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The Involution on the Equivariant Whitehead Group

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Abstract. For a finite group G we define an involution on the equivariant Whitehead group given by reversing the direction of an equivariant h-cobordism. It turns out that the involution is not compatible with the splitting of the equivariant Whitehead group into a direct sum of algebraic Whitehead groups, certain correction terms involving the transfer maps of the normal sphere bundles of the various fixed point sets come in. However, if the group has odd order, these transfer maps all vanish. We prove a duality formula for a G-homotopy equivalence $(f, \partial f): (M; \partial M) \to (N, \partial N)$ relating the equivariant Whitehead torsion of f and $(f, \partial f)$.

Key words. Equivariant Whitehead group, equivariant Whitehead torsion, involution, equivariant h-cobordism, duality formula.

0. Introduction

Let G be a finite group. We study the involution on the equivariant Whitehead group of a smooth G-manifold given by reversing the direction of an equivariant h-cobordism. This involution does not typically preserve the splitting of $\mathrm{Wh}_\rho^G(M)$ into non-equivariant groups. But we show it does preserve the splitting when G has odd order. We also give a general formula for it, and use this involution to compute the Whitehead torsion of a G-homotopy equivalence of pairs $(f, \partial f): (M, \partial M) \to (N, \partial N)$ from that of $f: M \to N$, if M and N are G manifolds.

Here are a few more details. The equivariant Whitehead group $\mathrm{Wh}^G(N)$ of a G-manifold N splits into algebraic Whitehead groups

$$\mathrm{Wh}^G(N) = \bigoplus_{(H)} \bigoplus_{C \in \pi_0(N^H)/\mathrm{WH}} \mathrm{Wh}(\pi_1(\mathrm{EWH}(C) \times_{\mathrm{WH}(C)} C))$$

where $\mathrm{WH}(C)$ is the isotropy group of $C \in \pi_0(N^H)$ under the WH-action. Let $\mathrm{Wh}_\rho^G(N)$ be the direct summand in $\mathrm{Wh}^G(N)$ corresponding to those components $C \in \pi_0(N^H)$ which contain an element $x \in C$ with isotropy group $G_x = H$. Then any element in $\mathrm{Wh}_\rho^G(N)$ can be realized as the Whitehead torsion of an equivariant h-cobordism over N, provided that certain codimension 3 conditions are satisfied. Hence, we can define an involution $*: \mathrm{Wh}_\rho^G(N) \to \mathrm{Wh}_\rho^G(N)$ by reversing the direction of h-cobordisms (see Section 2).

There is an algebraic involution on each of the summands in the splitting of $\mathrm{Wh}^G_\rho(N)$ coming from involutions on the group rings. There are some places in the literature where it is claimed that $*$ corresponds under the splitting to the direct sum of these involutions. But this is false. We do show that this is true if G has odd order (subject to a mild condition). See 4.2. In general, the involution on the split Whitehead group looks like a triangular matrix. Its entries on the diagonal are the algebraic involutions described above. The other entries are given by transfer homomorphisms associated with the spherical normal bundles of the various fixed point sets. We show that these transfer maps are nontrivial even for $G = \mathbb{Z}/2\mathbb{Z}$.

Consider a G-homotopy equivalence of G-manifold pairs $(f, \partial f): (M, \partial M) \to (N, \partial N)$. We will prove a formula:

$$\tau^G(f) = -*\tau^G(f, \partial f) - \Phi_f(\chi^G(N, \partial N)),$$

where $\Phi_f(\chi^G(N, \partial N))$ is a correction term depending only on the equivariant Euler characteristic $\chi^G(N, \partial N)$ and certain G_x-homotopy equivalences $\varphi_x: STM_x \to STN_{fx}$ associated with f for any $x \in M$. We show that Φ_f is zero if G is a product of a group of odd order and a 2-group and TM_x and TN_{fx} are linearly G_x-isomorphic for any $x \in M$. This formula is an important tool in the proof of the equivariant π-π-theorem in the simple category.

We have chosen to work in a smooth context. A simple group, $\mathrm{Wh}_G^{\mathrm{Top},\rho}(M)$, parametrizing topological G-h-cobordisms is defined by West and by Steinberger in [17]; this group has an analogous involution. In [17], the group we are using is denoted $\mathrm{Wh}_G^{PL,\rho}(M)$. We should also mention that results analogous to those here hold when G is a compact Lie group.

1. The Transfer Homomorphism

Let $\mathrm{Wh}^G(X)$ be the equivariant Whitehead group associated with the finite G-CW-complex X (see Illman [6]). Consider a G-$\mathrm{O}(n)$-vector bundle $p(\xi): \xi \downarrow X$ and sphere bundle $p(S\xi): S\xi \downarrow X$. Then $D\xi$ and $S\xi$ carry the structure of finite G-CW-complexes, unique up to simple G-homotopy equivalence, by the equivariant triangulation theorem (see Illman [7]) and we can define transfer homomorphisms,

1.1 $p(S\xi)^*: \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(S\xi)$

$p(D\xi)^*: \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(D\xi)$

as follows. They send an element in $\mathrm{Wh}^G(X)$ represented by the torsion $\tau^G(f)$ of a G-homotopy equivalence $f: Y \to X$ to $\tau^G(f_\#)$, where $f_\#$ is the bundle map given by the pull-back construction. If $p(S\xi)_*$ and $p(D\xi)_*$ are induced by the projections we want to study the compositions $p(D\xi)_* p(D\xi)^*$ and $p(S\xi)_* p(S\xi)^*: \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(X)$. We start by collecting some properties of the Whitehead torsion. Proofs can be found in Dovermann and Rothenberg [4], Hauschild [5], Illman [6], and Lück [13].

1.2. ADDITIVITY

Let (X_1, X_0) be a pair of finite G-GW-complexes and $i\colon X_0 \to X_2$ be a cellular G-map. Denote by X the finite G-CW-complex given by the G-pushout. Define (Y_1, Y_0), $j\colon Y_0 \to Y_2$ and Y similarly. Let $k_i\colon Y_i \to Y$ be the obvious map for $i = 0, 1, 2$. Consider a pair of G-homotopy equivalences $(f_1, f_0)\colon (X_1, X_0) \to (Y_1, Y_0)$ and a G-homotopy equivalence $f_2\colon X_2 \to Y_2$ such that $f_2 i = j f_0$. Let $f\colon X \to Y$ be the G-map given by the G-push-out property. Then f is a G homotopy equivalence. (see, e.g., [13], Lemma 2.13). We have:

$$\tau^G(f) = k_{1*}\tau^G(f_1) + k_{2*}\tau^G(f_2) - k_{0*}\tau^G(f_0).$$

1.3. COMPOSITION FORMULA

$$\tau^G(gf) = \tau^G(g) + g_*\tau^G(f).$$

1.4. PRODUCT FORMULA

If X is a G-space let $\{G/? \to X\}$ be the set of G-maps $x\colon G/H \to X$ for $G \supseteq H$. We call $x\colon G/H \to X$ and $y\colon G/K \to X$ equivalent if there is a G-isomorphism $\sigma\colon G/H \to G/K$ satisfying $y\sigma \simeq_G x$. Let $\{G/? \to X\}/\sim$ be the set of equivalence classes and $U^G(X)$ be the free abelian group generated by $\{G/? \to X\}/\sim$. If $X^H(x)$ is the path component of X^H containing $x(eH)$ we obtain a bijection $\{G/? \to X\}/\sim \;\to \coprod_{(H)} \pi_0(X^H)/\mathrm{WH}$ sending the class of x to the class of $X^H(x)$. In particular, $U^G(X)$ is $\oplus H_0(X^H)^{WH}$. Let WH(x) (resp. NH(x)) be the isotropy group of $X^H(x) \in \pi_0(X^H)$ under the WH-action (resp. NH-action). If X is a finite G-CW-complex, define the equivariant Euler characteristic, $\chi^G(X) \in U^G(X)$, by assigning to $[x\colon G/H \to X]$ the ordinary Euler characteristic,

$$\chi(X^H(x)/\mathrm{WH}(x), X^H(x) \cap X^{>H}/\mathrm{WH}(x)).$$

We get a natural pairing $U^G(X) \otimes \mathrm{Wh}^G(Y) \to \mathrm{Wh}^G(X \times Y)$ by sending $[x\colon G/H \to X] \otimes \tau^G(g)$ to $(x \times \mathrm{id})_*\tau^G(\mathrm{id} \times g)$ for a G-homotopy equivalence $g\colon Y' \to Y$. Then we have for two G-homotopy equivalences $f\colon X' \to X$ and $g\colon Y' \to Y$:

