Cobordism theory and the s-cobordism theorem

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Outline

- Cobordism theory
- The Pontrjagin-Thom construction
- The *s*-Cobordism Theorem
- Sketch of its proof
- The Whitehead group

Definition (Singular cobordism)

Let X be a CW-complex.

- Define the *n-th singular bordism group*

$$\Omega_n(X)$$

  by the oriented bordism classes of maps $f\colon M \to X$ with a closed oriented manifold as source.

- Addition comes from the disjoint union, the neutral element is represented by the map $\emptyset \to X$, and the inverse is given by changing the orientation.

- It becomes a covariant functor by composing the reference map to X with a map $f\colon X \to Y$.

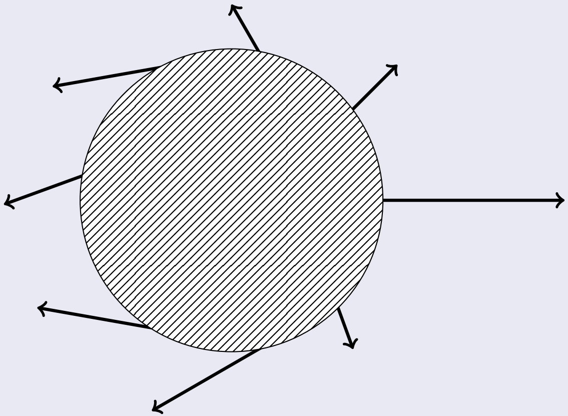
- We call $f_0 \colon M_0 \to X$ and $f_1 \colon M_1 \to X$ oriented bordant, if there is a compact oriented manifold W whose boundary is a disjoint union $\partial W = \partial_0 W \amalg \partial_1 W$, a map $F \colon W \to X$, and orientation preserving diffeomorphisms $u_i \colon M_i \xrightarrow{\cong} \partial_i W$ such that $F \circ u_i = f_i$.

- One can define $\Omega_n(X, A)$ also for pairs (X, A).

- We will orient the boundary of ∂W using the isomorphism $TW|_{\partial W} \cong \nu(\partial W, W) \oplus T\partial W$ and the orientation of $\nu(\partial W, W)$ coming from the outward normal vector field.

- This is consistent with the standard orientation on $D^2 \subseteq \mathbb{R}^2$ and on S^1.

Figure (Outward normal vector field)



Theorem (Singular bordism as homology theory)

We obtain by Ω_* a (generalized) homology theory.

- We get for its coefficient groups $\Omega_n = \Omega_n(\{\bullet\})$

Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9
$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

- Explicitly the isomorphism $\Omega_0 \xrightarrow{\mathbb{Z}}$ is given by counting the number of elements of a zero-dimensional closed manifold taking the orientation, which is essentially a sign $\pm$, into account. A generator of the infinite cyclic group Ω_4 is given by $(\{\bullet\}, +)$.

- Explicitly the isomorphism $\Omega_4 \xrightarrow{\cong} \mathbb{Z}$ is given by the signature. A generator of the infinite cyclic group Ω_0 is $\mathbb{CP}^2$.

Example (Low-dimensions)

Let X be a connected CW-complex. Let $\mathrm{pr}\, X \to \{\bullet\}$ be the projection. We conclude from the Atyiah-Hirzebruch spectral sequence:

- We obtain a bijection

$$\mathrm{pr}_* \colon \Omega_0(X) \xrightarrow{\cong} \Omega_0(\{\bullet\}) \cong \mathbb{Z};$$

- We get for $n = 1, 2, 3$ a bijection

$$c_n \colon \Omega_n(X) \xrightarrow{\cong} H_n(X; \mathbb{Z})$$

  where $c_n \colon \Omega_n(X) \xrightarrow{\cong} H_n(X; \mathbb{Z})$ sends the class of $f \colon M \to X$ to $f_*([M])$;

- We get a bijection

$$\mathrm{pr}_* \times c_4 \colon \Omega_4(X) \xrightarrow{\cong} \Omega_4(\{\bullet\}) \times H_4(X; \mathbb{Z}) \cong \mathbb{Z} \times H_4(X; \mathbb{Z}).$$

- The cartesian product implements the structure of an external product on Ω_*.

