Applications of the Farrell-Jones Conjecture (Lecture V)

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Outline

- Review of the Farrell-Jones Conjecture and of some applications.
- The Novikov Conjecture
- The Borel Conjecture
- The Moody's Conjecture
- The The Bass Conjecture
- Hyperbolic groups with spheres as boundary
- Computational aspects.

Review

Conjecture (The Farrell-Jones-Conjecture)

The Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly maps, which are induced by the projection $E_{\mathcal{VCYC}}(G) \to pt$*,*

$$\begin{aligned} H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) &\to H_n^G(pt, \mathbf{K}_R) = K_n(RG); \\ H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG), \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$*.*

- A more general version, the Full Farrell-Jones Conjecture, holds for large class of groups and has good inheritance properties.
- The Farrell-Jones Conjecture implies the following version for torsionfree groups

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for every $n \in \mathbb{Z}$.

Theorem (Some applications of the K-theoretic Farrell-Jones Conjecture)

The K-theoretic Farrell-Jones Conjecture implies:

- $K_n(\mathbb{Z}G) = 0$ *for* $n \le -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- *Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;*
- *Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;*
- *Serre's Conjecture: The group G is of type FF if and only if it is of type FP;*
- *Kaplansky's Idempotent Conjecture: Let R be an integral domain and let G be a torsionfree group. Then all idempotents of RG are trivial, i.e., equal to 0 or 1.*

The Novikov Conjecture

Definition (L-class)

The L-class of a closed manifold M is a certain rational polynomial in the rational Pontryagin classes

$$\mathcal{L}(M) = \mathcal{L}(p_1(M), p_2(M), \ldots) \in \bigoplus_{i \geq 0} H^{4i}(M;\mathbb{Q}).$$

Its i-th component is denoted by $\mathcal{L}_i(M) \in H^{4i}(M;\mathbb{Q})$.

$$\begin{aligned}
\mathcal{L}_1(M) &= \frac{1}{3} \cdot p_1(M); \\
\mathcal{L}_2(M) &= \frac{1}{45} \cdot \left(7 \cdot p_2(M) - p_1(M)^2\right); \\
\mathcal{L}_3(M) &= \frac{1}{945} \cdot \left(62 \cdot p_3(M) - 13 \cdot p_1(M) \cup p_2(M) + 2 \cdot p_1(M)^3\right).
\end{aligned}$$

Theorem (Signature Theorem, Hirzebruch)

If M is a $4k$-dimensional closed oriented manifold M, then we get for its signature

$$\operatorname{sign}(M) = \langle \mathcal{L}_k(M), [M] \rangle.$$

Exercise (Homotopy invariance of $\mathcal{L}_k(M)$ for $4k$)

Show that $\mathcal{L}_k(M)$ for $n = 4k$ is a homotopy invariant of closed orientable $4k$-dimensional manifolds.

- One can show that a polynomial in the Pontryagin classes gives a homotopy invariant if and only if it is a multiple of the k-th L-class.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0,f_0) = \operatorname{sign}_x(M_1,f_1).$$

Exercise (Novikov Conjecture for closed aspherical manifolds)

The Novikov Conjecture predicts for a homotopy equivalence $f\colon M \to N$ of closed aspherical manifolds

$$f^*(\mathcal{L}(N)) = \mathcal{L}(M).$$

- This is surprising since this is not true in general and in many cases one can detect that two specific closed homotopy equivalent manifolds cannot be diffeomorphic by the failure of this equality to be true.
- A deep theorem of Novikov predicts that $f^*(\mathcal{L}(N)) = \mathcal{L}(M)$ holds for a homeomorphism of closed manifolds.
- Hence an explanation, why the Novikov Conjecture may be true for closed aspherical manifolds, comes from the next conjecture.

The Borel Conjecture

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

In particular M and N are homeomorphic.

- This is the topological version of Mostow rigidity. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture is in general false in the smooth category. A counterexample is T^n for $n \geq 5$.

- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture. Namely, in the Borel Conjecture the fundamental group can be complicated but there are no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.

- The Borel Conjecture in dimension 1 and 2 is obviously true.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.

Exercise (The Borel Conjecture for T^2)

Prove the Borel Conjecture for the 2-torus T^2.

