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# SOME CRITERIA CONCERNING THE RATIONAL VANISHING OF WHITEHEAD GROUPS

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Abstract. We give several examples of finite groups $G$ for which the rank of the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)$ is or is not zero. This is motivated by an earlier theorem of the first author, which implies as a special case that when this group has nonzero rank, the Whitehead group of any other group (finite or infinite) that contains $G$ as a normal subgroup is rationally nontrivial.



## Introduction

This paper is motivated by the following question: for which groups $\Gamma$ does the Whitehead group $\mathrm{Wh}(\Gamma)$ vanish, integrally or rationally? When $\Gamma$ is finite, $\mathrm{Wh}(\Gamma)$ is always finitely generated, and its rank is determined in a theorem of Bass (see [Ba, Theorem 5]). So it is natural to begin studying the Whitehead group of an infinite group $\Gamma$ (integrally or rationally) by trying to compare it to the Whitehead groups of its finite subgroups.

One means of doing this is provided by the Farrell-Jones Conjecture for the algebraic $K$-theory of group rings. This conjecture is known to hold for a rather large class of groups, but is open in general. Among other things, it makes some predictions about the contributions of finite subgroups of a group $\Gamma$ to $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma)$ (see Theorem 1.1 below for one example of such results). Other known results, such as Theorem 1.2, depend instead on assumptions about the homology of centralizers of finite cyclic subgroups of $\Gamma$. But so far, there are very few results about Whitehead groups known to hold in all cases.

One exception to this is Theorem 1.3 below. This had originally been predicted by the Farrell-Jones Conjecture, but is now known to be true for all groups — independently of whether or not the conjecture holds. This theorem in turn implies Corollary 1.4(b), which says, for a group $\Gamma$ with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma) = 0$, that $\Gamma$ can contain a finite group $H$ as a normal subgroup only if $\mathbb{Z} \otimes_{\mathbb{Z}[\mathrm{Aut}(H)]} \mathrm{Wh}(H)$ is finite. Since the theorem and corollary hold for all groups, this could give some evidence that the Farrell-Jones Conjecture for the algebraic $K$-theory of group rings holds more generally.

In Section 1, we describe some of this background in more detail, ending with the statements of Theorem 1.3 and Corollary 1.4. This then motivates the results in the next two sections. In Section 2, we give a general formula for the rank of $\mathbb{Z} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)$, for a finite group $G$, in terms of numbers of classes of elements of $G$ under certain equivalence relations. This formula is then applied in Section 3, where we give a wide range of examples of finite groups $G$ for which $\mathrm{rk}(\mathbb{Z} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)) \neq 0$, and hence of groups that cannot occur as normal subgroups in a group $\Gamma$ with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma) = 0$.

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## 1. Some known results

We begin by describing some general information about the Whitehead group $G$ of a (discrete) group $\Gamma$, and the relation between $\mathrm{Wh}(\Gamma)$ and the Whitehead groups of finite subgroups of $\Gamma$.

A group $\Gamma$ is called a *Farrell-Jones group* if it satisfies the "Full Farrell-Jones Conjecture", as formulated, for example, in [Lu3, Conjecture 13.30]. The full statement is quite complicated and involves $L$-groups as well as $K$-theory, but one easily stated special case is that the Whitehead group of a torsion free group is always trivial.

The class $\mathcal{FJ}$ of Farrell-Jones groups is quite large, and in fact, no groups are known *not* to be in the class. It is known to contain all hyperbolic groups, finite-dimensional CAT(0)-groups, solvable groups, fundamental groups of manifolds of dimension $\leq 3$, and any lattice in a locally compact second countable Hausdorff group. Also, it is closed under taking subgroups, finite free products, finite direct products, and colimits of directed systems (with arbitrary structure maps). For more information about the Full Farrell-Jones Conjecture and its consequences, we refer to [Lu3, Chapter 15], and for a description of what is currently known about the class $\mathcal{FJ}$, to [Lu3, Chapter 16].

