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L^2-invariants: n years later

Mathematics – Monograph (English)

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Preface

This monograph can be thought of as part II or of a continuation of the book [18]. Since n **Comment 1 (by W.)**: This number has to be specified at the very end. Could it be 28? years have passed and many new interesting developments have occurred and a lot of new results have been achieved during these years, the time is ripe to give an update.

Comment 2 (by W.): Needs to be completed.

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Contents

Chapter 1
Introduction

1.1 The Organization of the Book and a User's Guide

We have written the text in a way such that one can read small units, e.g., a single chapter, independently from the rest, concentrate on certain aspects, and extract easily and quickly specific information. We hopefully have found the right mixture between definitions, theorems, examples, and remarks so that reading the book is entertaining and illuminating. We have successfully used parts of this book, sometimes a single chapter, for seminars, reading courses, and advanced lecture courses.

Each section comes with a review where we briefly recollect the basics needed from [18].

Comment 3 (by W.): Needs to be completed.

1.1.1 Supplement

The book contains a number of exercises. They are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text or improve her or his understanding by trying to solve the exercises. Hints to the solutions of the exercises are given in Chapter 9.

If one wants to find a specific topic, the extensive index of the monograph can be used to find the right spot for a specific topic. The index contains an item "Theorem", under which all theorems with their names appearing in the book are listed, and analogously there is an item "Conjecture".

Comment 4 (by W.): Needs to be completed.

1.1.2 Prerequisites

1.2 Notations and Conventions

We have tried to keep the notation consistent with [18].

Here is a briefing on our main conventions and notations. Details are of course discussed in the text.

- Ring will mean (not necessarily commutative) associative ring with unit unless explicitly stated otherwise;
- Module means always left module unless explicitly stated otherwise;

- Group means discrete group unless explicitly stated otherwise;
- We will always work in the category of compactly generated spaces, compare [24] and [26, I.4]. In particular every space is automatically Hausdorff;
- We use the standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\widehat{\mathbb{Z}_p}$, and $\widehat{\mathbb{Q}_p}$ for the integers, the rational numbers, the real numbers, the complex numbers, the p-adic numbers, and the p-adic rationals. Given a real number $r \in \mathbb{R}$ we write $\mathbb{R}_{\geq r} = \{s \in \mathbb{R} \mid s \geq r_0\}$. It is now selfexplanatory what the notions $\mathbb{R}_{>r}$, $\mathbb{R}_{\leq r}$ $\mathbb{R}_{<r}$ $\mathbb{Q}_{\geq r}$ $\mathbb{Q}_{>r}$, $\mathbb{Q}_{\leq r}$ $\mathbb{Q}_{<r}$ $\mathbb{Z}_{\geq r}$ $\mathbb{Z}_{>r}$, $\mathbb{Z}_{\leq r}$, and $\mathbb{Z}_{<r}$ mean. We also write $\mathbb{N} = \mathbb{Z}_{\geq 0}$ for the set $\{0, 1, 2, \ldots\}$ of natural numbers;
- We denote certain groups by:

symbol	name
$\mathbb{Z}/n$	finite cyclic group of order n
S_n	symmetric group of permutations of the set $\{1, 2, \ldots n\}$
A_n	alternating group of even permutations of the set $\{1, 2, \ldots, n\}$
D_∞	infinite dihedral group
D_{2n}	dihedral group of order $2n$

1.3 Acknowledgments

1.4 Notes

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Chapter 2
The Atiyah Conjecture

2.1 Introduction

2.2 Review

2.2.1 The Group von Neumann algebra

Let G be a (discrete) group. Let $l^2(G)$ be the (complex) Hilbert space of square summable formal sums $\sum_{g\in G}\lambda_g\cdot g$ for $\lambda_g\in\mathbb{C}$. The *group von Neumann algebra* $\mathcal{N}(G)$ is the $\mathbb{C}$-algebra $\mathcal{B}(l^2(G))$ of bounded G-equivariant operators $l^2(G)\to l^2(G)$. We will consider $\mathcal{N}(G)$ just as a ring with unit and ignore the topology on it. It becomes a ring with involution $*\colon\mathcal{N}(G)\to\mathcal{N}(G)$ by taking the adjoint of a bounded G-equivariant operator $l^2(G)\to l^2(G)$. **Comment 5 (by T.)**: We should specify once (best here in the def) our left/right convention and stick to it. **Comment 6 (by W.)**: At least we say in Section 1.2 that we work with left modules. Does this suffice? Otherwise we should say that we use the conventions of [18].

The so-called *von Neumann trace*

$$\operatorname{tr}_{\mathcal{N}(G)}\colon\mathcal{N}(G)\to\mathbb{C} \tag{2.1}$$

sends a bounded G-equivariant operator $f\colon l^2(G)\to l^2(G)$ to $\langle f(e),e\rangle_{l^2(G)}$ for $e\in G$ the unit element. This extends to matrices by the usual formula

$$\operatorname{tr}_{\mathcal{N}(G)}\colon\ \mathrm{M}_n(\mathcal{N}(G))\to\mathbb{C},\quad A\mapsto\sum_{i=1}^{n}\operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}). \tag{2.2}$$

Given a finitely generated projective $\mathcal{N}(G)$-module P, define its *Hattori-Stallings rank* or its *von Neumann dimension* to be

$$\dim_{\mathcal{N}(G)}(P):=\sum\operatorname{tr}_{\mathcal{N}(G)}(A)\in\mathbb{R}_{\geq 0}, \tag{2.3}$$

where A is an element in $\mathrm{M}_n(\mathcal{N}(G))$ with $A^2=A$ such that the image of the $\mathcal{N}(G)$-linear map $r_A\colon\mathcal{N}(G)^n\to\mathcal{N}(G)^n$ induced by right multiplication with A is $\mathcal{N}(G)$-isomorphic to P. One easily checks that this definition is independent of the choice of n and A and takes values in $\mathbb{R}_{\geq 0}$.

Exercise 2.4 Show that definition (2.3) is independent of the choice of n and A and takes values in $\mathbb{R}_{\geq 0}$.

Remark 2.5. The category of finitely generated projective $\mathcal{N}(G)$-modules is equivalent to the category of finitely generated Hilbert $\mathcal{N}(G)$-modules and under this identification the von Neumann dimension corresponds to the classical Murray-von Neumann dimension of finitely generated Hilbert $\mathcal{N}(G)$-modules which is defined in operator theoretic terms, see [18, Theorem 6.24 on page 249].

2.2.2 Dimension Theory

Next we review how one can extend the dimension $\dim_{\mathcal{N}(G)}$ for finitely generated projective $\mathcal{N}(G)$-modules of (2.3) to arbitrary $\mathcal{N}(G)$-modules.

Definition 2.6 (Closure of a submodule)**.** Let R be a ring. Consider an R-submodule M of the R-module N. Define the *closure* of M in N to be the R-submodule of N

$$\overline{M} = \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* = \hom_R(N, R) \text{ satisfying } M \subset \ker(f)\}.$$

For an R-module M define the R-submodule $\mathbf{T}M$ and the quotient R-module $\mathbf{P}M$ by

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \end{aligned}$$

Definition 2.7 (Extended dimension)**.** We define for a $\mathcal{N}(G)$-module M its *extended dimension*

$$\dim'_{\mathcal{N}(G)}(M) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$$

to be

$$\sup\{\dim_{\mathcal{N}(G)}(P) \mid P \subset M \text{ finitely generated projective submodule}\}.$$

(We will later drop the prime in $\dim'_{\mathcal{N}(G)}$, see Notation 2.9.)

Theorem 2.8 (Dimension function for arbitrary $\mathcal{N}(G)$-modules) *Let G be a group.*

(i) The ring $\mathcal{N}(G)$ is semihereditary*, i.e., any finitely generated submodule of a projective module is projective;*

(ii) For every finitely generated $\mathcal{N}(G)$-module M there exists $n \in \mathbb{Z}_{\geq 0}$ and an $\mathcal{N}(G)$-homomorphism $f\colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ and an exact sequence of $\mathcal{N}(G)$-modules

$$0 \to \mathcal{N}(G)^n \xrightarrow{f^*f} \mathcal{N}(G)^n \to M \to 0;$$

(iii) If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(G)$-module M, then $M/\overline{K}$ is finitely generated projective and $\overline{K}$ is a direct summand in M;

(iv) If M is a finitely generated $\mathcal{N}(G)$-module, then $\mathbf{P}M$ is finitely generated projective and

$$M \cong \mathbf{P}M \oplus \mathbf{T}M;$$

(v) The dimension $\dim'_{\mathcal{N}(G)}$ has the following properties:

(a) Extension Property

If M is a finitely generated projective $\mathcal{N}(G)$-module, then

$$\dim'_{\mathcal{N}(G)}(M) = \dim_{\mathcal{N}(G)}(M);$$

(b) Additivity

If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of $\mathcal{N}(G)$-modules, then

$$\dim'_{\mathcal{N}(G)}(M_1) = \dim_{\mathcal{N}(G)}(M_0) + \dim_{\mathcal{N}(G)}(M_2),$$

where for $r, s \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$ we define $r + s$ by the ordinary sum of two real numbers if both r and s are not ∞, and by ∞ otherwise;

(c) Cofinality

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e., $M = \bigcup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then

$$\dim'_{\mathcal{N}(G)}(M) = \sup\{\dim'_{\mathcal{N}(G)}(M_i) \mid i \in I\};$$

(d) Continuity

If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(G)$-module M, then

$$\dim'_{\mathcal{N}(G)}(K) = \dim'_{\mathcal{N}(G)}(\overline{K});$$

(e) If M is a finitely generated R-module, then

$$\dim'_{\mathcal{N}(G)}(M) = \dim_{\mathcal{N}(G)}(\mathbf{P}M);$$
$$\dim'_{\mathcal{N}(G)}(\mathbf{T}M) = 0;$$

(f) Uniqueness

The dimension $\dim'$ is uniquely determined by the Extension Property, Additivity, Cofinality, and Continuity.

Proof. See [18, Theorem 6.5 and Theorem 6.7 on page 239 and Lemma 6.28 on page 252]. □

Notation 2.9. In view of Theorem 2.8 we will not distinguish between $\dim'_{\mathcal{N}(G)}$ and $\dim_{\mathcal{N}(G)}$ in the sequel.

