Introduction to surgery theory (Lecture II)

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Outline

- State the existence problem and uniqueness problem in surgery theory.
- Explain the notion of Poincaré complex and of Spivak normal fibration.
- Introduce the surgery problem, the surgery step and the surgery obstruction.
- Explain the surgery exact sequence and its applications to topological rigidity.

The goal of surgery theory

Problem (Existence)

Let X be a space. When is X homotopy equivalent to a closed manifold?

Problem (Uniqueness)

Let M and N be two closed manifolds. Are they isomorphic?

- For simplicity we will mostly work with orientable connected closed manifolds.
- We can work with topological manifolds, PL-manifolds or smooth manifolds and then isomorphic means homeomorphic, PL-homeomorphic or diffeomorphic.
- We will begin with the existence problem. We will later see that the uniqueness problem can be interpreted as a relative existence problem thanks to the s-Cobordism Theorem.

Poincaré complexes

- A closed manifold carries the structure of a finite *CW*-complex. Hence we assume in the sequel in the existence problem that X itself is already a *CW*-complex.

- Fix a natural number $n \geq 4$. Then every finitely presented group occurs as fundamental group of a closed n-dimensional manifold. Since the fundamental group of a finite *CW*-complex is finitely presented, we get no constraints on the fundamental group.

Exercise (Fundamental groups of closed 3-manifolds)

Let G be the fundamental group of a closed 3-manifold. Show that then $\dim_{\mathbb{Q}}(H_2(G;\mathbb{Q})) \leq \dim_{\mathbb{Q}}(H_1(G;\mathbb{Q}))$ holds.

- Let M be a (connected orientable) closed n-dimensional manifold. Then $H_n(M;\mathbb{Z})$ is infinite cyclic. If $[M] \in H_n(M;\mathbb{Z})$ is a generator, then the cap product with $[M]$ yields for $k \in \mathbb{Z}$ isomorphisms

$$- \cap [M] \colon H^{n-k}(M;\mathbb{Z}) \xrightarrow{\cong} H_k(M;\mathbb{Z}).$$

  Obviously X has to satisfy the same property if it is homotopy equivalent to M.

- There is a much more sophisticated Poincaré duality behind the result above which we will explain next.

- Recall that a (not necessarily commutative) ring with involution R is ring R with an involution of rings

$$\bar{-} \colon R \to R, \quad r \mapsto \bar{r},$$

  i.e., a map satisfying $\bar{\bar{r}} = r$, $\overline{r+s} = \bar{r} + \bar{s}$, $\overline{r \cdot s} = \bar{s} \cdot \bar{r}$ and $\bar{1} = 1$ for $r, s \in R$.

- Our main example is the involution on the group ring $\mathbb{Z}G$ for a group G defined by sending $\sum_{g\in G} a_g \cdot g$ to $\sum_{g\in G} a_g \cdot g^{-1}$.

- Let M be a left R-module. Then $M^* := \hom_R(M, R)$ carries a canonical right R-module structure given by $(fr)(m) = f(m) \cdot r$ for a homomorphism of left R-modules $f\colon M \to R$ and $m \in M$. The involution allows us to view $M^* = \hom_R(M; R)$ as a left R-module, namely, define rf for $r \in R$ and $f \in M^*$ by $(rf)(m) := f(m) \cdot \overline{r}$ for $m \in M$.

- Given an R-chain complex of left R-modules C_* and $n \in \mathbb{Z}$, we define its dual chain complex C^{n-*} to be the chain complex of left R-modules whose p-th chain module is $\hom_R(C_{n-p}, R)$ and whose p-th differential is given by

$$\begin{aligned}(-1)^{n-p+1} \cdot \hom_R(c_{n-p+1}, \mathrm{id})\colon (C^{n-*})_p &= \hom_R(C_{n-p}, R) \\ \to (C^{n-*})_{p-1} &= \hom_R(C_{n-p+1}, R).\end{aligned}$$

Definition (Finite Poincaré complex)

A (connected) finite n-dimensional CW-complex X is a finite n-dimensional Poincaré complex if there is $[X] \in H_n(X;\mathbb{Z})$ such that the induced $\mathbb{Z}\pi$-chain map

$$- \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence.

- If we apply $\mathrm{id}_{\mathbb{Z}} \otimes_{\mathbb{Z}\pi} -$, we obtain a $\mathbb{Z}$-chain homotopy equivalence

  $$C^{n-*}(X) \to C_*(X)$$

  which induces after taking homology the Poincaré duality isomorphism $- \cap [X] \colon H^{n-k}(M;\mathbb{Z}) \xrightarrow{\cong} H_k(M;\mathbb{Z})$ from above.

