Universal torsion, L^2-invariants, polytopes and the Thurston norm

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Münster, December, 2015

Review of classical L^2-invariants

- Let $G \to \overline{X} \to X$ be a G-covering of a connected finite CW-complex X.
- The cellular chain complex of $\overline{X}$ is a finitely generated free $\mathbb{Z}G$-chain complex:

$$\cdots \xrightarrow{c_{n-1}} \bigoplus_{I_n} \mathbb{Z}G \xrightarrow{c_n} \bigoplus_{i_{n-1}} \mathbb{Z}G \xrightarrow{c_{n-1}} \cdots$$

- The associated L^2-chain complex

$$C_*^{(2)}(\overline{X}) := L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})$$

has Hilbert spaces with isometric linear G-action as chain modules and bounded G-equivariant operators as differentials

$$\cdots \xrightarrow{c_{n-1}^{(2)}} \bigoplus_{I_n} L^2(G) \xrightarrow{c_n^{(2)}} \bigoplus_{i_{n-1}} L^2(G) \xrightarrow{c_{n-1}^{(2)}} \cdots$$

Review of classical L^2-invariants

- Let $G \to \overline{X} \to X$ be a G-covering of a connected finite CW-complex X.
- The cellular chain complex of $\overline{X}$ is a finitely generated free $\mathbb{Z}G$-chain complex:

$$\cdots \xrightarrow{c_{n-1}} \bigoplus_{I_n} \mathbb{Z}G \xrightarrow{c_n} \bigoplus_{i_{n-1}} \mathbb{Z}G \xrightarrow{c_{n-1}} \cdots$$

- The associated L^2-chain complex

$$C_*^{(2)}(\overline{X}) := L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})$$

has Hilbert spaces with isometric linear G-action as chain modules and bounded G-equivariant operators as differentials

$$\cdots \xrightarrow{c_{n-1}^{(2)}} \bigoplus_{I_n} L^2(G) \xrightarrow{c_n^{(2)}} \bigoplus_{i_{n-1}} L^2(G) \xrightarrow{c_{n-1}^{(2)}} \cdots$$

Definition (L^2-homology and L^2-Betti numbers)

Define the *n-th L^2-homology* to be the Hilbert space

$$H_n^{(2)}(\overline{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define the *n-th L^2-Betti number*

$$b_n^{(2)}(\overline{X}) := \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(\overline{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

- The original notion is due to *Atiyah* and was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t\to\infty} \int_{\mathcal{F}} tr\left(e^{-t\cdot\overline{\Delta}_n}(\overline{x},\overline{x})\right) dvol_{\overline{M}} .$$

- If G is finite, we have

$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\left(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X};\mathbb{C})\right) \quad \in \mathbb{Z}.$$

- The original notion is due to *Atiyah* and was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t\to\infty} \int_{\mathcal{F}} tr\left(e^{-t\cdot\overline{\Delta}_n}(\overline{x},\overline{x})\right) d\mathrm{vol}_{\overline{M}} .$$

- If G is finite, we have

$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\left(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X};\mathbb{C})\right) \quad \in \mathbb{Z}.$$

- The original notion is due to *Atiyah* and was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t\to\infty} \int_{\mathcal{F}} tr\big(e^{-t\cdot\overline{\Delta}_n}(\overline{x},\overline{x})\big)\, d\mathrm{vol}_{\overline{M}}\,.$$

- If G is finite, we have

$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\big(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X};\mathbb{C})\big) \quad \in \mathbb{Z}.$$

- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in the case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in the case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in the case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing Reidemeister torsion.

- The definition of L^2-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for a bounded G-equivariant operator $f \colon L^2(G)^m \to L^2(G)^n$ to be the non-negative real number

$$\det^{(2)}(f) = \exp\left(\frac{1}{2} \cdot \int \ln(\lambda)\, d\nu_{f^*f}\right) \in \mathbb{R}^{>0}$$

where ν_{f^*f} is the spectral measure of the positive operator f^*f.

- If G is finite and $m = n$, then $\det^{(2)}(f) = |\det(f)|^{1/|G|}$.

- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing Reidemeister torsion.

