

Survey on approximating L^2 -invariants

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Bonn, December 2016

- I will assume that the audience is familiar with the basic notions and facts about L^2 -Betti-numbers, Fuglede-Kadison determinants, and L^2 -torsion.
- It is possible to follow the talk using the black box principle.
- Most of the material of this talk is taken from my survey article “*Survey on approximating L^2 -invariants by their classical counterparts: Betti numbers, torsion invariants and homological growth*”, arXiv:1501.07446 [math.GT], to appear in EMSS.
- The talk is thought as a **teaser** for this paper and will contain no references since they all can be found in this paper.

- Let G be a (discrete) group together with a (normal exhausting) chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

such that G_i is normal in G , the index $[G : G_i]$ is finite and $\bigcap_{i \geq 0} G_i = \{1\}$.

- Let $p: \bar{X} \rightarrow X$ be a G -covering. Put $X[i] := G_i \backslash \bar{X}$.
- We obtain a $[G : G_i]$ -sheeted covering $p[i]: X[i] \rightarrow X$. Its total space $X[i]$ inherits the structure of a finite CW-complex, a closed manifold or a closed Riemannian manifold respectively if X has such a structure

Basic problem

- Let α be a classical topological invariant such as the **Euler characteristic**, the **signature**, the **n th Betti number** with coefficients in the field \mathbb{Q} or \mathbb{F}_p , **torsion** in the sense of **Reidemeister** or **Ray-Singer**, or the **minimal number of generators** of the fundamental group.
- We want to study the sequence

$$\left(\frac{\alpha(X[i])}{[G : G_i]} \right)_{i \geq 0}.$$

Problem (**Approximation Problem**)

- 1 *Does the sequence converge?*
- 2 *If yes, is the limit independent of the chain?*
- 3 *If yes, what is the limit?*

- The hope is that the answer to the first two questions is yes and the limit turns out to be an L^2 -analogue $\alpha^{(2)}$ of α applied to the G -space \bar{X} , i.e., one can prove an equality of the type

$$\lim_{i \rightarrow \infty} \frac{\alpha(X[i])}{[G : G_i]} = \alpha^{(2)}(\bar{X}; \mathcal{N}(G)).$$

- Here $\mathcal{N}(G)$ stands for the group von Neumann algebra and is a reminiscence of the fact that the G -action on \bar{X} plays a role.
- The equality above is often used to compute the L^2 -invariant $\alpha^{(2)}(\bar{X}; \mathcal{N}(G))$ by its finite-dimensional analogues $\alpha(X[i])$. On the other hand, it implies the existence of finite coverings with large $\alpha(X[i])$, if $\alpha^{(2)}(\bar{X}; \mathcal{N}(G))$ is known to be positive.

Euler characteristic and signature

- The Approximation Problem has a positive answer for the Euler characteristic for the trivial reason that it is multiplicative under finite coverings.
- This applies also to the signature for smooth and topological manifolds.
- The signature for finite Poincaré complexes is not multiplicative under finite covering but nevertheless the Approximation Problem has a positive answer also in this setting.

Conjecture (Approximation in zero and prime characteristic for Betti numbers)

We get

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]} = b_n^{(2)}(\bar{X}; \mathcal{N}(G))$$

for all fields F and $n \geq 0$, provided that X is finite, and \bar{X} is contractible, or, equivalently, that X is aspherical, $G = \pi_1(X)$ and \bar{X} is the universal covering \tilde{X} .

- If the characteristic is zero, the conjecture above holds without the assumption contractible.
- The assumption that \overline{X} is contractible is in general necessary in the conjecture above if the characteristic is not zero.
- The following conjecture implies the one above.

Conjecture (Growth of number of generators of the homology)

We get for d the minimal number of generators:

$$\lim_{i \rightarrow \infty} \frac{d(H_n(X[i]; \mathbb{Z}))}{[G : G_i]} = b_n^{(2)}(\bar{X}; \mathcal{N}(G)),$$

provided that \bar{X} is contractible.

Question (Rank gradient, cost, first L^2 -Betti number and approximation)

Let G be a finitely presented group and let F be any field.
When do we have

$$\lim_{i \rightarrow \infty} \frac{b_1(G_i; F) - 1}{[G : G_i]} = b_1^{(2)}(G) - b_0^{(2)}(G) = \text{cost}(G) - 1 = \text{RG}(G; (G_i)_{i \geq 0}),$$

where $\text{cost}(G)$ denotes the *cost* and $\text{RG}(G; (G_i)_{i \geq 0})$ the *rank gradient*?