1.5. $\quad \tau^G(f \times g) = \chi^G(X) \otimes \tau^G(g) + \tau^G(f) \otimes \chi^G(Y).$

1.6 SPLITTING INTO ALGEBRAIC WHITEHEAD GROUPS

The equivariant Whitehead group splits into algebraic Whitehead groups as follows. For $G \supseteq H$ define $i(H)\colon \mathrm{Wh}^1(\mathrm{EWH} \times_{\mathrm{WH}} X^H) \to \mathrm{Wh}^G(X)$ as the composition, $\mathrm{Wh}^1(\mathrm{EWH} \times_{\mathrm{WH}} X^H)$

$$\xrightarrow{(1)} \mathrm{Wh}^{\mathrm{WH}}(\mathrm{EWH} \times X^H) \xrightarrow{(2)} \mathrm{Wh}^{\mathrm{WH}}(X^H) \xrightarrow{(3)} \mathrm{Wh}^{\mathrm{NH}}(X^H) \xrightarrow{(4)}$$
$$\mathrm{Wh}^G(G \times_{\mathrm{NH}} X^H) \xrightarrow{(5)} \mathrm{Wh}^G(X),$$

where (1) is given by the pull back construction, (2) by the projection, (3) by restriction,

(4) by induction and (5) by the map $G \times_{\mathrm{NH}} X^H \to X$ sending (g, x) to $g \cdot x$. We obtain an isomorphism.

$$\bigoplus_{(H)} i(H)\colon \bigoplus_{(H)} \mathrm{Wh}^1(\mathrm{EWH} \times_{\mathrm{WH}} X^H) \to \mathrm{Wh}^G(X).$$

If Z is a space $\mathrm{Wh}^1(Z)$ is isomorphic to $\bigoplus \mathrm{Wh}(\mathbb{Z}\pi_1(C))$, where C runs over $\pi_0(Z)$. Hence, we get an isomorphism

$$\bigoplus_{\{G/?\to X\}/\sim} \mathrm{Wh}(\mathbb{Z}\pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} X^H(x))) \to \mathrm{Wh}^G(X).$$

PROPOSITION 1.7. *For any G vector bundle ξ on a finite G-CW-complex $X, p(D\xi)_* p(D\xi)^*$ is the identity on* $\mathrm{Wh}^G(X)$.

Proof. We may as well assume X is a finite G simplicial complex. Let $f \cdot X' - X$ be a G-homotopy equivalence. By 1.3, we have

$$p(D\xi)_* p(D\xi)^*(\tau^G(f)) = p(D\xi)_*(\tau^G(f_\#)) = f_* \tau^G(p(Df^*\xi)) + \tau^G(f) - \tau^G(p(D\xi)).$$

Hence, it suffices to show $\tau^G(p(D\xi)) = 0$ for any bundle ξ. Because of the local triviality of ξ, 1.2 and 1.4, this reduces first to the case when X is G-contractible, and then to the obvious case $p\colon DV \to \{*\}$ for a G-representation V. □

The pairing in 1.4 induces a pairing $A(G) \otimes \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(X)$ if we identify the Burnside ring $A(G)$ with $U^G(\{*\})$ (the map sends $[G/H]$ to $x\colon G/H \to \{*\}$). Let $e^G(X) \in A(G)$ be the image of $\chi^G(X) \in U^G(X)$ under $\mathrm{pr}_*\colon U^G(X) \to U^G(\{*\}) = A(G)$. If $\beta(H, n)$ is the number of cells of type $G/H \times D^n$ in X we have

$$e^G(X) = \sum_{(H)} \sum_{n \geq 0} (-1)^n \cdot \beta(H, n) \cdot [G/H]$$

in $A(G)$. Formula 1.5 above now reduces to

$$(\pi_X)_* \tau^G(f \times 1_Y) = \tau^G(f) \cdot e^G(Y).$$

We derive from 1.4:

PROPOSITION 1.8. *If $\xi \downarrow X$ is the trivial G-$O(n)$-vector bundle $X \times V$ then $p(S\xi)_* p(S\xi)^*\colon \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(X)$ is multiplication by $e^G(SV)$.*

PROPOSITION 1.9. *Let $\xi \downarrow X$ be a G-$O(n)$-vector bundle and $f_\#\colon Sf^*\xi \to S\xi$ be given by the pull-back construction applied to a G-homotopy equivalence $f\colon Y \to X$ between finite G-CW-complexes, then we have $(f_\#)_* p(Sf^*\xi)^* = p(S\xi)^* f_*$. Similarly for the disc bundle, we have: $(f_\#)_* p(Df^*\xi)^* = p(D\xi)^* f_*$.*

Proof. Follows directly from the definitions. □

In Lück [12], the equivariant (unstable) first Stiefel–Whitney class $w\xi$ is defined for any locally linear G-S^n-fibration $S\xi$.

PROPOSITION 1.10. *Let ξ and η be G-$O(n)$-vector bundles over X with $w\xi = w\eta$. Then $p(S\xi)_* p(S\xi)^*$ and $p(S\eta)_* p(S\eta)^*$ agree.*

Proof. This follows from the algebraic description of $p(S\xi)^*$ given in Lück[13]. An alternative proof uses the notion of an equivariant Eilenberg–MacLane space introduced in Lück [11]. Let $\lambda \downarrow BF(G, n)$ be the classifying G-fibration and $\mathrm{BF}(G, n)$ the classifying space for locally linear G-S^n-fibrations. Let $b(\xi)$ and $b(\eta)\colon X \to \mathrm{BF}(G, n)$ be the classifying maps for ξ and η. Let $i\colon X \to K(\pi^G X, \mu, 1)$ and $j\colon \mathrm{BF}(G, n) \to K(\pi^G \mathrm{BF}(G, n), \mu, 1)$ be the canonical G-maps. We can interpret $w\xi$ and $w\eta$ as the G-homotopy classes of the G-maps

$$K(\pi^G X, \mu, 1) \to K(\pi^G \mathrm{BF}(G, n), \mu, 1)$$

induced by $jb(\xi)$ and $jb(\eta)$. By assumption we can find a G-map $k\colon K(\pi^G X, \mu, 1) \to K(\pi^G \mathrm{BF}(G, n), \mu, 1)$ representing both $w\xi$ and $w\eta$. Consider the G-homotopy pull-back,

$$\begin{array}{ccc} Z & \xrightarrow{\quad k_\# \quad} & \mathrm{BF}(G, n) \\ {\scriptstyle j_\#}\downarrow & & \downarrow{\scriptstyle j} \\ K(\pi^G X, \mu, 1) & \xrightarrow[\quad k \quad]{} & K(\pi^G \mathrm{BF}(G, n), \mu, 1). \end{array}$$

Since $jb(\xi) \simeq_G ki$, there is a G-map $a(\xi)\colon X \to Z$ satisfying $k_\# a(\xi) \simeq_G b(\xi)$ and $j_\# a(\xi) \simeq_G i$. Let $\zeta = (k_\#)^*\lambda$. By 1.6, i_* and j_* are isomorphisms of Whitehead groups. From Proposition 1.9 we get:

$$\begin{aligned} i_* p(S\xi)_* p(S\xi)^* i_*^{-1} &= (j_\#)_* a(\xi)_* p(S\xi)_* p(S\xi)^* i_*^{-1}. \\ &= (j_\#)_* p(S(\zeta))_* p(S(\zeta))^* a(\zeta)_* (i_*)^{-1} = (j_\#)_* p(S(\zeta))_* p(S(\zeta))^* j_{\#*}^{-1}. \end{aligned}$$

This is also true for η so that $i_* p(S\xi)_* p(S\xi)^* i_*^{-1} = i_* p(S\eta)_* p(S\eta)^* i_*^{-1}$ holds. But i_* is an isomorphism by 1.6. □

Remark. In the above argument (only) we make use of $\mathrm{Wh}^G(X)$ for an infinite G-complex X. This is defined exactly as in [6] by means of strong deformation retractions $Y \to X$, with the modest adjustment that we require only that Y-X have finitely many cells. (See Lück [13] for a full treatment.)

