Classifying spaces for families and the Farrell-Jones Conjecture

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Definition (G-CW-complex)

A G-CW-complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

Remark (*G-CW* in terms of *CW*)

A G-CW-complex X is the same as a G-CW-complex together with a cellular G-action such that for every open cell e and $g \in G$ *satisfying* $g \cdot e \cap e \neq \emptyset$ *we have* $gx = x$ *for every* $x \in e$*.*

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and the G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

Let G act properly and smoothly on a smooth manifold M.
Then M inherits the structure of a G-CW-complex.

Definition (Proper G-action)

A G-space X is called proper if for each pair of points x and y in X there are open neighbourhoods V_x of x and W_y of y in X such that set $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ is finite.

Lemma

1. *A proper G-space has always finite isotropy groups.*
2. *A G-CW-complex X is proper if and only if all its isotropy groups are compact.*

Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of G is a set of subgroups of G which is closed under conjugation and taking subgroups.

- A group G is called virtually cyclic if it is finite or contains $\mathbb{Z}$ as a subgroup of finite index.
- Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the classifying G-CW-complex for the family $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

- We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the universal G-CW-complex for proper G-actions.

- We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem (Homotopy characterisation of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$;*
- *Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if the H-fixed point set X^H is contractible for each $H \in \mathcal{F}$ and X^H is empty for $H \notin \mathcal{F}$.*

Remark ((Another) Homotopy characterisation of $E_{\mathcal{F}}(G)$)

Let X be a G-CW-complex whose isotropy groups belong to $\mathcal{F}$. Then X is a model for $E_{\mathcal{F}}(G)$ if and only if the two projections $X \times X \to X$ to the first and to the second factor are G-homotopic and for each $H \in \mathcal{F}$ there exists $x \in G_x$ with $H \subseteq G_x$.

- The G-space G/G is model for $E_{\mathcal{F}}(G)$ if and only if $\mathcal{F} = \mathcal{ALL}$.
- $EG \to BG := G\backslash EG$ is a model for the universal G-principal bundle for G-principal bundles over CW-complexes.
- A free G-CW-complex X is a model for EG if and only if X/G is an Eilenberg MacLane space of type $(G,1)$.

Example (Infinite dihedral group)

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group.
- A model for ED_∞ is the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- A model for $\underline{E}D_\infty$ is $\mathbb{R}$ with the obvious D_∞-action.

Remark (Maps between classifying spaces for families)

Let $\mathcal{F}$ and $\mathcal{G}$ be two families of subgroups of G. Then the following assertions are equivalent:

- *There is a G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$;*
- *The set $[E_{\mathcal{F}}(G), E_{\mathcal{G}}(G)]^G$ consists of precisely one element;*
- *The projection $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G) \to E_{\mathcal{F}}(G)$ is a G-homotopy equivalence;*
- *$\mathcal{F} \subseteq \mathcal{G}$.*

Special models for $\underline{E}G$

- We want to illustrate that the G-space $\underline{E}G$ often has very nice geometric models and appears naturally in many interesting situations.

Theorem (Simplicial Model)

Let $P_\infty(G)$ be the geometric realisation of the full simplicial complex on the set G with the obvious simplicial G-action.
Then its barycentric subdivision is a model for $\underline{E}G$.

Theorem

Consider the G-space

$$X_G = \left\{ f\colon G \to [0,1] \;\middle|\; f \text{ has finite support}, \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm. It is G-homotopy equivalent to $\underline{E}G$.

- The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.
- The identity map induces a G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism.

Theorem (Discrete subgroups of almost connected Lie groups)

Let L be a Lie group with finitely many path components. Let $G \subseteq L$ be a discrete subgroup of L.
Then L contains a maximal compact subgroup K, which is unique up to conjugation, and L/K with the obvious left G-action is a finite dimensional G-CW-model for $\underline{E}G$.

Theorem (Actions on CAT(0)-spaces)

Let X be a proper G-CW-complex. Suppose that X has the structure of a complete simply connected $\mathrm{CAT}(0)$-space for which G acts by isometries. Then X is a model for $\underline{E}G$.

- The result above contains as special case proper isometric G-actions on simply-connected complete Riemannian manifolds with non-positive sectional curvature and proper G-actions on trees.

- The Rips complex $P_d(G,S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realisation of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.

- The obvious G-action by simplicial automorphisms on $P_d(G,S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G,S)'$.