Exercise 2.10 Let P be a finitely generated projective $\mathcal{N}(G)$-module and let M be an arbitrary $\mathcal{N}(G)$-module. Prove:

(i) $P = 0 \Longleftrightarrow \dim_{\mathcal{N}(G)}(P) = 0$;
(ii) We have $\dim_{\mathcal{N}(G)}(M) = 0$ if and only if every finitely generated projective $\mathcal{N}(G)$-submodule of M is trivial;

Theorem 2.11 (Dimension and colimits) *Let $\{M_i \mid i \in I\}$ be a directed system of $\mathcal{N}(G)$-modules over the directed set I. For $i \leq j$ let $\phi_{i,j} \colon M_i \to M_j$ be the associated morphism of $\mathcal{N}(G)$-modules. For $i \in I$ let $\psi_i \colon M_i \to \operatorname{colim}_{i \in I} M_i$ be the canonical morphism of $\mathcal{N}(G)$-modules.*

(i) We get for the dimension of the $\mathcal{N}(G)$-module given by the colimit $\operatorname{colim}_{i \in I} M_i$

$$\dim_{\mathcal{N}(G)}\bigl(\operatorname{colim}_{i \in I} M_i\bigr) = \sup\bigl\{\dim(\operatorname{im}(\psi_i)) \mid i \in I\bigr\};$$

(ii) Suppose for each $i \in I$ that there is $i_0 \in I$ with $i \leq i_0$ such that $\dim(\operatorname{im}(\phi_{i,i_0})) < \infty$ holds. Then

$$\begin{aligned}&\dim\bigl(\operatorname{colim}_{i \in I} M_i\bigr)\\ &\quad = \sup\left\{\inf\left\{\dim(\operatorname{im}(\phi_{i,j} \colon M_i \to M_j)) \mid j \in I, i \leq j\right\} \mid i \in I\right\}.\end{aligned}$$

Proof. See [18, Theorem 6.13 on page 243]. □

Theorem 2.12 (Dimension and inverse limits) *Let $\{M_i \mid i \in I\}$ be an inverse system of $\mathcal{N}(G)$-modules over the directed set I. For $i \leq j$ let $\phi_{i,j} \colon M_j \to M_i$ be the associated morphism of $\mathcal{N}(G)$-modules. For $i \in I$ let $\psi_i \colon \lim_{i \in I} M_i \to M_i$ be the canonical map. Suppose that there is a countable sequence $i_1 \leq i_2 \leq \ldots$ such that for each $j \in I$ there is $n \geq 0$ with $j \leq i_n$.*

(i) We get for the dimension of the $\mathcal{N}(G)$-module given by the inverse limit $\operatorname{invlim}_{i \in I} M_i$

$$\dim_{\mathcal{N}(G)}\bigl(\operatorname{invlim}_{i \in I} M_i\bigr) = \sup\bigl\{\dim_{\mathcal{N}(G)}(\operatorname{im}(\psi_i \colon \lim_{i \in I} M_i \to M_i)) \mid i \in I\bigr\};$$

(ii) Suppose that for each index $i \in I$ there is an index $i_0 \in I$ with $i \leq i_0$ and $\dim_{\mathcal{N}(G)}(\operatorname{im}(\phi_{i,i_0})) < \infty$. Then

$$\dim_{\mathcal{N}(G)}\bigl(\operatorname{invlim}_{i \in I} M_i\bigr) = \sup\bigl\{\inf\bigl\{\dim_{\mathcal{N}(G)}(\operatorname{im}(\phi_{i,j})) \mid j \in I, i \leq j\bigr\} \mid i \in I\bigr\}.$$

Proof. See [18, Theorem 6.18 on page 244]. □

Let $i \colon H \to G$ be an injective group homomorphism. It induces a ring homomorphism $\mathcal{N}(i) \colon \mathcal{N}(H) \to \mathcal{N}(G)$, see [18, Definition 1.23 on page 29]. For an $\mathcal{N}(H)$-module M let i_*M be the $\mathcal{N}(G)$-module $\mathcal{N}(G) \otimes_{\mathcal{N}(H)} M$ obtained by induction with $\mathcal{N}(i)$.

Theorem 2.13 *Let $i \colon H \to G$ be an injective group homomorphism.*

(i) Induction with i is a faithfully flat functor from the category of $\mathcal{N}(H)$-modules to the category of $\mathcal{N}(G)$-modules, i.e., a sequence of $\mathcal{N}(H)$-modules $M_0 \to M_1 \to M_2$ is exact at M_1 if and only if the induced sequence of $\mathcal{N}(G)$-modules $i_*M_0 \to i_*M_1 \to i_*M_2$ is exact at i_*M_1;

(ii) For any $\mathcal{N}(H)$-module M we have:

$$\dim_{\mathcal{N}(H)}(M) = \dim_{\mathcal{N}(G)}(i_* M).$$

Proof. See [18, Theorem 6.29 on page 253]. □

2.2.3 L^2-Betti Numbers

Definition 2.14 (L^2-Betti numbers)**.** Let X be a (left) G-space. Equip $\mathcal{N}(G)$ with the obvious $\mathcal{N}(G)$-$\mathbb{Z}G$-bimodule structure. The *singular homology $H_p^G(X;\mathcal{N}(G))$ of X with coefficients in $\mathcal{N}(G)$* is the homology of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)$, where $C_*^{\mathrm{sing}}(X)$ is the singular chain complex of X with the induced $\mathbb{Z}G$-structure. Define the *n-th L^2-Betti number of X* by

$$b_n^{(2)}(X;\mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\bigl(H_n^G(X;\mathcal{N}(G))\bigr) \quad \in \mathbb{R}_{\geq 0} \amalg \{\infty\}.$$

Define for any (discrete) group G its *n-th L^2-Betti number* by

$$b_n^{(2)}(G) := b_n^{(2)}(EG;\mathcal{N}(G)) \quad \in \mathbb{R}_{\geq 0} \amalg \{\infty\}.$$

Theorem 2.15 (L^2-Betti numbers)

(i) Homology invariance
We have for a G-map $f\colon X \to Y$:

(a) Suppose for $k \in \mathbb{Z}_{\geq 0}$ that for each subgroup $H \subset G$ the induced map $f^H\colon X^H \longrightarrow Y^H$ is $\mathbb{C}$-homologically k-connected, i.e., the homomorphism $H_m^{\mathrm{sing}}(f^H;\mathbb{C})\colon H_m^{\mathrm{sing}}(X^H;\mathbb{C}) \to H_m^{\mathrm{sing}}(Y^H;\mathbb{C})$ induced by f^H on singular homology with complex coefficients is bijective for $m < k$ and surjective for $m = k$.
Then the induced map $H_n^G(f;\mathcal{N}(G))\colon H_n^G(X;\mathcal{N}(G)) \longrightarrow H_n^G(Y;\mathcal{N}(G))$ is bijective for $n < k$ and surjective for $n = k$ and we get

$$\begin{aligned} b_n^{(2)}(X;\mathcal{N}(G)) &= b_n^{(2)}(Y;\mathcal{N}(G)) \quad \text{for } n < k; \\ b_n^{(2)}(X;\mathcal{N}(G)) &\geq b_p^{(2)}(Y;\mathcal{N}(G)) \quad \text{for } n = k; \end{aligned}$$

(b) Suppose that for each subgroup $H \subset G$ the induced map $f^H\colon X^H \to Y^H$ is a $\mathbb{C}$-homology equivalence, i.e., $H_m^{\mathrm{sing}}(f^H;\mathbb{C})$ is bijective for $m \in \mathbb{Z}_{\geq 0}$. Then for all $n \in \mathbb{Z}_{\geq 0}$ the homomorphism $H_n^G(f;\mathcal{N}(G))\colon H_n^G(X;\mathcal{N}(G)) \to H_n^G(Y;\mathcal{N}(G))$ induced by f is bijective and we get

$$b_n^{(2)}(X;\mathcal{N}(G)) = b_n^{(2)}(Y;\mathcal{N}(G));$$

(ii) Comparison with the Borel construction
Let X be a G-CW-complex. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_m^{(2)}(G_x) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$.

Then we get for all $n \in \mathbb{Z}_{\geq 0}$.

$$b_n^{(2)}(X;\mathcal{N}(G)) = b_n^{(2)}(EG \times X;\mathcal{N}(G));$$

(iii) *Invariance under non-equivariant* $\mathbb{C}$*-homology equivalences*
Suppose that $f\colon X \to Y$ *is a* G*-equivariant map of* G*-CW-complexes such that the induced map* $H_m^{\mathrm{sing}}(f;\mathbb{C})$ *on singular homology with complex coefficients is bijective for* $m \in \mathbb{Z}_{\geq 0}$. *Suppose that for all* $x \in X$ *the isotropy group* G_x *is finite or satisfies* $b_m^{(2)}(G_x) = 0$ *for all* $m \in \mathbb{Z}_{\geq 0}$, *and analogously for all* $y \in Y$.
Then we get for for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X;\mathcal{N}(G)) = b_n^{(2)}(Y;\mathcal{N}(G));$$

(iv) *Independence of equivariant cells with infinite isotropy*
Let X *be a* G*-CW-complex. Let* $X[\infty]$ *be the* G*-CW-subcomplex consisting of those points whose isotropy subgroups are infinite.*
Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X;\mathcal{N}(G)) = b_n^{(2)}(X, X[\infty];\mathcal{N}(G));$$

(v) *Künneth formula*
Let X *be a* G*-space and* Y *be an* H*-space. Then* $X \times Y$ *is a* $G \times H$*-space and we get for all* $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y),$$

where we use the convention that $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ *for* $r \in \mathbb{R}_{>0} \amalg \{\infty\}$ *and* $r + \infty = \infty$ *for* $r \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$;

(vi) *Restriction*
Let $H \subset G$ *be a subgroup of finite index* $[G : H]$.

(a) *Let* M *be an* $\mathcal{N}(G)$*-module and* $\operatorname{res}_G^H M$ *be the* $\mathcal{N}(H)$*-module obtained from* M *by restriction.*
Then

$$\dim_{\mathcal{N}(H)}(\operatorname{res}_G^H M) = [G : H] \cdot \dim_{\mathcal{N}(G)}(M),$$

where $[G : H] \cdot \infty$ *is understood to be* ∞;

(b) *Let* X *be a* G*-space and let* $\operatorname{res}_G^H X$ *be the* H*-space obtained from* X *by restriction.*
Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(\operatorname{res}_G^H X;\mathcal{N}(H)) = [G : H] \cdot b_n^{(2)}(X;\mathcal{N}(G))$$

with the convention $[G : H] \cdot \infty = \infty$;

(vii) Induction

Let $i\colon H \to G$ be an inclusion of groups and let X be an H-space. Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$H_n^G(G \times_H X; \mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathcal{N}(H)} H_p^H(X; \mathcal{N}(H));$$
$$b_n^{(2)}(G \times_H X; \mathcal{N}(G)) = b_n^{(2)}(X; \mathcal{N}(H));$$

(viii) Zero-th homology and L^2-Betti number

Let X be a path-connected G-space. Then:

(a) There is an $\mathcal{N}(G)$-isomorphism $H_0^G(X; \mathcal{N}(G)) \xrightarrow{\cong} \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}$;
(b) We have $b_0^{(2)}(X; \mathcal{N}(G)) = |G|^{-1}$, where $|G|^{-1}$ is defined to be zero if the order $|G|$ of G is infinite;
(c) $H_0^G(X; \mathcal{N}(G))$ is trivial if and only if G is non-amenable;

Proof. See [18, Theorem 6.54 on page 265]. □

2.2.4 The Fundamental Square

Definition 2.16 (Properties of rings)**.** Let R be a ring.