- One can weaken or strengthen the condition that M is orientable.

- For instance, one can consider the unoriented bordism theory $\mathcal{N}_*(X)$. Its coefficient ring $\mathcal{N}_* = \mathcal{N}_*(\{\bullet\})$ is given by

$$\mathcal{N}_* \cong \mathbb{F}_2[\{x_i \mid i \in \mathbb{N}, i \neq 2^k - 1\}] = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$$

where x_i sits in degree i.There are explicite representatives for the x_i, for instance $\mathbb{RP}^i$ represents x_i for even i.

- One can also consider Spin-bordism Ω^{Spin}. We get for its coefficient groups $\Omega_n^{\mathrm{Spin}} = \Omega_n^{\mathrm{Spin}}(\{\bullet\})$

Ω_0^{Spin}	Ω_1^{Spin}	Ω_2^{Spin}	Ω_3^{Spin}	Ω_4^{Spin}	Ω_5^{Spin}	Ω_6^{Spin}	Ω_7^{Spin}	Ω_8^{Spin}
$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

The Pontrjagin-Thom construction

- All these various bordism theories can be obtained as special case from ξ-bordism for a k-dimensional vector bundle ξ with projection $p_\xi \colon E \to X$ over a space X.
- Recall that for an n-dimensional manifold M there exists an embedding $i \colon M \to \mathbb{R}^{k+n}$, which is unique up to isotopy, for k large enough. Furthermore i possesses a well-defined normal bundle $\nu(i)$.

Definition (ξ-bordism)

Let $\Omega_n(\xi)$ be the bordism group of quadruples $(M, i, f, \overline{f})$ consisting of a closed n-dimensional manifold M, an embedding $i \colon M \to \mathbb{R}^{n+k}$, and a map bundle map $\overline{f} \colon \nu(i) \to \xi$ covering a map $f \colon M \to X$.

Definition (Thom space)

The Thom space of a vector bundle $p_\xi \colon E \to X$ over a finite CW-complex is defined by DE/SE, or equivalently, by the one-point compactification $E \cup \{\infty\}$. It has a preferred base point $\infty = SE/SE$.

- For a finite-dimensional vector space V we denote the trivial vector bundle with fibre V by $\underline{V}$.
- There are homeomorphisms of pointed spaces

$$\begin{aligned} \mathrm{Th}(\xi \times \eta) &\cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta); \\ \mathrm{Th}(\xi \oplus \underline{\mathbb{R}^k}) &\cong \Sigma^k \,\mathrm{Th}(\xi). \end{aligned}$$

Theorem (Pontrjagin-Thom Construction)

Let $\xi\colon E \to X$ be a k-dimensional vector bundle over a CW-complex X. Then the map

$$P_n(\xi)\colon \Omega_n(\xi) \to \pi_{n+k}(\mathrm{Th}(\xi)),$$

which sends the bordism class of $(M, i, f, \overline{f})$ to the homotopy class of the composite $S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M)) \xrightarrow{\mathrm{Th}(\overline{f})} \mathrm{Th}(\xi)$, is a well-defined isomorphism, natural in ξ.

- We sketch the proof, the details can be found in Bröcker-tom Dieck [2].

