \mathbb{Z}\langle [x_0] \rangle \oplus \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [y] \rangle.$$

We conclude from (6.2)

$$\chi^D(\mathbb{R}) = [x_0] + [x_1] - [y].$$

This is consistent with the original Definition 6.1, since

$$\begin{aligned} \chi(W_D H_i \backslash (\mathbb{R}^{H_i}(x_i), \mathbb{R}^{>H_i}(x_i))) &= \chi(\{\text{pt.}\}) = 1; \\ \chi(W_D\{1\} \backslash (\mathbb{R}^{\{1\}}(y), \mathbb{R}^{>\{1\}}(y)) &= \chi([0, 1/2], \{0, 1/2\}) = -1. \end{aligned}$$

The character map (5.1) is given by

$$\begin{aligned} \chi^D(\mathbb{R}) = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{array} \right\} : U^D(\mathbb{R}) &= \mathbb{Z}\langle [x_0] \rangle \oplus \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [y] \rangle \\ &\to \mathbb{Z}\langle [x_0] \rangle \oplus \mathbb{Z}\langle [x_1] \rangle \oplus \mathbb{Z}\langle [y] \rangle \end{aligned}$$

since the D-set $\operatorname{mor}(y, x_i)$ is D/H_i and the sets $\operatorname{mor}(x_i, y)$ and $\operatorname{mor}(x_i, x_j)$ for $i \neq j$ are empty. The character map sends $\chi^D(\mathbb{R})$ to the various orbifold Euler characteristics which therefore must be

$$\begin{aligned} \chi^{\mathbb{Q}W_D H_i}(\mathbb{R}^{H_i}(x_i)) &= 1; \\ \chi^{\mathbb{Q}D}(\mathbb{R}) &= 0. \end{aligned}$$

One easily checks that this is consistent with the Definition 6.1 of the orbifold Euler characteristics.

Let Ξ be a vector field on $\mathbb{R}$. Under the standard identification $T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ this is the same as a function $\Xi\colon \mathbb{R} \to \mathbb{R}$. The vector field Ξ is transverse to the zero-section if and only if the function Ξ satisfies $\Xi(z) = 0 \Rightarrow \Xi'(z) \neq 0$. The vector field Ξ is D-equivariant if and only if $\Xi(-z) = -\Xi(z)$ and $\Xi(z) = \Xi(z+1)$ holds for all $z \in \mathbb{R}$. Let Ξ be a D-equivariant vector field transverse to the zero-section. For example, we can take $\Xi(z) = \sin(2\pi z)$.

We conclude $\Xi(0) = 0$ from $\Xi(-z) = -\Xi(z)$ and $\Xi(1/2) = 0$ from $\Xi(1-z) = -\Xi(z)$. Let $z_0 = 0 < z_1 < z_2 < \ldots < z_r = 1/2$ be the points $z \in [0, 1/2]$ for which $\Xi(z) = 0$. (The example of $\sin(2\pi z)$ shows the minimum value of r is 1.) For $i \in \{0, 1, \ldots, r\}$ put $\delta_i = \frac{\Xi'(z_i)}{|\Xi'(z_i)|}$. Since Ξ is different from zero on (z_i, z_{i+1}), we have $\delta_{i+1} = -\delta_i$ for $i \in \{0, 1, \ldots, r-1\}$. Let $d_{z_i}\Xi\colon T_{z_i}\mathbb{R} \to T_{z_i}\mathbb{R}$ be the map associated to Ξ at z_i (see (6.4)); this is simply multiplication by $\Xi'(z_i)$ under the standard identification $T_{z_i}\mathbb{R} = \mathbb{R}$. The degree of the map $(d_{z_i}\Xi)^c$ induced on $\mathbb{R}^c = S^1$ is δ_i. For $i = 0, r$ the isotropy group $D_{z_i} = \mathbb{Z}/2$ acts on $\mathbb{R}$ by $-\mathrm{id}$. Hence the degree $\deg\left((d_{z_i}\Xi)^{D_{z_i}})^c\right)$ is by definition 1 since $\dim(\mathbb{R}^{D_{z_i}}) = 0$. We conclude from (4.5) for $i = 0, r$

$$\operatorname{Deg}^{D_{z_i}}(d_{z_i}\Xi) \quad = \quad [H/H] + \frac{-1+\delta_i}{2} \cdot [H/\{1\}] \qquad \in A(D_{z_i})$$

and for $i \in \{1, 2, \ldots, r-1\}$

$$\operatorname{Deg}^{D_{z_i}}(d_{z_i}\Xi) \quad = \quad \delta_i \cdot [\{1\}] \qquad \in A(\{1\}).$$

