Classifying spaces for families (Lecture III)

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Outline

- We formulate the Farrell-Jones Conjecture for torsionfree groups and discuss some applications.
- We introduce the notion of a classifying space for a family of subgroups and explain its relevance for group theory and topology.
- We present some nice models for $\underline{E}G$.
- We discuss finiteness properties of EG, $\underline{E}G$ and $\underline{\underline{E}}G$.

The Farrell-Jones Conjectures for torsionfree groups

- Recall the following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$, which look similar:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

- If G and K are groups, then we have the following formulas, which look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G * K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G * K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

- This raises the question whether there is a generalized homology theory $\mathcal{H}_*$ such that $\mathcal{H}_n(BG) \cong K_n(RG)$ holds for $n \in \mathbb{Z}$.

- If this is true, we would get for the trivial group such that $\mathcal{H}_n(\mathrm{pt}) \cong K_n(R)$ holds for $n \in \mathbb{Z}$.

- Notice that there is a spectrum $\mathbf{K}_R$ satisfying $\pi_n(\mathbf{K}_R) \cong K_n(R)$ for $n \in \mathbb{Z}$ and for any spectrum $\mathbf{E}$ there is a generalized homology theory $H_*(-, \mathbf{E})$ satisfying $H_n(\mathrm{pt}; \mathbf{E}) \cong \pi_n(\mathbf{E})$ for $n \in \mathbb{Z}$.
  Hence the obvious candidate for $\mathcal{H}_*$ is $H_*(-; \mathbf{K}_R)$.

- Moreover, there exists a natural map $H_n(BG; \mathbf{K}_R) \to K_n(RG)$, called assembly map.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- The condition that R is regular is necessary. Recall that the Bass-Heller-Swan Theorem gives an isomorphism

$$K_n(R\mathbb{Z}) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R)$$

whereas the conjecture above predicts

$$K_n(R\mathbb{Z}) \cong K_n(R) \oplus K_{n-1}(R).$$

- Also the condition that G is torsionfree is necessary.
- If G is finite, the conjecture above implies that the change of rings map for $R \to RG$ induces an isomorphism

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- This is not true in general. Take for instance $R = \mathbb{Z}$, $n = 1$ and $G = \mathbb{Z}/5$. Then $K_1(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is zero, whereas $K_n(\mathbb{Z}[\mathbb{Z}/5]) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Wh}(\mathbb{Z}/5) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

Exercise (Failure for finite G)

Let G be a finite group G and F be a field F of characteristic zero. Show that the map $K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$ is bijective if and only if G is trivial.

Lemma

Let R be a regular ring. Suppose that G is torsionfree and satisfies the K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

Lemma

Suppose that G is torsionfree and satisfies the K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings. Then the Whitehead group $\mathrm{Wh}(G)$ *is trivial.*

Proof.

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG;\mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.

- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG;\mathbf{K}_R) \cong K_0(RG).$$

□

Proof (continued).

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$.

  We get an exact sequence

$$0 \to H_0(BG; \mathbf{K}_{\mathbb{Z}}) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G,G] \to 1.$$

- This implies

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0.$$

□

In particular, we get for a torsionfree group G satisfying the K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$.
- $\widetilde{K}_0(\mathbb{Z}G) = 0$.
- $\mathrm{Wh}(G) = 0$.
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex.
- Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial.

Exercise (Serre's Conjecture)

Suppose that the torsionfree group G satisfies the K-theoretic Farrell-Jones Conjecture for torsionfree groups. Then G is of type FF if and only if it is of type FP.

Conjecture (L-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$.
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R.
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$.
- If $\widetilde{K}_n(\mathbb{Z}G) = 0$ for $n \le 0$ and $\mathrm{Wh}(G) = 0$, then the decoration $\langle -\infty \rangle$ does not matter for $L_n(\mathbb{Z}G)$.

- We want to formulate a version of the Farrell-Jones Conjecture which works for all groups and rings.
- This requires as input the theory of classifying spaces for families of subgroups.
- These spaces are interesting in the own right for group theory and topology, and hence we spend the rest of this lecture about them.

Classifying spaces for families of subgroups

Definition (G-CW-complex)

A G-CW-complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

Exercise (*G*-*CW* in terms of *CW*)

A G-CW-complex X is the same as a G-CW-complex together with a cellular G-action such that for every open cell e and $g \in G$ *satisfying* $g \cdot e \cap e \neq \emptyset$ *we have* $gx = x$ *for every* $x \in e$.

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and the G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

Let G act properly and smoothly on a smooth manifold M. Then M inherits the structure of G-CW-complex.

- The obvious G-equivariant analogs of the Cellular Approximation Theorem and the Whitehead Theorem hold.

Definition (Proper G-action)

A G-space X is called proper if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that set $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ is finite.