We also make use of the transfer $p^*\colon \mathrm{Wh}^G(X) \to \mathrm{Wh}^G(S(\xi))$ for a locally linear G-S^n fibration ξ; the definition is analogous to that in 1.1, but the details are given in [13].

PROPOSITION 1.11. *Let ξ and η be G-$O(n)$-vector bundles over X. Suppose that G has odd order, and the nonequivariant first Stiefel–Whitney classes $w_1(\xi)$ and $w_1(\eta) \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ agree. Suppose also that for any $x \in X, S\xi_x \simeq_{G_x} S\eta_x$. Then $w\xi = w\eta$.*

Proof. See Lück [12]. □

THEOREM 1.12. *Let $\xi \downarrow X$ be a G-$O(n)$-vector bundle with trivial $w_1(\xi) \in H^1(X, \mathbb{Z}/2\mathbb{Z})$. Assume G has odd order and that there is some G representation V such that $S\xi_x \simeq_{G_x} SV$, for any $x \in X$. Then $p(S\xi)_* p(S\xi)^*\colon \mathrm{Wh}^G(X) \to Wh^G(X)$ is $(1 - (-1)^n)\cdot id$.*

Proof. Because of Propositions 1.10 and 1.11, we can assume that ξ is the trivial G-$O(n)$-vector bundle $X \times V$. Since G is odd, V/V^G is of complex type so that

$e^G(SV) \in A(G)$ is $(1 + (-1)^{\dim(SV)}) \cdot [G/G]$. Now apply Proposition 1.8. □

Next we consider G-$O(n)$-vector bundles ξ and η over X and a G-fibre homotopy equivalence,

$$\begin{array}{ccc} S\xi & \xrightarrow{F} & S\eta \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Define a homomorphism $\Phi_F\colon U^G(Y) \to \mathrm{Wh}^G(S\eta)$ as follows. Any base element of $U^G(Y)$ can be represented by $y = fx$ for some $x\colon G/H \to X$. Let $\Phi_F([y])$ be the image of $\tau^G(F \mid x^*S\xi\colon x^*S\xi \to y^*S\eta)$ under $(y_\#)_*\colon \mathrm{Wh}^G(y^*S\eta) \to \mathrm{Wh}^G(S\eta)$.

THEOREM 1.13. $\tau^G(F) = p(S\eta)^*(\tau^G(f)) + \Phi_F(\chi^G(Y))$.

Proof. Write F as the composite $S\xi \xrightarrow{F_1} f^*S\eta \xrightarrow{f_\#} S\eta$. Now $\tau^G(F) = \tau^G(f_\#) + f_{\#*}\tau^G(F_1)$ from 1.3. By definition, $\tau^G(f_\#) = p(S\eta)^*\tau^G(f)$ and we derive $f_{\#*}\tau^G(F_1) = \Phi_F(\chi^G(Y))$ from 1.2. □

COROLLARY 1.14. *Suppose that G has odd order and $w_1(\eta) \in H^1(Y; \mathbb{Z}/2)$ vanishes. Assume there is a G-module V with $S(\eta_y) \simeq_{G_y} S(V)$ for all $y \in Y$. If $\chi^G(Y) \in U^G(Y)$ is zero or if ξ_x and η_{fx} are linearly G_x-isomorphic for any $x \in X$, then $p(S(\eta))_*(\tau^G(f)) = (1 + (-1)^n)\tau^G(f)$.*

Proof. Apply Theorems 1.12 and 1.13 and the fact that any G_x-map $S\xi_x \to S\eta_{fx}$ is G_x-homotopic to one induced by a linear isomorphism, as $A(G_x)^* = \{\pm 1\}$ holds; in either case, $\Phi_F(\chi^G(Y)) = 0$. □

2. The Involution on the Equivariant Whitehead Group

Let M be a G-manifold, i.e., a smooth compact manifold possibly with boundary on which G acts smoothly. Denote

$$M_H = \{x \in M \mid G_x = H\}, \qquad M_{(H)} = \{x \in M \mid (G_x) = (H)\} \quad \text{and}$$
$$M^{(H)} = \{x \in M \mid (G_x) \geqslant (H)\}.$$

The isovariant Whitehead group is defined by

2.1. $\mathrm{Wh}^G_{\mathrm{Iso}}(M) = \bigoplus_{(H)} \mathrm{Wh}^1(M_H/\mathrm{WH})$.

Here $\mathrm{Wh}^1(M_H/\mathrm{WH})$ means the Whitehead group of the compact manifold obtained by removing an open regular neighborhood of $M^{>H}/\mathrm{WH}$ from M^H/WH.

An (isovariant) h-cobordism (W, M, N) is a G-manifold W with boundary $\partial W = M \cup N$, such that $\partial M = M \cap N = \partial N$ and the inclusions $M \to W$ and $N \to W$ are (isovariant) G-homotopy equivalences. We define the isovariant Whitehead torsion $\tau^G_{\mathrm{Iso}}(W, M, N)$ of an isovariant h-cobordism (W, M, N) inductively over the number of orbit types (H) with $H \in \mathrm{Iso}\, M = \{H \mid M_H \neq \emptyset\}$. Let (H) be maximal among these.

Then $M_{(H)} = M^{(H)}$ is a compact G-submanifold of M with normal G-vector bundle $\nu_M = \nu(M^{(H)}, M)$. Define ν_N and ν_W similarly. Note that $\nu_W | M = \nu_M$ and $\nu_W | N = \nu_N$. Sometimes we will denote by ν_M also the NH-normal bundle of M^H in M. Consider the G-manifolds $\bar{M} = M \backslash \operatorname{int} D\nu_M, \bar{N} = N \backslash \operatorname{int} D\nu_N \cup S\nu_W$ and $\bar{W} = W \backslash \operatorname{int} D\nu_W$. We get an isovariant h-cobordism $(\bar{W}, \bar{M}, \bar{N})$, since an h-cobordism is isovariant if and only if $M_H \to W_H$ and $N_H \to W_H$ are homotopy equivalences for each $H \in \operatorname{Iso} M$ (Hauschild [5]). By the induction hypothesis $\tau^G_{\text{Iso}}(\bar{W}, \bar{M}, \bar{N}) \in \text{Wh}^G_{\text{Iso}}(\bar{M})$ is defined as $(\bar{W}, \bar{M}, \bar{N})$ and has one orbit type less. Let $\tau(W_H/\text{WH}, M_H/\text{WH}, N_H/\text{WH}) \in \text{Wh}^1(M_H/\text{WH})$ be the Whitehead torsion of the nonequivariant h-cobordism $(W_H/\text{WH}, M_H/\text{WH}, N_H/\text{WH})$. The obvious map:

2.2. $s\colon \text{Wh}^G_{\text{Iso}}(\bar{M}) \oplus \text{Wh}^1(M_H/\text{WH}) \to \text{Wh}^G_{\text{Iso}}(M)$

is an isomorphism as $\bar{M}_K \to M_K$ is a WK-homotopy equivalence for any $K \in \operatorname{Iso} \bar{M}$. Define

2.3. $\tau^G_{\text{Iso}}(W, M, N) = s(\tau^G_{\text{Iso}}(\bar{W}, \bar{M}, \bar{N}) \oplus \tau(W_H/\text{WH}, M_H/\text{WH}, N_H/\text{WH}))$

THEOREM 2.4. (Equivariant s-cobordism Theorem): *Let M be a G-manifold such that $\dim(M_H) \geq 5$ for each $H \in \operatorname{Iso} M$.*

(i) *Two isovariant h-cobordisms (W, M, N) and (W', M, N') over M are G-diffeomorphic rel M if and only if $\tau^G_{\text{Iso}}(W, M, N)$ and $\tau^G_{\text{Iso}}(W', M, N')$ agree.*

(ii) *Any element in the isovariant Whitehead group $\text{Wh}^G_{\text{Iso}}(M)$ can be realized as $\tau^G_{\text{Iso}}(W, M, N)$ for some isovariant h-cobordism (W, M, N).*

Proof. See Browder and Quinn [1], Hauschild [5], Rothenberg [16]. □

A (not necessarily isovariant) h-cobordism (W, M, N) defines $\tau^G(W, M, N) \in \text{Wh}^G(M)$ by the formula: $j_* \tau^G(W, M, N) = \tau^G(j\colon M \to W)$. By Theorem 2.4, and the equivariant triangulation theorem, there is a map,

$$\Phi\colon \text{Wh}^G_{\text{Iso}}(M) \to \text{Wh}^G(M)$$

uniquely determined by the property that $\Phi(\tau^G_{\text{Iso}}(W, M, N)) = \tau^G(W, M, N)$, for any h-cobordism (W, M, N). Define the direct summand,