Theorem (The Farrell-Jones Conjecture implies the Borel Conjecture)

If the K-theoretic and the L-theoretic Farrell-Jones Conjecture hold for the group G, then the Borel Conjecture holds for any n-dimensional aspherical closed manifold with $\pi_1(M) \cong G$, provided that $n \geq 5$.

- Next we sketch the proof. Therefore we recall:

Conjecture (L-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The (topological) structure set $\mathcal{S}^{\mathrm{top}}(M)$ of a manifold M consists of equivalence classes of homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem (Topological rigidity and the structure set)

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

Theorem (Algebraic surgery sequence, Ranicki)

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\dots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$
$$\mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \dots$$

It can be identified with the classical geometric surgery exact sequence due to Sullivan and Wall in high dimensions.

- Here $\mathbf{L}\langle 1\rangle$ is the 1-connective cover of the L-theory spectrum $\mathbf{L}_{\mathbb{Z}}$. It comes with a natural map of spectra $\mathbf{i}\colon \mathbf{L}\langle 1\rangle \to \mathbf{L}_{\mathbb{Z}}$ which induces on π_i^s an isomorphism for $i \geq 1$, and we have $\pi_i^s(\mathbf{L}\langle 1\rangle) = 0$ for $i \leq 0$.

- There are natural identifications

$$\mathcal{N}(M) \cong [M, G/O] \cong H_n(M; \mathbf{L}\langle 1 \rangle).$$

- We can write A_n as the composite

$$\begin{aligned} A_n \colon \mathcal{N}(M) = H_n(M; \mathbf{L}\langle 1 \rangle) \\ \xrightarrow{H_n(\mathrm{id}_M; \mathbf{i})} H_n(M; \mathbf{L}_{\mathbb{Z}}) = H_n(BG; \mathbf{L}_{\mathbb{Z}}) = H_n^G(EG) \to L_n(\mathbb{Z}G), \end{aligned}$$

  where the second map is assembly map for the family $\mathcal{TR}$.

- The map A_n can be identified with the map given by the surgery obstruction in the geometric surgery exact sequence.

- This gives an interesting interpretation of the homotopy theoretic assembly map in geometric terms. Its proof is non-trivial.

- The analog statement about A_m holds in all degrees $m \geq n$.

- $\mathcal{S}^{\text{top}}(M)$ consists of one element if and only if A_{n+1} is surjective and A_n is injective.

- An easy spectral sequence argument shows that

$$H_m(\mathrm{id}_M; \mathbf{i})\colon H_n(M; \mathbf{L}\langle 1\rangle) \to H_m(M; \mathbf{L}_{\mathbb{Z}})$$

  is bijective for $m \geq n+1$ and injective for $m = n$.

- This finishes the proof since the Farrell-Jones Conjecture implies for $m = n, n+1$ the bijectivity of

$$H_n(M; \mathbf{L}_{\mathbb{Z}}) = H_n(BG; \mathbf{L}_{\mathbb{Z}}) = H_n^G(EG) \to L_m(\mathbb{Z}G).$$

- The K-theoretic Farrell-Jones Conjecture is needed in the proof above since it implies that $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq -1$ vanish and hence the decorations in the L-groups do not matter.

Theorem (The Farrell-Jones Conjecture implies the Novikov Conjecture)

Consider the following statements for a group G.

1. *The L-theoretic assembly map for the family* $\mathcal{VCYC}$

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

   is rationally injective.

2. *The L-theoretic assembly map for the family* $\mathcal{TR}$

$$H_n^G(E_{\mathcal{TR}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) = H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

   is rationally injective;

3. *For every* $n \geq 5$ *the Novikov Conjecture holds for G.*

Then: (1) $\implies$ (2) $\implies$ (3).

- The class of groups for which the Novikov Conjecture holds is larger than the class for which the L-theoretic Farrell-Jones Conjecture is known. For instance it contains all linear groups.

Exercise (Homotopy groups of G/TOP)

Show that $\pi_n(\mathrm{G}/\mathrm{TOP})$ *vanishes for odd* $n \geq 5$.