- Let $(N(M), \partial N(M))$ be a tubular neighbourhood of M. Recall that there is a diffeomorphism

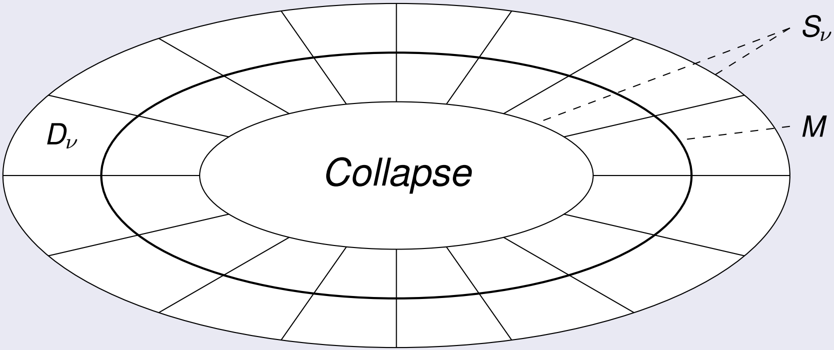
$$u\colon (D\nu(M), S\nu(M)) \to (N(M), \partial N(M)).$$

- The Thom collapse map

$$c\colon S^{n+k} = \mathbb{R}^{n+k} \amalg \{\infty\} \to \mathrm{Th}(\nu(M))$$

is the pointed map which is given by the diffeomorphism u^{-1} on the interior of $N(M)$ and sends the complement of the interior of $N(M)$ to the preferred base point ∞.

Figure (Pontrjagin-Thom construction)



- Thus we obtain a well-defined homomorphism

$$P_n(\xi)\colon \Omega_n(\xi) \to \pi_{n+k}(\mathrm{Th}(\xi)) \quad [M, i, f, \overline{f}] \mapsto [\mathrm{Th}(\overline{f}) \circ c].$$

- Next we define its inverse.
- Consider a pointed map $(S^{n+k}, \infty) \to (\mathrm{Th}(\xi), \infty)$.
- We can change f up to homotopy relative $\{\infty\}$ such that f becomes transverse to X. Notice that transversality makes sense although X is not a manifold, one needs only the fact that X is the zero-section in a vector bundle.
- Put $M = f^{-1}(X)$. The transversality construction yields a bundle map $\overline{f}\colon \nu(M) \to \xi$ covering $f|_M$. Let $i\colon M \to \mathbb{R}^{n+k} = S^{n+k} - \{\infty\}$ be the inclusion.
- Then the inverse of $P_n(\xi)$ sends the class of f to the class of $(M, i, f|_M, \overline{f})$.

- Let $p_{\xi_k}\colon E_k \to \mathrm{BSO}(k)$ be the universal oriented k-dimensional vector bundle.
- Let $\overline{j_k}\colon \xi_k \oplus \underline{\mathbb{R}} \to \xi_{k+1}$ be a bundle map covering a map $j_k\colon \mathrm{BSO}(k) \to \mathrm{BSO}(k+1)$. Up to homotopy of bundle maps this map is unique.
- Denote by γ_k the bundle $\mathrm{id}_X \times p_{\xi_k}\colon X \times E_k \to X \times \mathrm{BSO}(k)$.
- We get a map

$$\Omega_n(\overline{i_k})\colon \Omega_n(\gamma_k) \to \Omega_n(\gamma_{k+1})$$

which sends the class of $(M, i, f, \overline{f})$ to the class of the quadruple which comes from the embedding $j\colon M \xrightarrow{i} \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$ and the canonical isomorphism $\nu(i) \oplus \underline{\mathbb{R}} = \nu(j)$.

- Consider the homomorphism

$$V_k\colon \Omega_n(\gamma_k) \to \Omega_n(X)$$

which sends the class of $(M, i, f, \overline{f})$ to $(M, \mathrm{pr}_X \circ f)$, where pr_X is the projection $X \times \mathrm{BSO}(k) \to X$, and we equip M with the orientation determined by $\overline{f}$.