- The definition of L^2-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for a bounded G-equivariant operator $f\colon L^2(G)^m \to L^2(G)^n$ to be the non-negative real number

$$\det^{(2)}(f) = \exp\left(\frac{1}{2}\cdot\int \ln(\lambda)\, d\nu_{f^*f}\right) \in \mathbb{R}^{>0}$$

where ν_{f^*f} is the spectral measure of the positive operator f^*f.

- If G is finite and $m = n$, then $\det^{(2)}(f) = |\det(f)|^{1/|G|}$.

- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing Reidemeister torsion.

- The definition of L^2-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for a bounded G-equivariant operator $f \colon L^2(G)^m \to L^2(G)^n$ to be the non-negative real number

$$\det^{(2)}(f) = \exp\left(\frac{1}{2} \cdot \int \ln(\lambda) \, d\nu_{f^*f}\right) \in \mathbb{R}^{>0}$$

where ν_{f^*f} is the spectral measure of the positive operator f^*f.

- If G is finite and $m = n$, then $\det^{(2)}(f) = |\det(f)|^{1/|G|}$.

Definition (L^2-torsion)

Suppose that $\overline{X}$ is L^2-acyclic, i.e., all L^2-Betti numbers $b_n^{(2)}(\overline{X})$ vanish. Let $\Delta_n^{(2)}\colon C_n^{(2)}(\overline{X}) \to C_n^{(2)}(\overline{X})$ be the n-Laplace operator given by $c_{n+1}^{(2)} \circ \left(c_n^{(2)}\right)^* + \left(c_{n-1}^{(2)}\right)^* \circ c_n^{(2)}$.

Define the L^2-torsion

$$\rho^{(2)}(\overline{X}) := \frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln\left(\det{}^{(2)}(\Delta_n^{(2)})\right) \in \mathbb{R}.$$

Theorem (Lück-Schick)

Let M be a 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

Universal L^2-torsion

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced bounded G-equivariant $L^2(G)^m \to L^2(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.
- relations

$$\begin{aligned} \left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2]; \\ [g \circ f] &= [f] + [g]. \end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

Universal L^2-torsion

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced bounded G-equivariant $L^2(G)^m \to L^2(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.
- relations

$$\begin{aligned}\left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2];\\ [g \circ f] &= [f] + [g].\end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

Definition (Weak chain contraction)

Consider a $\mathbb{Z}G$-chain complex C_*. A weak chain contraction (γ_*, u_*) for C_* consists of a $\mathbb{Z}G$-chain map $u_*\colon C_* \to C_*$ and a $\mathbb{Z}G$-chain homotopy $\gamma_*\colon u_* \simeq 0_*$ such that $u_*^{(2)}\colon C_*^{(2)} \to C_*^{(2)}$ is a weak isomorphism for all $n \in \mathbb{Z}$ and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ holds for all $n \in \mathbb{Z}$.

Definition (Universal L^2-torsion)

Let C_* be a finite based free $\mathbb{Z}G$-chain complex such that $C_*^{(2)}$ is L^2-acyclic. Define its universal L^2-torsion

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(\mathbb{Z}G)$$

by

$$\rho_u^{(2)}(C_*) = [(uc+\gamma)_{\mathrm{odd}}] - [u_{\mathrm{odd}}],$$

where (γ_*, u_*) is any weak chain contraction of C_*.

Definition (Weak chain contraction)

Consider a $\mathbb{Z}G$-chain complex C_*. A weak chain contraction (γ_*, u_*) for C_* consists of a $\mathbb{Z}G$-chain map $u_* \colon C_* \to C_*$ and a $\mathbb{Z}G$-chain homotopy $\gamma_* \colon u_* \simeq 0_*$ such that $u_*^{(2)} \colon C_*^{(2)} \to C_*^{(2)}$ is a weak isomorphism for all $n \in \mathbb{Z}$ and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ holds for all $n \in \mathbb{Z}$.

