The Burnside Ring, Equivariant Stable Cohomotopy and the Segal Conjecture for Infinite Groups

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Outline and goal

- Long term goal: Extend notions about equivariant (co-)homotopy and (co-)homology for finite groups to infinite groups.
- Review for finite groups.
- Motivation and basic questions
- The notion of the Burnside ring for infinite groups.
- Stable cohomotopy for infinite groups.
- The Segal Conjecture for infinite groups.
- Rational computations of $K^*(BG)$.
- Outlook.

Review for finite groups

Definition (Burnside ring of a finite group)

The isomorphism classes of finite G-sets form a commutative associative semi-ring with unit under disjoint union and cartesian product. The Burnside ring $A(G)$ is the Grothendieck ring associated to this semi-ring.

Definition (Stable cohomotopy)

Let X be a G-CW-complex. Define for $n \in \mathbb{Z}$ its n-th stable cohomotopy group by

$$\pi_G^n(X) = \begin{cases} \operatorname{colim}_V [S^V \wedge S^{-n} \wedge X_+, S^V]^G & n \leq 0; \\ \operatorname{colim}_V [S^V \wedge X_+, S^n \wedge S^V]^G & n \geq 0. \end{cases}$$

where V runs through the orthogonal G-representations of G and $X_+ = X \amalg \{\bullet\}$.

Theorem (Segal (1971))

Let G be a finite group. Then we obtain an isomorphism of rings

$$\pi_G^0(\{\bullet\}) \xrightarrow{\cong} A(G)$$

Theorem (Atiyah-Segal (1969))

Let G be a finite group and let X be a finite G-CW-complex. Then there is an isomorphism

$$K_G^n(X)\widehat{\ }_I \xrightarrow{\cong} K^n(EG \times_G X)$$

where $I \subseteq R_{\mathbb{C}}(G)$ is the augmentation ideal.
In particular we obtain an isomorphism

$$R_{\mathbb{C}}(G)\widehat{\ }_I \xrightarrow{\cong} K^0(BG).$$

Theorem (Segal Conjecture, proved by Carlsson (1984))

The Segal Conjecture is true, i.e., for every finite group G and every finite G-CW-complex X there is an isomorphism

$$\pi_G^n(X)\widehat{_I} \xrightarrow{\cong} \pi^n(EG \times_G X),$$

where $I \subseteq A(G)$ is the augmentation ideal.
In particular we obtain an isomorphism

$$A(G)\widehat{_I} \xrightarrow{\cong} \pi_G^0(BG).$$

Motivation and basic questions

- Baum-Connes Conjecture and Farrell-Jones Conjecture.
- Computations of algebraic K- and L-groups of group rings or of topological K-theory of reduced C^*-algebras of infinite groups.
- Computations of (co)-homology or topological K-theory of the classifying space BG of an infinite group G.
- Can one extend classical results to this setting?
- Can one get new useful information in this new setting (here for infinite groups and their actions)?
- Are there interesting and promising open problems?

Classifying space for proper actions

- In the case of infinite groups on needs for geometric constructions the condition that the G-CW-complexes are proper, i.e., all isotropy groups are finite.
- Hence we cannot consider the one-point-space $\{\bullet\}$ and cannot assume that G-CW-complex has a base point which is fixed under the G-action if G is infinite.
- So we must find a replacement for $\{\bullet\}$.

Definition (Classifying space of proper G-actions)

A model for the classifying space for proper G-actions is a G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible if $H \subseteq G$ is finite and is empty if $H \subseteq G$ is infinite.

Theorem (tom Dieck (1971))

- *A model for* $\underline{E}G$ *exists;*
- *Two models are* G*-homotopy equivalent;*
- *The* G*-CW-complex* $\underline{E}G$ *is characterized uniquely up to* G*-homotopy by the property that for every proper* G*-CW-complex* X *there is up to* G*-homotopy precisely one* G*-map* $X \to \underline{E}G$.

- Obviously $\{\bullet\}$ is a model for $\underline{E}G$ if and only if G is finite.
- We have $EG = \underline{E}G$ if and only if G is torsionfree.
- The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small. For instance:
- Rips complex for word hyperbolic groups;
- Teichmüller space for mapping class groups;
- Outer space for the group of outer automorphisms of free groups;
- L/K for a connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;
- CAT(0)-spaces with proper isometric G-actions, e.g., Riemannian manifolds with non-positive sectional curvature or trees.