- The answer is yes if G contains an infinite normal amenable subgroup, since then all invariants are zero.
- All conditions appearing in the conjecture above are necessary.

Truncated Euler characteristic

- Let d be a natural number and let X be a space. Denote by $\mathcal{CW}_d(X)$ the set of CW -complexes Y which have a finite d -skeleton Y_d and are homotopy equivalent to X .
- Provided that $\mathcal{CW}_d(X)$ is not empty, define the d -th truncated Euler characteristic of X by

$$\chi_d^{\text{trun}}(X) := \begin{cases} \min\{\chi(Y_d) \mid Y \in \mathcal{CW}_d(X)\} & \text{if } d \text{ is even;} \\ \max\{\chi(Y_d) \mid Y \in \mathcal{CW}_d(X)\} & \text{if } d \text{ is odd,} \end{cases}$$

where $\chi(Y_d)$ is the Euler characteristic of the d -skeleton Y_d of Y .

Question (Asymptotic Morse equality for groups)

Let G be a group and let d be a natural number such that $\mathcal{CW}_d(BG)$ is not empty. When do we have

$$\lim_{i \rightarrow \infty} \frac{\chi_d^{\text{trun}}(BG_i)}{[G : G_i]} = \sum_{n=0}^d (-1)^n \cdot b_n^{(2)}(G)?$$

- The question above is in the case $d = 1$ precisely the question about rank gradient, cost, first L^2 -Betti number and approximation, since a group H is finitely generated if and only if there is a model for BH with finite 1-skeleton and in this case $\chi_1^{\text{trun}}(H) = 1 - d(H)$.

- In the case $d = 2$ the question above can be rephrased as the question, for a finitely presented group G , when do we have

$$\lim_{i \rightarrow \infty} \frac{1 - \text{def}(G_i)}{[G : G_i]} = b_2^{(2)}(G) - b_1^{(2)}(G) + b_0^{(2)}(G),$$

where $\text{def}(H)$ denotes for a finitely presented group H its **deficiency**, i.e., the maximum over the numbers $g - r$ for all finite presentations $H = \langle s_1, s_2, \dots, s_g \mid R_1, R_2, \dots, R_r \rangle$.

- Let G be a group with a finite model for BG . It is not hard to show that conjecture about approximation in zero and prime characteristic for Betti numbers is true for G , provided that the answer to the question above is positive for all $d \geq 0$.

Speed of convergence

- The speed of convergence of

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]} = b_n^{(2)}(\tilde{X})$$

(if it converges) can be arbitrarily slow for one chain and very fast for another chain in a sense that we shall now explain.

- Fix a prime p and two functions $F^s, F^f: \{i \in \mathbb{Z} \mid i \geq 1\} \rightarrow (0, \infty)$ such that

$$\lim_{i \rightarrow \infty} F^s(i) = 0;$$

$$\lim_{i \rightarrow \infty} F^f(i) = 0;$$

$$\lim_{i \rightarrow \infty} i \cdot F^f(i) = \infty.$$

Theorem

For every integer $n \geq 1$, there is a closed $(2n + 1)$ -dimensional Riemannian manifold X with non-positive sectional curvature and two chains $(G_i^s)_{i \geq 0}$ and $(G_i^f)_{i \geq 0}$ for $G = \pi_1(X)$ such that G_i^s and G_i^f are normal subgroups of G of finite p -power index, the intersections $\bigcap_{i \geq 0} G_i^s$ and $\bigcap_{i \geq 0} G_i^f$ are trivial, and we have for every field F

$$\lim_{i \rightarrow \infty} \frac{b_n(X^s[i]; F)}{[G : G_i^s]} = b_n^{(2)}(\tilde{X}) = 0;$$

$$\frac{b_n(X^s[i]; F)}{[G : G_i^s]} \geq F^s([G : G_i^s]);$$

$$\lim_{i \rightarrow \infty} \frac{b_n(X^f[i]; F)}{[G : G_i^f]} = b_n^{(2)}(\tilde{X}) = 0;$$

$$\frac{b_n(X^f[i]; F)}{[G : G_i^f]} \leq F^f([G : G_i^f]).$$

- The theorem above implies that one can find for any $0 < \epsilon$ two chains $(G_i^s)_{i \geq 0}$ and $(G_i^f)_{i \geq 0}$ satisfying

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i^s]} = 0;$$

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i^s]^{1-\epsilon}} = \infty;$$

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i^f]^\epsilon} = 0,$$

since we can take $F^s(i) = i^{-\epsilon/2}$ and $F^f(i) = i^{\epsilon/2-1}$.