Hence we get for the equivariant index

$$\begin{aligned} i^D(\Xi) \;&=\; [x_0] + \frac{-1+\delta_0}{2} \cdot [y] + [x_1] + \frac{-1+\delta_r}{2} \cdot [y] + \sum_{i=1}^{r-1} \delta_i \cdot [y] \\ &=\; [x_0] + [x_1] + \left(-1 + \frac{\delta_0}{2} + \frac{\delta_r}{2} + \sum_{i=1}^{r-1} \delta_i\right) \cdot [y] \\ &=\; [x_0] + [x_1] - [y] + \delta_0 \cdot \left(\frac{1}{2} + \frac{(-1)^r}{2} + \sum_{i=1}^{r-1} (-1)^i\right) \cdot [y_0] \\ &=\; [x_0] + [x_1] - [y] + \delta_0 \cdot 0 \cdot [y_0] \\ &=\; [x_0] + [x_1] - [y]. \end{aligned}$$

This is consistent with Theorem 6.6.

7. Constructing equivariant manifolds with given component structure and universal equivariant Euler characteristic

In this section we discuss the problem of whether there exists a proper smooth G-manifold M with prescribed sets $\pi_0(M^H)$ for $H \subseteq G$, and whether $\chi^G(M)$ can realize a given element in U^G.

Let $\mathsf{Or}(G;\mathcal{F}in)$ be the orbit category for the family of finite subgroups, i.e., objects are homogeneous spaces G/H for finite subgroups $H \subseteq G$ and morphisms are G-maps. A contravariant $\mathsf{Or}(G;\mathcal{F}in)$-set S is a contravariant functor from $\mathsf{Or}(G;\mathcal{F}in)$ to the category of sets. Given a G-space X, define a contravariant $\mathsf{Or}(G;\mathcal{F}in)$-set $\underline{\pi_0(X)}$ by sending G/H to $\pi_0(X^H) = \pi_0(\operatorname{map}_G(G/H,X))$. Next we investigate under which conditions a contravariant $\mathsf{Or}(G;\mathcal{F}in)$-set S arises from a finite proper G-CW-complex.

Lemma 7.1 *For a contravariant $\mathsf{Or}(G;\mathcal{F}in)$-set $S\colon \mathsf{Or}(G;\mathcal{F}in) \to \mathsf{Sets}$ the following assertions are equivalent:*

(a) *There are only finitely many elements $(H) \in \operatorname{consub}(H)$ with $S(G/H) \neq \emptyset$. For any finite subgroup $H \subseteq G$ the set $WH\backslash S(G/H)$ is finite and the isotropy group WH_s of each element $s \in S(G/H)$ is finitely generated;*

(b) *There is a proper finite G-CW-complex X such that there exists a natural equivalence $T\colon \underline{\pi_0(X)} \xrightarrow{\cong} S$.*

Proof : (b) $\Rightarrow$ (a) Since X is finite, there are finitely many elements (K_1), (K_2), ..., (K_m) in $\{(K) \in \operatorname{consub}(G) \mid |K| < \infty\}$ such that for each equivariant cell $G/H \times D^n$ there is $i \in \{1,2,\ldots,m\}$ with $(H) = (K_i)$. Hence any subgroup $H \subseteq G$ with $X^H \neq \emptyset$ is conjugate to a subgroup of one of the K_i's. Since a finite group has only finitely many subgroups, the set $\{(H) \in \operatorname{consub}(G) \mid X^H \neq \emptyset\}$ is finite.

Since X is finite and proper and $WH\backslash(G/K^H)$ is finite for each finite group $H \subseteq G$ and subgroup $K \subseteq G$, the WH-CW-complex X^H is finite proper for each finite subgroup $H \subseteq G$. Hence the quotient space $WH\backslash X^H$ is a finite CW-complex and has only finitely many components. Since $WH\backslash\pi_0(X^H) \cong \pi_0(WH\backslash X^H)$, the set $WH\backslash\pi_0(X^H)$ is finite.