- Let $\operatorname{colim}_{k\to\infty} \Omega_n(\gamma_k)$ be the colimit of the directed system indexed by $k \geq 0$

$$\ldots \xrightarrow{\Omega_n(\overline{i_{k-1}})} \Omega_n(\gamma_k) \xrightarrow{\Omega_n(\overline{i_k})} \Omega_n(\gamma_{k+1}) \xrightarrow{\Omega_n(\overline{i_{k+1}})} \ldots$$

- We obtain a bijection

$$V\colon \operatorname*{colim}_{k\to\infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \Omega_n(X).$$

- We see a sequence of spaces $\mathrm{Th}(\gamma_k)$ together with maps

$$\mathrm{Th}(\overline{i_k})\colon \Sigma\,\mathrm{Th}(\gamma_k) = \mathrm{Th}(\gamma_k \oplus \underline{\mathbb{R}}) \to \mathrm{Th}(\gamma_{k+1}).$$

- We obtain homomorphisms

$$s_k \colon \pi_{n+k}(\mathrm{Th}(\gamma_k)) \to \pi_{n+k+1}(\Sigma\,\mathrm{Th}(\gamma_k)) \xrightarrow{\pi_{n+k+1}(\mathrm{Th}(\overline{i_k}))} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})),$$

where the first map is the suspension homomorphism.

- We now define the group $\operatorname{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k))$ to be the colimit of the directed system

$$\cdots \xrightarrow{s_{k-1}} \pi_{n+k}(\mathrm{Th}(\gamma_k)) \xrightarrow{s_k} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})) \xrightarrow{s_{k+1}} \cdots .$$

- From the theorem above we obtain a bijection

$$P\colon \operatorname*{colim}_{k\to\infty} \Omega_n(\gamma_k) \xrightarrow{\cong} \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

Theorem (Pontrjagin-Thom Construction and Oriented Bordism)

There is an isomorphism of abelian groups natural in X

$$P\colon \Omega_n(X) \xrightarrow{\cong} \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

- Notice that the sequence of Thom spaces above yields the so called Thom spectrum $\mathbf{Th}(\gamma)$ and the right handside in the isomorphism above is $\pi_n^s(\mathbf{Th}(\gamma))$.
- Analogously one gets for framed bordism $\Omega^{\mathrm{fr}}(X)$ an isomorphism

$$\Omega_n^{\mathrm{fr}}(X) \xrightarrow{\cong} \pi_n^s(X)$$

where π_*^s denotes stable homotopy.

The s-Cobordism Theorem

Definition (h-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

- The next result is due to Barden, Mazur, Stallings, see [1, 7]. Its topological version was proved by Kirby and Siebenmann [6, Essay II].
- More information about the s-cobordism theorem can be found for instance in [5], [9], [10].

Theorem (*s*-Cobordism Theorem)

Let M_0 be a closed connected smooth manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M_0)$. Then

1. *Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion taking values in the Whitehead group*

$$\tau(W, M_0) \in \operatorname{Wh}(\pi)$$

   vanishes;

2. *For any $x \in \operatorname{Wh}(\pi)$ there is an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 with $\tau(W, M_0) = x \in \operatorname{Wh}(\pi)$;*
3. *The function assigning to an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of h-cobordisms over M_0 to the Whitehead group $\operatorname{Wh}(\pi)$.*

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.

Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

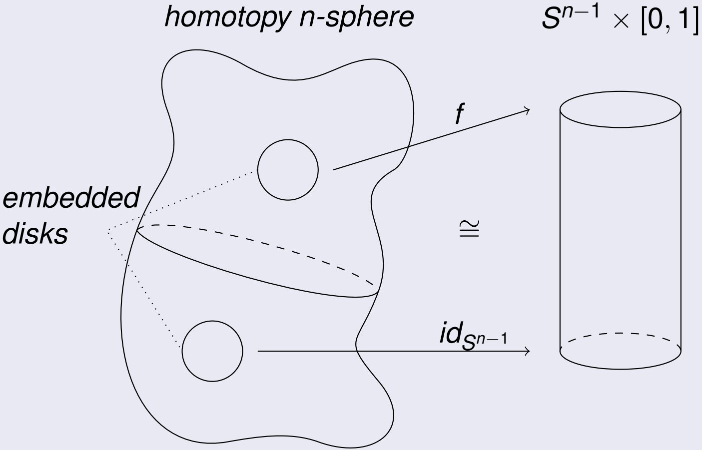
Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

□

Figure (Proof of the Poincaré Conjecture)



- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$ since the Alexander trick does not work smoothly.
- Indeed, there exist so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the Surgery Program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall, which we will explain later.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

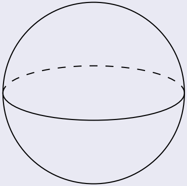
Sketch of the proof of the s-Cobordism Theorem

- We follow the exposition which will appear in Crowley-Lück-Macko [3].