2.5. $\text{Wh}^G_\rho(M) \subset \text{Wh}^G(M)$

to be the image of

$$\bigoplus_{I\{G/? \to M\}/\sim} \text{Wh}(\mathbb{Z}\pi_1(\text{EWH} \times_{\text{WH}(x)} M^H(x)))$$

under the isomorphism of 1.6, where $I\{G/? \to M\}/\sim$ is the subset of $\{G/? \to M\}/\sim$ represented by elements $[x\colon G/H \to X]$ with $M^H(x)_H \neq \emptyset$. In other words, we consider only components C of M^H containing a point $x \in C$ with $G_x = H$. For each $x \in M$ with $G_x = H$, WH(x) acts freely on $M_H(x)$, so Φ sends the summand corresponding to $\text{Wh}^1(M_H(x)/\text{WH}(x))$ to the summand of $\text{Wh}^G(M)$ corresponding, via 1.6, to

$\mathrm{Wh}(\mathbb{Z}\pi_1(\mathrm{EWH} \times_{\mathrm{WH}(x)} M^H(x)))$. Therefore, Φ is a map

2.6. $\Phi\colon \mathrm{Wh}^G_{\mathrm{Iso}}(M) \to \mathrm{Wh}^G_p(M)$.

For the rest of this paper we make the following assumption.

ASSUMPTION 2.7. M has codimension 3 gaps. That is to say, dim D-dim $C \neq 1$ and 2, when $C \in \pi_0(M^K), D \in \pi_0(M^H), C \subset D, H \subset K$. Moreover, $\dim(M_H) \geqslant 5$ holds for any $H \in \mathrm{Iso}\, M$.

THEOREM 2.8. (i) *Φ is an isomorphism of abelian groups.* (ii) *Any h-cobordism over M is isovariant.*

Proof. (i) The proof is done inductively over the number of orbit types. Choose (H) so that $H \in \mathrm{Iso}\, M$ is maximal. Consider an isovariant h-cobordism (W, M, N) and define $(\bar{W}, \bar{M}, \bar{N})$ as above. Let

2.9. $\mathrm{trf}\colon \mathrm{Wh}^1(M_H/\mathrm{WH}) \to \mathrm{Wh}^G_p(\bar{M})$

be the composition:

$$\mathrm{Wh}^1(M_H/\mathrm{WH}) \to \mathrm{Wh}^{\mathrm{WH}}(M_H) \to \mathrm{Wh}^{\mathrm{NH}}(M_H) \xrightarrow{p(S\nu_M)^*} \mathrm{Wh}^{\mathrm{NH}}(S\nu_M) \to$$
$$\mathrm{Wh}^G(G \times_{\mathrm{NH}} S\nu_M) \to \mathrm{Wh}^G_p(\bar{M}).$$

We claim that the following diagram commutes if k, r, and s are the obvious isomorphisms.

2.10.
$$\begin{array}{ccc}
\mathrm{Wh}^G_{\mathrm{Iso}}(\bar{M}) \oplus \mathrm{Wh}^1(M_H/\mathrm{WH}) & \xrightarrow[\cong]{s} & \mathrm{Wh}^G_{\mathrm{Iso}}(M) \\
\Big\downarrow \begin{bmatrix} \Phi_{\bar{M}} & -\mathrm{trf} \\ 0 & k \end{bmatrix} & & \Big\downarrow \Phi_M \\
\mathrm{Wh}^G_p(\bar{M}) \oplus \mathrm{Wh}^1(\mathrm{EWH} \times_{\mathrm{WH}} M_H) & \xrightarrow{r} & \mathrm{Wh}^G_p(M)
\end{array}$$

(k is an isomorphism by 1.6).

By definition, $\Phi_M s(\tau^G(\bar{W}, \bar{M}, \bar{N}) \oplus \tau^1(W_H/\mathrm{WH}, M_H/\mathrm{WH}, N_H/\mathrm{WH}))$ is $\tau^G(W, M, N)$. The following calculation in $\mathrm{Wh}^G_p(M)$ is a consequence of 1.2. The phrase 'in $\mathrm{Wh}^G_p(M)$' means that all torsion elements are mapped to $\mathrm{Wh}^G_p(M)$ by a homomorphism which is obvious from the context.

$$\begin{aligned}
\tau^G(M \subset W) &= \tau^G(M - \mathrm{int}\, D\nu_M \subset W - \mathrm{int}\, D\nu_W) - \tau^G(S\nu_M \subset S\nu_W) \\
&\quad + \tau^G(D\nu_M \subset D\nu_W) = \tau^G(\bar{M} \subset \bar{W}) - \mathrm{trf}(\tau^1(M_H/\mathrm{WH}) \subset W_H/\mathrm{WH}) \\
&\quad + \tau^1(M_H/\mathrm{WH} \subset W_H/\mathrm{WH})
\end{aligned}$$

Hence, 2.10 is commutative. Since $\Phi_{\bar{M}}$ is an isomorphism of abelian groups by induction hypothesis, the same is true for Φ_M.

(ii) Notice that $M_K \to M^K$ is 2-connected for $K \in \mathrm{Iso}\, M$ and (W, M, N) is isovariant if and only if $M_K \to W_K$ and $N_K \to W_K$ are weak homotopy equivalences for $K \in \mathrm{Iso}\, M$ (see Hauschild [5]). The details of the induction over the orbit types is left to the reader. □

Next we define maps $*$ such that the following diagram commutes:

2.11.
$$\begin{array}{ccc} \mathrm{Wh}^G_p(M) & \xrightarrow{*} & \mathrm{Wh}^G_p(M) \\ \downarrow{\scriptstyle \Phi_M} & & \downarrow{\scriptstyle \Phi_M} \\ \mathrm{Wh}^G_{\mathrm{Iso}}(M) & \xrightarrow{*} & \mathrm{Wh}^G_{\mathrm{Iso}}(M) \end{array}$$

Namely, $*$ sends $\tau^G_{\mathrm{Iso}}(W, M, N)$ (resp. $\tau^G(W, M, N)$) to $j(M)_*^{-1}j(N)_*\tau^G_{\mathrm{Iso}}(W, N, M)$ (resp. $j(M)_*^{-1}j(N)_*\tau^G(W, N, M)$). Here $j(N)$ and $j(M)$ denote the obvious inclusions. This is well defined by Theorems 2.4 and 2.8.

We want to express $*$ on $\mathrm{Wh}^G_p(M)$ in terms of nonequivariant Whitehead groups and show that $*$ is an involution of abelian groups. Again we use induction over the orbit types starting with the case where M has only one orbit type (H). Let C be a component of $M/G = M_H/\mathrm{WH}$. Let $w_1(C)\colon \pi_1(C) \to \{\pm 1\}$ be its first Stiefel–Whitney class and $n(C)$ its dimension. Equip $\mathbb{Z}\pi_1(C)$ with the involution $\Sigma\lambda_g \cdot g \to \Sigma\lambda_g \cdot w_1(C)(g)\cdot g^{-1}$. It induces an involution on $\mathrm{Wh}(\pi_1(C))$. Multiplying it with the sign $(-1)^{n(C)}$ we get an involution $*(C)$. Then the following diagram commutes, where C runs over $\pi_0(M/G)$ (see Milnor [15]).

$$\begin{array}{ccc} \mathrm{Wh}^G_p(M) & \xrightarrow{\quad *\quad} & \mathrm{Wh}^G_p(M) \\ \cong\downarrow & & \cong\downarrow \\ \bigoplus \mathrm{Wh}(\pi_1(C)) & \xrightarrow{\oplus *(C)} & \bigoplus \mathrm{Wh}(\pi_1(C)) \end{array}$$

This finishes the initial step. In the induction step choose (H), $H \in \mathrm{Iso}\, M$ maximal. Next we prove the commutativity of the diagram