Conjecture (Homological growth and L^2 -torsion for aspherical closed manifolds)

Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Let \tilde{M} be its universal covering. Then

- For any natural number n with $2n \neq d$ we get

$$b_n^{(2)}(\tilde{M}) = 0.$$

If $d = 2n$, we have

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\tilde{M}) \geq 0.$$

If $d = 2n$ and M carries a Riemannian metric of negative sectional curvature, then

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\tilde{M}) > 0;$$

Conjecture (continued)

- Let $(G_i)_{i \geq 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G : G_i]$ and trivial intersection $\bigcap_{i \geq 0} G_i = \{1\}$.
Then we get for any natural number n and any field F

$$b_n^{(2)}(\tilde{M}) = \lim_{i \rightarrow \infty} \frac{b_n(M[i]; F)}{[G : G_i]} = \lim_{i \rightarrow \infty} \frac{d(H_n(M[i]; \mathbb{Z}))}{[G : G_i]};$$

and for $n = 1$

$$\begin{aligned} b_1^{(2)}(\tilde{M}) &= \lim_{i \rightarrow \infty} \frac{b_1(M[i]; F)}{[G : G_i]} = \lim_{i \rightarrow \infty} \frac{d(G_i/[G_i, G_i])}{[G : G_i]} \\ &= RG(G, (G_i)_{i \geq 0}) = \begin{cases} 0 & d \neq 2; \\ -\chi(M) & d = 2; \end{cases} \end{aligned}$$

Conjecture (continued)

- We get for the truncated Euler characteristic $m \geq 0$

$$\lim_{i \rightarrow \infty} \frac{\chi_m^{\text{trun}}(M[i])}{[G : G_i]} = \begin{cases} \chi(M) & \text{if } d \text{ is even and } 2m \geq d; \\ 0 & \text{otherwise;} \end{cases}$$

- If $d = 2n + 1$ is odd, we have

$$(-1)^n \cdot \rho_{\text{an}}^{(2)}(\tilde{M}) \geq 0;$$

If $d = 2n + 1$ is odd and M carries a Riemannian metric with negative sectional curvature, we have

$$(-1)^n \cdot \rho_{\text{an}}^{(2)}(\tilde{M}) > 0;$$

Conjecture (continued)

- Let $(G_i)_{i \geq 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G : G_i]$ and trivial intersection $\bigcap_{i \geq 0} G_i = \{1\}$. Put $M[i] = G_i \backslash \tilde{M}$.

Then we get for any natural number n with $2n + 1 \neq d$

$$\lim_{i \rightarrow \infty} \frac{\ln (|\text{tors}(H_n(M[i]))|)}{[G : G_i]} = 0,$$

and we get in the case $d = 2n + 1$

$$\lim_{i \rightarrow \infty} \frac{\ln (|\text{tors}(H_n(M[i]))|)}{[G : G_i]} = (-1)^n \cdot \rho_{\text{an}}^{(2)}(\tilde{M}) \geq 0.$$

Example (Hyperbolic 3-manifolds)

Suppose that M is a closed hyperbolic 3-manifold. Then $\rho_{\text{an}}(\tilde{M})$ is known to be $-\frac{1}{6\pi} \cdot \text{vol}(M)$, hence the conjecture above predicts

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_1(G_i))|)}{[G : G_i]} = \frac{1}{6\pi} \cdot \text{vol}(M).$$

- Since the volume is always positive, the equation above implies that $|\text{tors}(H_1(G_i))|$ growth exponentially in $[G : G_i]$.

Definition (Integral torsion)

Define for a finite \mathbb{Z} -chain complex D_* its **integral torsion**

$$\rho^{\mathbb{Z}}(D_*) := \sum_{n \geq 0} (-1)^n \cdot \ln (|\text{tors}(H_n(D_*))|) \in \mathbb{R},$$

where $|\text{tors}(H_n(D_*))|$ is the order of the torsion subgroup of the finitely generated abelian group $H_n(D_*)$.

Given a finite CW-complex X , define its **integral torsion**

$$\rho^{\mathbb{Z}}(X) \in \mathbb{R}$$

by $\rho^{\mathbb{Z}}(C_*(X))$, where $C_*(X)$ is its cellular \mathbb{Z} -complex.

Conjecture (Approximation Conjecture for integral torsion)

Let X be a finite connected CW-complex. Suppose that $b_n^{(2)}(\bar{X}; \mathcal{N}(G))$ vanishes for all $n \geq 0$. Then

$$\rho_{\text{top}}^{(2)}(\bar{X}; \mathcal{N}(G)) = \lim_{i \rightarrow \infty} \frac{\rho^{\mathbb{Z}}(X[i])}{[G : G_i]}.$$

- Notice that in the conjecture above the assumption aspherical does not appear.