Moody's Induction Conjecture

Conjecture (Moody's Induction Conjecture)

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

Theorem (Bredon homology)

Consider any covariant functor

$$M\colon \mathrm{Or}G \to \Lambda\text{-}\mathrm{Modules}.$$

Then there is up to natural equivalence of G-homology theories precisely one G-homology theory $H_*^G(-, M)$, called Bredon homology, with the property that the covariant functor

$$H_n^G\colon \mathrm{Or}G \to \Lambda\text{-}\mathrm{Modules}, \quad G/H \mapsto H_n^G(G/H)$$

is trivial for $n \neq 0$ and naturally equivalent to M for $n = 0$.

- Bredon homology plays the role of cellular of singular homology in the equivariant setting.

- Let $\mathcal{H}_*^G$ be a G-homology theory. Then there is an equivariant version of the Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}_{p+q}^G(X)$. Its E^2-term

$$E^2_{p,q} = H_p^G(X; \mathcal{H}_q^G(?))$$

is given by the Bredon homology associated to the covariant functor

$$\mathrm{Or}G \to \Lambda\text{-}\mathrm{Modules}, \quad G/H \mapsto \mathcal{H}_q^G(G/H).$$

- Let M be the constant functor with value the Λ-module A. Then we get for every G-CW-complex X

$$H_n^G(X; M) \cong_\Lambda H_n(X/G; A).$$

- We have $H_0(E_{\mathcal{F}}(G); M) \cong \mathrm{colim}_{\mathrm{Or}_{\mathcal{F}}(G)} M$.

Theorem (Farrell-Jones implies Moody)

The K-theoretic Farrell-Jones Conjecture implies Moody's Induction Conjecture.

Proof.

- The Transitivity Principle implies that the canonical maps

$$\begin{aligned} H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_F)[1/p] &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_F)[1/p], \end{aligned}$$

  are bijective.

- Hence the Farrell-Jones Conjecture implies

$$\begin{aligned} H_0^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) &\to K_0(RG); \\ H_0^G(E_{\mathcal{FIN}}(G); \mathbf{K}_F)[1/p] &\to K_0(FG)[1/p]. \end{aligned}$$

□

Proof continued.

- Since $K_n(RH)$ and $K_n(FH)][1/p]$ vanish for $n \leq -1$, the equivariant Atiyah-Hirzebruch spectral sequence implies

$$\begin{array}{rcl} H_0^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) & \to & H_0^G(E_{\mathcal{FIN}}(G);K_0(R?)); \\ H_0^G(E_{\mathcal{FIN}}(G);\mathbf{K}_F)[1/p] & \to & H_0^G(E_{\mathcal{FIN}}(G);K_0(F?))[1/p], \end{array}$$

for the covariant functors from $\mathrm{Or}G$ to $\mathbb{Z}\text{-}\mathrm{Modules}$ and $\mathbb{Z}[1/p]\text{-}\mathrm{Modules}$ respectively sending G/H to $K_0(RH)$ and $K_0(FH)[1/p]$ respectively.

- Now the claim follows from the identifications

$$\begin{array}{rcl} H_0^G(E_{\mathcal{FIN}}(G);\mathbf{K}(R?)) & \cong & \operatorname{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(R?); \\ H_0^G(E_{\mathcal{FIN}}(G);\mathbf{K}(F?))[1/p] & \cong & \operatorname{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(F?)[1/p]. \end{array}$$

$\square$

The Bass Conjecture

- Let $\mathrm{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$.
- Let R be a commutative ring. Let $\mathrm{class}(G, R)$ be the free R-module with the set $\mathrm{con}(G)$ as basis. This is the same as the R-module of R-valued class functions $f\colon G \to R$ for which $\{(g) \in \mathrm{con}(G) \mid f(g) \neq 0\}$ is finite.
- Define the universal R-trace

$$\mathrm{tr}_{RG}^{u}\colon RG \to \mathrm{class}(G, R), \quad \sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} r_g \cdot (g).$$

It extends to a function $\mathrm{tr}_{RG}^{u}\colon M_n(RG) \to \mathrm{class}(G, R)$ on (n, n)-matrices over RG by taking the sum of the traces of the diagonal entries.

- Let P be a finitely generated projective RG-module. Choose a matrix $A \in M_n(RG)$ such that $A^2 = A$ and the image of the RG-map $r_A \colon RG^n \to RG^n$ given by right multiplication with A is RG-isomorphic to P. Define the Hattori-Stallings rank of P to be

$$\mathrm{HS}_{RG}(P) = \mathrm{tr}^u_{RG}(A) \in \mathrm{class}(G, R).$$

- The Hattori-Stallings rank depends only on the isomorphism class of the RG-module P.