Theorem (Rips complex of a hyperbolic group, Meintrup-Schick)

Let G be a discrete group with a finite symmetric set of generators. Suppose that (G, S) is δ-hyperbolic for the real number $\delta > 0$. Let d be a natural number with $d \geq 16\delta + 8$.
Then the barycentric subdivision of the Rips complex $P_d(G, S)'$ is a finite G-CW-model for $\underline{E}G$.

- Let $\Gamma_{g,r}^{s}$ be the mapping class group of an orientable compact surface F of genus g with s punctures and r boundary components.
- We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative.
- It is well-known that the associated Teichmüller space $\mathcal{T}_{g,r}^{s}$ is a contractible space on which $\Gamma_{g,r}^{s}$ acts properly.

Theorem (Teichmüller space)

The $\Gamma_{g,r}^{s}$-space $\mathcal{T}_{g,r}^{s}$ is a model for $\underline{E}\Gamma_{g,r}^{s}$.

Example ($SL_2(\mathbb{Z})$)

- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{Z})$.

- The group $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup. Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{Z})$.

- Since the 2-dimensional hyperbolic space $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant -1 and $SL_2(\mathbb{Z})$ acts proper on it by Moebius transformations, the $SL_2(\mathbb{Z})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$.

- The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on $\mathbb{H}^2$ by Moebius transformations. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Hence the $SL_2(\mathbb{Z})$-spaces $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{Z})$-diffeomorphic.

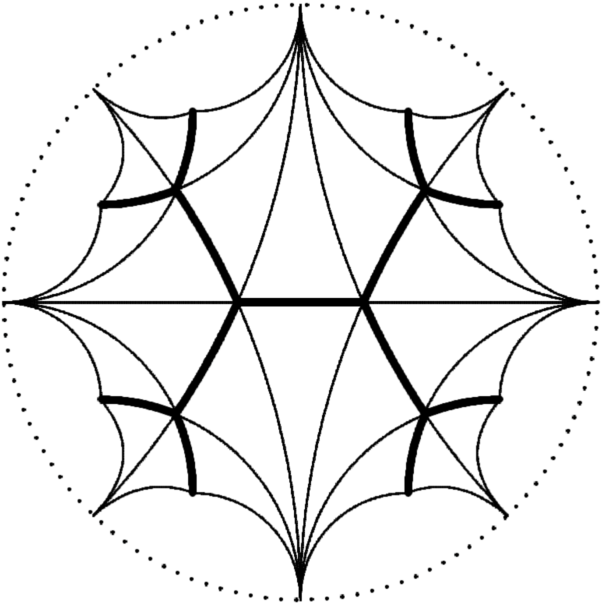
Example (continued)

- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilisers. The tree has alternately two and three edges emanating from each vertex.
  This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.

- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. Here is a concrete description of such a $SL_2(\mathbb{Z})$-homotopy equivalence.

Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action.
- Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior.
- Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action which is the tree model above.
- The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.



The Farrell-Jones and the Baum-Connes Conjectures

Conjecture (K-theoretic Farrell-Jones Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in the ring R for the group G predicts that the assembly map, which is the map induced by the projection $E_{\mathcal{VCYC}}(G) \to G/G$,

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(G/G, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $H_*^G(-; \mathbf{K}_R)$ is a G-homology theory satisfying $H_n^G(G/H, \mathbf{K}_R) \cong K_n(RH)$ for $n \in \mathbb{Z}$.
- There is also an L-theoretic version.

- The basic idea is to understand the K-theory of RG in terms of its values on RV for all virtually cyclic subgroups V and just reduce the computation for general G to the virtually cyclic subgroups $V \subseteq G$.

- In general the right hand side is the hard part and the left side is the more accessible part, since for equivariant homology theories there are methods for its computations available, for instance spectral sequences and equivariant Chern characters.

- Often the assembly maps have a more structural geometric or analytic description, which are more sophisticated and harder to construct, but link the Farrell-Jones Conjecture to interesting problems in geometry, topology, algebra or operator theory and are relevant for proofs.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture with coefficients in the ring R for the group G predicts that the assembly map, which is the map induced by the projection $\underline{E}G \to G/G$,

$$K_n^G(\underline{E}G, \mathbf{K}_R) \to K_n^G(G/G, \mathbf{K}_R) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_*^G(-)$ is equivariant topological homology. It satisfies $K_n^G(G/H) \cong K_n(C_*^r(H))$, where $K_n(C_r^*(H))$ is the topological K-theory of the reduced group C^*-algebra of H.
- The importance of these conjectures is that they imply prominent conjectures due to Bass, Borel, Kaplansky, Novikov, Serre, ... and have meanwhile been proved for large classes of groups.