Lemma

1. *A proper G-space has always finite isotropy groups.*
2. *A G-CW-complex X is proper if and only if all its isotropy groups are compact.*

Exercise (Non-proper action)

Construct a closed manifold with a free $\mathbb{Z}$-action which is not proper.

Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of G is a set of subgroups of G which is closed under conjugation and taking subgroups

- A group G is called virtually cyclic if it is finite or contains $\mathbb{Z}$ as a subgroup of finite index.
- Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the classifying G-CW-complex for the family $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

- We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the universal G-CW-complex for proper G-actions.
- We also write $EG = E_{\mathcal{TR}}(G)$ and $\underline{\underline{E}}G := E_{\mathcal{VCYC}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$;*
- *Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if the H-fixed point set X^H is contractible for each $H \in \mathcal{F}$ and X^H is empty for $H \notin \mathcal{F}$.*

Exercise ((Another) Homotopy characterisation of $E_{\mathcal{F}}(G)$)

Let X be a G-CW-complex whose isotropy groups belong to $\mathcal{F}$. Then X is a model for $E_{\mathcal{F}}(G)$ if and only if the two projections $X \times X \to X$ to the first and to the second factor are G-homotopic and for each $H \in \mathcal{F}$ there exists $x \in G_x$ with $H \subseteq G_x$.

Exercise (Some models for $E_{\mathcal{F}}(G)$)

Show

1. *The G-spaces G/G is model for $E_{\mathcal{F}}(G)$ if and only if $\mathcal{F} = \mathcal{ALL}$.*
2. *$EG \to BG := G\backslash EG$ is a model for the universal G-principal bundle for G-principal bundles over CW-complexes.*
3. *A free G-CW-complex X is a model for EG if and only if X/G is an Eilenberg MacLane space of type $(G, 1)$.*

Example (Infinite dihedral group)

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group.
- A model for ED_∞ is the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- A model for $\underline{E}D_\infty$ is $\mathbb{R}$ with the obvious D_∞-action.

Exercise (Contractible $\underline{E}G/G$)

Construct an infinite group G such that $\underline{E}G/G$ is contractible. Can such a group G be torsionfree?

Exercise (Maps between classifying spaces for families)

Let $\mathcal{F}$ and $\mathcal{G}$ be two families of subgroups of G. Show that the following assertions are equivalent

1. *There is a G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$;*
2. *The set $[E_{\mathcal{F}}(G), E_{\mathcal{G}}(G)]^G$ consists of precisely one element;*
3. *The projection $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G) \to E_{\mathcal{F}}(G)$ is a G-homotopy equivalence;*
4. $\mathcal{F} \subseteq \mathcal{G}$.

Special models for $\underline{E}G$

- We want to illustrate that the G-space $\underline{E}G$ often has very nice geometric models and appears naturally in many interesting situations.

Theorem (Simplicial Model)

Let $P_\infty(G)$ be the geometric realization of the full simplicial on G. This is a model for $\underline{E}G$.

Theorem

Consider the G-space

$$X_G = \left\{ f\colon G \to [0,1] \;\middle|\; f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm. It is G-homotopy equivalent to $\underline{E}G$.

- The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.
- The identity map induces a G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism.

Theorem (Discrete subgroups of almost connected Lie groups)

Let L be a Lie group with finitely many path components. Let $G \subseteq L$ be a discrete subgroup of L.
Then L contains a maximal compact subgroup K, which is unique up to conjugation, and L/K with the obvious left G-action is a finite dimensional G-CW-model for $\underline{E}G$.

Theorem (Actions on CAT(0)-spaces)

Let X be a proper G-CW-complex. Suppose that X has the structure of a complete simply connected $\mathrm{CAT}(0)$-space for which G acts by isometries. Then X is a model for $\underline{E}G$.

- The result above contains as special case proper isometric G-actions on simply-connected complete Riemannian manifolds with non-positive sectional curvature and proper G-actions on trees.

- The Rips complex $P_d(G, S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realization of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.
- The obvious G-action by simplicial automorphisms on $P_d(G, S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G, S)'$.

Theorem (Rips complex)

Let G be a discrete group with a finite symmetric set of generators. Suppose that (G,S) is δ-hyperbolic for the real number $\delta > 0$. Let d be a natural number with $d \geq 16\delta + 8$.
Then the barycentric subdivision of the Rips complex $P_d(G,S)'$ is a finite G-CW-model for $\underline{E}G$.

Exercise (Rational homology of hyperbolic groups)

Let G be a hyperbolic group. Show that there exists a natural number N such that $H_n(G;\mathbb{Q}) = 0$ holds for $n \geq N$.

- Let $\Gamma_{g,r}^s$ be the mapping class group of an orientable compact surface F of genus g with s punctures and r boundary components.
- We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative.
- It is well-known that the associated Teichmüller space $\mathcal{T}_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly.