(i) It is called *Noetherian* if any submodule of a finitely generated R-module is again finitely generated;
(ii) It is called *regular* if it is Noetherian and every R-module has a projective resolution of finite dimension;
(iii) It is called *semihereditary*, if any finitely generated submodule of a projective module is projective;
(iv) It is called *von Neumann regular* if any finitely presented R-module is projective;
(v) It is called semisimple if every R-module is projective.

Recall that a ring is Noetherian if and only if any ascending sequence of ideas $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq R$ becomes stationary, i.e., there exists $n \in \mathbb{Z}_{\geq 0}$ satisfying $I_n = I_j$ for every $j \in \mathbb{Z}_{\geq n}$.

Proposition 2.17. *The following statements are equivalent for a ring R:*

(i) It is semisimple:
(ii) Every R-module is injective;
(iii) It is of the form $\prod_{i=1}^m \mathrm{M}_{n_i}(D_i)$ for $m, n_1, n_2, \ldots, n_m \in \mathbb{Z}_{\geq 1}$ and skew fields D_i.
(iv) Every short exact sequence splits;
(v) It is Noetherian and von Neumann regular.

Proof. See [23, page 604]. □

Proposition 2.18. *The following statements are equivalent for a ring R:*

(i) *R is von Neumann regular;*
(ii) *For any $r \in R$ there exists $s \in R$ with $rsr = r$;*
(iii) *Every principal ideal in R is generated by an idempotent;*
(iv) *Every finitely generated submodule of a finitely generated projective R-module is a direct summand;*
(v) *Every finitely presented R-module is projective;*
(vi) *Every R-module is R-flat.*

Proof. See [22, Lemma 4.15, Theorem 4.16 and Theorem 9.15], [25, Theorem 4.2.9 on page 98]. □

Associated to the von Neumann algebra $\mathcal{N}(G)$ is the *algebra of affiliated operators* $\mathcal{U}(G)$ which contains $\mathcal{N}(G)$. Its functional analytic defintion can be found in [18, Section 8.1]. For us the following properties will be relevant.

Proposition 2.19.

(i) *The ring $\mathcal{U}$ is the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-zero divisors;*
(ii) *$\mathcal{U}$ is flat as an $\mathcal{N}(G)$-module;*
(iii) *$\mathcal{U}(G)$ is von Neumann regular;*
(iv) *The inclusion $i\colon \mathcal{N}(G) \to \mathcal{U}(G)$ induces an isomorphism*

$$K_0(i)\colon K_0(\mathcal{N}(G)) \xrightarrow{\cong} K_0(\mathcal{U}(G)).$$

Proof. See [18, Theorem 8.2 on page 327 and Theorem 9.20 (i) on page 345]. □

One can define

$$\dim_{\mathcal{U}(G)}(M) \in \mathbb{R}_{\geq 0} \amalg \{\infty\} \tag{2.20}$$

for an arbitrary $\mathcal{U}(G)$-module M such that for any $\mathcal{N}(G)$-module N we have

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} N) = \dim_{\mathcal{N}(G)}(N). \tag{2.21}$$

Moreover, $\dim_{\mathcal{U}(G)}$ shares all the good properties of $\dim_{\mathcal{N}(G)}$ such as Cofinality and so on. For a projective $\mathcal{U}(G)$-module P we have $\dim_{\mathcal{U}(G)}(P) = 0 \iff P = \{0\}$. For all of these claims see [18, Definition 8.28 and Theorem 8.29 on page 330].

Let G be a group and F a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.

Now one can consider the so called *division closure* $\mathcal{D}_F(G)$ of FG in $\mathcal{U}(G)$, i.e., the smallest ring $\mathcal{D}_F(G)$ satisfying $FG \subset \mathcal{D}_F(G) \subset \mathcal{U}(G)$ with the property that, if $x \in \mathcal{D}_F(G)$ is invertible in $\mathcal{U}(G)$, its inverse is already contained in $\mathcal{D}_F(G)$.

Sometimes one views also the *rational closure* $\mathcal{R}_F(G)$, i..e., the smallest ring $\mathcal{R}_F(G)$ satisfying $FG \subset \mathcal{R}_F(G) \subset \mathcal{U}(G)$ with the property that, if

$A \in \mathrm{M}_n(\mathcal{R}_F(G))$ is invertible in $\mathrm{M}_n(\mathcal{U}(G))$, its inverse is already contained in $\mathrm{M}_n(\mathcal{R}_F(G))$.

A subring $R \subseteq \mathcal{U}(G)$ is called *$*$-regular* if it is closed under the involution $*$ on $\mathcal{U}(G)$ and is von Neumann regular. Let $\mathcal{REG}_F(G)$ be the *$*$-regular closure of FG in $\mathcal{U}(G)$*, i.e, the smallest $*$-regular subring of $\mathcal{U}(G)$ containing FG.

Proposition 2.22. *Let G be a group and F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

(i) We have the inclusions $FG \subseteq \mathcal{D}_F(G) \subseteq \mathcal{R}_F(G) \subseteq \mathcal{REG}_F(G)$;
(ii) If $\mathcal{D}_F(G)$ is von Neumann regular, then $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$:
(iii) If $\mathcal{REG}_F(G)$ is a skewfield, then $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$.

Proof. (i) The inclusions $FG \subseteq \mathcal{D}_F(G) \subseteq \mathcal{R}_F(G)$ are obvious. Since $\mathcal{REG}_F(G)$ is von Neumann regular, it is not hard to check that it is division closed in $\mathcal{U}(G)$, see [20, Proposition 13.15 on page 103]. Hence $\mathcal{R}_F(G) \subseteq \mathcal{REG}_F(G)$.

(ii) If $\mathcal{D}_F(G)$ is von Neumann regular, it is $*$-regular and hence $\mathcal{REG}_F(G) \subseteq \mathcal{D}_F(G)$. Now the claim follows from assertion (i).

(iii) Suppose that $\mathcal{REG}_F(G)$ is a skewfield. Then every element in $\mathcal{D}_F(G)$ different from 0 is a unit in $\mathcal{U}(G)$ and hence is a unit in $\mathcal{D}_F(G)$. Therefore $\mathcal{D}_F(G)$ is a skewfield. Now apply assertion (ii). □

The so-called *fundamental square* is given by the square of inclusions of rings

$$\begin{array}{ccc} FG & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{U}(G). \end{array} \tag{2.23}$$

where S is a $*$-regular ring, e.g., $S = \mathcal{D}_F(G)$, $\mathcal{D}_F(G)$, or $\mathcal{REG}_F(G)$.

2.2.5 K_0 and G_0 of Rings

Definition 2.24 (Projective class group $K_0(R)$ and $G_0(R)$)**.** Let R be ring. Define its *projective class group* $K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.

We denote by $\widetilde{K}_0(R)$ and $\widetilde{G}_0(R)$ respectively the quotient of $K_0(R)$ and $G_0(R)$ respectively by the subgroup generated by the class $[R]$ of R.

Every ring homomorphism $\varphi\colon R \to S$ induces a homomorphism of abelian groups $K_0(\varphi)\colon K_0(R) \to K_0(S)$ by sending the class $[P]$ of a finitely generated projective R-module to the class $[\varphi_* P]$ of the finitely generated projective S-module $\varphi_* P = S \otimes_R P$. This is not true in general for G_0, one needs the extra condition that the functor sending a finitely generated R-module M to the finitely generated S-module $\varphi_* M$ is exact. This condition is automatically satisfied if R is semisimple, e.g. $R = FH$ for a finite group H and a field F of characteristic zero. This has the effect that the inclusion $FG \to \mathcal{D}_F(G)$ does *not* induce a homomorphism from $G_0(FG)$ to $G_0(\mathcal{D}_F(G))$ in general. If G contain the free group $\mathbb{Z} * \mathbb{Z}$ as subgroup, the class of $[\mathbb{C}G]$ in $G_0(\mathbb{C}G)$ is trivial, whereas the class of $[\mathbb{C}G]$ in $K_0(\mathbb{C}G)$ generates an infinite cyclic subgroup. The map $K_0(R) \to G_0(R)$ is in general not injective or surjective, but is a bijection if R is regular or semisimple.

If R is semisimple, it is of the form $R = \prod_{i=1}^{k} \mathrm{M}_{l_i}(D_i)$ for skew fields D_i and one obtains explicit isomorphisms

$$K_0(R) \xrightarrow{\cong} \prod_{i=1}^{k} K_0(\mathrm{M}_{l_i}(D_i)) \xrightarrow{\cong} \prod_{i=1}^{k} K_0(D_i) \xrightarrow{\cong} \prod_{i=1}^{k} \mathbb{Z}. \tag{2.25}$$

In particular $K_0(R)$ is a finitely generated free abelian group. For more information about $K_0(R)$ and $G_0(R)$ we refer for instance to [19, Chapter 2].

2.3 The Statement of the Atiyah Conjecture

Conjecture 2.26 (Atiyah Conjecture)**.** *Let G be a group. Let F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let Λ be an abelian group satisfying $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{R}$.*

We say that G satisfies the Atiyah Conjecture of order Λ with coefficients in F *if for any matrix $A \in \mathrm{M}_{m,n}(FG)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\big) \in \Lambda.$$

Define the abelian group

$$\mathbb{Z} \subseteq \frac{1}{\mathcal{FIN}(G)}\mathbb{Z} \subseteq \mathbb{Q} \tag{2.27}$$

to be the additive subgroup of $\mathbb{Q}$ generated by the set $\{|H|^{-1} \mid H \subseteq G, |H| < \infty\}$. We will explain in Remark 2.38 that $\frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$ is the smallest possible choice for Λ in Conjecture 2.26. So the most interesting case of Conjecture 2.26 is $\Lambda = \frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$.

The case $F = \mathbb{Q}$ is the most relevant one for applications to topology as explained in Proposition 2.32, whereas for applications in algebra the case

$F = \mathbb{C}$ is the desired one. If we take $F = \mathbb{C}$ and $\Lambda = \frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$ in Conjecture 2.26, we obtain the so-called Strong Atiyah Conjecture.

Conjecture 2.28 (Strong Atiyah Conjecture)**.** *A group G satisfies the strong* Strong Atiyah Conjecture *if for any matrix $A \in \mathrm{M}_{m,n}(\mathbb{C}G)$ the von Neumann dimension of the kernel of $\mathcal{N}(G)$-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}\bigl(\ker(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\bigr) \in \frac{1}{\mathcal{FIN}(G)}\mathbb{Z}.$$

In the special case that G is torsionfree, we will consider the following versions of the two conjectures above.