Theorem (Closed manifolds are Poincaré complexes)

A closed n-dimensional manifold M is a finite n-dimensional Poincaré complex.

- We conclude that a finite n-dimensional CW-complex X is homotopy equivalent to a closed n-dimensional manifold only if it is up to homotopy a finite n-dimensional Poincaré complex.

Exercise (Poincaré chain homotopy equivalence for S^1)

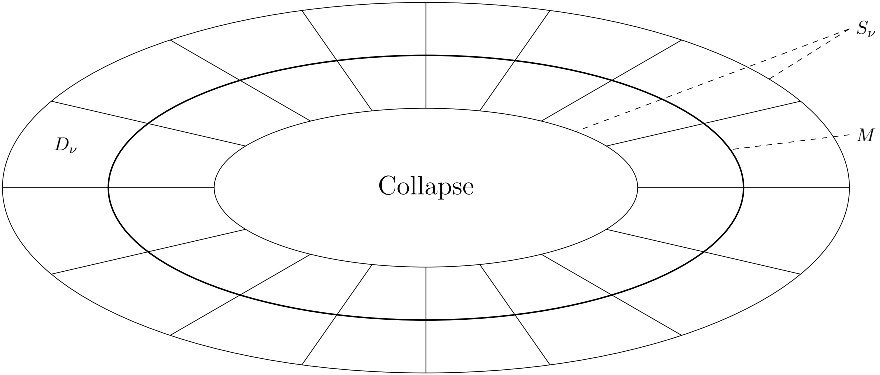
Equip S^1 with the CW-structure with two cells. Write down explicitly a $\mathbb{Z}[\pi_1(S^1)]$-chain homotopy equivalence

$$C^{1-*}(\widetilde{S^1}) \to C_*(\widetilde{S^1})$$

after appropriate identifications of the chain modules with $\mathbb{Z}[\pi_1(S^1)]$.

The Spivak normal fibration

- We briefly recall the Pontryagin-Thom construction for a closed n-dimensional manifold M.
- Choose an embedding $i\colon M \to S^{n+k}$ normal bundle $\nu(M)$.
- Choose a tubular neighborhood $N \subseteq S^{n+k}$ of M. This is a compact manifold with boundary ∂N with $M \subseteq \operatorname{int}(N)$ and comes with a diffeomorphism $f\colon (D\nu(M), S\nu(M)) \xrightarrow{\cong} (N, \partial N)$ which is the identity on the zero section.
- Let $c\colon S^{n+k} \to \operatorname{Th}(\nu(M))$ be the collaps map onto the Thom space $\operatorname{Th}(\nu(M)) := D\nu(M)/S\nu(M)$ which is given by f^{-1} on $\operatorname{int}(N)$ and sends any point outside $\operatorname{int}(N)$ to the base point.
- Then the Hurewicz homomorphism $\pi_{n+k}(Th(M)) \to H_{n+k}(\operatorname{Th}(M))$ sends $[c]$ to a generator of the infinite cyclic group $H_{n+k}(\operatorname{Th}(M))$.


S_ν
M
D_ν
Collapse

- The normal bundle is stably independent of the choice of the embedding.
- Next we describe the homotopy theoretic analog of the normal bundle for a finite n-dimensional Poincaré complex X.

Definition (Spivak normal structure)

A Spivak normal $(k-1)$-structure is a pair (p, c) where $p\colon E \to X$ is a $(k-1)$-spherical fibration called the Spivak normal fibration, and $c\colon S^{n+k} \to \mathrm{Th}(p)$ is a map such that the Hurewicz homomorphism $h\colon \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p))$ sends $[c]$ to a generator of the infinite cyclic group $H_{n+k}(\mathrm{Th}(p))$.

Theorem (Existence and Uniqueness of Spivak Normal Fibrations)

1. *If k is a natural number satisfying $k \geq n+1$, then there exists a Spivak normal $(k-1)$-structure (p, c);*
2. *For $i = 0, 1$ let $p_i \colon E_i \to X$ and $c_i \colon S^{n+k_i} \to \mathrm{Th}(p_i)$ be Spivak normal (k_i-1)-structures for X.*
   Then there exists an integer k with $k \geq k_0, k_1$ such that there is up to strong fibre homotopy precisely one strong fibre homotopy equivalence
   $$(\mathrm{id}, \overline{f}) \colon p_0 * \underline{S^{k-k_0}} \to p_1 * \underline{S^{k-k_1}}$$
   for which $\pi_{n+k}(\mathrm{Th}(\overline{f}))(\Sigma^{k-k_0}([c_0])) = \Sigma^{k-k_1}([c_1])$ holds.