2.12.
$$\begin{array}{ccc} \mathrm{Wh}^G_p(\bar{M}) \oplus \mathrm{Wh}^1(M_H/\mathrm{WH}) & \xrightarrow{r} & \mathrm{Wh}^G_p(M) \\ \downarrow{\scriptstyle \begin{bmatrix} * & -\mathrm{trf}* \\ 0 & * \end{bmatrix}} & & \downarrow{\scriptstyle *} \\ \mathrm{Wh}^G_p(\bar{M}) \oplus \mathrm{Wh}^1(M_H/\mathrm{WH}) & \xrightarrow{r} & \mathrm{Wh}^G_p(M) \end{array}$$

This is a consequence of the following calculations in $\mathrm{Wh}^G_p(M)$:

$$\begin{aligned} \tau^G(W, M, N) &= \tau^G(\bar{M} \cup Sv_M Dv_M \subset \bar{W} \cup Sv_W Dv_W) \\ &= \tau^G(\bar{M} \subset \bar{W}) - \tau^G(Sv_M \subset Sv_W) + \tau^G(Dv_M \subset Dv_W) \\ &= \tau^G(\bar{W}, \bar{M}, \bar{N}) - \mathrm{trf}(\tau^1(W_H/\mathrm{WH}, M_H/\mathrm{WH}, N_H/\mathrm{WH})) \\ &\quad + \tau^1(W_H/\mathrm{WH}, M_H/\mathrm{WH}, N_H/\mathrm{WH}), \end{aligned}$$

and, using 1.3,

2.13.
$$\begin{aligned} *\tau^G(W, M, N) &= \tau^G(N \subset W) = \tau^G(N \subset N \cup Dv_N Dv W) + \tau^G(\bar{N} \subset \bar{W}) \\ &= \tau^1(W_H/\mathrm{WH}, N_H/\mathrm{WH}, M_H/\mathrm{WH}) + \tau^G(\bar{N} \subset \bar{W}). \end{aligned}$$

Hence $*\colon \mathrm{Wh}^G_p(M) \to \mathrm{Wh}^G_p(M)$ is an isomorphism of abelian groups. It remains to show that $*$ is an involution. In the sequel, all torsion elements are understood to be mapped into $\mathrm{Wh}_p^G(M)$. Represent $x \in \mathrm{Wh}_p^G(M)$ by $\tau^G(W, M, N)$ and $*(x)$ by $-\tau^G(\hat{W}, M, \hat{N})$. Then we have, by definition,

$$*(x) = \tau^G(W, N, M) \quad \text{and} \quad **(x) = -\tau^G(\hat{W}, \hat{N}, M).$$

By assumption

$$\tau^G(W \bigcup \hat{W}, N, \hat{N}) = \tau^G(W, N, M) + \tau^G(\hat{W}, M, \hat{N}) = *(x) - *(x) = 0.$$

Therefore $\tau^G(W \bigcup \hat{W}, \hat{N}, N) = 0$ also, by Theorems 2.4 and 2.8. Therefore, $\tau^G(\hat{W}, \hat{N}, M) + \tau^G(W, M, N) = -**(x) + x$ vanishes. This finishes the proof that the maps $*$ in 2.11 are involutions of abelian groups.

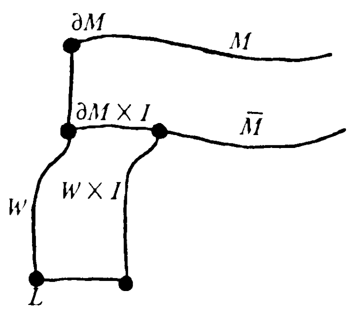
Since there is an algebraic description of trf in Lück [13], we obtain, all in all, an algebraic description of $*\colon \mathrm{Wh}_p^G(M) \to \mathrm{Wh}_p^G(M)$.

We close this section by collecting some elementary properties of $*$ and $p(S(\xi))^*$.

LEMMA 2.14. *Let $i\colon \partial M \to M$ be the inclusion of the boundary of the G-manifold M. Then we have $*i_* = -i_**$*

Proof. Let $(W, \partial M, L)$ be a h-cobordism on ∂M. Identify $\partial M \times I \subset M = M \times \{1\}$ with a collar. Up to straightening the angle, we have an h-cobordism (V, M, N) where V is $M \times I \bigcup_{\partial M \times I} W \times I$, M is $M \times \{0\}$ and N is ∂V-int M. We want to compute $\tau^G(N \subset V)$ in $\mathrm{Wh}^G(M)$. Let $\bar{M}$ be M-int$(\partial M \times I)$.

$$\begin{aligned}\tau^G(N \subset V) &= \tau^G(L \times I \textstyle\bigcup_{L \times \partial I} W \times \partial I \bigcup_{\partial M \times \partial I}(\bar{M} \bigcup \partial M \times I) \subset W \times I \bigcup_{\partial M \times I} M \times I)\\ &= \tau^G(L \times I \textstyle\bigcup_{L \times \partial I} W \times \partial I \subset W \times I)\\ &= \tau^G(L \times I \subset W \times I) - \tau^G(L \times \partial I \subset W \times \partial I) + \tau^G(W \times \partial I \subset W \times \partial I)\\ &= \tau^G(L \subset W) - 2 \cdot \tau^G(L \subset W) = -\tau^G(L \subset W).\end{aligned}$$



We also have in $\mathrm{Wh}^G(M)$:

$$\begin{aligned}\tau^G(M \subset V) &= \tau^G(M \subset W \times I \textstyle\bigcup_{\partial M \times I} M)\\ &= \tau^G(\partial M \times I \textstyle\bigcup_{\partial M \times I} M \subset W \times I \bigcup_{\partial M \times I} M)\\ &= \tau^G(\partial M \times I \subset W \times I) = \tau^G(\partial M \subset W).\end{aligned}$$

Hence, we get

$$* i_*(\tau^G(W, \partial M, L)) = *(\tau^G(V, M, N)) = \tau^G(V, N, M) = -i_* \tau^G(L \subset W)$$
$$= i_* *(\tau^G(W, \partial M, L)). \qquad \square$$

LEMMA 2.15. *Let $\xi \downarrow M$ be a G-O(n)-vector bundle on a closed G-manifold M, and let $i: S\xi \to D\xi$ be the inclusion. The we have*

(i) $* p(D\xi)^* = p(D\xi)^* * - i_* p(S\xi)^* *$,
(ii) $* p(S\xi)^* = p(S\xi)^* *$,
(iii) $i_* * = - * i_*$.

Proof. (i) Consider the h-cobordism (W, M, N) over M. Let $\eta \downarrow W$ be a G-O(n)-vector bundle with $\eta|_M = \xi$. Then $(D\eta, D\xi, D\eta|_N \cup S\eta)$ is an h-cobordism over $D\xi$, and we have

$$\tau^G(D\eta, D\xi, D\eta \mid N \cup S\eta) = p(D\xi)^* \tau^G(W, M, N).$$

We get in $\mathrm{Wh}^G(D\xi)$:

$$\begin{aligned} * p(D\xi)^* \tau^G(W, M, N) & \\ &= \tau^G(D\eta, D\eta|_N \cup S\eta, D\xi) \\ &= \tau^G(D\eta|_N \subset D\eta) - \tau^G(S\eta|_N \subset S\eta) \\ &= p(D\xi)^* \tau^G(N \subset W) - p(S\xi)^* \tau^G(N \subset W) \\ &= p(D\xi)^*(*\tau^G(W, M, N)) - p(S\xi)^*(*\tau^G(W, M, N)), \end{aligned}$$

which proves (i). Property (ii) is verified similarly, and (iii) follows from Lemma 2.14. $\square$

3. Maps between G-Manifolds

Let $(f, \partial f): (M, \partial M) \to (N, \partial N)$ be a G-homotopy equivalence of pairs of G-manifolds. We define a homomorphism

3.1. $\Phi_f: U^G(N) \to Wh_\rho^G(N)$

as follows. Any base element $[y: G/H \to N]$ in $U^G(N)$ can be represented by $f \circ x: G/H \to N$ for some $x: G/H \to M$. Let $\varphi: tp_M \to f^* tp_N$ be the OrG-equivalence uniquely determined by $\mathrm{DEG}(f, \varphi^{-1}) = 1$ (see Lück [12]). From φ, we get for any $x \in M^H$ and $H \subset G$, an H-homotopy equivalence

$$\varphi(G/H)(x)_{eH}: TM_x^c \to TN_{fx}^c$$

between the one point compactifications of the tangent spaces. In the sequel, the only input we need from φ is the desuspension of $\varphi(G/H)(x)_{eH}$ denoted by $\varphi_x: STM_x \to STN_y$. Recall that the Burnside ring $A(H)$ acts on $\mathrm{Wh}^H(STN_y)$. Let $\Phi_f([y])$ denote the image of

$$(1 - e^H(STN_y))\cdot \tau^H(\varphi_x) \in \mathrm{Wh}^H(STN_y).$$

under the composition

$$\mathrm{Wh}^H(STN_y) \xrightarrow{pr_*} \mathrm{Wh}^H(\{*\}) \xrightarrow{\text{ind}} \mathrm{Wh}^G(G/H) \xrightarrow{y_*} \mathrm{Wh}^G(N).$$

One easily verifies that $\Phi_f([y]) \in \mathrm{Wh}^G_\rho(N)$.