Conjecture 2.29 (Atiyah Conjecture with coefficients in F for torsionfree groups)**.** *Let G be a torsionfree group. Let F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

We say that G satisfies the Atiyah Conjecture with coefficients in F for the torsionfree group G *if for any matrix $A \in \mathrm{M}_{m,n}(FG)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}\bigl(\ker(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\bigr) \in \mathbb{Z}.$$

Conjecture 2.30 (Strong Atiyah Conjecture for torsionfree groups)**.** *A torsionfree group G satisfies the* Strong Atiyah Conjecture *if for any matrix $A \in \mathrm{M}_{m,n}(\mathbb{C}G)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}\bigl(\ker(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n)\bigr) \in \mathbb{Z}.$$

Obviously Conjecture 2.29 holds for all field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ if and only if Conjecture 2.30 is true.

Notation 2.31. A group G has property (B) if there is an upper bound on the orders of the finite subgroups of G.

We will explain in Section 2.8 that there are counterexamples to Conjecture 2.26 and Conjecture 2.28 unless one makes the assumption that G has property (B) introduced in Notation 2.31. No counterexamples are known for Conjecture 2.29 and Conjecture 2.30.

2.4 Some Reformulations and Variants of the Atiyah Conjecture

We can reformulate the Atiyah Conjecture 2.26 for $F = \mathbb{Q}$ in terms of L^2-Betti numbers as follows.

Proposition 2.32. *Let G be a group. Consider any $d \in \mathbb{Z}_{\geq 2}$. Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be an additive subgroup of $\mathbb{Q}$. Then the following statements are equivalent:*

(i) For every cocompact free proper G-manifold M and every $n \in \mathbb{Z}_{\geq 0}$, we get

$$b_n^{(2)}(M; \mathcal{N}(G)) \in \Lambda;$$

(ii) For every cocompact free proper G-manifold M without boundary of dimension $(2d+4)$ we get

$$b_d^{(2)}(M; \mathcal{N}(G)) \in \Lambda;$$

(iii) For every free G-CW-complex X of finite type and every $n \in \mathbb{Z}_{\geq 0}$, we get

$$b_n^{(2)}(X; \mathcal{N}(G)) \in \Lambda;$$

(iv) For every $(d+1)$-dimensional free G-CW-complex X, we get

$$b_d^{(2)}(X; \mathcal{N}(G)) \in \Lambda;$$

(v) The Atiyah Conjecture 2.26 of order Λ with coefficients in $\mathbb{Q}$ is true for G.

Proof. The proof is similar to the one of [18, Lemma 10.5 on page 371]. □

One can rephrase the Atiyah Conjecture also in a more module-theoretic fashion.

Notation 2.33. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Define $\Lambda(G,F)_{\mathrm{fgp}}$, $\Lambda(G,F)_{\mathrm{fp}}$, $\Lambda(G,F)_{\mathrm{fg}}$, or $\Lambda(G,F)_{\mathrm{all}}$ respectively to be the additive subgroup of $\mathbb{R}$ given by differences

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_1) - \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_0),$$

where M_0 and M_1 run through all finitely generated projective FG-modules, finitely presented FG-modules, finitely generated FG-modules, or all FG-modules with $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_i) < \infty$ for $i = 0, 1$ respectively.

Proposition 2.34. *A group G satisfies the Atiyah Conjecture of order Λ with coefficients in F if and only if $\Lambda(G,F)_{\mathrm{fp}} \subseteq \Lambda$ holds, or, equivalently, for any finitely presented FG-module M we have*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M) \in \Lambda.$$

Proof. See [18, Lemma 10.7 on page 372]. □

Proposition 2.35. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

(i) We have $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \subseteq \Lambda(G,F)_{\mathrm{fgp}}$.
If the assembly map

$$\operatorname{colim}_{H\subseteq G,|H|<\infty} K_0(FH) \to K_0(FG)$$

given by the various inclusions $H \subseteq G$ *is surjective, then we have* $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} = \Lambda(G,F)_{\mathrm{fgp}}$*;*
(ii) We have $\Lambda(G,F)_{\mathrm{fgp}} \subseteq \Lambda(G,F)_{\mathrm{fp}} \subseteq \Lambda(G,F)_{\mathrm{fg}} \subseteq \Lambda(G,F)_{\mathrm{all}}$*;*
(iii) $\Lambda(G,F)_{\mathrm{fg}} \subseteq \overline{\Lambda(G,F)_{\mathrm{fp}}}$*, where the closure is taken in* $\mathbb{R}$*;*
(iv) We have

$$\begin{aligned}\overline{\Lambda(G,F)_{\mathrm{fp}}} &= \overline{\Lambda(G,F)_{\mathrm{fg}}};\\ \Lambda(G,F)_{\mathrm{all}} &= \overline{\Lambda(G,F)_{\mathrm{fg}}} \amalg \{\infty\}.\end{aligned}$$

Proof. See [18, Lemma 10.10 on page 373] □

Remark 2.36. Suppose that G satisfies property (B). Then the least common multiple $\operatorname{lcm}(G)$ of the orders of finite subgroups is defined and

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} = \{r \in \mathbb{R} \mid \operatorname{lcm}(G)\cdot r \in \mathbb{Z}\}.$$

Hence by Proposition 2.34 and Proposition 2.35 the strong Atiyah Conjecture 2.28 is equivalent to the equality

$$\{r \in \mathbb{R} \mid \operatorname{lcm}(G)\cdot r \in \mathbb{Z}\} = \Lambda(G,\mathbb{C})_{\mathrm{fgp}} = \Lambda(G,\mathbb{C})_{\mathrm{fp}} = \Lambda(G,\mathbb{C})_{\mathrm{fg}} = \Lambda(G,\mathbb{C})_{\mathrm{all}}.$$

In particular the strong Atiyah Conjecture 2.30 for a torsionfree group G is equivalent to the equality

$$\mathbb{Z} = \Lambda(G,\mathbb{C})_{\mathrm{fgp}} = \Lambda(G,\mathbb{C})_{\mathrm{fp}} = \Lambda(G,\mathbb{C})_{\mathrm{fg}} = \Lambda(G,\mathbb{C})_{\mathrm{all}}.$$

Exercise 2.37 Show that the following assertions for a group G are equivalent:

(i) The group G has property (B);
(ii) The subgroup $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ of $\mathbb{R}$ is discrete;
(iii) The subgroup $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ of $\mathbb{R}$ is closed.

Remark 2.38. If the Atiyah Conjecture 2.26 of order Λ with coefficients in F holds for G, then we must have $\frac{1}{|\mathcal{FIN}(G)|}\cdot\mathbb{Z} \subseteq \Lambda$ because of Proposition 2.35.

2.5 The K-Theoretic Atiyah Conjecture

Next we give a purely K-theoretic version of the Atiyah-Conjecture. We need some notation for certain subgroups of the projective class group $K_0(\mathcal{U}(G))$ of $\mathcal{U}(G)$.

Notation 2.39. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Define the subgroups $\kappa(G,F)_{\mathrm{fgp}}$, and $\kappa(G,F)_{\mathrm{fp}}$, respectively of $K_0(\mathcal{U}(G))$ to be the subgroup given by differences

$$[\mathcal{U}(G) \otimes_{FG} M_1] - [\mathcal{U}(G) \otimes_{FG} M_0],$$

where M_0 and M_1 run through all finitely generated projective FG-modules or finitely presented FG-modules respectively.

Define the subgroup $\kappa(G;F)_{\mathrm{fin}}$ of $K_0(\mathcal{U}(G))$ to be the image of the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(j)} K_0(\mathcal{U}(G))$$

where the first map is induced by the various inclusions of finite subgroups $H \subseteq G$ and the second map is induced by the inclusion $j\colon FG \to \mathcal{U}(G)$.

The definition of $\kappa(G,F)_{\mathrm{fp}}$ make senses, since for a finitely presented FG-module M the $\mathcal{U}(G)$-module $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M$ is a finitely presented $\mathcal{U}(G)$-module and hence a finitely generated projective $\mathcal{U}(G)$-module, since $\mathcal{U}(G)$ is von Neumann regular by Proposition 2.19 (iii).

Conjecture 2.40 (The K-theoretic Atiyah Conjecture)**.** *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then*

$$\kappa(G,F)_{\mathrm{fin}} = \kappa(G,F)_{\mathrm{fp}}.$$

Proposition 2.41. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

(i) We have $\kappa(G,F)_{\mathrm{fin}} \subseteq \kappa(G,F)_{\mathrm{fgp}} \subseteq \kappa(G,F)_{\mathrm{fp}}$;
(ii) If G satisfies the Full Farrell-Jones Conjecture, see [19, Conjecture 13.30 on page 401], then $\kappa(G,F)_{\mathrm{fin}} = \kappa(G,F)_{\mathrm{fgp}}$.

Proof. (i) This is obvious since $\kappa(G,F)_{\mathrm{fgp}}$ agrees the image of the homomorphism $K_0(j)\colon K_0(FG) \to K_0(\mathcal{U}(G))$.

(ii) The Full Farrell-Jones Conjecture implies that the map

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG)$$

is surjective, see [19, Theorem 13.65 (Xii) on page 421]. ◻

Remark 2.42. The Full Farrell-Jones Conjecture is known for a large class of groups, see [19, Theorem 16.1 on page 499]. For example, it contains all hyperbolic groups, finite-dimensional CAT(0)-groups, lattices in path connected second countable locally compact Hausdorff groups, fundamental groups of manifolds of dimension ≤ 3, and S-arithmetic groups. Furthermore, it is closed under taking subgroups, passing to over group of finite index, and passing to colimits over directed systems of groups with arbitrary structure maps. Its precise statement is not relevant for this monograph and therefore omitted.

Remark 2.43. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. The K-theoretic Atiyah Conjecture 2.40 implies the Atiyah Conjecture 2.26 of order $\Lambda = \frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$ with coefficients in F by the following argument. The dimension function $\dim_{\mathcal{N}(G)}$ of (2.3) defines a homomorphism

$$\dim_{\mathcal{N}(G)} \colon K_0(\mathcal{N}(G)) \to \mathbb{R}. \tag{2.44}$$

The homomorphism $K_0(i) \colon K_0(\mathcal{N}(G)) \xrightarrow{\cong} K_0(\mathcal{U}(G))$ is bijective by Proposition 2.19 (iv). One easily checks using (2.21) that the image of $\kappa(G,F)_{\mathrm{fp}}$ and $\kappa(G,F)_{\mathrm{fin}}$ under the composite $K_0(\mathcal{U}(G)) \xrightarrow{K_0(i)^{-1}} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathbb{R}$ is $\Lambda(G,F)_{\mathrm{fp}}$ and $\frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$ respectively.