Generalizations of the representation and the Burnside ring

- Before we try to extend the notion of the Burnside ring to finite group, we review the possible generalizations of the representation ring over a field F of characteristic zero to infinite groups. This will be a guide line.

Definition (Generalizations of the representation ring)

- Let $\mathrm{Sw}^f(G;F)$ be the Grothendieck group of finite-dimensional F-vector spaces with linear G-action. (This is word by word the classical definition).
- Let $K_0(FG)$ be the projective class group.
- Put

$$\begin{aligned} R_{\mathrm{cov},F}(G) &:= \operatorname*{colim}_{H\subseteq G, |H|<\infty} R_F(H); \\ R_{\mathrm{inv},F}(G) &:= \operatorname*{invlim}_{H\subseteq G, |H|<\infty} R_F(H). \end{aligned}$$

- Let $K^0_G(\underline{E}G)$ and $K_0^G(\underline{E}G)$ respectively be the zero-th equivariant topological K-theory group and equivariant topological K-homology group of $\underline{E}G$.

- Notice that for a finite group all the notions in the definition above reduce to $R_F(G)$.
- For infinite groups all of these notions are different.
- One cannot say which is the right one. The possible choice depends on the problem one is studying. All of these notions have been studied and applied to various problems.
- The definitions above suggest the following definitions for possible generalizations of the Burnside ring.
- The dictionary between the generalizations for the Burnside ring and for the representation ring come from the passage from a G-set S to its permutation module, i.e., the F-vector space FS with S as basis.

Definition (Generalizations of the Burnside ring)

- Define $\overline{A}(G)$ to be the Grothendieck group of finite sets with G-action. (This is word by word the classical definition.)
- Define $\underline{A}(G)$ to be the Grothendieck group of proper cofinite G-sets.
- Put

$$\begin{aligned} A_{\text{cov}}(G) &:= \operatorname*{colim}_{H \subseteq G, |H| < \infty} A(H); \\ A_{\text{inv}}(G) &:= \operatorname*{invlim}_{H \subseteq G, |H| < \infty} A(H). \end{aligned}$$

- Let $\pi_G^0(\underline{E}G)$ and $\pi_0^G(\underline{E}G)$ respectively be the zero-th equivariant stable cohomotopy and homotopy group respectively of the classifying space for proper G-actions $\underline{E}G$.

We have the following dictionary

$R_F(G)$	$A(G)$	key words
$\mathrm{Sw}^f(G;F)$	$\overline{A}(G)$	induction theory, Green functors, profinite groups
$K_0(FG)$	$\underline{A}(G)$	universal additive invariant, equivariant Euler characteristic, L^2-Euler characteristic
$R_{\mathrm{cov},F}(G)$	$A_{\mathrm{cov}}(G)$	collecting all values for finite subgroups with respect to induction
$R_{\mathrm{inv},F}(G)$	$A_{\mathrm{inv}}(G)$	collecting all values for finite subgroups with respect to restriction
$K_G^0(\underline{E}G)$	$\pi_G^0(\underline{E}G)$	completion theorems, equivariant vector bundles,
$K_0^G(\underline{E}G)$	$\pi_0^G(\underline{E}G)$	representation theory, Baum-Connes Conjecture

Equivariant cohomology theories

Definition (G-cohomology theory)

A G-cohomology theory $\mathcal{H}_G^*$ is a contravariant functor $\mathcal{H}_G^*$ from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded R-modules together with natural transformations

$$\delta_G^n(X,A) : \mathcal{H}_G^n(A) \to \mathcal{H}_G^{n+1}(X,A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant cohomology theory)

An equivariant cohomology theory $\mathcal{H}^?_*$ consists of a G-cohomology theory $\mathcal{H}^*_G$ for every group G together with the following so called induction structure: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}^n_H(X, A) \;\to\; \mathcal{H}^n_G(\operatorname{ind}_\alpha(X, A))$$

satisfying:

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms
- Functoriality in α
- Compatibility with conjugation

- Here are some examples for equivariant cohomology theories $\mathcal{H}^*_?$
- Quotients
  Let $\mathcal{K}^*$ be a non-equivariant cohomology theory. Define $\mathcal{H}^*_?$ by

$$\mathcal{H}^*_G(X) \ := \ \mathcal{K}^*(G\backslash X).$$

- Borel homology
  Let $\mathcal{K}^*$ be a non-equivariant homology theory. Define $\mathcal{H}^*_?$ by

$$\mathcal{H}^*_G(X) \ := \ \mathcal{K}^*(EG \times_G X).$$

- Equivariant topological K-theory K^*_G for proper G-CW-complexes is constructed by Lück-Oliver (2001) in terms of equivariant spectra. Let $H \subseteq G$ be a finite group. Then $K^G_n(G/H) = K^n_H(\{\bullet\})$ is $R_{\mathbb{C}}(\{\bullet\})$ for even n and $\{0\}$ for odd n. It agrees with the construction of Kasparov in terms of Kasparov cycles.

- An Ω-spectrum $\mathbf{E}$ defines a cohomology theory by sending a space X to $\pi_*^s(\mathrm{map}(X_+, \mathbf{E}))$. This generalizes to the equivariant setting as follows.

Theorem (Equivariant cohomology theories and spectra Lück(2004))

Consider a contravariant functor

$$\mathbf{E}\colon \ \mathrm{GROUPOIDS} \to \Omega - \mathrm{SPECTRA}$$

sending equivalences of groupoids to weak equivalences of spectra. Then there exists an equivariant cohomology theory $\mathcal{H}^*_?(-; \mathbf{E})$ with the property that for every group G, subgroup $H \subseteq G$ and $n \in \mathbb{Z}$

$$\mathcal{H}^n_G(G/H) \ = \ \mathcal{H}^n_H(\{\bullet\}) \ = \ \pi_{-n}(\mathbf{E}(H)).$$

Theorem (Equivariant stable cohomotopy in terms of equivariant vector bundles, Lueck(2005))

Equivariant stable cohomotopy $\pi_?^*$ *is defined and yields an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes.*
In particular for every finite subgroup H of the group G we have

$$\pi_G^n(G/H) \cong \pi_H^n(\{\bullet\})$$

and there are isomorphisms of rings

$$\pi_G^0(G/H) \cong \pi_H^0(\{\bullet\}) \cong A(H).$$

If G is finite, this definition coincides with the classical one.

- Here is a sketch of its construction.
- Let X be a finite proper G-CW-complex.
- An element in $\pi_G^n(X)$ is represented by a fiber preserving and fiberwise basepoint preserving G-map

$$u\colon S^{\xi \oplus \underline{\mathbb{R}^k}} \to S^{\xi \oplus \underline{\mathbb{R}^{k+n}}}$$

where ξ is a G-vector bundle over X, we denote by $\underline{\mathbb{R}^k}$ is the trivial G-vector bundle $X \times \mathbb{R}^k \to X$ for the trivial G-representation $\mathbb{R}^k$ and k is some integer satisfying $k + n \geq 0$.

- Addition comes from a fiberwise pinching construction. The multiplicative structure can be defined by a fiberwise smash product or by composition.

- The class $[u] \in \pi_G^n(X)$ of u does not change if
- We alter u by a homotopy of such maps;
- We replace u by the following stabilization with a G-vector bundle μ

$$S^{(\xi\oplus\mu)\oplus\underline{\mathbb{R}^k}} = S^{\xi\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\mu} \xrightarrow{u\wedge_X \mathrm{id}} S^{\xi\oplus\underline{\mathbb{R}^{k+n}}} \wedge_X S^{\mu} = S^{(\xi\oplus\mu)\oplus\underline{\mathbb{R}^{k+n}}};$$

- We conjugate u by an isomorphism of G-vector bundle $v\colon \xi \to \xi'$, i.e., we replace u by the composition

$$S^{\xi'\oplus\underline{\mathbb{R}^k}} \xrightarrow{S^{v^{-1}}\oplus\mathrm{id}} S^{\xi\oplus\underline{\mathbb{R}^k}} \xrightarrow{u} S^{\xi\oplus\underline{\mathbb{R}^{k+n}}} \xrightarrow{S^{v}\oplus\mathrm{id}} S^{\xi'\oplus\underline{\mathbb{R}^{k+n}}}.$$