Recall that all G-manifolds are supposed to satisfy Assumption 2.7.

Let $\tau^G(f, \partial f) \in \mathrm{Wh}^G(N)$ be $\tau^G(f) - i_*\tau^G(\partial f)$ and let $\chi^G(N, \partial N) \in U^G(N)$ denote $\chi^G(N) - i_*\chi^G(\partial N)$, where $i: \partial N \to N$ is the inclusion map. It is easy to verify that $\operatorname{Iso} M = \operatorname{Iso} N$ and that f induces an isomorphism $\pi_0(M^H) \approx \pi_0(N^H)$ for each H in $\operatorname{Iso} M$. It follows that $\tau^G(f, \partial f)$ and $\tau^G(f)$ lie in $\mathrm{Wh}^G_\rho(N)$. The involution $*$ on $\mathrm{Wh}^G_\rho(N)$ was introduced in the last section.

THEOREM3.2. *Let $(f, \partial f): (M, \partial M) \to (N, \partial N)$ be a G-homotopy equivalence of pairs of G-manifolds. Then, $\tau^G(f) = -*\tau^G(f, \partial f) - *\Phi_f(\chi^G(N, \partial N))$.*

Proof. By Kawakubo [8], we can assume that $(M, \partial M)$ and $(N, \partial N)$ are embedded in (DV, SV) for a G-representation V with normal bundles ν_M and ν_N and the following is true.

3.3. (i) There is a pair of G-fibre homotopy equivalences $(\beta, S\beta): (D\nu_M, S\nu_M) \to (D\nu_N, S\nu_N)$ covering f.

(ii) There is an embedding $b: (D\nu_M, S\nu_M) \to (D\nu_N, D\nu_N - \operatorname{int} \frac{1}{2}D\nu_N)$ such that the G-maps $(D\nu_M, S\nu_M) \to (D\nu_N, D\nu_N - \operatorname{int} \frac{1}{2}D\nu_N)$ induced by β and b are G homotopic. Moreover, $\frac{1}{2}D\nu_N \subset b(D\nu_M)$ holds. The homotopy sends $D\nu_M \mid \partial M$ to $D\nu_N \mid \partial N$.

(iii) The inclusion $S(\nu_N)^H \subset D(\nu_N)^H$ is 2-connected for all $H \subset G$.

Moreover, $S\nu_N$ and $D\nu_N$ satisfy Assumption 2.7.

To achieve (iii), one may have to enlarge V. Consider the following cobordism $(W, \frac{1}{2}S\nu_N, X)$ given by $(b(D\nu_M)\text{-int}\,\frac{1}{2}D\nu_N, \frac{1}{2}S\nu_N, b(S\nu_M) \cup b(D\nu_M \mid \partial M)\text{-int}\,\frac{1}{2}(D\nu_N \mid \partial N))$. Since $b(D\nu_M) \subset D\nu_N$ is a G-homotopy equivalence and $W \subset b(D\nu_N)$ and $D\nu_N\text{-int}\,\frac{1}{2}D\nu_N \subset D\nu_N$ induce 2-connected maps on the H-fixed point sets for any $H \subset G$, the inclusion $W \subset D\nu_N\text{-int}\,\frac{1}{2}D\nu_N$ is a G-homotopy equivalence by excision. Moreover, because $b: D\nu_M \to D\nu_N$ is G-homotopic to β, we get, in $\mathrm{Wh}^G_\rho(D\nu_N)$:

3.4.
$$\begin{aligned}&\tau^G(W \subset D\nu_N\text{-int}\,\tfrac{1}{2}D\nu_N)\\ &= \tau^G(b(D\nu_M) \subset D\nu_N)\\ &= \tau^G(b: D\nu_M \to D\nu_N) = \tau^G(\beta: D\nu_M \to D\nu_N) = p(D\nu_N)^*\tau^G(f).\end{aligned}$$

Since $\frac{1}{2}S\nu_N \subset D\nu_N\text{-int}\,\frac{1}{2}D\nu_N$ is a simple G-homotopy equivalence, $\frac{1}{2}S\nu_N \subset W$ is a G homotopy equivalence, and in $\mathrm{Wh}^G_\rho(D\nu_N)$ we obtain, by means of 3.4:

3.5. $\tau^G(\frac{1}{2}S\nu_N \subset W) = -p(D\nu_N)^*\tau^G(f)$.

Now $h: Sv_M \to h(Sv_M)$ is a simple G-homotopy equivalence and its composition with $h(Sv_M) \subset Dv_N\text{-int}\,\frac{1}{2}Dv_N$ is G-homotopy equivalent to $S\beta: Sv_M \to Dv_N\text{-int}\,\frac{1}{2}Dv_N$. Hence $h(Sv_M) \subset Dv_N\text{-int}\,\frac{1}{2}Dv_N$ is a G-homotopy equivalence. By Theorem 1.13, where $\Phi_{S\beta}$ was defined, we get, in $\mathrm{Wh}^G(Sv_N)$:

3.6. $\tau^G(h(Sv_M) \subset Dv_N\text{-int}\,\frac{1}{2}Dv_N) = \tau^G(S\beta: Sv_M \to Sv_N) = p(Sv_N)^*\tau^G(f) + \Phi_{S\beta}(\chi^G(N))$.

Let $j: Sv_N \to Dv_N$ denote inclusion. Because of 3.4 and 3.6, we obtain in $\mathrm{Wh}^G_\rho(Dv_N)$:

$$
\begin{aligned}
3.7\quad \tau^G(h(Sv_M) \subset W) &= \tau^G(h(Sv_M) \subset Dv_N - \mathrm{int}\,\tfrac{1}{2}Dv_N) \\
&\quad - \tau^G(W \subset Dv_N - \mathrm{int}\,\tfrac{1}{2}Dv_N) \\
&= j_*p(Sv_N)^*\tau^G(f) + j_*\Phi_{S\beta}(\chi^G(N)) - p(Dv_N)^*\tau^G(f).
\end{aligned}
$$

Similarly, we get in $\mathrm{Wh}^G_\rho(Dv_N)$:

$$
\begin{aligned}
3.8.\quad &\tau^G(h(Sv_M \mid \partial M) \subset h(Dv_M \mid \partial M) - \mathrm{int}\,\tfrac{1}{2}Dv_N \mid \partial N) \\
&= j_*p(Sv_N)^*i_*\tau^G(\partial f) + j_*\Phi_{S\beta}(i_*\chi^G(\partial N)) - p(Dv_N)^*i_*\tau^G(\partial f).
\end{aligned}
$$

Combining 3.7 and 3.8, we obtain, in $\mathrm{Wh}^G_\rho(Dv_N)$:

$$
\begin{aligned}
3.9.\quad \tau^G(X \subset W) &= \tau^G(h(Sv_M) \subset W) - \tau^G(h(Sv_M \mid \partial M) \subset h(Dv_M \mid \partial M) \\
&\quad - \mathrm{int}\,\tfrac{1}{2}Dv_N \mid \partial N) \\
&= j_*p(Sv_N)^*\tau^G(f, \partial f) + j_*\Phi_{S\beta}(\chi^G(N, \partial N)) - p(Dv_N)^*\tau^G(f, \partial f).
\end{aligned}
$$

Let $k: X \to Sv_N$ be the obvious homotopy equivalence, and now identify $\frac{1}{2}Sv_N$ with Sv_N. Because of Lemma 2.15, for the torsion of $(W, \frac{1}{2}Sv_N, X)$ in $\mathrm{Wh}^G_\rho(Dv_N)$ we obtain:

3.10. $j_*\tau^G(W, \frac{1}{2}Sv_N, X) = j_* * k_*\tau^G(W, X, \frac{1}{2}Sv_N) = -*j_*k_*\tau^G(W, X, \frac{1}{2}Sv_N)$.