Theorem (Teichmüller space)

The $\Gamma_{g,r}^s$-space $\mathcal{T}_{g,r}^s$ is a model for $\underline{E}\Gamma_{g,r}^s$.

- Let F_n be the free group of rank n.
- Denote by $\mathrm{Out}(F_n)$ the group of outer automorphisms.
- Culler-Vogtmann have constructed a space X_n called outer space on which $\mathrm{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.
- The space X_n contains a spine K_n which is an $\mathrm{Out}(F_n)$-equivariant deformation retraction.
  This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\mathrm{Out}(F_n)$-action is by simplicial automorphisms and cocompact.

Theorem (Spine of outer space)

The barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\,\mathrm{Out}(F_n)$.

Example ($SL_2(\mathbb{Z})$)

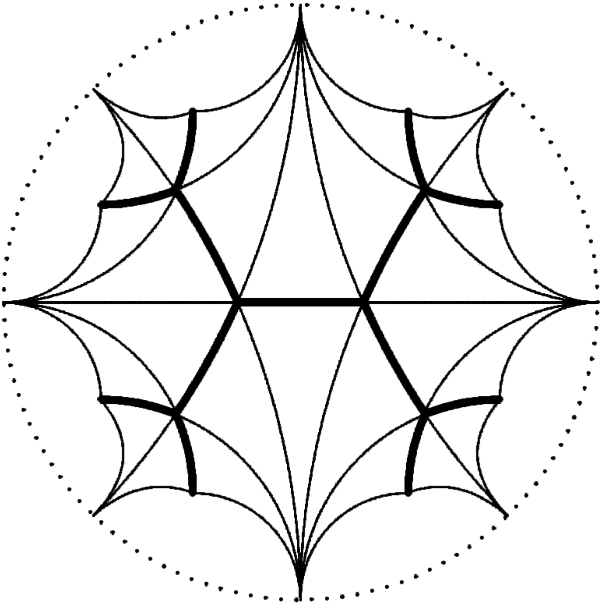
- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{Z})$.
- The group $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup. Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{Z})$.
- Since the 2-dimensional hyperbolic space $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant -1 and $SL_2(\mathbb{Z})$ acts proper on it by Moebius transformations, the $SL_2(\mathbb{Z})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$.
- The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on $\mathbb{H}^2$ by Moebius transformations. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Hence the $SL_2(\mathbb{Z})$-spaces $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{Z})$-diffeomorphic.

Example (continued)

- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex.
  This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.
- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. Here is a concrete description of such a $SL_2(\mathbb{Z})$-homotopy equivalence.

Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action.
- Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior.
- Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action which is the tree model above.
- The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.



Finiteness properties

- Finiteness properties of the spaces EG and $\underline{E}G$ have been intensively studied in the literature. We mention a few examples and results.
- If EG has a finite-dimensional model, the group G must be torsionfree. There are often finite models for $\underline{E}G$ for groups G with torsion.
- Often geometry provides small models for $\underline{E}G$ in cases, where the models for EG are huge. These small models can be useful for computations concerning BG itself.

Exercise (Models of finite type)

Show: If there is a model for $\underline{E}G$ of finite type, then the same is true for EG and BG.

Exercise (Finitely generated homology)

Suppose that G is a hyperbolic group, a mapping class group, $\operatorname{Out}(F_n)$ or a cocompact discrete subgroup of a connected Lie group.
Show that then G is finitely presented and that $H_i(G;\mathbb{Z})$ is finitely generated for all $i \geq 0$.

Theorem (Discrete subgroups of Lie groups)

Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup K. Let $G \subseteq L$ be a discrete subgroup.

1. *Then L/K with the left G-action is a model for $\underline{E}G$.*
2. *Suppose additionally that G is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index.*
   Then we have for its virtual cohomological dimension

   $$\mathrm{vcd}(G) \leq \dim(L/K).$$

   Equality holds if and only if $G\backslash L$ is compact.

Theorem (A criterion for 1-dimensional models for *BG*, Stallings, Swan)

The following statements are equivalent:

- *There exists a* 1*-dimensional model for EG;*
- *There exists a* 1*-dimensional model for BG;*
- *The cohomological dimension of G is less or equal to one;*
- *G is a free group.*

Theorem (A criterion for 1-dimensional models for $\underline{E}G$, Dunwoody, Karras-Pietrowsky-Solitar)

- *There exists a* 1*-dimensional model for* $\underline{E}G$ *if and only if the cohomological dimension of* G *over the rationals* $\mathbb{Q}$ *is less or equal to one.*
- *Suppose that* G *is finitely generated. Then there exists a* 1*-dimensional model for* $\underline{E}G$ *if and only if* B *is virtually finitely generated free.*

Theorem (Virtual cohomological dimension and $\dim(\underline{E}G)$, Lück)

Let G be virtually torsionfree.