As one simple example of the role played by Farrell-Jones groups, we note the following criterion for the rational vanishing of $K$-theory in degree at most 1. When $H \leq \Gamma$ are groups, we let $\mathrm{Aut}_\Gamma(H)$ be the group of automorphisms of $H$ of the form $(x \mapsto gxg^{-1})$ for $g \in N_\Gamma(H)$.

**Theorem 1.1** ([Lu3, Theorem 17.4])**.** *Let $\Gamma$ be a Farrell-Jones group. Consider the following conditions on $\Gamma$:*

(P) *The order of every finite cyclic subgroup $C \leq \Gamma$ is a prime power.*

(A) *For every finite cyclic subgroup $1 \neq C \leq \Gamma$, the automorphism group $\mathrm{Aut}(C)$ is generated by $\mathrm{Aut}_\Gamma(C)$ and the automorphism $(x \mapsto x^{-1})$.*

*Then*

(a) $K_n(\mathbb{Z}\Gamma) = 0$ *for* $n \leq -2$*;*

(b) $\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(\mathbb{Z}\Gamma) = 0$ *if and only if condition* (P) *holds;*

(c) $\mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Z}\Gamma) = 0$ *if condition* (P)*; and*

(d) *conditions* (P) *and* (A) $\implies$ $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma) = 0$ $\implies$ *condition* (A)*.*

The next theorem illustrates the sort of homological condition that can be used to get such results, without assuming $G$ is in $\mathcal{FJ}$.

**Theorem 1.2** ([LRRV, Theorem 1.1])**.** *Let $\Gamma$ be a group. Assume, for every finite cyclic subgroup $C \leq \Gamma$, that the homology groups $H_1(BC_\Gamma(C);\mathbb{Z})$ and $H_2(BC_\Gamma(C);\mathbb{Z})$ of the centralizer $C_\Gamma(C)$ are finitely generated. Then the canonical map*

$$\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(\Gamma)} \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(H) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma)$$

*is injective.*

Here $\mathsf{Sub}_{\mathcal{FIN}}(\Gamma)$ is the category whose objects are the finite subgroups of $\Gamma$, and where a morphism from $H$ to $K$ is the class, modulo $\mathrm{Inn}(K)$, of a group homomorphism $f\colon H \longrightarrow K$ of the form $(x \mapsto gxg^{-1})$ for some $g \in \Gamma$.

Note that the homology condition appearing in Theorem 1.2 holds whenever there is a model for the classifying space $E_{\mathcal{FIN}}(\Gamma)$ for proper $\Gamma$-actions whose 2-skeleton is cocompact (i.e., the orbit space of the 2-skeleton is finite). This is proved using arguments similar to those in the proofs of Lemmas 1.3 and 4.1 in [Lu1]. Examples of groups which satisfy the hypotheses of Theorem 1.2 but are not known to be Farrell-Jones groups are $\mathrm{Out}(F_n)$ (where $F_n$ is a free group on $n$ letters) and Thompson's group.

In contrast to Theorems 1.1 and 1.2, the next theorem holds for *all* (discrete) groups. Let $i\colon G \to \Gamma$ be the inclusion of a finite normal subgroup $G$ in a group $\Gamma$, and let $i_*\colon \mathrm{Wh}(G) \longrightarrow \mathrm{Wh}(\Gamma)$ be the induced homomorphism. The conjugation actions of $\Gamma$ on $G$ and $\Gamma$ induce $\Gamma$-actions on $\mathrm{Wh}(G)$ and $\mathrm{Wh}(\Gamma)$, of which the latter is trivial. Hence $i_*$ induces a homomorphism

$$\overline{\iota_*}\colon \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathrm{Wh}(G) \longrightarrow \mathrm{Wh}(\Gamma).$$

**Theorem 1.3** ([Lu2, Theorem 9.38]). *Let $i\colon G \to \Gamma$ be the inclusion of a finite normal subgroup $G$ into an arbitrary group $\Gamma$. Then the homomorphism*

$$\overline{\iota_*}\colon \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathrm{Wh}(G) \longrightarrow \mathrm{Wh}(\Gamma)$$

*induced by $i$ has finite kernel.*

**Corollary 1.4.** *Let $\Gamma$ be a group such that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma) = 0$. Then the following hold.*