Consider a finite subgroup $H \subseteq G$ and a component $C \in \pi_0(X^H)$. Let WH_C be the isotropy group C. Then C is a connected proper finite WH_C-CW-complex. The long exact homotopy sequence of the fibration

$$C \to EWH_C \times_{WH_C} C \to BWH_C$$

shows that WH_C is a quotient of $\pi_1(EWH_C \times_{WH_C} C)$. Since for any finite subgroup $K \subseteq WH_C$ the WH_C-space $EWH_C \times_{WH_C} WH_C/K$ is homotopy equivalent to BK and hence to a CW-complex of finite type (a CW-complex for which each skeleton is finite), C is built out of finitely many cells $G/K \times D^i$, $K \subseteq G$

finite, and $EWH_C \times_{WH_C} WH_C/K$ has the homotopy type of CW-complex of finite type. This implies that $\pi_1(EWH_C \times_{WH_C} C)$ is finitely generated. Hence WH_C is finitely generated.

(a) $\Rightarrow$ (b) Choose an ordering $(H_1), (H_2), \ldots, (H_r)$ on the set $\{(H) \in \text{consub}(G) \mid S(G/H) \neq \emptyset\}$ such that H_i is subconjugate to H_j only if $i \geq j$ holds. Define

$$X_0 \ := \ \coprod_{i=1}^{s} \coprod_{WH_i \backslash S(G/H_i)} G/H_i.$$

Define a transformation $\phi_0 \colon \underline{\pi_0(X_0)} \to S$ as follows. Fix an object $G/K \in \mathsf{Or}(G;\mathcal{F}in)$. Then $\underline{\pi_0(X_0)}$ evaluated at this object G/K is given by

$$\coprod_{i=1}^{s} \coprod_{WH_i \backslash S(G/H)} \text{map}_G(G/K, G/H_i)$$

since $\pi_0(G/H_i^K) = \text{map}_G(G/K, G/H_i)$. Now require that $\phi_0(G/K)$ sends a G-map $\sigma \in \text{map}_G(G/K, G/H_i)$ belonging to the summand for $WH_i \cdot s \in WH_i \backslash S(G/H_i)$ to $S(\sigma)(s)$. One easily checks that $\phi_0(G/H) \colon \pi_0(X^H) \to S(G/H)$ is surjective for all $H \subseteq G$.

In the next step we attach equivariant one-cells to X_0 to get a G-CW-complex X together with a transformation $\phi \colon \underline{\pi_0(X)} \to S$, such that the composition of ϕ with the transformation $\underline{\pi_0(X_0)} \to \underline{\pi_0(X)}$ induced by the inclusion is ϕ_0, and $\phi(G/H)$ is bijective for all $H \subseteq G$. We do this by constructing by induction a sequence of proper cocompact G-CW-complexes $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_r$, together with transformations $\phi_i \colon \underline{\pi_0(X_i)} \to S$ such that the composition of ϕ_i with the transformation $\underline{\pi_0(X_{i-1})} \to \underline{\pi_0(X_i)}$ induced by the inclusion is ϕ_{i-1}, $\phi_i(G/H_j)$ is bijective for all $j \leq i$, and X_i is obtained from X_{i-1} by attaching finitely many equivariant cells of the type $G/H_i \times D^1$. Then we can take $X = X_r$ and $\phi = \phi_r$.

The induction beginning is X_0 together with ϕ_0. The induction step from $i-1$ to i is done as follows. Consider $WH_i \cdot s \in WH_i \backslash S(G/H_i)$. Let $(WH_i)_s$ be the isotropy group of s under the WH_i-action on $S(G/H_i)$. The preimage of s under $\phi_{i-1}(G/H_i) \colon \pi_0(X_{i-1}^{H_i}) \to S(G/H_i)$ consists of finitely many $(WH_i)_s$-orbits since $WH_i \backslash \pi_0(X_{i-1}^{H_i})$ is finite by the implication (b) $\Rightarrow$ (a) which we have already proved. Let $u(s)_1$, $u(s)_2$, $\ldots, u(s)_v$ be a system of generators of $(WH_i)_s$ which contains the unit element $1 \in (WH_i)_s$. Fix elements $C(s)_1$, $C(s)_2$, $\ldots$, $C(s)_{n(s)}$ in $\phi_{i-1}^{-1}(G/H_i)(s)$ such that