- Obvious question: Why do we consider G-vector bundles ξ instead of G-representations V?
- Why we cannot just use the word by word extensions of the classical definition?
- The proof that π_G^* is a G-cohomology theory with a multiplicative structure would go through and for finite groups we would get the classical notion.
- The problem is that the induction structure does not exists anymore as the following example will show.
- So a key idea is to replace representations or, equivalently, trivial G-vector bundles by arbitrary G-vector bundles.
- For infinite groups there are not enough representations but enough equivariant vector bundles.

Example (Groups without non-trivial representations)

- *The exists infinite simple groups G.*
- *For such a group every (finite-dimensional) G-representation is trivial.*
- *Then the word by word extension of the classical definition to a proper G-CW-complex X would just lead to $\pi^n(G\backslash X)$.*
- *In particular $\pi_G^n(G/H)$ is the non-equivariant stable cohomotopy group $\pi_s^n(\{\bullet\})$ for all finite subgroups $H \subseteq G$.*
- *On the other hand the existence of an induction structure would predict for $X = G/H$ that $\pi_G^n(G/H)$ is isomorphic to $\pi_H^n(\{\bullet\})$, which is in general different from $\pi_s^n(\{\bullet\})$.*

- There is a spectrum version of equivariant stable cohomotopy for arbitrary proper G-CW-complexes which reduces to the one above for finite proper G-CW-complexes Barcenaz (2008).
- Rationally stable cohomotopy is singular cohomology with rational coefficients. This result extends to the equivariant setting as follows.

Theorem (Rational Computation of π_*^G, Lueck(2005))

There are isomorphisms

$$\pi_G^n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H),H\subseteq G} H^n\left(W_G H\backslash X^H;\mathbb{Q}\right)$$

for all $n \in \mathbb{Z}$ and all finite proper G-CW-complexes X.
They are compatible with the obvious multiplicative structures and induction structures.

Completion Theorems

Theorem (Segal Conjecture for infinite groups, Lueck (2008))

Let X be a finite proper G-CW-complex and let L be a proper finite dimensional G-CW-complex such that there is an upper bound on the order of its isotropy groups. Let $f\colon X \to L$ be a G-map.
Then there is an isomorphism of pro-$\mathbb{Z}$-modules

$$\{\pi_G^m(X)/\mathbb{I}_G(L)^n \cdot \pi_G^m(X)\}_{n\geq 1} \to \{\pi_s^m\left((EG \times_G X)_{(n-1)}\right)\}_{n\geq 1}.$$

In particular we obtain an isomorphism

$$\pi_s^m(EG \times_G X) \;\cong\; \pi_G^m(X)\widehat{\;}_{\mathbb{I}_G(L)}.$$

Corollary

Suppose that there is a finite G-CW-model for $\underline{E}G$. We define the homotopy theoretic Burnside ring $A_{\mathrm{ho}}(G)$ by $\pi_G^0(\underline{E}G)$. Let $I \subseteq A_{\mathrm{ho}}(G)$ be the augmentation ideal. It is the kernel of the map sending $[u]$ to the degree of u_x for any $x \in \underline{E}G$. Then we obtain an isomorphism

$$\pi_s^m(BG) \cong \pi_G^m(\underline{E}G)\widehat{_I}.$$

In dimension zero we get an isomorphism

$$\pi_s^0(BG) \cong A_{\mathrm{ho}}(G)\widehat{_I}.$$

Theorem (Atiyah-Segal Completion Theorem for infinite groups, Lück-Oliver (2001))

The analogue of all these results for the Atiyah-Segal Completion

- The proofs of these completion theorems use the fact that they have already been proved for finite groups.
- In the Atiyah-Segal case the main problem is to construct a certain family of elements in the various representation rings of the finite subgroups of G which satisfy certain compatibility conditions coming from inclusion and conjugation of finite subgroups. The prime deal structure of the representation rings do play an important role
- In the Segal case an analogous problem arises but one has to replace the representation rings by Burnside rings.
- However, the methods of proofs are rather different as already the proofs of the Atyah-Segal Completion Theorem and of the Segal Conjecture for finite groups are rather different.