(a) *Every finite cyclic subgroup of the center $Z(\Gamma)$ has order 1, 2, 3, 4, or 6.*

(b) *For each finite normal subgroup $G \trianglelefteq \Gamma$, the group $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} \mathrm{Wh}(G)$ and hence also $\mathbb{Z} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)$ is finite, where $\mathrm{Aut}(G)$ acts in the canonical way on $\mathrm{Wh}(G)$ and trivially on $\mathbb{Z}$.*

Motivated by Corollary 1.4(b), in the rest of the paper, we look for finite groups $G$ for which $\mathrm{rk}(\mathbb{Z} \otimes_{\mathbb{Z}[\mathrm{Aut}(G)]} \mathrm{Wh}(G)) > 0$, since no such group can occur as a normal subgroup in $\Gamma$ if $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\Gamma) = 0$. Note that this condition on $\mathbb{Z} \otimes_{\mathbb{Z}[\mathrm{Aut}(G)]} \mathrm{Wh}(G)$ is independent of $\Gamma$.

## 2. Whitehead groups and conjugacy classes

From now on, we focus attention on Whitehead groups of finite groups.

When $K$ is a field of characteristic 0 and $n > 0$ is an integer, we let $\mu_n$ be the group of $n$-th roots of unity in an algebraic closure of $K$, and regard the Galois group $\mathrm{Gal}(K(\mu_n)/K)$ as a subgroup of $(\mathbb{Z}/n)^\times$. Thus each automorphism $\gamma \in \mathrm{Gal}(K(\mu_n)/K)$ is identified with the unique class $a+n\mathbb{Z} \in (\mathbb{Z}/n)^\times$ such that $\gamma(\zeta) = \zeta^a$ for $\zeta \in \mu_n$. For example,

$$\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) = (\mathbb{Z}/n)^\times, \quad \mathrm{Gal}(\mathbb{R}(\mu_n)/\mathbb{R}) = \{\pm 1\} \text{ if } n \geq 3, \quad \text{and} \quad \mathrm{Gal}(\mathbb{C}(\mu_n)/\mathbb{C}) = 1.$$

**Definition 2.1.** Let $K$ be a field of characteristic 0. When $G$ is a finite group, two elements $g, h \in G$ of the same order $n$ are *$K$-$G$-conjugate* if $g$ is $G$-conjugate to $h^a$ for some $a + n\mathbb{Z} \in \mathrm{Gal}(K(\mu_n)/K)$. More generally, when $A \leq \mathrm{Aut}(G)$ is a subgroup containing $\mathrm{Inn}(G)$, the elements $g, h \in G$ of order $n$ are *$K$-$A$-conjugate* if $g$ is in the $A$-orbit of $h^a$ for some $a + n\mathbb{Z} \in \mathrm{Gal}(K(\mu_n)/K)$.

Thus $g, h \in G$ are $\mathbb{R}$-$A$-conjugate if $g$ is in the $A$-orbit of $h$ or $h^{-1}$, while they are $\mathbb{Q}$-$A$-conjugate if the cyclic subgroups $\langle g\rangle$ and $\langle h\rangle$ are in the same $A$-orbit. In particular, the number of $\mathbb{Q}$-$G$-conjugacy classes is equal to the number of conjugacy classes of cyclic subgroups of $G$.

As usual, by the *rank* of a finitely generated abelian group $B$, we mean the rank of its free part; i.e., the order of its largest $\mathbb{Z}$-linearly independent subset. Thus $\mathrm{rk}(B) = \dim(\mathbb{Q} \otimes_{\mathbb{Z}} B)$.