$$(WH_i)_s \backslash \phi_{i-1}(G/H_i)^{-1}(s) \ = \ \{(WH_i)_s \cdot C(s)_i \mid i = 1, 2 \ldots, n(s)\}$$

and $(WH_i)_s \cdot C(s)_j = (WH_i)_s \cdot C(s)_k$ implies $j = k$. Now attach for each $WH_i \cdot s \in WH_i \backslash S(G/H_i)$, each generator $u(s)_i$, each $C(s)_j$ an equivariant cell $G/H_i \times D^1$ to X_{i-1} such that $\{1H_i\} \times D^1$ connects $C_1(s)$ and $u(s)_i \cdot C_j(s)$. The resulting G-CW-complex is the desired G-CW-complex X_i, one easily constructs the desired transformation ϕ_i out of ϕ_{i-1}. ■

Notice that for a finite group G the statement (a) in Lemma 7.1 is equivalent to the statement that $S(G/H)$ is finite for all subgroups $H \subseteq G$. If we take as $\mathsf{Or}(G;\mathcal{F}in)$-set the functor S which sends $G/\{1\}$ to $\{*\}$ and G/H for $H \neq \{1\}$ to $\emptyset$, then Lemma 7.1 boils down to the statement that there exists a connected finite free G-CW-complex X if and only if G is finitely generated.

Now given a contravariant $\mathsf{Or}(G;\mathcal{F}in)$-set S, we define $U^G(S)$ to be the free abelian group on $\coprod_{(H)\in \mathrm{consub}(G)} WH\backslash S(G/H)$. If S satisfies the equivalent conditions of Lemma 7.1, then clearly this is naturally isomorphic to $U^G(X)$, with X as in part (b) of Lemma 7.1.

Next we prove that is X is a finite proper G-CW-complex, then any element $u \in U^G(X)$ can be realized from χ^G of a manifold. More precisely, there is a G-map $f\colon M \to X$ with M a G-manifold, such that $U^G(f)$ is an isomorphism sending $\chi^G(M)$ to u. We are grateful to Tammo tom Dieck for pointing out to us the use of the multiplicative induction in the proof of the next result.

Lemma 7.2 *Let X be a finite proper G-CW-complex and $u \in U^G(X)$. Then there is a proper cocompact G-manifold M without boundary together with a G-map $f\colon M \to X$ with the following properties:*

For any $x \in M$ the G_x-representation T_xM is a multiple of the regular G_x-representation $\mathbb{R}[G_x]$ for G_x the isotropy group of $x \in X$. The dimensions of the components $C \in \pi_0(M)$ are all equal. The components of M^H are orientable manifolds for each $H \subseteq G$. The induced map $\pi_0(f^H)\colon \pi_0(M^H) \to \pi_0(X^H)$ is bijective for each finite subgroup $H \subseteq G$. The induced map

$$U^G(f)\colon U^G(M) \xrightarrow{\cong} U^G(X)$$

is bijective and sends $\chi^G(M)$ to u.

Proof : In the first step we want to reduce the claim to the case, where X is a finite proper 1-dimensional G-CW-complex such that for any $x\colon G/H \to X$ there is a zero cell G/H which meets X^H.

If the G-map $f\colon X \to Y$ of proper G-CW-complexes induces bijections $\pi_0(f^H)\colon \pi_0(X^H) \xrightarrow{\cong} \pi_0(Y^H)$ for all finite subgroups $H \subseteq G$, then the induced map $U^G(f)\colon U^G(X) \to U^G(Y)$ is bijective by (3.3). In particular the inclusion of the 1-skeleton $i_1\colon X_1 \to X$ induces a bijection $U^G(i_1)$.