Finiteness properties

- Finiteness properties of the spaces EG and $\underline{E}G$ have been intensively studied in the literature. We mention a few examples and results.
- If EG has a finite-dimensional model, the group G must be torsionfree. There are often finite models for $\underline{E}G$ for groups G with torsion.
- Often geometry provides small models for $\underline{E}G$ in cases, where the models for EG are huge. These small models can be useful for computations concerning BG itself.

Theorem (A criterion for 1-dimensional models for *BG*, Stallings, Swan)

The following statements are equivalent:

- *There exists a* 1*-dimensional model for* EG*;*
- *There exists a* 1*-dimensional model for* BG*;*
- *The cohomological dimension of* G *is less or equal to one;*
- G *is a free group.*

Theorem (A criterion for 1-dimensional models for $\underline{E}G$, Dunwoody, Karras-Pietrowsky-Solitar)

- *There exists a* 1*-dimensional model for* $\underline{E}G$ *if and only if the cohomological dimension of G over the rationals* $\mathbb{Q}$ *is less or equal to one.*
- *Suppose that G is finitely generated. Then there exists a* 1*-dimensional model for* $\underline{E}G$ *if and only if B is virtually finitely generated free.*

Theorem (Virtual cohomological dimension and $\dim(\underline{E}G)$, Lück)

Let G be virtually torsionfree.

- *Then*

  $$\operatorname{vcd}(G) \leq \dim(\underline{E}G)$$

  for any model for $\underline{E}G$.

- *Let $l \geq 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \leq l$.*
  Then there is a model for $\underline{E}G$ of dimension $\max\{3, \operatorname{vcd}(G)\} + l$.

Problem (Brown)

For which groups G, which are virtually torsionfree, does there exist a G-CW-model for $\underline{E}G$ *of dimension* $\operatorname{vcd}(G)$*?*

- The results above do give some evidence for a positive answer.
- However, Leary-Nucinkis have constructed groups, where the answer is negative.
- They even show that the upper bounds given above are optimal.

Theorem (Leary-Nucinkis)

Let X be a CW-complex. Then there exists a group G with $X \simeq G\backslash \underline{E}G$.

Groups with special maximal finite subgroups

- Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$.
  Assume that G satisfies the following assertions:
  - (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;
  - (NM) $M \in \mathcal{MFIN}, M \neq \{1\} \Rightarrow N_G M = M$.
- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible.
- Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$.
- By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.
- Define X as the G-pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow u_1 & & \downarrow f_1 \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & X \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

Theorem

The G-space X is a model for $\underline{E}G$.

- This small model is very useful for computation of all kind of K- and L-groups of RG, provided that the Farrell-Jones Conjecture is true. These computations have interesting applications to questions about the classification of manifolds and of certain C^*-algebras.
- The potential of these models is already interesting for ordinary group (co-)homology as illustrated next.

- Consider the pushout obtained from the G-pushout above by dividing out the G-action

$$\begin{array}{ccc} \coprod_{i \in I} BM_i & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \coprod_{i \in I} \mathrm{pt} & \longrightarrow & G\backslash \underline{E}G \end{array}$$

- The associated Mayer-Vietoris sequence yields

$$\ldots \to \widetilde{H}_{p+1}(G\backslash \underline{E}G) \to \bigoplus_{i \in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG) \to \widetilde{H}_p(G\backslash \underline{E}G) \to \ldots$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E}G) + 1$

$$\bigoplus_{i \in I} H_p(M_i) \xrightarrow{\cong} H_p(G).$$

- Let G be one relator-group. Then G has a model for $\underline{E}G$ of dimension 2 and contains up to conjugacy precisely one maximal subgroup M. The subgroup M is isomorphic to $\mathbb{Z}/m$ for some $m \geq 1$.
  Hence we get for $n \geq 3$

$$H_n(\mathbb{Z}/m) \xrightarrow{\cong} H_n(G).$$

Outlook

- We have only presented a tip of an iceberg.
- There are many applications or appearances of the classifying spaces for families we have not mentioned at all.
- They play a prominent role in equivariant homotopy theory.
- Their definition makes also sense for topological groups, where they have prominent appearances in the theory of Lie groups or of reductive p-adic groups.
- Often there are nice geometric models for $\underline{E}G$ also for topological groups, for instance the Bruhat-Tits building for reductive p-adic groups, the space of Riemannian metrics on a closed smooth manifold M with its action of the group of selfdiffeomorphisms of M, ...