2.6 The Center-Valued Atiyah Conjecture

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Using the center-valued universal trace and the associated center-valued universal dimension function, one obtains homomorphisms

$$\dim^u_{\mathcal{N}(G)} \colon K_0(\mathcal{N}(G)) \to \operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \operatorname{center}(\mathcal{N}(G)) \mid a = a^*\},$$

and

$$\dim^u_{\mathcal{U}(G)} \colon K_0(\mathcal{U}(G)) \to \operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \operatorname{center}(\mathcal{N}(G)) \mid a = a^*\},$$

where $\operatorname{center}(\mathcal{N}(G))$ is the center of the von Neumann algebra and the group structure on $\operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2}$ comes from the addition. The composite of $\dim^u_{\mathcal{U}(G)}$ with the isomorphism $K_0(j) \colon K_0(FG) \to K_0(\mathcal{U}(G))$ is $\dim^u_{\mathcal{N}(G)}$. The homomorphism $\dim_{\mathcal{N}(G)}$ is always injective. It is bijective if $\mathcal{N}(G)$ is of type II_1. All these claims follow from [18, Theorem 9.13 (2) on page 342]. If G is finitely generated and not virtually finitely generated abelian, then $\mathcal{N}(G)$ is of type II_1, see [18, Lemma 9.4 (3) on page 338]. If G is torsionfree, $\mathcal{N}(G) = \mathcal{N}(G)^{\mathbb{Z}/2} = \mathbb{R}$ holds by [18, Lemma 9.4 (4) on page 338].

Notation 2.45. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, which is closed under complex conjugation. Define $\Lambda^u(G,F)_{\mathrm{fgp}}$, and $\Lambda^u(G,F)_{\mathrm{fp}}$, respectively to be the additive subgroup of $\operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2}$ given by differences

$$\dim^u_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_1) - \dim^u_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_0),$$

where M_0 and M_1 run through all finitely generated projective FG-modules and finitely presented FG-modules for $i = 0, 1$ respectively. Let $\Lambda^u(G,F)_{\mathrm{fin}}$ be the image of the composite

$$\operatorname{colim}_{H\subseteq G,|H|<\infty} K_0(FH) \to K_0(FG) \to K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2}.$$

Conjecture 2.46 (The center-valued Atiyah Conjecture)**.** *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then* the center-valued Atiyah Conjecture *predicts*

$$\Lambda^u(G,F)_{\mathrm{fin}} = \Lambda^u(G,F)_{\mathrm{fp}}.$$

Proposition 2.47. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then the K-theoretic Atiyah Conjecture 2.40 and the center-valued Atiyah Conjecture 2.46 are equivalent.*

Proof. This follows from the definitions and the fact that the homomorphism $\dim^u_{\mathcal{N}(G)} \colon K_0(\mathcal{N}(G)) \to \operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2}$ is injective. □

2.7 Some Implications and Applications of the Atiyah Conjecture

We briefly describe a few applications of the Atyiah Conjecture-many more will be discussed later in this monograph. **Comment 7 (by W.)**: Maybe add references later.

2.7.1 The Zero-Divisor Conjecture of Kaplansky and the Embedding Conjecture of Malcev

We have the following prominent conjectures due to Kaplansky and Malcev.

Conjecture 2.48 (Zero-Divisor Conjecture)**.** *Let F be a field of characteristic zero and G be a torsionfree group. Then FG contains no non-trivial zero-divisors.*

Conjecture 2.49 (Embedding Conjecture)**.** *Let F be a field of characteristic zero and G be a torsionfree group. Then FG embeds into a skewfield.*

Obviously the Embedding Conjecture 2.49 implies the Zero-Divisor Conjecture 2.48.

Remark 2.50. We conclude from Theorem 2.69 (i) that the Strong Atiyah Conjecture for torsionfree groups, see Conjecture 2.30, implies the Embedding Conjecture 2.49 if $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ holds. The Strong Atiyah Conjecture for torsionfree groups, see Conjecture 2.30, implies the Zero-Divisor Conjecture 2.48 for all fields F of characteristic zero because of [19, Remark 2.84 on page 57].

Remark 2.51. Let F be a field and G be a group such that FG has non-nontrivial zero-divisors. Then G is amenable if and only if FG is an Ore domain, i.e., satisfies the Ore condition with respect to the multiplicative subgroup $FG \setminus \{0\}$, see [3, Theorem A.1].

Suppose that G be an amenable group. Then the strong Atiyah Conjecture 2.28 holds for G if and only if the Zero-Divisor Conjecture 2.48 holds for $\mathbb{C}G$, see [18, Lemma 10.16 on page 376]. In this case $\mathcal{D}_{\mathbb{C}}(G) = \mathcal{R}_{\mathbb{C}}(G) = \mathcal{REG}_{\mathbb{C}}(G)$ agrees with the skewfield given by the Ore localization of $\mathbb{C}G$ with respect the multiplicative subgroup $FG \setminus \{0\}$.

2.7.2 Computation of L^2-Betti Numbers

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Given a matrix $A \in \mathrm{M}_{m,n}(FG)$ one can consider the FG-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A and asks for the computation of $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}))$. If one has an algorithm to solve the word problem in G, one can define a monotone increasing sequence $(c_p(A))_{p \geq 0}$ of elements in $\mathbb{R}_{\geq 0}$ such that each element is computable and one has

$$\dim_{\mathcal{N}(G)}(\ker(r_A)) = \lim_{p \to \infty} c_p(A).$$

This is proved in [18, Section 3.7]. Löh and Uschold [17] investigate the computability degree of this limit.

Now suppose that G satisfies condition (B) and that the Atiyah Conjecture of order $\Lambda = \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ with coefficients in F, see Conjecture 2.26, holds. If there is a p such that $c_p(A) < \operatorname{lcm}(G)^{-1}$ holds, then we conclude from Remark 2.36 that we have $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = 0$, and one does not have to compute all the elements of the sequence $(c_p(A))_{p \geq 0}$.

It is clear that this can be very useful to show the vanishing of L^2-Betti numbers, see [18, Remark 3.173 on page 195] and to compute L^2-invariant in general.

2.7.3 Finite Generation of Projective Modules

Question 2.52 (Finite generation of projective modules over group rings and their von Neumann dimension)**.** *Let G be a group satisfying condition (B). Consider a ring R with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$. Let P be a projective RG-module such that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \infty$ holds. Is then P finitely generated?*

This question is only interesting in the case, where G has property (B).

Exercise 2.53 Consider a field F with $\mathbb{Z} \subseteq F \subseteq \mathbb{C}$. Construct an abelian group G such that for any $\epsilon \in \mathbb{R}_{>0}$ there is a projective FG-module P such that P is not finitely generated and $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \infty$ holds.

Here is a partial answer to Question 2.52.

Proposition 2.54. *Let G be a group satisfying condition (B). Consider a ring R and a field F with $\mathbb{Z} \subseteq R \subseteq F \subseteq \mathbb{C}$. Suppose that the Atiyah Conjecture of order $\Lambda = \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ with coefficients in F, see Conjecture 2.26, holds. Let P be a projective RG-module. Then the following assertions are equivalent:*

(i) The RG-module P is finitely generated;
(ii) We have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \infty$*;*

It is a direct consequence of the following two facts whose proof can be found in [13, Lemma 4 and Lemma 5].

Consider the situation of Proposition 2.54. Then

- Suppose that G satisfies condition (B) and that the Atiyah Conjecture of order $\Lambda = \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ with coefficients in F, see Conjecture 2.26, holds. Let N be an RG-module. Suppose that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} N) < \infty$. Then there exists a finitely generated RG-submodule $M \subset N$ satisfying $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} N/M) = 0$;
- Let P be a projective RG-module such that for some finitely generated RG-submodule $M \subset P$ we have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P/M) = 0$. Then P is finitely generated.

Recall that a group G is of type FP_n or FF_n respectively if there exists an exact sequence of $\mathbb{Z}G$-modules $0 \to P_n \to P_{n-1} \to \cdots \to P_n \to \mathbb{Z}$ for $\mathbb{Z}$ equipped with the trivial G-action such that each module P_i is finitely generated projective or finitely generated free respectively. We omit the elementary proof that the next Proposition 2.55 follows from Proposition 2.54.

Proposition 2.55. *Let G be a finitely generated group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) < \infty$. Suppose that G satisfies the Atiyah Conjecture with coefficients in $F = \mathbb{Q}$ for the torsionfree group G, see Conjecture 2.29*

Then G is of type FP_2*.*

If G satisfies the Full Farrell-Jones Conjecture, then we can replace FP_2 *by* FF_2 *in the statement above.*

Theorem 2.56 (Vanishing of top L^2-Betti numbers and passing to subgroups) *Let H be a subgroup of G. Let $\overline{X}$ be a d-dimensional proper G-CW-complex which we can also view as d-dimensional proper H-CW-complex by restriction. Suppose $b_d^{(2)}(\overline{X};\mathcal{N}(G)) = 0$.*

Then we get $b_d^{(2)}(\overline{X};\mathcal{N}(H)) = 0$.

Proof. For any $\mathbb{C}G$-module M be a obtain an $\mathcal{N}(H)$-homomorphism, natural in M,

$$T(M)\colon \mathcal{N}(H) \otimes_{\mathbb{C}H} M \to \mathcal{N}(G) \otimes_{\mathbb{C}G} M, \quad u \otimes x \mapsto i(u) \otimes x. \tag{2.57}$$

Next we want to show that it is injective for projective $\mathbb{C}G$-modules M. As T is compatible with direct sums over arbitrary index sets, it suffices to prove the claim for $M = \mathbb{C}G$.