Since X_1 is finite and proper, $WH\backslash\pi_0(X^H)$ is finite for all finite subgroups $H \subseteq G$ and the set $\{(H) \in \mathrm{consub}(G) \mid X^H \neq \emptyset\}$ is finite (see Lemma 7.1). We conclude from (3.3) that $\mathrm{Is}\,\pi_0(G, X_1)$ is finite. Fix for any $[x\colon G/H \to X_1]$ a representative $x\colon G/H \to X$ whose image lies in X_0. This is possible by the equivariant cellular approximation theorem. Define a finite proper G-CW-complex Y by the G-pushout diagram

$$\begin{array}{ccc}
\coprod_{\substack{[x\colon G/H\to X]\in\\ \operatorname{Is}\pi_0(G,X)}} G/H\times\{0\} & \xrightarrow{\coprod_{\substack{[x\colon G/H\to X]\in\\ \operatorname{Is}\pi_0(G,X)}} x} & X_1\\
{\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i_3}\\
\coprod_{\substack{[x\colon G/H\to X]\in\\ \operatorname{Is}\pi_0(G,X)}} G/H\times[0,1] & \longrightarrow & Y
\end{array}$$

where i_2 is the inclusion. Since i_2 is a G-homotopy equivalence, i_3 is a G-homotopy equivalence. Let $i_3^{-1}\colon Y\to X_1$ be a G-homotopy inverse. Then $U^G(i_1\circ i_3^{-1})\colon U^G(Y)\to U^G(X)$ is a bijection and Y is a finite proper G-CW-complex such that for each G-map $y\colon G/H\to Y$ there is a zero cell G/H which meets $Y^H(y)$, namely $G/H\times\{1\}\subseteq G/H\times[0,1]$ for the corresponding $[x]\in\operatorname{Is}\Pi_0(G,X)$ in the pushout diagram above. Hence we can assume without loss of generality $X=Y$ in the sequel.

Fix a number n such for any $H\subseteq G$ with $X^H\neq\emptyset$ the order $|H|$ divides n. Let G/H_1, G/H_2, ..., G/H_r be the equivariant zero-cells of X. Let N_i be a $4n/|H_i|$-dimensional closed oriented manifold. Let $\prod_{H_i}N_i$ be the closed H_i-manifold with the H_i-action coming from permuting the factors. This is called the *multiplicative induction* or *coinduction* of N_i. One easily checks that for $K\subseteq H_i$ the K-fixed point sets of $\prod_{H_i}N_i$ is diffeomorphic to $\prod_{k=1}^{|H_i/K|}N_i$ and hence a closed connected orientable manifold which is non-empty. Moreover, the $(H_i)_x$-representation $T_x\left(\prod_{H_i}N_i\right)$ is a multiple of the regular real $(H_i)_x$-representation for all $x\in\prod_{H_i}N_i$. The manifolds N_i will be specified later. Given an orthogonal H-representation V of a finite group H, we denote by DV and SV the unit disk and the unit sphere and by $\operatorname{int}(DV)=DV-SV$ the interior of DV. Define the $4n$-dimensional proper cocompact G-manifold M_0 to be

$$M_0 \;=\; \coprod_{i=1}^{r} G\times_{H_i}\left(\prod_{H_i}N_i\right).$$

Let $f_0\colon M_0\to X$ be the map which is on $G\times_{H_i}\left(\prod_{H_i}N_i\right)$ given by the canonical projection onto the cell $G/H_i=G\times_{H_i}\{*\}$.

Fix a G-pushout describing, how $X=X_1$ is obtained from X_0 by attaching equivariant one-cells

$$\begin{array}{ccc}
\coprod_{j=1}^{s} G/K_j\times S^0 & \xrightarrow{q} & X_0\\
\downarrow & & \downarrow\\
\coprod_{j=1}^{s} G/K_j\times D^1 & \xrightarrow[Q]{} & X
\end{array}$$