Rational computations of $K^*(BG)$

- A good theory of equivariant Chern characters has been developed and has been applied to several instances.
- In particular they play an important role in the computation of algebraic K- and L-groups of group rings and the topological K-theory of the reduced group C^*-algebra based on the Baum-Connes Conjecture and the Farrel-Jones Conjecture.
- As an illustration we mention the following result which aims in a different direction, namely, the topological K-theory of BG.
- It is a typical example of the successful method to make computations about BG using $\underline{E}G$.

Theorem (Rational computation of $K^*(BG)$, Lueck(2007))

Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions. Then there is a $\mathbb{Q}$-isomorphism

$$\overline{\mathrm{ch}}^n_{G,\mathbb{Q}} \colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong}$$

$$\left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \mathrm{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right),$$

where $\mathrm{con}_p(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order p^d for some integer $d \geq 1$ and $C_G\langle g\rangle$ is the centralizer of the cyclic subgroup $\langle g\rangle$ generated by g.

- The map above is in general not compatible with the obvious multiplicative structures. If we complexify, we obtain isomorphisms compatible with the multiplicative structures.
- There is a formula for $K^*(BG)$ for finite groups

$$\begin{aligned} K^0(BG) &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}_p} \\ &\cong \mathbb{Z} \times \prod_{p \text{ prime}} (\widehat{\mathbb{Z}_p})^{|\operatorname{con}_p(G)|}; \\ K^1(BG) &\cong 0. \end{aligned}$$

- For infinite groups on cannot expect a general integral answer. The main new input is the topological K-theory of the orbifold $G\backslash \underline{E}G$. Certain computations will appear in a paper joint with Joachim.

Theorem (Multiplicative structure, Lueck(2007))

Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions. Then there is a $\mathbb{C}$-isomorphism

$$\overline{\mathrm{ch}}^n_{G,\mathbb{C}} \colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong}$$

$$\left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{C})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \mathrm{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle; \widehat{\mathbb{Q}_p} \otimes_{\mathbb{Q}} \mathbb{C})\right).$$

It is compatible with the standard multiplicative structure on $K^*(BG)$ and the natural one on the target which is given by

$$\left(a, u_{p,(g)}\right) \cdot \left(b, v_{p,(g)}\right) = \left(a \cdot b, \left(a \cdot v_{p,(g)} + b \cdot u_{p,(g)} + u_{p,(g)} \cdot v_{p,(g)}\right)\right)$$

Open problems

- Construction of a stable homotopy category including a Quillen model structure and smash products (joint project with Schwede)
- Extend the theory to Lie groups.
- At last some wild speculation:
- There are examples of topological groups which are not locally compact (and in particular not Lie groups) but which have a Lie-compact-subgroup-structure, i.e., every compact subgroup is a Lie group.
- Examples are diffeomorphism groups of closed smooth manifolds, loop groups and Kac-Moody groups.
- These often have interesting models for the space $\underline{E}G$ for proper G-actions.
- For instance for a closed smooth manifold M the space of Riemannian metrics is a model for $\underline{E}G$ for the diffeomorphism group of M acting in the obvious way.

- One should give precise definition of the equivariant K-homology of proper G-CW-complexes for topological groups with a Lie-compact-subgroup-structure.
- This would yield a precise definition of the source of the Baum-Connes Conjecture in this setting.
- However since the groups G are not necessarily locally compact, there exists no Haar measure and we cannot make sense of $L^2(G)$ or $C_r^*(G)$. So we have no definition for the target of the Baum-Connes assembly map.
- Nevertheless there is some vague indication that such a Baum-Connes Conjecture may make sense.

- Kitchloo (2008) computed $K_*^G(\underline{E}G)$ using a nice model for $\underline{E}G$ and assuming the existence of the homology theory K_*^G for some loop groups. The answer is in terms of the representation theory of the loop group.
- Notice that $K_*(C_r^*(G))$ is designed to capture the representation theory of a topologial group G.