- The Pontryagin-Thom construction yields a Spivak normal $(k-1)$-structure on a closed manifold M with the sphere bundle $S\nu(M)$ as the spherical $(k-1)$ fibration.

- Hence a finite n-dimensional Poincaré complex is homotopy equivalent to a closed manifold only if the Spivak normal fibration has (stably) a vector bundle reduction.

- There exists a finite n-dimensional Poincaré complex whose Spivak normal fibration does not possess a vector bundle reduction and which therefore is not homotopy equivalent to a closed manifold.

- Hence we assume from now on that X is a (connected oriented) finite n-dimensional Poincaré complex which comes with a vector bundle reduction ξ of the Spivak normal fibration.

Normal maps

Definition (Normal map of degree one)

A normal map of degree one with target X consists of:

- A closed (oriented) n-dimensional manifold M;
- A map of degree one $f\colon M \to X$;
- A $(k+n)$-dimensional vector bundle ξ over X;
- A bundle map $\overline{f}\colon TM \oplus \underline{\mathbb{R}^k} \to \xi$ covering f.

- A vector bundle reduction yields a normal map of degree one with X as target as explained next.
- Let η be a vector bundle reduction of the Spivak normal fibration.
- Let $c\colon S^{n+k} \to \mathrm{Th}(p)$ be the associated collaps map. Make it transversal to the zero-section in $\mathrm{Th}(p)$.
- Let M be the preimage of the zero-section. This is a closed submanifold of S^{n+k} and comes with a map $f\colon M \to X$ of degree one covered by a bundle map $\nu(M \subseteq S^{n+k}) \to \eta$.
- Since $TM \oplus \nu(M \subseteq S^{n+k})$ is stably trivial, we can construct from these data a normal map of degree one from M to X.

Problem (Surgery problem)

Let $(f, \overline{f})\colon M \to X$ be a normal map of degree one. Can we modify it without changing the target such that f becomes a homotopy equivalence?

Exercise (Existence of normal maps)

Suppose that X is homotopy equivalent to a closed manifold M. Show that then there exists a normal map of degree one from M to X whose underlying map $f\colon M \to X$ is a homotopy equivalence.

The surgery step

- Suppose that M is a closed manifold of dimension n, X is a CW-complex and $f\colon M \to X$ is a k-connected map. Consider $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ \Big\downarrow j & & \Big\downarrow f \\ D^{k+1} & \xrightarrow{Q} & X. \end{array}$$

  We want to kill ω.

- In the category of CW-complexes this can be achieved by attaching cells. But attaching a cell destroys in general the structure of a closed manifold, so we have to do a more sophisticated modification.

- Suppose that the map $q\colon S^k \to M$ extends to an embedding

$$q^{\text{th}}\colon S^k \times D^{n-k} \hookrightarrow M.$$

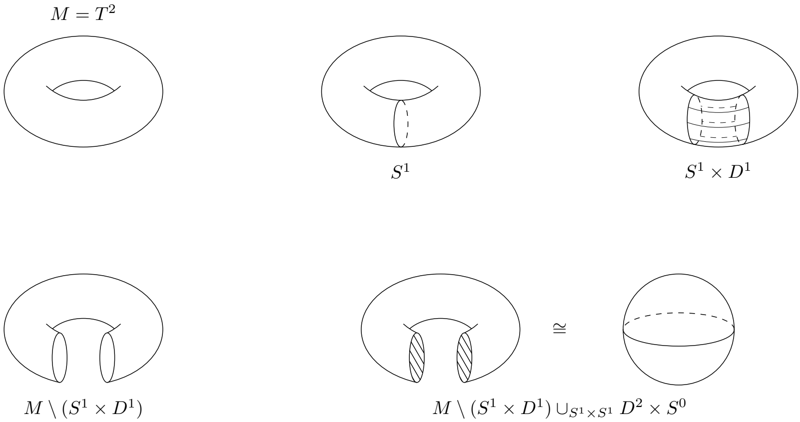
- Let $\text{int}(\text{im}(q^{\text{th}}))$ be the interior of the image of q^{th}. Then $M - \text{int}(\text{im}(q^{\text{th}}))$ is a manifold with boundary $\text{im}(q^{\text{th}}|_{S^k \times S^{n-k-1}})$.