- The Hattori-Stallings rank extends the notion of a character of a representation of a finite group to infinite groups.

Theorem (Hattori-Stallings rank and characters)

Let G be a finite group and F be a field of characteristic zero. Let V be a finitely generated FG-module which is the same as a finite-dimensional G-representation over F. Let χ_V be its character. If $C_G(g)$ denotes the centralizer of the element $g \in G$, then show

$$\chi_V(g^{-1}) = |C_G(g)| \cdot \mathrm{HS}_{FG}(P)(g).$$

- The Hattori-Stallings rank induces an R-homomorphism, the Hattori-Stallings homomorphism,

$$\mathrm{HS}_{RG} \colon K_0(RG) \otimes_{\mathbb{Z}} R \to \mathrm{class}(G, R), \quad [P] \otimes r \mapsto r \cdot \mathrm{HS}_{RG}(P).$$

Exercise (Bijectivity of the Hattori-Stallings rank)

Show that for a group satisfying Moody' Induction Conjecture, the map induced by the Hattori-Stallings rank

$$\mathrm{HS}_{\mathbb{C}G} \colon K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{class}(G, \mathbb{C})$$

is injective and has as image the class functions which vanish on elements of infinite order.

Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.

Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

Exercise (The character of a rationalized finitely generated $\mathbb{Z}G$-module)

Let G be a finite group and M be a finitely generated projective $\mathbb{Z}G$-module. Suppose that G satisfies the Bass Conjecture.

Show that $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is a finitely generated free $\mathbb{Q}G$-module.

- Actually, the conclusion appearing in the exercise above is a theorem of Swan which is older than and was one motivation for the Bass Conjecture.

Theorem (Farrell-Jones implies Bass)

The K-theoretic Farrell-Jones Conjecture implies the Bass Conjecture

Proof.

- We give only the proof in the case $R = \mathbb{Z}$ using the fact that the K-theoretic Farrell-Jones Conjecture implies Moody's Induction Conjecture.
- The claim follows from Moody's Induction Conjecture applied to $\mathbb{Q}$ if g has infinite order, since the Hattori-Stallings rank is compatible with induction.
- In the case, where g is finite and not invertible in R, Moody's Induction Conjecture applied to $\mathbb{Q}$ reduces the claim to the special case, where G is finite, which follows from Swan's Theorem.

□

Word hyperbolic groups with spheres as boundary

Conjecture (Gromov)

Let G be a torsionfree hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

For $n \geq 6$ the Conjecture is true, and the manifold M is unique up to homeomorphism.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

Theorem (Ferry-Lück-Weinberger (in preparation))

Let G be a torsionfree hyperbolic group with S^2 as boundary and $l \geq 3$ be natural number.

- *Then there is a closed aspherical $(k+l)$-dimensional manifold M with an isomorphism $u_M \colon \pi_1(M) \xrightarrow{\cong} G \times \mathbb{Z}^l$.*
- *If M' is another closed aspherical manifold M' with an isomorphism $u_{M'} \colon \pi_1(M') \xrightarrow{\cong} G \times \mathbb{Z}^l$, then there is a homeomorphism $f \colon M \to M'$ with $\pi_1(f) = u_{M'}^{-1} \circ u_M$.*

- The Cannon Conjecture is now equivalent to the statement that there exists a closed 3-manifold N such that M is homeomorphic to $N \times T^l$.

Equivariant Chern characters

Theorem (Dold (1962))

Let $\mathcal{H}_*$ be a generalized homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$.

Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X;\Lambda) \otimes_\Lambda \mathcal{H}_q(pt) \xrightarrow{\cong} \mathcal{H}_n(X).$$

- This means that the Atiyah-Hirzebruch spectral sequence collapses in the strongest sense.
- The assumption $\mathbb{Q} \subseteq \Lambda$ is necessary.