We now conclude from Lemmas 2.15, 3.5, 3.9, and 3.10, that in $\mathrm{Wh}^G_\rho(Dv_N)$:

$$
\begin{aligned}
3.11.\quad &-p(Dv_N)^*\tau^G(f) \\
&= -*j_*p(Sv_N)^*\tau^G(f, \partial f) - *j_*\Phi_{S\beta}(\chi^G(N, \partial N)) + *p(Dv_N)^*\tau^G(f, \partial f) \\
&= j_*p(Sv_N)^* * \tau^G(f, \partial f) - *j_*\Phi_{S\beta}(\chi^G(N, \partial N)) + p(Dv_N)^* * \tau^G(f, \partial f) \\
&\quad - j_*p(Sv_N)^* * \tau^G(f, \partial f) \\
&= p(Dv_N)^* * \tau^G(f, \partial f) - *j_*\Phi_{S\beta}(\chi^G(N, \partial N)).
\end{aligned}
$$

Hence we get, by applying $p(Dv_H)_*$ to 3.11, and using Proposition 1.7, that, in $\mathrm{Wh}^G_\rho(Dv_N)$:

3.12. $\tau^G(f) = -*\tau^G(f, \partial f) + p(Dv_N)_* * j_*\Phi_{S\beta}(\chi^G(N, \partial N))$.

Therefore it suffices to verify

3.13. $*\Phi_f = -p(Dv_N)_* * j_*\Phi_{S\beta}: U^G(N) \to \mathrm{Wh}^G_\rho(N)$.

For this we need the following lemma.

LEMMA 3.14. *Let $f\colon SV' \to SV$ and $g\colon SW' \to SW$ be G-homotopy equivalences between spheres of G-representations. Then we have in* $\mathrm{Wh}^G(\{*\})$:

$$(1-\chi^G(SV * SW))\cdot\tau^G(f * g) = (1-\chi^G(SV))\cdot\tau^G(f) + (1-\chi^G(SW))\cdot\tau^G(g)$$

Proof. The join $X * Y$ is defined as $\mathrm{Cone}(X) \times Y \bigcup_{X\times Y} X \times \mathrm{Cone}(Y)$. Now the result follows from 1.2 and 1.4 □

To verify 3.13 we have to show:

3.15. $\Phi_f(y) = -*p(D\nu_N)_* *j_*\Phi_{S\beta}(y)$

for any $y\colon G/H \to N$ of the form $y = f\circ x$ where $x\colon G/H \to M$ is a G-map.

Since $*$ commutes with codimension zero embeddings, we can replace N and $D\nu_N$ by DT and $DT \times DW$, where T and W denote the fibers at y of TN and νN. Since $*$ commutes with $\mathrm{ind}_H^G\colon \mathrm{Wh}^H(X) \to \mathrm{Wh}^G(G\times_H X)$, we can also assume $H = G$.

To establish 3.15, we have to prove

3.16. $(1-e^G(ST))\iota'_*\tau^G(\varphi_x) = -*p_1 * \iota_*\tau^G(S\beta_x)$

where $\iota\colon SW \to DT \times DW$ and $\iota'\colon ST \to DT\times DW$ are inclusions and $p_1\colon DT\times DW \to DT$ is projection. In view of Lemma 3.14, this reduces to proving

3.17. $\iota'_*\tau^G(\varphi_x) = *p_1 *(1-e^G(SW))\iota'_*\tau^G(\varphi_x)$

because the join of $S\beta_x$ and φ_x is homotopic to the identity.

But according to 1.8 and 1.9, the map

$$(\times DW - \times SW)\colon \mathrm{Wh}^G(DT) \to \mathrm{Wh}^G(DT\times DW)$$

sends any element $p_{1*}(a)$ to $(1-e^G(SW))a$, and the map $\times DW\colon \mathrm{Wh}^G(DT) \to \mathrm{Wh}^G(DT\times DW)$ is inverse to p_{1*}.

So to prove 3.17, it suffices to show that the diagram below commutes:

$$\begin{array}{ccc}
\mathrm{Wh}^H_\rho(DT) & \xrightarrow{\;*\;} & \mathrm{Wh}^H_\rho(DT) \\
\Big\downarrow{\scriptstyle (\times DW - \times SW)} & & \Big\downarrow{\scriptstyle \times DW} \\
\mathrm{Wh}^H_\rho(DT\times DW) & \xrightarrow{\;*\;} & \mathrm{Wh}^H_\rho(DT\times DW)
\end{array}$$

But this is clear from 2.15(i).

This completes the proof of 3.2. □

4. Examples and Applications

We begin with some illustrations of the results of Section 2 by computing the involution on $\mathrm{Wh}^G_\rho(N)$ in the case of a semi-free action. Namely, let N be a G-manifold such that G and $\{1\}$ are the only isotropy groups. For simplicity, we assume that N and N^G are connected. Assume $n = \dim(N^G)$ and $n + k = \dim(N)$. Assumption 2.7 reduces to: $n \geqslant 5$ and $k \geqslant 3$. Write $\pi = \pi_1(N^G)$ and $\Gamma = \pi_1(N)$. Since N has a fixed point, G

acts on Γ and we can consider the semi-direct product $\Gamma \times_s G$. One easily verifies $\Gamma \times_s G = \pi_1(EG \times_G N)$. Let $w_1(N^G): \pi \to \{\pm 1\}$ be the first Stiefel–Whitney class of N^G. We equip π with the $w_1(N^G)$-twisted involution, $\Sigma\lambda_g \cdot g \to \Sigma\lambda_g \cdot w_1(N^G)(g) \cdot g^{-1}$. Let $*: \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi)$ be the induced involution multiplied with the sign $(-1)^n$. Define $*: \mathrm{Wh}(\Gamma \times_s G) \to \mathrm{Wh}(\Gamma \times_s G)$ analogously using $w_1(N - N^G/G)$ and $(-1)^{n+k}$. Consider the normal G-vector bundle $v = v(N^G, N)$ of N^G in N and the induced fibre bundle $p: Sv/G \to N^G$. Notice that $\pi_1(Sv/G) = \pi \times G$. In Lück [9], the transfer $p^*: \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi \times G)$ is defined algebraically. The obvious map $i: \pi \times G \to \Gamma \times_s G$ induces $i_*: \mathrm{Wh}(\pi \times G) \to \mathrm{Wh}(\Gamma \times_s G)$. Then the following diagram commutes by the results of Section 2.

$$\begin{array}{ccc} \mathrm{Wh}_p^G(N) & \longrightarrow & \mathrm{Wh}(\pi) \oplus \mathrm{Wh}(\Gamma \times_s G) \\ \downarrow * & & \downarrow \begin{bmatrix} * & 0 \\ -i_* p^* * & * \end{bmatrix} \\ \mathrm{Wh}_p^G(N) & \longrightarrow & \mathrm{Wh}(\pi) \oplus \mathrm{Wh}(\Gamma \times_s G) \end{array}$$

Let V be the normal G-slice of N^G in N. Then p has SV/G as typical fibre. The algebraic transfer depends only on the pointed transport of the pointed fibre $\sigma(p): \pi \times G \to [SV/G, SV/G]^+$, i.e. a homomorphism into the monoid of pointed homotopy classes of pointed self-maps of SV/G. Now suppose that G has odd order. Then any self-G-homotopy equivalence $SV \to SV$ is G-homotopic to the identity as $V^G = 0$ and $A(G)^* = \{\pm 1\}$ holds. If $q: N^G \times V \to N^G$ is the trivial $G - \mathbb{R}^k$-vector bundle, then $\sigma(p) = \sigma(q)$ and, hence, the transfer maps p^* and q^* agree. By the product formula q^* and p^* vanish, as $\chi(SV/G)$ is zero. Hence $(-i_* p^*)*$ is trivial and the involution on $\mathrm{Wh}_p^G(N)$ is given by the direct sum of the involutions on $\mathrm{Wh}(\pi)$ and $\mathrm{Wh}(\Gamma \times_s G)$ described above (compare with Theorem 4.2).