- We can get rid of the boundary by attaching $D^{k+1} \times S^{n-k-1}$ along $q^{\text{th}}|_{S^k \times S^{n-k-1}}$. Denote the resulting manifold

$$M' := \left(D^{k+1} \times S^{n-k-1}\right) \cup_{q^{\text{th}}|_{S^k \times S^{n-k-1}}} \left(M - \text{int}(\text{im}(q^{\text{th}}))\right).$$

- The manifold M' is said to be obtained from M by surgery along q^{th}.

- Let $f\colon T^2 \to S^2$ be a Hopf collapse map. We fix $y_0 \in S^1$ so that $S^1 := S^1 \times \{y_0\} \subset T^2$ satisfies $f(S^1) = x_0$. We define $\omega \in \pi_2(f)$ by extending $f|_{S^1}$ to the constant map at x_0 on all of D^2. The following diagram illustrates the effect of surgery on the source.



Exercise (Surgery on $T^2 \to S^2$)

Show that the map $f' \colon S^2 \to S^2$ obtained by carrying out the surgery step on the Hopf collapse map $f \colon T^2 \to S^2$ as described above is a homotopy equivalence.

Exercise (Euler characteristic as surgery obstruction)

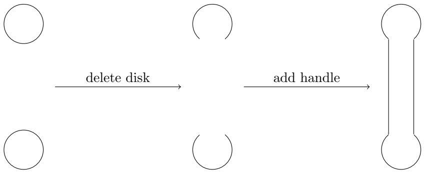
Consider a map $f \colon M \to X$ from a closed n-dimensional manifold M to a finite CW-complex X. Suppose that it can be converted by a finite sequence of surgery steps to a homotopy equivalence $f' \colon M' \to X$. Show that then $\chi(M) - \chi(X) \equiv 0 \mod 2$.

- It is important to notice that the maps $f\colon M \to X$ and $f'\colon M' \to X$ are bordant as manifolds with reference map to X.
- The relevant bordism is given by

$$W = \left(D^{k+1} \times D^{n-k}\right) \cup_{q^{\text{th}}} (M \times [0,1]),$$

where we think of q^{th} as an embedding $S^k \times D^{n-k} \to M \times \{1\}$. In other words, W is obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k}$ to $M \times \{1\}$.
- Then M appears in W as $M \times \{0\}$ and M' as other component of the boundary of W.
- The manifold W is called the trace of surgery along the embedding q^{th}.

Surgery step



Normal bordism

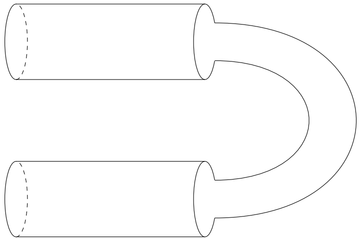


Figure 1: normal bordism

- Notice that the inclusion $M - \mathrm{int}(\mathrm{im}(q^{\mathrm{th}})) \to M$ is $(n-k-1)$-connected since $S^k \times S^{n-k-1} \to S^k \times D^{n-k}$ is $(n-k-1)$-connected. Hence $\pi_l(f) = \pi_l(f')$ for $l \leq k$ and there is an epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ whose kernel contains ω, provided that $2(k+1) \leq n$.

- The condition $2(k+1) \leq n$ can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality also forces a change in dimension $(n-l)$. This phenomenon is the reason why there are surgery obstructions to converting any map $f \colon M \to X$ into a homotopy equivalence in a finite number of surgery steps for odd dimension n.

- The bundle data ensure that the thickening q^{th} exists when we are doing surgery below the middle dimension. If one carries out the thickening in a specific way, the bundle data extend to the resulting normal map of degree one and we can continue the process.

Theorem (Making a normal map highly connected)

Given a normal map of degree one, we can carry out a finite sequence of surgery steps so that the resulting $f' \colon N \to X$ is k-connected, where $n = 2k$ or $n = 2k + 1$.

Exercise (Criterion for homotopy equivalence)

Show that a normal map of degree one which is $k + 1$-connected, where $n = 2k$ or $n = 2k + 1$, is a homotopy equivalence.

- Hence we have to make a normal map, which is already k-connected, $k+1$-connected in order to achieve a homotopy equivalence, where $n = 2k$ or $n = 2k+1$. Exactly here the surgery obstruction occurs.