Consider a G-map $\sigma\colon G/K\to G/H$ for finite subgroups $H,K\subseteq G$. Suppose that $|H|$ and $|K|$ divide n. Choose $g\in G$ such that $g^{-1}Kg\subseteq H$ and σ sends $g'K$ to $g'gH$. Let $c_g\colon K\to H$ be the injective group homomorphism $g'\mapsto g^{-1}g'g$. The K-representations $\mathbb{R}[K]^{4n/|K|}$ and $c_g^*\mathbb{R}[H]^{4n/|H|}$, which is obtained from the H-representation $\mathbb{R}[H]^{4n/|H|}$ by restriction with c_g, are isomorphic. Choose an isometric c_g-equivariant linear isomorphism

$$\phi\colon \mathbb{R}[K]^{4n/|K|} \to \mathbb{R}[H]^{4n/|H|}.$$

Choose a small number $\epsilon > 0$ and an element $w \in \mathbb{R}[H]^{4n/|H|}$ such that $||w|| = 1$, the H-isotropy group of w is $g^{-1}Kg$ and the distance of two distinct points in the H-orbit through w is larger than 3ϵ. The following map is a G-embedding

$$\psi\colon G \times_K D\mathbb{R}[K]^{4n/|K|} \to G \times_H \mathbb{R}[H]^{4n/|H|}, \qquad (g', v) \;\mapsto\; (g'g, \epsilon \cdot \phi(v) + w)$$

such that the following diagram with the canonical projections as vertical arrows commutes:

$$\begin{array}{ccc} G \times_K D\mathbb{R}[K]^{4n/|K|} & \xrightarrow{\psi} & G \times_H \mathbb{R}[H]^{4n/|H|} \\ \mathrm{pr}\big\downarrow & & \big\downarrow\mathrm{pr} \\ G/K & \xrightarrow[\sigma]{} & G/H \end{array}$$

Using the construction above with appropriate choices of w and ϵ we can find a G-embedding

$$\Psi\colon \coprod_{j=1}^{s} G \times_{K_j} \left(D\mathbb{R}[K_j]^{4n/|K_j|} \times S^0\right) \to M_0$$

such that the following diagram commutes:

$$\begin{array}{ccc} \coprod_{j=1}^{s} G \times_{K_j} \left(D\mathbb{R}[K_j]^{4n/|K|} \times S^0\right) & \xrightarrow{\Psi} & M_0 \\ \mathrm{pr}\big\downarrow & & \big\downarrow f_0 \\ \coprod_{j=1}^{s} G/K_j \times S^0 & \xrightarrow[q]{} & X_0 \end{array}$$

where pr is the obvious projection. Let M_0' be the proper cocompact G-manifold with boundary

$$M_0' \;=\; M_0 - \Psi\left(\coprod_{j=1}^{s} G \times_{K_j} \left(\mathrm{int}\left(D\mathbb{R}[K_j]^{4n/|K_j|}\right) \times S^0\right)\right).$$

Define a smooth G-manifold M by the following pushout

$$\begin{array}{ccc} \coprod_{j=1}^{s} G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K_j|} \times S^0\right) & \xrightarrow{\Psi|} & M_0' \\ \big\downarrow & & \big\downarrow \\ \coprod_{j=1}^{s} G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K_j|} \times D^1\right) & \xrightarrow[\overline{\Psi|}]{} & M \end{array}$$

where $\Psi|$ denotes the restriction of Ψ. The map $f_0\colon M_0 \to X_0$ defines a map $f_0'\colon M_0' \to X_0$ by restriction. Since $(S\mathbb{R}[K_j]^{4n/|K|})^L$ for each $L \subseteq K_j$ and $\left(\prod_{H_i} N_i\right)^L$ for each $L \subseteq H_i$ are non-empty and connected, the maps f_0' and the projections $\mathrm{pr}\colon \coprod_{j=1}^s G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K|} \times S^0\right) \to \coprod_{j=1}^s G/K_j \times S^0$ and $\mathrm{pr}\colon \coprod_{j=1}^s G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K|} \times D^1\right) \to \coprod_{j=1}^s G/K_j \times D^1$ induce on the L-fixed point set 1-connected maps for each $L \subseteq G$. Hence the G-map $f\colon M \to X$, which is induced by these three maps and the pushout property, induces a 1-connected map on the L-fixed point sets for each $L \subseteq G$. In particular $\pi_0(f^L)\colon \pi_0(M^L) \to \pi_0(X^L)$ is bijective for each $L \subseteq G$ and the map $U^G(f)\colon U^G(M) \to U^G(X)$ is bijective by (3.4). Obviously the G_x-representation T_xM for any point $x \in M$ is a multiple of the regular G_x-representation $\mathbb{R}[G_x]$.