Let T be a transversal of the projection $\mathrm{pr}\colon G \to H\backslash G$, i.e., T is a subset of G such that $\mathrm{pr}|_T\colon T \to H\backslash G$ is a bijection. We will assume $e \in T$ for the unit $e \in G$. We obtain an isomorphism of left $\mathbb{C}H$-modules

$$\bigoplus_{t \in G/H} \mathbb{C}H \xrightarrow{\cong} \mathbb{C}G, \quad (x_t)_{t\in T} \mapsto \sum_{t \in T} x_t \cdot t.$$

It induces a bijection

$$\alpha \colon \bigoplus_{t\in T} \mathcal{B}(L^2(H))^H \xrightarrow{\cong} \mathcal{B}(L^2(H))^H \otimes_{\mathbb{C}H} \mathbb{C}G, \quad (f_t)_{t\in T} \mapsto \sum_{t\in T} f_t \otimes t$$

and an H-equivariant isomorphism of Hilbert spaces

$$\xi \colon \overline{\bigoplus}_{t\in T} l^2(H) \xrightarrow{\cong} l^2(G), \quad (u_t)_{t\in T} \mapsto \sum_{t\in T} u_t \cdot t,$$

where the source is the Hilbert space completion of the pre Hilbert space $\bigoplus_{t\in G/H} l^2(H)$. We have the isomorphism

$$\beta \colon \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \xrightarrow{\cong} \mathcal{B}(L^2(G))^G, \quad f \otimes \left(\sum_{g\in G} \lambda_g \cdot g\right) \mapsto \sum_{g\in G} \overline{\lambda_g} \cdot \left(f \circ r_{g^{-1}}\right)$$

where $r_{g^{-1}} \colon l^2(G) \to l^2(G)$ is right multiplication with g^{-1}. Hence it remains to show that

$$\beta \circ T(\mathbb{C}G) \circ \alpha \colon \bigoplus_{t\in G/H} \mathcal{B}(l^2(H))^H \to \mathcal{B}(l^2(G))^G$$

is injective. Given $s \in T$ and an element f_s in the copy of $\mathcal{B}(l^2(H))^H$ belonging to s in the source of α, we get a commutative diagram of operators of Hilbert spaces

$$\begin{tikzcd}
\overline{\bigoplus}_{t\in T} l^2(H) \arrow[r, "\xi", "\cong"'] \arrow[d, "\sigma_s"'] & l^2(G) \arrow[d, "r_{s^{-1}}"] \\
\overline{\bigoplus}_{t\in T} l^2(H) \arrow[r, "\xi", "\cong"'] \arrow[d, "\bigoplus_{t\in T} f_s"'] & l^2(G) \arrow[d, "i(f_s)"] \\
\overline{\bigoplus}_{t\in T} l^2(H) \arrow[r, "\xi", "\cong"'] & l^2(G)
\end{tikzcd} \tag{2.58}$$

where for $t_0, t_1 \in T$ the operator $(\sigma_s)_{t_0,t_1} \colon l^2(H) \to l^2(H)$ is trivial if $\mathrm{pr}(t_0 s^{-1}) \neq \mathrm{pr}(t_1)$ and is right multiplication with the element $t_0 s^{-1} t_1^{-1} \in H$ if $\mathrm{pr}(t_0 s^{-1}) = \mathrm{pr}(t_1)$.

Now suppose that the element $(f_s)_{s\in T} \in \bigoplus_{t\in G/H} \mathcal{B}(l^2(H))^H$ lies in the kernel of $\beta \circ T(\mathbb{C}G) \circ \alpha$. Then $\xi^{-1} \circ \left(\sum_{s\in T} i(f_s) \circ r_{s^{-1}}\right) \circ \xi \colon \overline{\bigoplus}_{t\in T} l^2(H) \to \overline{\bigoplus}_{t\in T} l^2(H)$ is trivial.

We get for $t \in T$ that $(\sigma_s)_{t,e}$ is zero for $s \neq t$ since $\mathrm{pr}(ts^{-1}) = \mathrm{pr}(e) \implies ts^{-1} \in H \implies \mathrm{pr}(t) = \mathrm{pr}(s) \implies s = t$. Fix $t \in T$. Let $j_t \colon l^2(H) \to \overline{\bigoplus}_{t \in T} l^2(H)$ be the inclusion of the summand belonging to $t \in T$ and $\mathrm{pr}_e \colon \overline{\bigoplus}_{t \in T} l^2(H) \to l^2(H)$ be the projection onto the summand belonging to $e \in T$. Then the composite

$$\mathrm{pr}_e \circ \xi^{-1} \circ \left(\sum_{s \in T} i(f_s) \circ r_{s^{-1}} \right) \circ \xi \circ j_t \colon l^2(H) \to l^2(H)$$

is zero and sends u to $f_t(u)$ because of the commutative diagram (2.58). Hence $f_t = 0$ for all $t \in T$. This finishes the proof that the natural map $T(M)$ of (2.57) is bijective for every free $\mathbb{C}G$-module M.

If $c_d \colon C_*(\overline{X}) \to C_*(\overline{X})$ is the d-th differential in the cellular $\mathbb{C}G$-chain complex $C_*(\overline{X};\mathbb{C})$, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{N}(H) \otimes_{\mathbb{C}H} C_d(\overline{X};\mathbb{C}) \arrow[r, "T(C_d(\overline{X};\mathbb{C}))"] \arrow[d, "\mathrm{id}_{\mathcal{N}(H)} \otimes_{\mathbb{C}H} c_d"'] & \mathcal{N}(G) \otimes_{\mathbb{C}G} C_d(\overline{X};\mathbb{C}) \arrow[d, "\mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{C}G} c_d"] \\
\mathcal{N}(H) \otimes_{\mathbb{C}H} C_{d-1}(\overline{X};\mathbb{C}) \arrow[r, "T(C_{d-1}(\overline{X};\mathbb{C}))"'] & \mathcal{N}(G) \otimes_{\mathbb{C}G} C_{d-1}(\overline{X};\mathbb{C})
\end{tikzcd}$$

whose horizontal arrows are injective. Hence the left vertical arrow is injective if the right vertical arrow is injective. Now the claim follows from the fact that a projective $\mathcal{N}(G)$-module P is trivial if and only if $\dim_{\mathcal{N}(G)}(P)$ vanishes, see [18, Lemma 6.28 (3) on page 252]. ◻

Theorem 2.56 has been proved for proper G-actions on simplicial complexes using orbit equivalence in [6, Theorem 1.5].

Proposition 2.55 and Theorem 2.56 imply the next proposition which has already been proved by Jaikin-Zapirain-Linton [9, Theorem 3.7].

Proposition 2.59. *Let G be a group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) = 0$. Suppose that G satisfies the Atiyah Conjecture with coefficients in $F = \mathbb{Q}$ for the torsionfree group G, see Conjecture 2.29.*

Then G is almost coherent in the sense that every finitely generated subgroup $H \subseteq G$ is of type FP_2.

If G satisfies the Full Farrell-Jones Conjecture, then we can replace FP_2 *by* FF_2 *in the statement above.*

Proposition 2.59 is interesting in connection with the following question.

Question 2.60 (Coherent groups)**.** *Let G be a group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) = 0$. Is then G coherent in the sense that any finitely generated subgroup $H \subseteq G$ has a finite 2-dimensional model for BH?*

Comment 8 (by W.): Discuss here or later the relation to the existence of non-positive immersions and the vanishing of the second L^2-Betti number.

See in particular Jaikin-Zapirain-Linton [9, Theorem 1.2].

Or one shifts parts of this subsection to Chapter 5.

2.8 Counterexamples to the Atiyah Conjecture

Dropping the condition (B) that there is an upper bound on the orders of finite subgroups, one might still ask if $\Lambda(G;\mathbb{Q})_{\mathrm{fp}} = \mathbb{Q}$ holds, motivated by the observation that the image of the composite

$$\bigoplus_{H \subseteq G, |H|<\infty} K_0(\mathbb{Q}H) \to K_0(\mathbb{Q}G) \to K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

is contained in $\mathbb{Q}$. This goes back to Atiyah's original question [1, page 72], who asked for the rationality of the L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ of every closed manifold M. Austin [2, Corollary 1.2] gave the first example of a finitely generated group G, where for some matrix $A \in \mathrm{M}_{m,n}(\mathbb{Q}G)$ the dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ is irrational. Grabowski [7, Theorem 1.3] proved, using Turing machines, that any non-negative real number can occur in this way for some finitely generated group G and some matrix A. **Comment 9 (by T.)**: Mention also first examples with finitely presented group? **Comment 10 (by W.)**: I agree. Can Thomas formulate a sentence? Löh and Uschold [17] investigate the computability degree of real numbers arising as L^2-Betti numbers or L^2-torsion of groups, parametrised over the Turing degree of the word problem. Roughly speaking, the complexity of the computation of L^2-invariants of a group is the same as the complexity of the word problem.

2.9 Positive Results about the Atiyah Conjecture

The notions of elementary amenable groups and amenable groups are explained for instance in [18, Subsection 6.4.1].

Definition 2.61 (Class of groups $\mathcal{C}$)**.** Let $\mathcal{C}$ be the smallest class of groups satisfying the following conditions:

(i) $\mathcal{C}$ contains all free groups;
(ii) If $\{G_i \mid i \in I\}$ is a directed system of subgroups directed by inclusion such that each G_i belongs to $\mathcal{C}$, then $G = \bigcup_{i \in I} G_i$ belongs to $\mathcal{C}$;
(iii) Let $1 \to K \to G \to Q \to 1$ be an extension of groups such that K belongs to $\mathcal{C}$ and Q is elementary amenable, then G belongs to $\mathcal{C}$.

Definition 2.62 (Class of groups $\mathcal{A}_F$)**.** Consider a field F satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Denote by $\mathcal{A}_F$ the collection of all groups which have property (B) **Comment 11 (by T.)**: We should also add hyperlinks for such properties;

however, at important points I'd prefer to explicitly state this property (not too long) **Comment 12 (by W.)**: I agree. Can Thomas do this at this place. I am not quire sure about the latex command. I can implement then this at all other places. and satisfy the Atiyah Conjecture 2.26 with coefficients in F.

Definition 2.63 (Class of groups $\mathcal{D}_F$)**.** Consider a field F satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let $\mathcal{D}_F$ be the smallest class of groups satisfying the following conditions:

(i) The trivial group belongs to $\mathcal{D}_F$;
(ii) If $\{G_i \mid i \in I\}$ is a filtered system of groups in $\mathcal{D}_F$ (with arbitrary structure maps), then its colimit again belongs to $\mathcal{D}_F$;
(iii) If $\{G_i \mid i \in I\}$ is a cofiltered system of groups in $\mathcal{D}_F$ (with arbitrary structure maps), then its limit again belongs to $\mathcal{D}_F$;
(iv) If G belongs to $\mathcal{D}_F$ and $H \subseteq G$ is a subgroup, then $H \in \mathcal{D}_F$;
(v) If $p\colon G \to A$ is an epimorphism of a torsionfree group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}_F$ for every finite group $B \subseteq A$, then $G \in \mathcal{D}_F$.

Note that each element in $\mathcal{D}_F$ is a torsionfree group, and the class $\mathcal{D}_F$ contains all residually (torsionfree elementary amenable) groups.

A group is called *locally indicable*, if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Locally indicable groups are torsionfree. Examples for locally indicable groups are torsionfree one-relator groups.