- In odd dimension $n = 2k+1$ the surgery obstruction comes from the previous observation that by Poincare duality modifications in the $(k+1)$-th homology cause automatically (undesired) changes in the k-th homology.

- In even dimension $n = 2k$ one encounters the problem that the bundle data only guarantee that one can find an immersion with finitely many self-intersection points

$$q^{\mathrm{th}} \colon S^k \times D^k \to M.$$

  The surgery obstruction is the algebraic obstruction to get rid of the self-intersection points. If $n \geq 5$, its vanishing is indeed sufficient to convert q^{th} into an embedding.

- One prominent necessary surgery obstruction is given in the case $n = 4k$ by the difference of the signatures $\text{sign}(X) - \text{sign}(M)$ since the signature is a bordism invariant and a homotopy invariant.

- If $\pi_1(M)$ is simply connected and $n = 4k$ for $k \geq 2$, then the vanishing of $\text{sign}(X) - \text{sign}(M)$ is indeed sufficient.

- If $\pi_1(M)$ is simply connected and n is odd and $n \geq 5$, there are no surgery obstructions.

Theorem (Existence problem in the simply connected case)

Let X be a simply connected finite Poincaré complex of dimension n

1. *Suppose that n is odd and $n \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle.*
2. *Suppose $n = 4k \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi \colon E \to X$ such that*
$$\langle \mathcal{L}(\xi), [X] \rangle = \operatorname{sign}(X).$$
3. *Suppose that $n = 4k + 2 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle such that the Arf invariant of the associated surgery problem, which takes values in $\mathbb{Z}/2$, vanishes.*

Algebraic L-groups

- In general there are surgery obstructions taking values in the so called L-groups $L_n(\mathbb{Z}[\pi_1(M)])$.
- In even dimensions $L_n(R)$ is defined for a ring with involution in terms of quadratic forms over R, where the hyperbolic quadratic forms always represent zero. In odd dimensions $L_n(R)$ is defined in terms of automorphisms of hyperbolic quadratic forms, or, equivalently, in terms of so called formations.
- The L-groups are easier to compute than K-groups since they are 4-periodic, i.e., $L_n(R) \cong L_{n+4}(R)$.
- We have

$$L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 4k; \\ \mathbb{Z}/2 & \text{if } n = 4k+2; \\ \{0\} & \text{if } n = 2k+1. \end{cases}$$

- The surgery obstruction is defined in all dimensions and is always a necessary condition to solve the surgery problem.

- In dimension $n \geq 5$ the vanishing of the surgery obstruction is sufficient.

- In dimension 4 the methods of proof of sufficiency break down because the so called Whitney trick is not available anymore which relies in higher dimensions on the fact that two embedded 2-disks can be made disjoint by transversality.

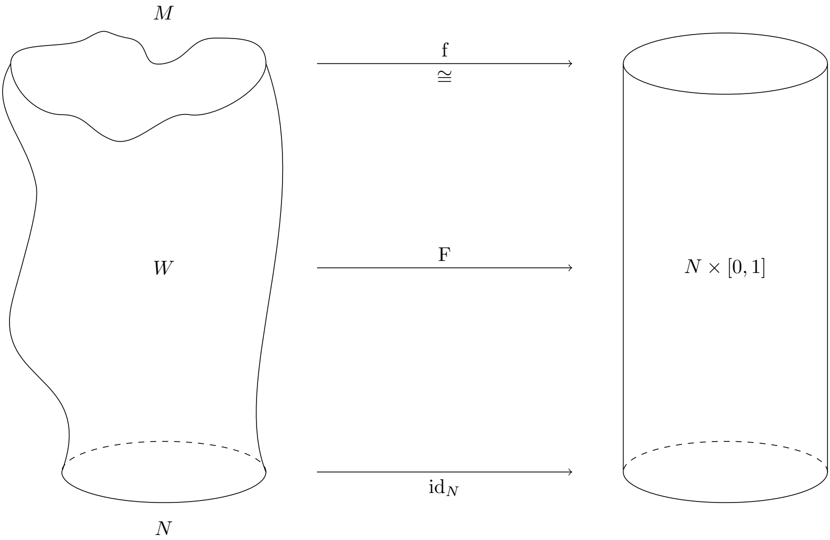
- In dimension 3 problems occur concerning the effect of surgery on the fundamental group.

The surgery program

- The surgery program addresses the uniqueness problem whether two closed manifolds M and N are diffeomorphic.