Definition (Universal L^2-torsion)

Let C_* be a finite based free $\mathbb{Z}G$-chain complex such that $C_*^{(2)}$ is L^2-acyclic. Define its universal L^2-torsion

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(\mathbb{Z}G)$$

by

$$\rho_u^{(2)}(C_*) = [(uc + \gamma)_{\text{odd}}] - [u_{\text{odd}}],$$

where (γ_*, u_*) is any weak chain contraction of C_*.

- An additive L^2-torsion invariant (A, a) consists of an abelian group A and an assignment which associates to a finite based free $\mathbb{Z}G$-chain complex C_*, for which $C_*^{(2)}$ is L^2-acyclic, an element $a(C_*) \in A$ such that for any based exact short sequence of such $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ we get

$$a(D_*) = a(C_*) + a(E_*),$$

and we have $a\bigl(\cdots \to 0 \to \mathbb{Z}G \xrightarrow{\pm \operatorname{id}} \mathbb{Z}G \to 0 \to \cdots\bigr) = 0$.

- We call an additive L^2-torsion invariant (U, u) universal if for every additive L^2-torsion invariant (A, a) there is precisely one group homomorphism $f\colon U \to A$ satisfying $f(u(C_*)) = a(C_*)$ for any such $\mathbb{Z}G$-chain complex.

- Then $(K_1^w(\mathbb{Z}G), \rho_u^{(2)})$ is the universal additive L^2-torsion invariant.

- An additive L^2-torsion invariant (A, a) consists of an abelian group A and an assignment which associates to a finite based free $\mathbb{Z}G$-chain complex C_*, for which $C_*^{(2)}$ is L^2-acyclic, an element $a(C_*) \in A$ such that for any based exact short sequence of such $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ we get

$$a(D_*) = a(C_*) + a(E_*),$$

  and we have $a\bigl(\cdots \to 0 \to \mathbb{Z}G \xrightarrow{\pm \mathrm{id}} \mathbb{Z}G \to 0 \to \cdots\bigr) = 0$.

- We call an additive L^2-torsion invariant (U, u) universal if for every additive L^2-torsion invariant (A, a) there is precisely one group homomorphism $f\colon U \to A$ satisfying $f(u(C_*)) = a(C_*)$ for any such $\mathbb{Z}G$-chain complex.

- Then $(K_1^w(\mathbb{Z}G), \rho_u^{(2)})$ is the universal additive L^2-torsion invariant.

- An additive L^2-torsion invariant (A, a) consists of an abelian group A and an assignment which associates to a finite based free $\mathbb{Z}G$-chain complex C_*, for which $C_*^{(2)}$ is L^2-acyclic, an element $a(C_*) \in A$ such that for any based exact short sequence of such $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ we get

$$a(D_*) = a(C_*) + a(E_*),$$

and we have $a\bigl(\cdots \to 0 \to \mathbb{Z}G \xrightarrow{\pm \mathrm{id}} \mathbb{Z}G \to 0 \to \cdots\bigr) = 0$.

- We call an additive L^2-torsion invariant (U, u) universal if for every additive L^2-torsion invariant (A, a) there is precisely one group homomorphism $f \colon U \to A$ satisfying $f(u(C_*)) = a(C_*)$ for any such $\mathbb{Z}G$-chain complex.

- Then $(K_1^w(\mathbb{Z}G), \rho_u^{(2)})$ is the universal additive L^2-torsion invariant.

- The universal L^2-torsion is a simple homotopy invariant.
- It satisfies useful sum formulas and product formulas. There are also formulas for appropriate fibrations and S^1-actions.
- If G is finite, we rediscover essentially the classical Reidemeister torsion.
- We have $\rho^{(2}(\widetilde{S^1}) = (z-1)$ in $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^{\times}/\{\pm z^n \mid n \in \mathbb{Z}\}$.
- We have $\rho^{(2}(\widetilde{T^n}) = 0$ for $n \geq 2$.

- The universal L^2-torsion is a simple homotopy invariant.
- It satisfies useful sum formulas and product formulas. There are also formulas for appropriate fibrations and S^1-actions.
- If G is finite, we rediscover essentially the classical Reidemeister torsion.
- We have $\rho^{(2}(\widetilde{S^1}) = (z-1)$ in $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^{\times}/\{\pm z^n \mid n \in \mathbb{Z}\}$.
- We have $\rho^{(2}(\widetilde{T^n}) = 0$ for $n \geq 2$.