- *Then*

$$\operatorname{vcd}(G) \leq \dim(\underline{E}G)$$

  for any model for $\underline{E}G$.

- *Let $l \geq 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \leq l$.*
  Then there is a model for $\underline{E}G$ of dimension $\max\{3, \operatorname{vcd}(G)\} + l$.

Problem (Brown)

For which groups G, which are virtually torsionfree, does there exist a G-CW-model for $\underline{E}G$ of dimension $\operatorname{vcd}(G)$?

- The results above do give some evidence for a positive answer.
- However, Leary-Nucinkis have constructed groups, where the answer is negative.

Theorem (Leary-Nucinkis)

Let X be a CW-complex. Then there exists a group G with $X \simeq G\backslash\underline{E}G$.

Groups with special maximal finite subgroups

- Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$.
  Assume that G satisfies the following assertions:
  - (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;
  - (NM) $M \in \mathcal{MFIN}, M \neq \{1\} \Rightarrow N_G M = M$.
- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible.
- Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$.
- By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.
- Define X as the G-pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow{u_1} & & \downarrow{f_1} \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & X \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

Theorem

The G-space X is a model for $\underline{E}G$.

Proof.

- Obviously X is a G-CW-complex with finite isotropy groups.
- We have to show for $H \subseteq G$ finite that $\underline{E}G^H$ contractible.
- We begin with the case $H \neq \{1\}$.

  Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$ and we get

$$\left(\coprod_{i \in I} G/M_i\right)^H = \left(G/M_{i_0}\right)^H = \text{pt}.$$

  Hence $\underline{E}G^H = \text{pt}$.

Proof continued.

- It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence $\underline{E}G$ is contractible.

□

- This small model is very useful for computation of all kind of K- and L-groups of RG, provided that the Farrell-Jones Conjecture is true. These computations have interesting applications to questions about the classification of manifolds and of certain C^*-algebras.

- The potential of these models is already interesting for ordinary group (co-)homology as illustrated next.

- Consider the pushout obtained from the G-pushout above by dividing out the G-action

$$\begin{array}{ccc}
\coprod_{i\in I} BM_i & \longrightarrow & BG \\
\big\downarrow & & \big\downarrow \\
\coprod_{i\in I} \mathrm{pt} & \longrightarrow & G\backslash \underline{E}G
\end{array}$$

- The associated Mayer-Vietoris sequence yields

$$\ldots \to \widetilde{H}_{p+1}(G\backslash \underline{E}G) \to \bigoplus_{i\in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG) \\ \to \widetilde{H}_p(G\backslash \underline{E}G) \to \ldots$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E}G) + 1$

$$\bigoplus_{i \in I} H_p(M_i) \xrightarrow{\cong} H_p(G).$$

- Let G be one relator-group. Then G has a model for $\underline{E}G$ of dimension 2, contains up to conjugacy precisely one maximal subgroup M, and M is isomorphic to $\mathbb{Z}/m$ for some $m \geq 1$. Hence we get for $n \geq 3$

$$H_n(\mathbb{Z}/m) \xrightarrow{\cong} H_n(G).$$

Information about $\underline{\underline{E}}G$

- We will be forced when dealing with the Farrell-Jones Conjecture to deal with $\underline{\underline{E}}G$ instead of $\underline{E}G$. In general $\underline{\underline{E}}G$ is much more complicated and does not have such nice models as $\underline{E}G$.
- The following conjecture is known to be true for many groups, e.g., hyperbolic groups.

Conjecture (Finite models for $\underline{\underline{E}}G$)

There is a model of finite type for $\underline{\underline{E}}G$ if and only if G is virtually cyclic.

Theorem (Lück-Weiermann)

1. $\operatorname{mindim}^G(\underline{E}G) \leq 1 + \operatorname{mindim}^G(\underline{\underline{E}}G)$;
2. *If G is virtually $\mathbb{Z}^n$ for $n \geq 2$, then* $\operatorname{mindim}^G(\underline{\underline{E}}G) = n + 1$ *and* $\operatorname{mindim}^G(\underline{E}G) = n$;
3. *There exists an extension* $1 \to \mathbb{Z} \to H \to \mathbb{Z}^n \to 1$ *and an automorphism ϕ of H such that the semidirect product $G = H \rtimes_\phi \mathbb{Z}$ satisfies*

$$\begin{aligned} \operatorname{mindim}^G(\underline{\underline{E}}G) &= n + 1; \\ \operatorname{mindim}^G(\underline{E}G) &= n + 2. \end{aligned}$$

In particular we get

$$\operatorname{mindim}^G(\underline{\underline{E}}G) < \operatorname{mindim}^G(\underline{E}G).$$

To be continued

Stay tuned