- The idea is to construct an h-cobordism $(W; M, N)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M and hence M and N are diffeomorphic.

- So the surgery program is:
  1. Construct a homotopy equivalence $f\colon M \to N$;
  2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id})\colon (W; M, N) \to (N \times [0,1]; N \times \{0\}, N \times \{1\})$;
  3. Modify W and F relative boundary by surgery such that F becomes a homotopy equivalence and thus W becomes an h-cobordism.
  4. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial. Or one knows already that $\mathrm{Wh}(\pi_1(M))$ vanishes.


M
f
$\cong$
W
F
$N \times [0,1]$
id_N
N

The surgery exact sequence

Definition (The structure set)

Let N be a closed topological manifold of dimension n. We call two simple homotopy equivalences $f_i \colon M_i \to N$ from closed topological manifolds M_i of dimension n to N for $i = 0, 1$ equivalent if there exists a homeomorphism $g \colon M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0.

The structure set $\mathcal{S}_n^{\mathrm{top}}(N)$ of N is the set of equivalence classes of simple homotopy equivalences $M \to X$ from closed topological manifolds of dimension n to N. This set has a preferred base point, namely the class of the identity $\mathrm{id} \colon N \to N$.

- If we assume $\mathrm{Wh}(\pi_1(N)) = 0$, then every homotopy equivalence with target N is automatically simple.
- There is an obvious version, where topological and homeomorphism are replaced by smooth and diffeomorphism.

Definition (Topological rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Exercise (Topological rigidity)

Show for a closed topological manifold M that it is topologically rigid if and only if the structure set $\mathcal{S}_n^{\mathrm{top}}(M)$ consists of exactly one point.

Exercise (The sphere is topological rigidity)

Show that the Poincaré Conjecture implies that S^n is topologically rigid.

Theorem (The topological Surgery Exact Sequence)

For a closed n-dimensional topological manifold N with $n \geq 5$, there is an exact sequence of abelian groups, called surgery exact sequence,

$$\ldots \xrightarrow{\eta} \mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}_n^{\mathrm{top}}(N) \xrightarrow{\eta} \mathcal{N}_n^{\mathrm{top}}(N) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi)$$

- $L_n^s(\mathbb{Z}\pi)$ is the algebraic L-group of the group ring $\mathbb{Z}\pi$ for $pi = \pi_1(N)$ (with decoration s).
- $\mathcal{N}_n^{\mathrm{top}}(N)$ is the set of normal bordism classes of normal maps of degree one with target N.
- $\mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\})$ is the set of normal bordism classes of normal maps $(M, \partial M) \to (N \times [0,1], N \times \{0,1\}$ of degree one with target $N \times [0,1]$ which are simple homotopy equivalences on the boundary.

- The map σ is given by the surgery obstruction.
- The map η sends $f\colon M \to N$ to the normal map of degree one for which $\xi = (f^{-1})^* TN$.
- The map ∂ sends an element $x \in L_{n+1}(\mathbb{Z}\pi)$ to $f\colon M \to N$ if there exists a normal map $F\colon (W, \partial W) \to (N \times [0,1], N \times \{0,1\}$ of degree one with target $N \times [0,1]$ such that $\partial W = N \amalg M$, $F|_N = \mathrm{id}_N$, $F|_M = f$, and the surgery obstruction of F is x.
- There is a space G/TOP together with bijections

$$\begin{array}{rcl} [N, \mathrm{G/TOP}] & \xrightarrow{\cong} & \mathcal{N}_n^{\mathrm{top}}(N); \\ [N \times [0,1]/N \times \{0,1\}, \mathrm{G/TOP}] & \xrightarrow{\cong} & \mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\}). \end{array}$$

- There is an analog of the surgery exact sequence in the smooth category except that it is only an exact sequence of pointed sets and not of abelian groups.

Corollary

A topological manifold of dimension $n \geq 5$ is topologically rigid if and only if the map $\mathcal{N}^{\mathrm{top}}_{n+1}(N \times [0,1], N \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi)$ is surjective and the map $\mathcal{N}^{\mathrm{top}}_n(N) \to L^s_n(\mathbb{Z}\pi)$ is injective.

- Notice the following formulas which look similar:

$$\begin{aligned} L_n(\mathbb{Z}[\mathbb{Z}]) &\cong L_n(\mathbb{Z}) \oplus L_{n-1}(\mathbb{Z}); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

Question (*L*-theory of group rings and group homology)

Is there a relation between $L_n(RG)$ and group homology of G?

To be continued

Stay tuned