- The universal L^2-torsion is a simple homotopy invariant.
- It satisfies useful sum formulas and product formulas. There are also formulas for appropriate fibrations and S^1-actions.
- If G is finite, we rediscover essentially the classical Reidemeister torsion.
- We have $\rho^{(2}(\widetilde{S^1}) = (z-1)$ in $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^\times/\{\pm z^n \mid n \in \mathbb{Z}\}$.
- We have $\rho^{(2}(\widetilde{T^n}) = 0$ for $n \geq 2$.

Theorem (Jaco-Shalen-Johannson decomposition)

Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group whose boundary is empty or toroidal. Let $M_1, M_2, \ldots, M_r$ be its pieces in the Jaco-Shalen-Johannson decomposition. Let $j_i \colon \pi_1(M_i) \to \pi_1(M)$ be the injection induced by the inclusion $M_i \to M$.

Then each M_i and M are L^2-acyclic and we have

$$\rho_u^{(2)}(\widetilde{M}) = \sum_{i=1}^{r} (j_i)_*\big(\rho_u^{(2)}(\widetilde{M_i})\big).$$

- Many other invariants come from the universal L^2-torsion by applying a homomorphism $K_1^w(\mathbb{Z}G) \to A$ of abelian groups.
- For instance, the Fuglede-Kadison determinant defines a homomorphism

$$\det^{(2)} \colon \operatorname{Wh}^w(\mathbb{Z}G) \to \mathbb{R}$$

which maps the universal L^2-torsion $\rho_u^{(2)}(\overline{X})$ to the (classical) L^2-torsion $\rho^{(2)}(\overline{X})$.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]_{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If G is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]_{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.
- The Atiyah Conjecture implies for a torsionfree group G that the rational group ring has no non-trivial zero-divisors.
- Notice that the Farrell-Jones Conjecture implies for a torsionfree group G that the group ring over any field of characteristic zero has no non-trivial idempotents.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.
- The Atiyah Conjecture implies for a torsionfree group G that the rational group ring has no non-trivial zero-divisors.
- Notice that the Farrell-Jones Conjecture implies for a torsionfree group G that the group ring over any field of characteristic zero has no non-trivial idempotents.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.

- The Atiyah Conjecture implies for a torsionfree group G that the rational group ring has no non-trivial zero-divisors.

- Notice that the Farrell-Jones Conjecture implies for a torsionfree group G that the group ring over any field of characteristic zero has no non-trivial idempotents.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.*
   Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture, actually even over $\mathbb{C}$.
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This theorem and results by Waldhausen show for the fundamental group π of a 3-manifold (with the exception of some graph manifolds) that it satisfies the Atiyah Conjecture and that $\mathrm{Wh}(\pi)$ vanishes.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.*
   Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture, actually even over $\mathbb{C}$.
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This theorem and results by Waldhausen show for the fundamental group π of a 3-manifold (with the exception of some graph manifolds) that it satisfies the Atiyah Conjecture and that $\mathrm{Wh}(\pi)$ vanishes.

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.
- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.
- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.

- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.
- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Q}(z^{\pm 1})^\times$$

- It turns out that in the case $G = \mathbb{Z}$ the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by

$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$

- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion

$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g\in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

Twisting L^2-invariants

- Consider a *CW*-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

Twisting L^2-invariants

- Consider a *CW*-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by

$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$

- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion

$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g\in G} \lambda_g \cdot g \mapsto \sum_{g\in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

- Define ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

by sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

- Its value at $t = 1$ is just the L^2-torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

- Define ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

by sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

- Its value at $t = 1$ is just the L^2-torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_{\pi} \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

- Define ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

by sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

- Its value at $t = 1$ is just the L^2-torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

Theorem (Lück)

Suppose that $\widetilde{X}$ is L^2-acyclic.