- There are elementary examples on the level of chain complexes which show that there is no relationship between the differentials and homology in each degree. The equality has only to be a chance to be true after one passes to the alternating sum.
- The condition of L^2 -acyclicity appearing in the Approximation Conjecture for integral torsion is necessary.

Theorem

Let C_* and D_* be two finite free $\mathbb{Z}G$ -chain complexes. Suppose that $C_* \otimes_{\mathbb{Z}} \mathbb{Q}$ and $D_* \otimes_{\mathbb{Z}} \mathbb{Q}$ are $\mathbb{Q}G$ -chain homotopy equivalent and that $C_*^{(2)}$ is L^2 -acyclic. Then $D_*^{(2)}$ is L^2 -acyclic and

$$\rho^{(2)}(D_*^{(2)}) - \rho^{(2)}(C_*^{(2)}) = \lim_{i \in I} \frac{\rho^{\mathbb{Z}}(D[i]_*) - \rho^{\mathbb{Z}}(C[i]_*)}{[G : G_i]}.$$

Theorem

Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1 -action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$ and fields F

$$\begin{aligned}\lim_{i \rightarrow \infty} \frac{b_n(M[i]; F)}{[G : G_i]} &= 0; \\ \lim_{i \rightarrow \infty} \frac{d(H_n(M[i]; \mathbb{Z}))}{[G : G_i]} &= 0; \\ \lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_n(M[i]))|)}{[G : G_i]} &= 0; \\ \lim_{i \rightarrow \infty} \frac{\rho_{\text{an}}(M[i]; \mathcal{N}(\{1\}))}{[G : G_i]} &= 0; \\ \lim_{i \rightarrow \infty} \frac{\rho^{\mathbb{Z}}(M[i])}{[G : G_i]} &= 0;\end{aligned}$$

Theorem (Continued)

$$b_n^{(2)}(\tilde{M}) = 0;$$
$$\rho_{\text{an}}^{(2)}(\tilde{M}) = 0.$$

In particular many of the conjectures above are known to be true for $G = \pi_1(M)$ and $X = M$.

The Approximation Conjecture for Fuglede-Kadison determinants

- Consider a \mathbb{C} -homomorphism of finite-dimensional Hilbert spaces $f: V \rightarrow W$. It induces an endomorphism $f^*f: V \rightarrow V$. We have $\ker(f) = \ker(f^*f)$. Denote by $\ker(f)^\perp$ the orthogonal complement of $\ker(f)$. Then f^*f induces an automorphism $(f^*f)^\perp: \ker(f)^\perp \rightarrow \ker(f)^\perp$. Define

$$\det'(f) := \sqrt{\det((f^*f)^\perp)}.$$

- If $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ are the non-zero eigenvalues (listed with multiplicity) of the positive operator $f^*f: V \rightarrow V$, then

$$\det'(f) = \prod_{j \geq 1} \sqrt{\lambda_j}.$$

- If f is an isomorphism, then $\det'(f)$ reduces to $\sqrt{\det(f^*f)}$.

Conjecture (Approximation Conjecture for Fuglede-Kadison determinants)

Consider a matrix $A \in M_{r,s}(\mathbb{Q}G)$. Then we get

$$\ln(\det_{\mathcal{N}(G)}^{(2)}(r_A^{(2)})) = \lim_{i \rightarrow \infty} \frac{\ln(\det'(r_{A[i]}^{(2)}))}{[G : G_i]}.$$

- The conjecture above does not hold if one replaces \mathbb{Q} by \mathbb{C} .
- The only non-trivial torsionfree group, for which is known, is $G = \mathbb{Z}$.

The Approximation Conjecture for L^2 -torsion

Conjecture (Approximation Conjecture for analytic torsion)

Let X be a closed Riemannian manifold. Then

$$\rho_{\text{an}}^{(2)}(\bar{X}; \mathcal{N}(G)) = \lim_{i \rightarrow \infty} \frac{\ln(\rho_{\text{an}}(X[i]))}{[G : G_i]}.$$

- There are topological counterparts which we will denote by $\rho_{\text{top}}(X[i])$ and $\rho_{\text{top}}^{(2)}(\bar{X})$ which agree with their analytic versions. So the conjecture above is equivalent to its topological counterpart.