Now suppose that G is $\mathbb{Z}/2\mathbb{Z}$. Then two pointed homotopy equivalences f and g: $SV/G \to SV/G$ are pointed homotopic, if and only if $\deg(f^\sim) = \deg(g^\sim) \in \{\pm 1\}$ holds for the lifts $f^\sim$ and $g^\sim$. Therefore we can interpret $\sigma(p)$ as a homomorphism $\pi \times G \to \{\pm 1\}$. Let $w_1(Sv) \in H^1(N^G; \mathbb{Z}/2\mathbb{Z})$ be the first Stiefel–Whitney class of $S(v) \downarrow N^G$. If we write $G = \{\pm 1\}$ then $\sigma(p)$ sends $(v, g) \in \pi \times G$ to $w_1(Sv)(v) \cdot g$. Now consider $p_* p^*: \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi)$. Let $Sw(\pi)$ be the Grothendieck-group of $\mathbb{Z}\pi$-modules which are finitely generated free over $\mathbb{Z}$. It acts on $\mathrm{Wh}(\pi)$ by $\otimes_{\mathbb{Z}}$. If $\mathbb{Z}^w$ stands for $\mathbb{Z}$, equipped with the π-action coming from $w_1(Sv)$ and $\mathbb{Z}$ is the trivial $\mathbb{Z}\pi$-module then $p_* p^*$ is multiplication with $[\mathbb{Z}] + (-1)^k[\mathbb{Z}^w] \in Sw(\pi)$ (see Lück [10]). Hence, $p_* p^*$ and p^* are not trivial in general. If k is even and $w_1(Sv) = 0$ $p_* p^*$ is multiplication by 2. Even if k is odd and $\chi(SV/G) = \chi(SV) = 0$, $p_* p^*$ and p^* can be non-trivial for appropriate π and $w_1(Sv)$.

One can also give examples of a group G and a G-manifold N such that N has two orbit types, all fixed point sets are empty or simply connected, and the involution on $\mathrm{Wh}_p^G(N)$ is not the direct sum of involutions on the summands. However, in some favorable cases the involution on $\mathrm{Wh}_p^G(N)$ splits in such a simple fashion. Namely, make the following assumptions. Let the order of G be odd. Consider a connected G-manifold

N such that for any $x: G/H \to N$ there is an NH(x)-representation V such that $\mathrm{res}_H^{\mathrm{NH}(x)}(V)$ and the normal H-slice $\nu(N^H(x), N)_x$ are H-homotopy equivalent. (This condition is always fullfilled for abelian G.) Since

$$1 \to \pi_1(N^H(x)) \xrightarrow{i_*} \pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x)) \to \mathrm{WH}(x) \to 1$$

is exact and WH(x) has odd order, there is a homomorphism $v(x): \pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x)) \to \{\pm 1\}$ uniquely determined by the property $v(x)i_* = w_1(N) \mid N^H(x)$.

Let

$$*: \mathrm{Wh}(\pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x))) \to \mathrm{Wh}(\pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x)))$$

be the $v(x)$-twisted involution multiplied with the sign $(-1)^{\dim(N)}$.

THEOREM 4.2. *With the assumptions above, the following diagram commutes where the sum runs over* $I\{G/? \to N\}/\sim$

$$\begin{array}{ccc} \mathrm{Wh}_\rho^G(N) & \xrightarrow{\approx} & \oplus\, \mathrm{Wh}(\pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x))) \\ \downarrow * & & \downarrow \oplus * \\ \mathrm{Wh}_\rho^G(N) & \xrightarrow{\approx} & \oplus\, \mathrm{Wh}(\pi_1(\mathrm{EWH}(x) \times_{\mathrm{WH}(x)} N^H(x))). \end{array}$$

Proof. This follows from 2.13 and Theorem 1.12. □

This splitting result is quite helpful in the calculation of $H^*(\mathbb{Z}/2\mathbb{Z}; \mathrm{Wh}_G^{\mathrm{Top}}(M))$ in Connolly and Kozniewski [2], where crystallographic manifolds corresponding to crystallographic groups Γ with holonomy group G of odd order are examined.

THEOREM 4.3. *Consider a G-homotopy equivalence* $(f, \partial f): (M, \partial M) \to (N, \partial N)$ *between G-manifolds. Suppose* $\pi_0(i^H): \pi_0(\partial N^H) \to \pi_0(N^H)$ *and* $\pi_1(i^H, x): \pi_1(\partial N^H, x) \to \pi_1(N^H, x)$ *are bijective for any* $H \subset G$ *and* $x \in \partial N^H$. *Assume any one of the following conditions:*

(i) *The map* $\Phi_f: U^G(N) \to \mathrm{Wh}^G(N)$ *appearing in 3.1 is zero.*

(ii) *If* $\varphi: tp_M \to f^*tp_N$ *denotes the unique* OrG*-equivalence with* $\mathrm{DEG}(f, \varphi) = 1$, *then* $\varphi(G/H)(x)_{eH}: TM_x^c \to TN_y^c$ *is a simple H homotopy equivalence for any* $H \subset G$ *and* $x \in M^H$.

(iii) *For* $x \in M^H$ *the* G_x*-representations* TM_x *and* TN_{fx} *are linearly* G_x*-isomorphic and G is the product of a group of odd order and a 2-group.*

(iv) $\chi^G(N) \in U^G(N)$ *or* $\chi^G(N, \partial N) \in U^G(N)$ *vanishes.*

Then: If one of the elements, $\tau^G(f) \in \mathrm{Wh}^G(N)$ *or* $\tau^G(f, \partial f) \in \mathrm{Wh}^G(N)$ *vanishes, then all the elements* $\tau^G(\partial f) \in \mathrm{Wh}^G(\partial N)$, $\tau^G(f) \in \mathrm{Wh}^G(N)$, *and* $\tau^G(f, \partial f) \in \mathrm{Wh}^G(N)$ *are zero.*

Proof. If (i) or (iv) is true, this follows from the formula $\tau^G(f) = -*\tau^G(f, \partial f) - *\Phi_f(\chi^G(N, \partial N))$ of Theorem 3.2. Notice that $i_*: \mathrm{Wh}^G(\partial N) \to \mathrm{Wh}^G(N)$ is bijective by assumption and 1.6. Obviously, (ii) implies (i) Moreover, under condition (iii), any G homotopy equivalence $TM_x^c \to TN_{fx}^c$ is G homotopic to a G map induced by a linear G-isomorphism and, hence, is simple. This follows from a result of Tornhave [18]. The proof is carried out in detail in Dovermann and Rothenberg [4]. □

This theorem is an important tool for the proof of the equivariant π-π-theorem for G-manifolds and simple G-homotopy equivalence (see Dovermann and Rothenberg [4], Lück and Madsen [14]).

Finally, we want to illustrate by an example the appearance of the correction term $\Phi_f(\chi^G(N, \partial N))$ in the formula of Theorem 3.2. Notice that it does not appear in the nonequivariant case. Namely, consider a G-homotopy equivalence $\partial f: SV \to SW$ between spheres of G-representations. Define $(f, \partial f): (DV, SV) \to (DW, SW)$ by coning. Suppose for simplicity that SV^G is nonempty. Then $U^G(DW)$ is just $A(G)$ and $\varphi_x: ST(DV)_x \to ST(DW)_{fx}$ is given by $\mathrm{res}_H^G(\partial f: SV \to SW)$ for $H \subset G$ and $x \in M^H$. Moreover, $\Phi_f: A(G) \to \mathrm{Wh}^G(DW)$ sends the base element $[G/H]$ to $\mathrm{ind}_H^G \mathrm{res}_H^G(1 - \chi^G(SW)) \cdot \tau^G(\partial f)$.

Hence we have

4.4. $\Phi_f(\chi^G(DW, SW)) = \chi^G(DW, SW) \cdot (1 - \chi^G(SW)) \cdot \tau^G(\partial f) = \tau^G(\partial f)$.

Now, from Theorem 3.2. we get

4.5. $\tau^G(f) = -*\tau^G(f, \partial f) - *\Phi_f(\chi^G(DW, SW))$

Obviously, $\tau^G(f)$ is zero. Hence, 4.5 reduces to

4.6. $0 = *\tau^G(\partial f) - *\Phi_f(\chi^G(DW, SW))$

But 4.4 and 4.6 match up. So $\tau^G(f) \neq -*\tau^G(f, \partial f)$ in general.

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