1. *The L^2torsion function $\rho^{(2)} := \rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$ is well-defined.*
2. *The limits $\limsup_{t\to\infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\liminf_{t\to 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

   $$\deg(X;\phi) \in \mathbb{R}$$

   to be their difference.
3. *There is a ϕ-twisted Fuglede-Kadison determinant*

   $$\det^{(2)}_{\mathrm{tw},\phi}\colon K_1^w(\mathbb{Z}G) \to \mathrm{map}((0,\infty),\mathbb{R})$$

   which sends $\rho_u^{(2)}(\widetilde{X})$ to $\rho^{(2)}(\widetilde{X};\phi)$.

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_-(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle with connected closed surface $F \ncong S^2$ and $\phi = \pi_1(p)$, then

$$x_M(\phi) = -\chi(F).$$

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_-(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle with connected closed surface $F \ncong S^2$ and $\phi = \pi_1(p)$, then

$$x_M(\phi) = -\chi(F).$$

Theorem (Friedl-Lück, Liu)

Let M be a 3-manifold. Then for every $\phi \in H^1(M;\mathbb{Z})$ we get the equality

$$\deg(M;\phi) = x_M(\phi).$$

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

- The Newton polytope
$$N(p) \subseteq \mathbb{R}^n$$
of a polynomial
$$p(t_1, t_2, \ldots, t_n) = \sum_{i_1,\ldots,i_n} a_{i_1,i_2,\ldots,i_n} \cdot t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$
in n variables $t_1, t_2, \ldots, t_n$ is defined to be the convex hull of the elements $\{(i_1, i_2, \ldots i_n) \in \mathbb{Z}^n \mid a_{i_1,i_2,\ldots,i_n} \neq 0\}$;
- One has
$$N(p \cdot q) = N(p) + N(q).$$

- The Newton polytope

$$N(p) \subseteq \mathbb{R}^n$$

of a polynomial

$$p(t_1, t_2, \ldots, t_n) = \sum_{i_1, \ldots, i_n} a_{i_1, i_2, \ldots, i_n} \cdot t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

in n variables $t_1, t_2, \ldots, t_n$ is defined to be the convex hull of the elements $\{(i_1, i_2, \ldots i_n) \in \mathbb{Z}^n \mid a_{i_1, i_2, \ldots, i_n} \neq 0\}$;

- One has

$$N(p \cdot q) = N(p) + N(q).$$

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f \colon A \to A'$, we obtain a homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f) \colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f \colon A \to A'$, we obtain a homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f) \colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f\colon A \to A'$, we obtain a homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f)\colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f\colon A \to A'$, we obtain a homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f)\colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

  and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Example ($A = \mathbb{Z}$)

- An integral polytope in $\mathbb{Z}_{\mathbb{R}}$ is just an interval $[m, n]$ for $m, n \in \mathbb{Z}$ satisfying $m \leq n$.
- The Minkowski sum becomes $[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$.
- One obtains isomorphisms of abelian groups

$$\begin{array}{rcll} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^2 & [[m,n]] \mapsto (n-m, m). \\ \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}, & [[m,n]] \mapsto n-m. \end{array}$$

- We obtain an injection

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- It implies that $\mathcal{P}_{\mathbb{Z}}(A)$ is torsionfree and not divisible.
- Conjecturally $\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is always a free abelian group.
- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

We want to generalize it to the so called polytope homomorphism.

- We obtain an injection

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- It implies that $\mathcal{P}_{\mathbb{Z}}(A)$ is torsionfree and not divisible.

- Conjecturally $\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is always a free abelian group.

- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

We want to generalize it to the so called polytope homomorphism.

- We obtain an injection

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- It implies that $\mathcal{P}_{\mathbb{Z}}(A)$ is torsionfree and not divisible.
- Conjecturally $\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is always a free abelian group.
- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

We want to generalize it to the so called polytope homomorphism.

- We obtain an injection

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- It implies that $\mathcal{P}_{\mathbb{Z}}(A)$ is torsionfree and not divisible.
- Conjecturally $\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is always a free abelian group.
- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

We want to generalize it to the so called polytope homomorphism.