The following diagram commutes

$$\begin{array}{ccccc} U^G(M_0) & \xleftarrow[\cong]{U^G(i_0)} & U^G(M_0') & \xrightarrow{U^G(i_4)} & U^G(M) \\ {\scriptstyle U^G(f_0)}\downarrow\cong & & {\scriptstyle U^G(f_0')}\downarrow\cong & & {\scriptstyle U^G(f)}\downarrow\cong \\ U^G(X_0) & \xrightarrow[\cong]{\mathrm{id}} & U^G(X_0) & \xrightarrow{U^G(i_5)} & U^G(X) \end{array}$$

where i_0, i_4 and i_5 denote the inclusions.

Next we compute $U^G(f)(\chi^G(M))$. We get by the sum formula for the universal equivariant Euler characteristic [7, Theorem 5.4 on page 100]

$$\begin{aligned}
&U^G(f)(\chi^G(M)) \\
&\quad = U^G(i_5 \circ f_0)\left(\chi^G(M_0)\right) \\
&\qquad - U^G(i_5 \circ f_0 \circ \Psi)\left(\chi^G\left(\coprod_{j=1}^s G \times_{K_j} \left(\left(D\mathbb{R}[K_j]^{4n/|K_j|}\right) \times S^0\right)\right)\right) \\
&\qquad + U^G(i_5 \circ f_0 \circ \Psi \circ i_6)\left(\chi^G\left(\coprod_{j=1}^s G \times_{K_j} \left(\left(S\mathbb{R}[K_j]^{4n/|K_j|}\right) \times S^0\right)\right)\right) \\
&\qquad - U^G(i_5 \circ f_0' \circ \Psi|)\left(\chi^G\left(\coprod_{j=1}^s G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K_j|} \times S^0\right)\right)\right) \\
&\qquad + U^G(f \circ \overline{\Psi|})\left(\chi^G\left(\coprod_{j=1}^s G \times_{K_j} \left(S\mathbb{R}[K_j]^{4n/|K_j|} \times D^1\right)\right)\right),
\end{aligned}$$

where i_6 is the inclusion. Since $(S\mathbb{R}[K_j]^{4n/|K_j|})^L$ is an odd-dimensional sphere and hence has vanishing (non-equivariant) Euler characteristic for each $L \subseteq K_j$, the element $\chi^{K_j}\left(S\mathbb{R}[K_j]^{4n/|K_j|}\right)$ in $U^{K_j}(\{*\}) = A(K_j)$ is sent to zero under the injective map $\chi_0^{K_j}\colon A(K_j) \to \prod_{(L)\in\mathrm{consub}(K_j)} \mathbb{Z}$. This implies

$$\chi^{K_j}\left(S\mathbb{R}[K_j]^{4n/|K_j|}\right) = 0$$

and hence

$$\chi^G\left(\coprod_{j=1}^{s} G\times_{K_j}\left(S\mathbb{R}[K_j]^{4n/|K_j|}\times S^0\right)\right) \quad = \quad 0.$$

The space $D\mathbb{R}[K_j]^{4n/|K_j|}$ is K_j-homotopy equivalent to $\{*\}$. Hence

$$\begin{aligned} & U^G(i_5\circ f_0\circ\Psi)\left(\chi^G\left(\coprod_{j=1}^{s} G\times_{K_j}\left(D\mathbb{R}[K_j]^{4n/|K_j|}\times S^0\right)\right)\right) \\ &= U^G(i_5\circ f_0\circ\Psi\circ i_7)\left(\chi^G\left(\coprod_{j=1}^{s} G\times_{K_j} S^0\right)\right) \\ &= 2\cdot\sum_{j=1}^{s} U^G(x_j)(\chi^G(G/K_j)), \end{aligned}$$

where i_7 is the inclusion and $x_j^1\colon G/K_j\to X$ is the restriction of the characteristic map of the one cell $G/K_j\times D^1$ to $G/K_j\times\{1/2\}$. If $x_i^0\colon G/H_i\to X$ is the characteristic map of the 0-cell G/H_i and $\alpha_i\colon H_i\to G$ the inclusion, we conclude

$$\begin{aligned} U^G(f)(\chi^G(M)) \quad = \quad & \sum_{i=1}^{r} U^G(x_i^0)\circ(\alpha_i)_*\left(\chi^{H_i}\left(\prod_{H_i} N_i\right)\right) \\ & - \ 2\cdot\sum_{j=1}^{s} U^G(x_j)(\chi^G(G/K_j)). \end{aligned} \tag{7.3}$$