Theorem 2.64 (Status of the Atiyah Conjecture 2.26 with coefficients in F) *Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

(i) If G belongs to $\mathcal{C}$ and has property (B), then $G \in \mathcal{A}_F$;
(ii) If G belongs to $\mathcal{D}_F$, then G is torsionfree and $G \in \mathcal{A}_F$;
(iii) If $1 \to H \to G \to Q \to 1$ is an extension of groups, H is torsionfree and belongs to $\mathcal{A}_F$, and Q is locally indicable, then G is torsionfree and belongs to $\mathcal{A}_F$;
(iv) If $G \in \mathcal{A}_F$ and $H \subseteq G$ is a subgroup with $\operatorname{lcm}(H) = \operatorname{lcm}(G)$, then $H \in \mathcal{A}_F$;
(v) If G is the directed union $\bigcup_{i \in I} G_i$ of subgroups G_i directed by inclusion and each G_i belongs to $\mathcal{A}_F$, then G belongs to $\mathcal{A}_F$;
(vi) The group G belongs to $\mathcal{A}_F$ if and only if all its finitely generated subgroups belong to $\mathcal{A}_F$;
(vii) If $1 \to K \to G \to Q \to 1$ is an extension of groups such that K is finite and G belongs to $\mathcal{A}_F$, then Q belongs to $\mathcal{A}_F$;
(viii) Let M be a connected (not necessarily compact) d-dimensional manifold (possibly with non-empty boundary) such that $d \leq 3$ and its fundamental group $\pi_1(M)$ is torsionfree, then $\pi_1(M) \in \mathcal{C}$ and hence $\pi_1(M) \in \mathcal{A}_F$;
(ix) If the group G has property (B) and belongs to one of the following classes below, then G belongs to $\mathcal{A}_F$:

(a) Residually $\{$torsionfree elementary amenable$\}$ groups;

(b) Free by elementary amenable groups;
(c) Braid groups;
(d) Right-angled Artin and Coxeter groups;
(e) Torsionfree p-adic analytic pro-p-groups;
(f) Locally indicable groups;
(g) One-relator groups.

Proof. (i) This is due to Linnell, see for instance [16] or [18, Theorem 10.19 on page 378].

(ii) This follows from [8, Corollary 1.2], which is based on on [4, Theorem 1.4].

(iii) This follows from [10, Proposition 6.5].

(iv) This follows from [18, Theorem 6.29 (2) on page 253].

(v) See [18, Lemma 10.4 on page 371].

(vi) This follows from assertions (iv) and (v).

(vii) This follows from [18, Lemma 13.45 on page 473]. **Comment 13 (by T.)**: Eventually, here we could add the relevant computations of the lcms, or give a reference which includes this? **Comment 14 (by W.)**: This is a general statement and one may not say anything about the lcm here. Figuring out the lcm depends on the special case one is looking at.

(viii) This follows from [11, Theorem 1.1] for $d = 3$. The case $d = 2$ can be reduced to the case $d = 3$ by crossing with S^1 and assertion (iv) or just use the fact that the epimorphism the fundamental group of a connected manifold of dimension ≤ 2 onto its abelianization has free group as kernel.

(ix) This follows from the other assertions or from [8, Theorem 1.1 and Corollary 1.2] and [10, Corollary 1.3] using [14, Theorem 2] and [5, Theorem 1.1]. □

Remark 2.65. The class $\mathcal{A}_F$ is very large by aforementioned results. Nevertheless, we do not know whether the Atiyah Conjecture 2.26 holds for all hyperbolic groups or for all amenable groups.

Theorem 2.66 (Status of the K-theoretic Atiyah Conjecture 2.40) *The K-theoretic Atiyah Conjecture 2.40 holds for a group G and $F = \mathbb{C}$ if G belongs to the class $\mathcal{C}$ and has property (B).*

Proof. This follows essentially by inspecting the proof of Linnell for the Atiyah Conjecture, see [12, Theorem 1.11]. □

There are partial results on the difficult question, whether the Atiyah Conjecture 2.26 holds for a group G if it holds for a subgroup of finite index, see for instance [15]. **Comment 15 (by T.)**: So, in the "passage to subgroups" of Theorem 2.66 you had restrictions in mind. How should we formulate them? **Comment 16 (by W.)**: I mean the general results Thomas worked together with Linnell in [15]. If I remember correctly, one does not know that this passage to overgroups of finite index works in general. I think that the interested reader should just consult that paper and we do not have to say more at this place unless there is a nice positive example.

2.10 Strategy for a Proof of the K-Theoretic Atiyah Conjecture Following Linnell

We have already explained in Section 2.8 that the Atiyah Conjecture has only a chance to be true if the group G has property (B), i.e., has a bound on the order of its finite subgroups. The strategy of proof of the Atiyah Conjecture for groups G which have property (B) and belong to the class $\mathcal{C}$ due to Linnell [16] has been elaborated on and expanded and finally carried out in details for the class $\mathcal{C}$ in [18, Chapter 10] and [21]. We summarize and record some old ideas and some new developments concerning it in this section. We first give a brief summary in the torsionfree case in Subsection 2.10.1 and in the case, where G has property (B), in Subsection 2.10.2. Then we give more details for the interested reader in Subsection 2.10.3.

We have introduced the subrings ring $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ of $\mathcal{U}(G)$ and explained that $\dim_{\mathcal{N}(G)}$ extends to $\dim_{\mathcal{U}(G)}$ in Subsection 2.2.4.

2.10.1 Summary in the Torsionfree Case

Let G be a torsionfree group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then the following assertions are equivalent:

(i) The K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(ii) The Atiyah Conjecture with coefficients in F for torsionfree groups, see Conjecture 2.29, holds for G.
(iii) There is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$, which is von Neuman regular and satisfies $\widetilde{K}_0(S) = \{0\}$;
(iv) The ring $\mathcal{REG}_F(G)$ satisfies $\widetilde{K}_0(S) = \{0\}$;
(v) The ring $\mathcal{REG}_F(G)$ is a skewfield;
(vi) The ring $\mathcal{D}_F(G)$ is von Neumann regular and satisfies $\widetilde{K}_0(S) = \{0\}$;
(vii) We have $\widetilde{G}_0(\mathcal{D}_F(G)) = \{0\}$;
(viii) The ring $\mathcal{D}_F(G)$ is a skewfield;
(ix) The ring $\mathcal{R}_F(G)$ is a skewfield.

To prove the K-theoretic Atiyah Conjecture 2.40 or the equivalent Atiyah Conjecture with coefficients in F for torsionfree groups, see Conjecture 2.29, the most promising attempt seems to be to prove assertion (iii) or (iv). Assertion (vii) looks also promising but we have explained in Subsection 2.2.5 that it is better to work with $K_0(R)$ than with $G_0(R)$. We will also prove that in the case that one of the equivalent assertions above is true, the three rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree. Without involving K_0 or G_0 we do not know a strategy to prove assertions (v), (viii) and (ix) directly.

2.10.2 Summary in the Case Where There a Bound on the Orders of Finite Subgroups

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

Notation 2.67. We say that a ring S with $FG \subseteq S \subseteq \mathcal{U}(G)$ has *property (K)* if the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(k)} K_0(S)$$

for $k\colon FG \to S$ the inclusion is surjective.

Then the following assertions are equivalent if G has property (B):

(i) The K-theoretic Atiyah Conjecture 2.40 holds for G and F
(ii) There is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K);
(iii) The ring $\mathcal{REG}_F(G)$ has property (K);
(iv) The ring $\mathcal{D}_F(G)$ is von Neumann regular and has property (K);
(v) The composite $\bigoplus_{H \subseteq G, |H| < \infty} G_0(FH) \to G_0(\mathcal{D}_F(G))$ induced by the various inclusions $FH \to \mathcal{D}_F(G)$ is surjective;
(vi) The ring $\mathcal{R}_F(G)$ is von Neumann regular and has property (K);

To prove the K-theoretic Atiyah Conjecture 2.40 the most promising attempt seems to be to prove assertion (ii) and (iii). Assertion (v) looks also promising but we have explained in Subsection 2.2.5 that it is better to work with $K_0(R)$ than with $G_0(R)$. We will also prove that in the case that one of the equivalent assertions above is true, the three rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree.

2.10.3 More Information and Details

Theorem 2.68 (Strategy for a proof of the K-theoretic Atiyah Conjecture) *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

(i) If there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K), then the K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(ii) If G satisfies the Full-Farrell-Jones Conjecture and there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and for which $K_0(k)\colon K_0(FG) \to K_0(S)$ is surjective, then the K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(iii) If $\mathcal{REG}_F(G)$ has property (K), then the K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(iv) If G satisfies the Full-Farrell-Jones Conjecture and and the homomorphism $K_0(k)\colon K_0(FG) \to K_0(\mathcal{REG}_F(G))$ is surjective, then the K-theoretic Atiyah Conjecture 2.40 holds for G and F;

(v) If G is torsionfree, the following statements are equivalent:

(a) The K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(b) The Atiyah Conjecture with coefficients in F for G, see Conjecture 2.29, is true;
(c) There is a skewfield D with $FG \subseteq D \subseteq \mathcal{U}(G)$;
(d) $\mathcal{D}_F(G)$ is a skew-field;
(e) $\mathcal{R}_F(G)$ is a skew-field;
(f) $\mathcal{REG}_F(G)$ is a skew-field;

(vi) Let S be a ring satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K). Then the following assertions are equivalent:

(a) The group G has property (B);
(b) The ring S is semisimple;

(vii) Let S be a ring satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K). Then the following assertions are equivalent:

(a) G is torsionfree;
(b) The ring S is a skewfield;

(viii) If G is torsionfree and there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K), then the rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree and are skewfields.

Proof. (i) Let M be a finitely presented FG-module. Then $S \otimes_{FG} M$ is a finitely presented S-module. As S is von Neumann regular, $S \otimes_{FG} M$ is a finitely generated projective S-module and hence defines a class $[S \otimes_F GM] \in K_0(S)$. Because of property (K) this class lies in the image of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(k)} K_0(S)$. Hence $\mathcal{U}(G) \otimes_{FG} M$ is a finitely generated projective $\mathcal{U}(G)$-module whose class in the image of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G))$ for the inclusion l.

(ii) This follows from assertion (i) since the Full-Farrell-Jones Conjecture implies the surjectivity of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG)$, see [19, Theorem 13.65 (xii) on page 421].

(iii) This follows from assertion (i), since $\mathcal{REG}_F(G)$ is von Neumann regular.

(iv) This follows from assertion (ii) since $\mathcal{REG}_F(G)$ is von Neumann regular.

(v) We get the implication (va) $\implies$ (vb) from Remark 2.43. The implication (vb) $\implies$ (vd) is proved in [18, Lemma 10.39 on page 388], where only the case $F = \mathbb{C}$ is treated, but the argument carries directly over to the general case $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. The implication (vd) $\implies$ (vc) is obvious. The implication (vc) $\implies$ (va) follows from assertion (ii). Hence we have proved that the assertions (va), (vb), (vc), and (vd) are equivalent. Now assertion (v) follows from assertion (viii).

(vi) Suppose that S is semisimple. Then $K_0(S)$ is finitely generated, see (2.25). Hence the image of the composite

$$\bigoplus_{H\subseteq G, |H|<\infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(k)} K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

for the inclusion $l\colon S \to \mathcal{U}(G)$ is finitely generated. This image is $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \subseteq \Lambda$. Hence G has property (B).

Suppose G has property (B). Because of Proposition 2.17 it suffices to show that S is Noetherian.

Let $\operatorname{lcm}(G)$ be the least common multiple of the orders of finite subgroups of G. Since S has property (K), the image of the composition

$$K_0(S) \xrightarrow{l} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

lies in $\frac{1}{\operatorname{lcm}(G)}\mathbb{Z} = \{r \in \mathbb{R} \mid \operatorname{lcm}(G)\cdot r \in \mathbb{Z}\}$. Next we show that for any chain of ideals $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ of $\mathcal{S}(G)$ with $I_i \neq I_{i+1}$ we have $r \leq \operatorname{lcm}(G)$.