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

Polytope homomorphism

- Consider the projection
$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$
Let K be its kernel.
- After a choice of a set-theoretic section of pr we get isomorphisms
$$\begin{array}{rcl} \mathbb{Z}K * H_1(G)_f & \xrightarrow{\cong} & \mathbb{Z}G; \\ S^{-1}\bigl(\mathcal{D}(K) * H_1(G)_f\bigr) & \xrightarrow{\cong} & \mathcal{D}(G), \end{array}$$
where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H_1(M;\mathbb{R}).$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H_1(M;\mathbb{R}).$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H_1(M; \mathbb{R}).$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\mathrm{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\mathrm{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H_1(M;\mathbb{R}).$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- The composite

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times \xrightarrow{\cong} \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^\times \xrightarrow{P'} \mathcal{P}_{\mathbb{Z}}(H_1(G)_f)$$

factories to the polytope homomorphism

$$P\colon \operatorname{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z},\operatorname{Wh}}(H_1(G)_f).$$

Definition (Dual Thurston polytope)

Let M be a 3-manifold. Define the dual Thurston polytope to be subset of $H_1(M;\mathbb{R})$

$$T(M) := \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

- Thurston has shown that the dual Thurston polytope is always an integral polytope.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)\}.$$

Definition (Dual Thurston polytope)

Let M be a 3-manifold. Define the dual Thurston polytope to be subset of $H_1(M;\mathbb{R})$

$$T(M) := \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

- Thurston has shown that the dual Thurston polytope is always an integral polytope.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)\}.$$

Definition (Dual Thurston polytope)

Let M be a 3-manifold. Define the dual Thurston polytope to be subset of $H_1(M;\mathbb{R})$

$$T(M) := \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \le x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

- Thurston has shown that the dual Thurston polytope is always an integral polytope.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)\}.$$

Definition (Dual Thurston polytope)

Let M be a 3-manifold. Define the dual Thurston polytope to be subset of $H_1(M;\mathbb{R})$

$$T(M) := \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

- Thurston has shown that the dual Thurston polytope is always an integral polytope.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)\}.$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then the image of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ under the polytope homomorphism

$$P\colon \operatorname{Wh}^w(\pi_1(M)) \to \mathcal{P}_{\mathbb{Z},\operatorname{Wh}}(H_1(\pi_1(M))_f)$$

is represented by the dual of the Thurston polytope.

Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey and Cochran, provided that G occurs in the rational derived series of $\pi_1(M)$.
- At least the degree of these polynomials is a well-defined invariant of M and G.
- We can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey and Cochran, provided that G occurs in the rational derived series of $\pi_1(M)$.
- At least the degree of these polynomials is a well-defined invariant of M and G.
- We can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.
- It has nice properties, e.g., it depends only on the conjugacy class of f, satisfies a sum formula and a formula for exact sequences.
- If G is amenable, it vanishes.
- If G is the fundamental group of a compact surface F and f comes from an automorphism $a\colon F \to F$, then T_f is a 3-manifold and a lot of the material above applies.
- For instance, if a is irreducible, $\rho_u^{(2)}(f)$ detects whether a is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

Let

$$P(f) \in \mathcal{P}_{\mathbb{Z}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}_{\mathbb{Z}}(\mathrm{pr})} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

  Let

$$P(f) \in \mathcal{P}_{\mathbb{Z}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

  be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}_{\mathbb{Z}}(\mathrm{pr})} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest.

L^2-Euler characteristic

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its L^2-Euler characteristic

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0}(-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

L^2-Euler characteristic

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its L^2-Euler characteristic

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} (-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f}; \phi) = \chi(X).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\widetilde{M}; \phi) = x_M(\phi).$$

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f};\phi) = \chi(X).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\widetilde{M};\phi) = x_M(\phi).$$

- Suppose that G is torsionfree and satisfies the Atiyah Conjecture. Consider $\phi\colon G \to \mathbb{Z}$.

  Then there is a homomorphism

  $$\chi_\phi^{(2)}\colon \mathrm{Wh}^w(G) \to \mathbb{Z}$$

  which sends the universal L^2-torsion $\rho_u^{(2)}(\widetilde{X})$ to $\chi^{(2)}(\widetilde{X};\phi)$.