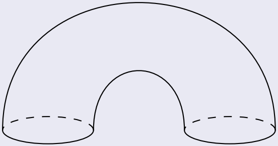
Definition (Handlebody)

- The n-dimensional handle of index q or briefly q-handle is $D^q \times D^{n-q}$.
- Its core is $D^q \times \{0\}$. The boundary of the core is $S^{q-1} \times \{0\}$.
- Its cocore is $\{0\} \times D^{n-q}$ and its transverse sphere is $\{0\} \times S^{n-q-1}$.

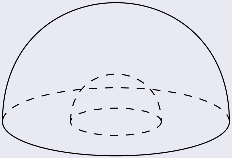
Figure (Handlebody)



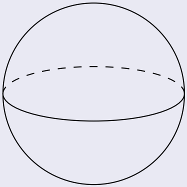
$D^0 \times D^3$



$D^1 \times D^2$



$D^2 \times D^1$



$D^3 \times D^0$

Definition (Attaching a handle)

Consider an n-dimensional manifold M with boundary ∂M. If $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial M$ is an embedding, then we say that the manifold

$$M + (\phi^q) := M \cup_{\phi^q} D^q \times D^{n-q}$$

is obtained from M by attaching a handle of index q by ϕ^q.

- One should think of a handle $D^q \times D^{n-q}$ as a q-cell $D^q \times \{0\}$ which is thickened to $D^q \times D^{n-q}$.
- Attaching a q-handle $D^q \times D^{n-q}$ along $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial M$ correspond to attaching a q-cell $D^q \times \{0\}$ along $\phi^q|_{S^{q-1} \times \{0\}}$

- Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then we want to construct W from $\partial_0 W \times [0,1]$ by attaching handles as follows.

- If $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_0 W \times \{1\}$ is an embedding, we get by attaching a handle the compact manifold $W_1 = \partial_0 W \times [0,1] + (\phi^q)$. Notice we have not change $\partial_0 W = \partial_0 W \times \{0\}$.

- Now we can iterate this process and we obtain a compact manifold with boundary

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r}),$$

- We call a description of W as above a handlebody decomposition of W relative $\partial_0 W$.

- From Morse theory, see [4, Chapter 6], [8, part I] we obtain the following lemma.

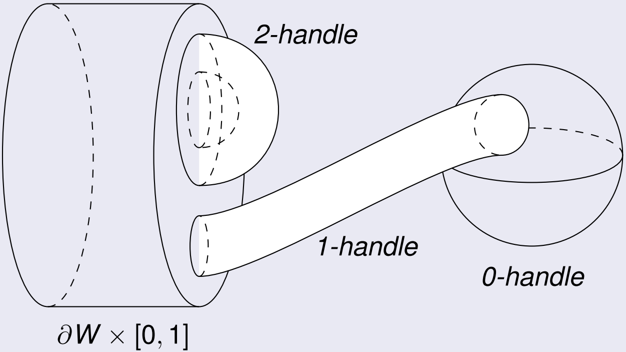
Lemma

Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$.

Then W possesses a handlebody decomposition relative $\partial_0 W$, i.e., W is up to diffeomorphism relative $\partial_0 W = \partial_0 W \times \{0\}$ of the form

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r}).$$

Figure (Handlebody decomposition)



Lemma (Isotopy Lemma)

Let W be an n-dimensional compact manifold, whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Let $\phi^q, \psi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be isotopic embeddings.

Then there is a diffeomorphism

$$W + (\phi^q) \xrightarrow{\cong} W + (\psi^q)$$

relative $\partial_0 W$.