Dolds' Chern character for a CW-complex X is given by the following composite

$$\mathrm{ch}_n\colon \bigoplus_{p+q=n} H_p(X;\mathcal{H}_q(*)) \xleftarrow{\alpha} \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*) \xleftarrow[\cong]{\bigoplus_{p+q=n} \mathrm{hur} \otimes \mathrm{id}} \bigoplus_{p+q=n} \pi_p^s(X_+,*) \otimes_{\mathbb{Z}} \mathcal{H}_q(*) \xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X).$$

- We want to extend this to the equivariant setting.
- This requires an extra structure on the coefficients of an equivariant homology theory $\mathcal{H}^?_*$.
- We define a covariant functor called induction

$$\mathrm{ind}\colon \mathrm{FGINJ} \to \Lambda\text{-}\mathrm{Mod}$$

  from the category FGINJ of finite groups with injective group homomorphisms as morphisms to the category of Λ-modules as follows.

  It sends G to $\mathcal{H}^G_n(\mathrm{pt})$ and an injection of finite groups $\alpha\colon H \to G$ to the morphism given by the induction structure

$$\mathcal{H}^H_n(\mathrm{pt}) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}^G_n(\mathrm{ind}_\alpha \mathrm{pt}) \xrightarrow{\mathcal{H}^G_n(\mathrm{pr})} \mathcal{H}^G_n(\mathrm{pt}).$$

Definition (Mackey extension)

We say that $\mathcal{H}_*^?$ has a Mackey extension if for every $n \in \mathbb{Z}$ there is a contravariant functor called restriction

$$\mathrm{res}\colon \mathrm{FGINJ} \to \Lambda\text{-}\mathrm{Mod}$$

such that these two functors ind and res agree on objects and satisfy the double coset formula, i.e., we have for two subgroups $H, K \subset G$ of the finite group G

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G = \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$.

- In every case we will consider such a Mackey extension does exist and is given by an actual restriction.
- For instance for $K_0^G(?) = R_{\mathbb{C}}$ induction functor is just the classical induction of representations. The restriction functor is given by the classical restriction of representations.
- Analogous statements hold for $K_n(R?)$ and $L_n^{\langle -\infty \rangle}(R?)$.

Theorem (Lück)

Let $\mathcal{H}^?_*$ be a proper equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}^?_*$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.

Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}^G_n \colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H;\Lambda) \otimes_{\Lambda[W_GH]} S_H\left(\mathcal{H}^H_q(*)\right) \xrightarrow{\cong} \mathcal{H}^G_n(X)$$

- C_GH is the centralizer and N_GH the normalizer of $H \subseteq G$;
- $W_GH := N_GH/H \cdot C_GH$ (This is always a finite group);
- $S_H\left(\mathcal{H}^H_q(*)\right) := \operatorname{cok}\left(\bigoplus_{\substack{K\subset H\\ K\neq H}} \operatorname{ind}^H_K \colon \bigoplus_{\substack{K\subset H\\ K\neq H}} \mathcal{H}^K_q(*) \to \mathcal{H}^H_q(*)\right)$.
- $\mathrm{ch}^?_*$ is an equivalence of equivariant homology theories.

Theorem (Lück)

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.

Exercise (Infinite dihedral group D_∞)

Compute $K_n(\mathbb{C}D_\infty) \otimes_{\mathbb{Z}} \mathbb{C}$ for $n \leq 1$.

Hyperbolic groups

Theorem (Hyperbolic groups)

- *Let G be a hyperbolic group. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtual cyclic subgroups of G.*
- *For $n \in \mathbb{Z}$ there is an isomorphism*

$$H_n^G(\underline{E}G; \mathbf{K}_R) \oplus \bigoplus_{V \in \mathcal{M}} \mathcal{H}_n^V(\underline{E}V \to pt; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG);$$

- *For $n \in \mathbb{Z}$ there is an isomorphism*

$$H_n(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus \bigoplus_{V \in \mathcal{M}} \mathcal{H}_n^V(\underline{E}V \to pt; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

Theorem (Torsionfree hyperbolic groups)

If G is a torsionfree hyperbolic group, then we get isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C\subseteq G, C\neq 1\\ C \text{ maximal cyclic}}} \left(\mathrm{NK}_n(R)\oplus \mathrm{NK}_n(R)\right)\right) \xrightarrow{\cong} K_n(RG),$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG).$$

Exercise (Fundamental groups of hyperbolic closed 3-manifolds)

Let G be the fundamental group of a hyperbolic closed 3-manifold. Compute $L_n^{\epsilon}(\mathbb{Z}G)$ for all decorations ϵ in terms of $H_1(M;\mathbb{Z})$.

The end of these lecture series

Thank you for your attention!