Choose $x_i \in I_i$ with $x_i \notin I_{i-1}$ for $1 \leq i \leq r-1$. Let J_i be the ideal generated by $x_1, x_2, \ldots, x_i$ for $1 \leq i \leq r-1$. Then we obtain a sequence of finitely generated ideals of the same length $\{0\} = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_r = \mathcal{S}(G)$ of $\mathcal{S}(G)$ with $J_i \neq J_{i+1}$. Since S is von Neumann regular, Proposition 2.18 implies we get a direct sum decompositions $J_i = J_{i-1} \oplus K_i$ for $i = 1, 2, \ldots, r$ for finitely generated projective non-trivial S-modules K_1, K_2, $\ldots$, K_r. Choose an idempotent $p_i \in M_{n_i}(S)$ representing K_i. Then p_i considered as an element $M_{n_i}(\mathcal{U}(G))$ represents $\mathcal{U}(G) \otimes_{\mathcal{S}(G)} K_i$ and is non-trivial. Hence $\mathcal{U}(G) \otimes_{\mathcal{S}(G)} K_i$ is a non-trivial finitely generated projective $\mathcal{U}(G)$-module. We conclude from [18, Theorem 8.29 on page 330] that $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{U}(S)} K_i) > 0$ holds for $i = 1, 2, \ldots, r$. Hence we get

$$\begin{aligned} 0 < \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_1) &< \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_2) \\ &< \cdots < \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_{r-1}) < 1. \end{aligned}$$

Since $\operatorname{lcm}(G) \cdot \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_S J_i)$ is an integer for $i = 1, 2, \ldots, r-1$, we get $r \leq \operatorname{lcm}(G)$.

(vii) Suppose that G is torsionfree. We have seen in the proof of assertion (vi) that S has no non-trivial ideal. This implies that S is a skewfield.

Suppose that S is a skewfield. Then $\mathbb{Z} \to K_0(S)$ sending n to $n \cdot [S]$ is bijective. Hence the image of $K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$ is $\mathbb{Z}$. Therefore the image of the composite

$$\bigoplus_{H\subseteq G, |H|<\infty} K_0(FH) \to K_0(FG) \xrightarrow{K_0(k)} K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

is $\mathbb{Z}$. Since this image is $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \subseteq \Lambda$, the group G is torsionfree.

(viii) Since $\mathcal{D}_F(G)$ is division closed in $\mathcal{U}(G)$ and hence division closed in the skewfield S, it is itself a skewfield. Now the equality $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$ follows from Proposition 2.22 (ii). This finishes the proof of Theorem 2.68. ◻

Next we record the special case, where G is torsionfree.

Theorem 2.69 (The Atiyah Conjecture for torsionfree groups and skewfields) *Let G be a torsionfree group. Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

(i) The following statements are equivalent:

- *(a) The K-theoretic Atiyah Conjecture 2.40 holds for G and F;*
- *(b) The group G satisfies the Atiyah Conjecture with coefficients in F for torsionfree groups, see Conjecture 2.29;*
- *(c) There exists a skewfield S with $FG \subseteq S \subseteq \mathcal{U}(G)$;*
- *(d) The ring $\mathcal{D}_F(G)$ is a skew field;*
- *(e) The ring $\mathcal{R}_F(G)$ is a skew field;*
- *(f) The ring $\mathcal{REG}_F(G)$ is a skew field;*
- *(g) The subrings rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree and are skew fields;*

(ii) Suppose that G satisfies the Atiyah Conjecture with coefficients in F for torsionfree groups, see Conjecture 2.29;
Then we get for every projective FG-chain complex C_* and every $n \geq 0$

$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{FG} C_*\big) = \dim_{\mathcal{D}(G)}\big(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big).$$

In particular $b_n^{(2)}\big(\mathcal{N}(G) \otimes_{FG} C_*\big)$ is an integer or ∞.

Proof. (i) Assertions (ia) and (ib) are equivalent by Proposition 2.47, since for torsionfree groups we have $\operatorname{center}(\mathcal{N}(G))^{\mathbb{Z}/2} = \mathbb{R}$ and under this identification we have $\dim^u_{\mathcal{N}(G)} = \dim_{\mathcal{N}(G)}$. The assertions (ia) (ic), (id) (ie), and (if) are equivalent by Theorem 2.68 (v). We get the implication (id) $\Longrightarrow$ (ig) from Proposition 2.22 (ii).

(ii) This follows from Proposition 2.19 (ii) and (2.21). ◻

Consider the situation of Theorem 2.68 (vi). One can say actually more about S than just that it is semisimple. Recall that we have seen in the proof of assertion (vi) that for any chain of ideals $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ of S with $I_i \neq I_{i+1}$ we have $r \leq \operatorname{lcm}(G)$.

Theorem 2.70 (The ring $\mathcal{D}_F(G)$ is Atiyah-expected)
Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. Suppose G has property (B). Then the following assertions are equivalent:

(i) The ring $\mathcal{D}_F(G)$ is semisimple ring, which is Atiyah-expected in the sense of [12, Definition 3.6];
(ii) The ring $\mathcal{D}_F(G)$ is semisimple and has property (K);
(iii) The composite

$$\bigoplus_{H \subseteq G, |H| < \infty} G_0(FH) \to G_0(\mathcal{D}_F(G))$$

induced by the various inclusions $FH \to \mathcal{D}_F(G)$ is surjective;
(iv) The K-theoretic Atiyah Conjecture 2.40 holds for G and F;
(v) The center-valued Atiyah Conjecture 2.46 holds for G and F.

Proof. The equivalence of the statements (i), (ii), (iii), and (v) is proved in [12, Theorem 3.7]. The equivalence of the statements (iv) and (v) is proved in Proposition 2.47. □

The notion of Atiyah-expected in the sense of [12, Definition 3.6] makes precise predictions about the numbers k and l_i appearing in the decomposition of $\mathcal{D}_F(G)$ as product $\prod_{i=1}^{k} \mathrm{M}_{l_i}(D_i)$ for appropriate division rings D_i in terms of G and F.

2.11 Strategy for a Proof of the K-Theoretic Atiyah Conjecture in the Torsionfree Case Following Jaikin-Zapirain

2.12 Notes

For more information about the Atiyah Conjecture we refer for instance to [8], [9], and [18, Chapter 10], **Comment 17 (by W.)**: This list has to be completed.

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Chapter 3
L^2-torsion

3.1 Introduction

3.2 Review

3.3 Title of the Section

3.4 Notes

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Prime Characteristic

4.1 Introduction

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Applications to Group Theory

5.1 Introduction

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Measurable Group Theory

6.1 Introduction

6.2 Review

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7.1 Introduction

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Chapter 8
Further Topics

8.1 Introduction

8.2 Review

8.3 Title of the Section

8.4 Errata to Part I

Comment 18 (by W.): Add a list Errata correcting misprints and wrong statement appearing in [18].

8.5 Notes

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Chapter 9
Solutions of the Exercises

Chapter 2

2.4. Let R be a ring coming with a trace $\operatorname{tr}_R\colon R \to G$ with values in an abelian group G. This trace extends to matrices by the usual formula $\operatorname{tr}_R(A) = \sum_{i=1}^n \operatorname{tr}_R(a_{i,i})$ for $A \in \mathrm{M}_n(R)$. Then the Hattori-Stallings rank $\dim_R(P) \in G$ of a finitely generated projective R-module P is defined to be $\operatorname{tr}_R(A)$ where A is an element in $\mathrm{M}_n(R)$ with $A^2 = A$ such that the image of the R-linear map $r_A\colon R^n \to R^n$ induced by right multiplication with A is R-isomorphic to P. Let $B \in \mathrm{M}_m(R)$ be another such matrix. By taking the block sum with trivial square matrix one can arrange $m = n$. Then one can find $U \in \mathrm{GL}_n(R)$ with $UAU^{-1} = B$. Now the trace property implies $\operatorname{tr}_R(B) = \operatorname{tr}_R(UAU^{-1}) = \operatorname{tr}_R(AU^{-1}U) = \operatorname{tr}_R(A)$.

If we apply this to $R = \mathcal{N}(G)$ and $\operatorname{tr}_R = \operatorname{tr}_{\mathcal{N}(G)}$, we obtain a well-defined element $\dim_{\mathcal{N}(G)}(P) \in \mathbb{C}$. It remains to prove that it actually lies in $\mathbb{R}_{\geq 0}$. This follows from the fact that we can alway find $A \in \mathrm{M}_n(\mathcal{N}(G))$ satisfying $A = A^2$, $\operatorname{im}(r_A) \cong_{\mathcal{N}(G)} P$, and $A = B^*B$ for some $B \in \mathrm{M}_n(\mathcal{N}(G))$. This follows from Remark 2.5 and taking orthogonal projections. This implies $\dim_{\mathcal{N}(G)}(P) = \operatorname{tr}_{\mathcal{N}(G)}(B^*B) \in \mathbb{R}_{\geq 0}$.

2.10. The first assertion follows from [18, Theorem 1.9 (3) on page on page 18] and obviously implies the second assertion.

2.37. The implication (i) $\Longrightarrow$ (ii) follows from Remark 2.36. The implication (ii) $\Longrightarrow$ (iii) is obvious. The implication (iii) $\Longrightarrow$ (i) is proved as follows. Let $r = \inf\{q \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \mid q > 0\}$. As the subgroup $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ is countable and closed in $\mathbb{R}$, we must have $r \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ and $r > 0$. This implies $r \leq |H|^{-1}$ for every finite subgroup $H \subseteq G$. Hence r^{-1} is an upper bound on the orders of the finite subgroups of G.

2.53. Take $G = \bigoplus_{i=1}^{\infty} Z/2^k$. Choose $n \in \mathbb{Z}_{\geq 1}$ such that $2^{1-n} \leq \epsilon$ holds. Put $P = \bigoplus_{k=n}^{\infty} F[G/\mathbb{Z}_{2^k}]$, where $F[G/\mathbb{Z}_{2^k}]$ is the permutation FG-module given by the FG-set $G/\mathbb{Z}_{2^k}$ obtained from G by dividing out the summand $\mathbb{Z}/2^k$. Then P is projective but not finitely generated and we conclude from Theorem 2.11

$$\dim_{\mathcal{N}(G)}(P) = \sum_{k=n}^{\infty} \dim_{\mathcal{N}(G)}(F[G/\mathbb{Z}/2^k]) = \sum_{k=n}^{\infty} \dim_{\mathcal{N}(\mathbb{Z}/2^k)}(F)$$
$$= \sum_{k=n}^{\infty} 2^{-k} = 2^{1-n} < \epsilon.$$

Chapter 3

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Notation

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