Lemma (Diffeomorphism Lemma)

Let W resp. W' be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$ resp. $\partial_0 W' \coprod \partial_1 W'$. Let $F\colon W \to W'$ be a diffeomorphism which induces a diffeomorphism $f_0\colon \partial_0 W \to \partial_0 W'$. Let $\phi^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding.

Then there is an embedding $\overline{\phi}^q\colon S^{q-1} \times D^{n-q} \to \partial_1 W'$ and a diffeomorphism

$$F'\colon W + (\phi^q) \to W' + (\overline{\phi}^q)$$

which induces f_0 on $\partial_0 W$.

Lemma (Cancellation Lemma)

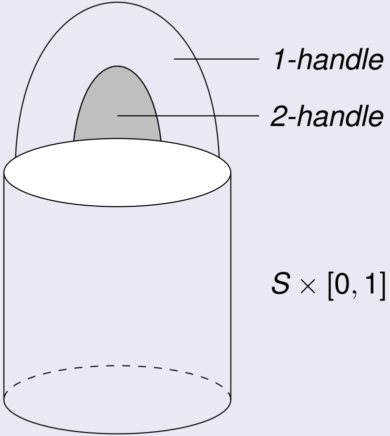
Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Let $\phi^q \colon S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Let $\psi^{q+1} \colon S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ be an embedding. Suppose that $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of the handle (ϕ^q) and meets the transverse sphere in exactly one point.

Then there is a diffeomorphism

$$W \xrightarrow{\cong} W + (\phi^q) + (\psi^{q+1})$$

relative $\partial_0 W$.

Figure (Handle cancellation)



Lemma

Let W be an n-dimensional manifold for $n \geq 6$ whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$. Then the following statements are equivalent

1. *The inclusion $\partial_0 W \to W$ is 1-connected;*
2. *We can find a diffeomorphism relative $\partial_0 W$*

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \cdots + \sum_{i=1}^{p_n} (\overline{\phi}_i^n).$$

Lemma (Normal Form Lemma)

Let $(W; \partial_0 W, \partial_1 W)$ be a compact h-cobordism of dimension $n \geq 6$. Let q be an integer with $2 \leq q \leq n-3$.

Then there is a handlebody decomposition which has only handles of index q and $(q+1)$, i.e., there is a diffeomorphism relative $\partial_0 W$

$$W \quad \cong \quad \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

- Suppose that W is in normal form.
- Let $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ be the $\mathbb{Z}\pi$-chain complex of the pair of universal coverings of W and $\partial_0 W$. Since W is an h-cobordism, it is acyclic.
- The two non-trivial $\mathbb{Z}\pi$ chain modules comes with $\mathbb{Z}\pi$-bases determined by the handles.
- Thus the only non-trivial differential is a $\mathbb{Z}\pi$-isomorphism and is described by an invertible matrix A over $\mathbb{Z}\pi$.
- If A is the empty matrix, then W is diffeomorphic relative $\partial_0 W$ to $\partial_0 W \times [0, 1]$.

- Next we define an abelian group $\mathrm{Wh}(\pi)$ as follows.
- It is the set of equivalence classes of invertible matrices of arbitrary size with entries in $\mathbb{Z}\pi$, where we call an invertible (m, m)-matrix A and an invertible (n, n)-matrix B over $\mathbb{Z}\pi$ equivalent, if we can pass from A to B by a sequence of the following operations:
  1. B is obtained from A by adding the k-th row multiplied with x from the left to the l-th row for $x \in \mathbb{Z}\pi$ and $k \neq l$;
  2. B is obtained by taking the direct sum of A and the $(1, 1)$-matrix $I_1 = (1)$, i.e., B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
  3. A is the direct sum of B and I_1;
  4. B is obtained from A by multiplying the i-th row from the left with a trivial unit , i.e., with an element of the shape $\pm\gamma$ for $\gamma \in \pi$;
  5. B is obtained from A by interchanging two rows or two columns.
- The sum is given by the block sum, the neutral element is represented by the empty matrix, inverses are given by taking the inverse of a matrix.