Theorem (Relating the Approximation Conjectures for Fuglede-Kadison determinant and torsion invariants)

Suppose that X is a closed Riemannian manifold such that $b_n^{(2)}(\bar{X}, \mathcal{N}(G))$ vanishes for all $n \geq 0$.

If G satisfies the Approximation Conjecture for Fuglede-Kadison determinants, then the Approximation Conjecture for analytic (or topological) torsion holds for X .

- Notice that Approximation Conjecture for Fuglede-Kadison determinants is just a statement about matrices, whereas the Approximation Conjecture for analytic torsion involve Riemannian manifolds.

Dropping the assumption finite index

- In many of the conjectures have analogues in the more general case, where one can drop the condition such that $[G : G_i]$ is finite. For instance:

Conjecture (Approximation Conjecture for Fuglede-Kadison determinants)

For any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant

$$\begin{aligned} \det_{\mathcal{N}(G)}(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s) \\ = \lim_{i \rightarrow \infty} \det_{\mathcal{N}(G/G_i)}(r_{A[i]}^{(2)} : L^2(G/G_i)^r \rightarrow L^2(G/G_i)^s), \end{aligned}$$

Strategy for a proof for the Approximation Conjecture for Fuglede-Kadison determinants

Theorem

The conjecture above is true if the *uniform integrability condition* is satisfied, i.e., there exists $\epsilon > 0$ such that

$$\int_{0+}^{\epsilon} \sup \left\{ \frac{F[i](\lambda) - F[i](0)}{\lambda} \mid i \in I \right\} d\lambda < \infty.$$

where $F[i]$ is the spectral density function of $r_{A[i]}^{(2)} : L^2(G/G_i)^r \rightarrow L^2(G/G_i)^s$.

- This condition is **not** automatically satisfied by the following result.

Lemma

There is a sequence of functions $f_n: [0, 1] \rightarrow [0, 1]$

- The function $f_n(\lambda)$ is monotone non-decreasing and continuous for $n \geq 1$;
- $f_n(0) = 0$ and $f_n(1) = 1$ for all $n \geq 1$;
- $\lim_{n \rightarrow \infty} f_n(\lambda) = \lambda$ for $\lambda \in [0, 1]$;
- We have for all $n \geq 1$ and $\lambda \in [0, 1)$

$$\lambda \leq f_n(\lambda) \leq \frac{1}{-\ln(\lambda)} + \lambda \leq \frac{2}{-\ln(\lambda)};$$

- We have for $\lambda \in [0, e^{-1}]$

$$\sup\{f_n(\lambda) \mid n \geq 0\} = \frac{1}{-\ln(\lambda)} + \lambda;$$

Lemma (Continued)

- We have

$$\int_{0+}^1 \frac{\sup\{f_n(\lambda) \mid n \geq 0\}}{\lambda} d\lambda = \infty;$$

- We get for all $n \geq 1$

$$\int_{0+}^1 \frac{f_n(\lambda)}{\lambda} d\lambda \geq \ln(2) + 1;$$

- We have

$$\int_{0+}^1 \lim_{n \rightarrow \infty} \frac{f_n(\lambda)}{\lambda} d\lambda < \liminf_{n \rightarrow \infty} \int_{0+}^1 \frac{f_n(\lambda)}{\lambda} d\lambda \leq \limsup_{n \rightarrow \infty} \int_{0+}^1 \frac{f_n(\lambda)}{\lambda} d\lambda;$$

- We get for all $n \geq 1$

$$\int_{0+}^1 \frac{f_n(\lambda)}{\lambda} d\lambda \leq 4.$$

- The sequence of functions $(f_n)_{n \geq 0}$ has an exotic behaviour close to zero. It is very unlikely that such a sequence $(f_n)_{n \geq 0}$ actually occurs as the sequence of spectral density functions of the manifolds $G_i \backslash M$ for some smooth manifold M with proper free cocompact G -action and G -invariant Riemannian metric.
- The example above shows that we need to have more information about such sequences of spectral density functions using the fact that they come from geometry.

Theorem

Suppose that there exists constants $C > 0$, $0 < \epsilon < 1$ and $\delta > 0$ independent of i such that for all λ with $0 < \lambda \leq \epsilon$ and all $i \in I$ we have

$$F[i](\lambda) - F[i](0) \leq \frac{C}{(-\ln(\lambda))^{1+\delta}}.$$

Then the uniform integrability condition appearing above is satisfied.

- One always has in the geometric situation

$$F[i](\lambda) - F[i](0) \leq \frac{C}{-\ln(\lambda)}.$$

- Probably such $\delta > 0$ exists but will depend on the concrete manifold we are looking at.