For a field $K$ of characteristic 0 and a finite group $G$, we let $\mathrm{Irr}_K(G)$ be the set of isomorphism classes of irreducible $KG$-modules, regarded as a finite $\mathrm{Aut}(G)$-set. Via character theory, one shows the following:

**Proposition 2.2.** *Let $K$ be a field of characteristic 0. For each finite group $G$, and each group of automorphisms $A \le \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$,*

$$|\mathrm{Irr}_K(G)/A| = \#\bigl\{K\text{-}A\text{-conjugacy classes of elements of } G\bigr\}.$$

*Proof.* Let $R_K(G)$ be the representation ring for $KG$-representations: thus a free abelian group with basis $\mathrm{Irr}_K(G)$. Let $\mathrm{Cl}_K(G)$ be the space of all maps $G \longrightarrow K$ that are constant on $K$-$G$-conjugacy classes, regarded as a vector space over $K$. By [Se, §12.4, Corollary 2], the characters of the elements in $\mathrm{Irr}_K(G)$ form a $K$-basis for $\mathrm{Cl}_K(G)$. Hence the sums of the characters in each $A$-orbit form a basis for $\mathrm{Fix}(A, \mathrm{Cl}_K(G))$, and so

$$|\mathrm{Irr}_K(G)/A| = \dim_K(\mathrm{Fix}(A, \mathrm{Cl}_K(G))) = \#\bigl\{K\text{-}A\text{-conjugacy classes of elements of } G\bigr\}. \qquad \square$$

We refer to [Ol, Section I.2a] for the definition of reduced norms for finite dimensional semisimple $\mathbb{Q}$-algebras.

**Proposition 2.3.** *Let $G$ be a finite group. Set $K = Z(\mathbb{Q}G)$, a product of fields, and let $R \le K$ be its unique maximal order (the product of the rings of integers in the factors). The reduced norm induces a homomorphism*

$$\mathrm{nr}\colon K_1(\mathbb{Z}G) \longrightarrow R^\times$$

*that commutes with the actions of $\mathrm{Out}(G)$ on $K_1(\mathbb{Z}G)$ and on $R^\times$, and whose kernel and cokernel are both finite. In particular, for each $A \le \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$,*

$$\mathrm{rk}(\mathbb{Z} \otimes_{\mathbb{Z}A} \mathrm{Wh}(G)) = \mathrm{rk}(\mathrm{Fix}(A, \mathrm{Wh}(G))) = \mathrm{rk}(\mathrm{Fix}(A, K_1(\mathbb{Z}G))) = \mathrm{rk}(\mathrm{Fix}(A, R^\times)).$$

*Proof.* The fact that the reduced norm induces a homomorphism whose kernel and cokernel are finite was shown by Swan (see, e.g., [Sw, Chapter 8] or [Ol, Theorem I.2.5(ii)]).

Clearly, nr commutes with the actions of $A$. So $\mathrm{Fix}(A, K_1(\mathbb{Z}G))$ and $\mathrm{Fix}(A, R^\times)$ have the same rank. $\square$

We also need Dirichlet's units theorem in the following form.

**Lemma 2.4.** *Let $K \supseteq \mathbb{Q}$ be a finite extension, and let $R \subseteq K$ be the ring of integers. Then*

$$\mathrm{rk}(R^\times) = \#\bigl\{\text{field factors in } \mathbb{R} \otimes_{\mathbb{Q}} K\bigr\} - 1.$$

*Proof.* The ring $\mathbb{R} \otimes_{\mathbb{Q}} K$ is isomorphic to a product of one copy of $\mathbb{R}$ for each embedding of $K$ into $\mathbb{R}$, and one copy of $\mathbb{C}$ for each pair of conjugate embeddings of $K$ into $\mathbb{C}$ with image not in $\mathbb{R}$. So the statement follows from Dirichlet's theorem in its usual formulation (see [ST, Theorem B.6]). $\square$

**Proposition 2.5.** *Let $G$ be a finite group. Then for each $A \le \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, we have*

$$\mathrm{rk}(\mathbb{Z} \otimes_{\mathbb{Z}A} \mathrm{Wh}(G)) = \mathrm{rk}(\mathrm{Fix}(A, \mathrm{Wh}(G))) = |\mathrm{Irr}_{\mathbb{R}}(G)/A| - |\mathrm{Irr}_{\mathbb{Q}}(G)/A|.$$

*Proof.* Set $K = Z(\mathbb{Q}G)$, a product of fields, and let $R \le K$ be its unique maximal order (the product of the rings of integers in the factors). Then $\mathrm{rk}(\mathrm{Fix}(A, \mathrm{Wh}(G))) = \mathrm{rk}(\mathrm{Fix}(A, R^\times))$ by Proposition 2.3, and it remains to describe the rank of $\mathrm{Fix}(A, R^\times)$ in terms of representations of $G$.