The element $\chi^{H_i}\left(\prod_{H_i} N_i\right)\in A(H_i)$ is sent under the injective character map

$$\mathrm{ch}_0^{H_i}\colon A(H_i)\to\prod_{(K)\,\mathrm{consub}(H_i)}\mathbb{Z},\qquad [S]\mapsto|S^K|$$

to $(\chi(N_i)^{|H/K|}\mid(K)\in\mathrm{consub}(H_i)\}$. This implies

$$\begin{aligned} \chi^{H_i}\left(\prod_{H_i} N_i\right) \quad = \quad & \chi(N_i)\cdot[H_i/H_i] \\ & + \sum_{\substack{(K)\in\mathrm{consub}(H_i),\\ K\neq H_i}} \lambda_{(K)}(\chi(N_i))\cdot[H_i/K] \end{aligned} \tag{7.4}$$

for appropriate functions $\lambda_{(K)}\colon\mathbb{Z}\to\mathbb{Z}$. We can order the equivariant zero-cells G/H_i of X_0 such that H_i is subconjugate to H_j only if $i\geq j$ holds. We conclude from (7.4) that for an appropriate map μ, the composition

$$\bigoplus_{i=1}^{r}\mathbb{Z}\xrightarrow{\mu}\bigoplus_{i=1}^{r}\mathbb{Z}=U^G(X_0)\xrightarrow{U^G(i_8)}U^G(X_1)$$

sends $\{\chi(N_i) \mid i = 1, 2 \ldots, r\}$ to $\sum_{i=1}^{r} U^G(x_i^0) \circ (\alpha_i)_* \left(\chi^{H_i}\left(\prod_{H_i} N_i\right)\right)$, where $i_8 \colon X_0 \to X$ is the inclusion and μ is given by

$$\mu(a_1, a_2, \ldots, a_r) \ = \ \left(\sum_{j=1}^{r} \mu_{1,j}(a_j), \sum_{j=1}^{r} \mu_{2,j}(a_j), \ldots, \sum_{j=1}^{r} \mu_{r,j}(a_j), \right)$$

for (not necessarily linear) maps $\mu_{i,j} \colon \mathbb{Z} \to \mathbb{Z}$ for which $\mu_{i,j} = 0$ for $i > j$ and $\mu_{i,i} = \mathrm{id}$. The map $U^G(i_8)$ is surjective and the map μ bijective. Since for any integer k and any positive integer l there is a closed connected $4l$-dimensional manifold N with $\chi(N) = k$, we can find appropriate N_i with the right Euler characteristics $\chi(N_i)$ such that for a given element $u \in U^G(X)$

$$\sum_{i=1}^{r} U^G(x_i^0) \circ (\alpha_i)_* \left(\chi^{H_i}\left(\prod_{H_i} N_i\right)\right) \ = \ u + 2 \cdot \sum_{j=1}^{s} U^G(x_j)(\chi^G(G/K_j)). \tag{7.5}$$

We conclude from (7.3) and (7.5) that

$$U^G(f)(\chi^G(M)) \ = u.$$

This finishes the proof of Lemma 7.2. ■

Putting Lemmas 7.1 and 7.2 together gives the following result on the realization problem:

Theorem 7.6 *Let S be a contravariant $\mathsf{Or}(G; \mathcal{F}in)$-set, and suppose that there are only finitely many elements $(H) \in \mathrm{consub}(H)$ with $S(G/H) \neq \emptyset$. Also assume that for any finite subgroup $H \subseteq G$, the set $WH \backslash S(G/H)$ is finite, and that the isotropy group WH_s of each element $s \in S(G/H)$ is finitely generated. Let $u \in U^G(S)$. Then there is a proper cocompact G-manifold M (without boundary) and there is a natural equivalence $T \colon \underline{\pi_0(M)} \xrightarrow{\cong} S$ sending $\chi^G(M)$ to u.*

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