Lemma

1. *Let* $(W, \partial_0 W, \partial_1 W)$ *be an n-dimensional compact h-cobordism for* $n \geq 6$ *and A be the matrix defined above. If* $[A] = 0$ *in* $\mathrm{Wh}(\pi)$*, then the h-cobordism W is trivial relative* $\partial_0 W$*;*
2. *Consider an element* $u \in \mathrm{Wh}(\pi)$*, a closed manifold M of dimension* $n - 1 \geq 5$ *with fundamental group* π *and an integer q with* $2 \leq q \leq n - 3$*. Then we can find an h-cobordism of the shape*

$$W = M \times [0, 1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

such that $[A] = u$*.*

- The idea of proof of the lemma above is to realize any of the operations on A geometrically by modifying the handle body decomposition geometrically. These are

1. handle slides;
2. Adding trivially a pair of q-handle and a $q + 1$-handle.
3. Deleting a pair of a q-handle and a $q + 1$-handle using the Elimination Lemma.
4. Changing the orientation of a handle and the lift of it to the universal coverings.
5. Changing the enumeration of the handles.

- The handle slide is possible and has the desired effect due to the following lemma which we state without further explanations.

Lemma (Modification Lemma)

Let $f\colon S^q \to \partial_1^\circ W_q$ be an embedding and let $x_j \in \mathbb{Z}\pi$ be elements for $j = 1, 2 \ldots, p_{q+1}$. Then there is an embedding $g\colon S^q \to \partial_1^\circ W_q$ with the following properties:

1. *f and g are isotopic in $\partial_1 W_{q+1}$;*
2. *For a given lift $\widetilde{f}\colon S^q \to \widetilde{W_q}$ of f one can find a lift $\widetilde{g}\colon S^q \to \widetilde{W_q}$ of g such that we get in $C_q(\widetilde{W})$*

$$[\widetilde{g}] = [\widetilde{f}] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi_j^{q+1}],$$

where d_{q+1} is the $(q+1)$-th differential in $C_*(\widetilde{W}, \widetilde{\partial_0 W})$.

- We give a different more conceptual definition of the abelian group $\mathrm{Wh}(\pi)$ later.
- By definition the matrix A from above determines an element in $\mathrm{Wh}(\pi)$, which turns out independent of the choice of the normal form and hence gives a well-defined element in $\mathrm{Wh}(\pi)$ depending only the diffeomorphism type of W relative $\partial_0 W$.
- Actually, this element can be described intrinsically by the so called Whitehead torsion.
- Putting these statements together, finishes the proof of the s-Cobordism Theorem.

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- $K_1(R)$ is isomorphic to $GL(R)/[GL(R), GL(R)]$.

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the reduced K_1-group

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

  which in general is not bijective.

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

Proof.

- The ring $\mathbb{Z}$ possesses an Euclidean algorithm.
- Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1).
- This implies that any element in $K_1(\mathbb{Z})$ is represented by ± 1.

□

- Let G be a finite group. Let F be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
- Define $r_F(G)$ to be the number of irreducible F-representations of G.
- The Whitehead group $\mathrm{Wh}(G)$ is a finitely generated abelian group of rank $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- The torsion subgroup of $\mathrm{Wh}(G)$ is the kernel of the map $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.
- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable.

Exercise (Non-vanishing of $\mathrm{Wh}(\mathbb{Z}/5)$)

Using the ring homomorphism $f\colon \mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$ which sends the generator of $\mathbb{Z}/5$ to $\exp(2\pi i/5)$ and the norm of a complex number, define a homomorphism of abelian groups

$$\phi\colon \mathrm{Wh}(\mathbb{Z}/5) \to \mathbb{R}^{>0}.$$

Show that the class of the unit $1 - t - t^{-1}$ in $\mathrm{Wh}(\mathbb{Z}/5)$ is an element of infinite order.

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