Set $K_0 = \mathrm{Fix}(A, K)$ and $R_0 = \mathrm{Fix}(A, R)$: the subgroups of elements fixed by $A$. By Wedderburn's theorem, there is a natural bijection of $A$-sets from $\mathrm{Irr}_{\mathbb{R}}(G)$ to the set of simple factors in $\mathbb{R}G$, and hence to the set of field factors in $Z(\mathbb{R}G) \cong \mathbb{R} \otimes_{\mathbb{Q}} K$. Also, by Galois theory, if a subgroup $A_0 \le A$ sends a field factor to itself, then the set of elements in that field fixed by $A_0$ is a subfield. Thus the number of field factors in $\mathbb{R} \otimes_{\mathbb{Q}} K_0 = \mathrm{Fix}(A, \mathbb{R} \otimes_{\mathbb{Q}} K)$ is equal to $|\mathrm{Irr}_{\mathbb{R}}(G)/A|$.

Clearly, $R_0$ contains the product of the rings of integers in the field factors of $K_0$, with equality since for each field factor $L$ of $K_0$, the image of $R_0$ under projection to $L$ is a finitely generated subring and hence contained in the ring of integers (see, e.g., [ST, Lemma 2.8]). So by Lemma 2.4,

$$\begin{aligned}\mathrm{rk}(\mathrm{Fix}(A, R^\times)) = \mathrm{rk}((R_0)^\times) &= \#\big\{\text{field factors in } \mathbb{R} \otimes_{\mathbb{Q}} K_0\big\} - |\mathrm{Irr}_{\mathbb{Q}}(G)/A| \\ &= |\mathrm{Irr}_{\mathbb{R}}(G)/A| - |\mathrm{Irr}_{\mathbb{Q}}(G)/A|. \qquad \square\end{aligned}$$

**Theorem 2.6.** *For each finite group $G$, and each subgroup $A \le \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$,*

$$\begin{aligned}\dim_{\mathbb{Q}}\big((\mathbb{Q} \otimes_{\mathbb{Z}A} \mathrm{Wh}(G))\big) = \#\big\{&\mathbb{R}\text{-}A\text{-conjugacy classes of elements in } G\big\} \\ &- \#\big\{\mathbb{Q}\text{-}A\text{-conjugacy classes of elements in } G\big\}.\end{aligned}$$

*Proof.* When $A = \mathrm{Inn}(G)$, this is a theorem of Bass [Ba, Theorem 5]. For arbitrary $A \le \mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, it follows from Propositions 2.5 and 2.2. $\square$

## 3. Examples

Throughout this section, it will be convenient to define, for each finite group $G$,

$$\mathscr{N}_G = \#\big\{\mathbb{R}\text{-Aut}(G)\text{-conjugacy classes in } G\big\} - \#\big\{\mathbb{Q}\text{-Aut}(G)\text{-conjugacy classes in } G\big\}.$$

Thus by Theorem 2.6,

$$\mathscr{N}_G = \mathrm{rk}(\mathbb{Z} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}\mathrm{Aut}(G)} \mathrm{Wh}(G)).$$

So by Corollary 1.4(b), if $\mathscr{N}_G > 0$, then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\varGamma) \ne 0$ for every group $\varGamma$ that contains $G$ as a normal subgroup.

We now construct a wide variety of examples of finite groups $G$ with $\mathscr{N}_G > 0$, including some small metacyclic groups, metacyclic $p$-groups, and simple groups. For example, we show that the smallest group with $\mathscr{N}_G > 0$ is the nonabelian group of order 55, and that $\mathscr{N}_G > 0$ when $G$ is the nonabelian group of order $p^3$ and exponent $p^2$ and $p \ge 5$ is prime. We also determine exactly which of the sporadic simple groups, and which of the linear groups $PSL_n(q)$, satisfy $\mathscr{N}_G > 0$.

We first need some tools for constructing automorphisms.

**Lemma 3.1.** (a) *Let $H \trianglelefteq G$ be a pair of finite groups such that $H$ is abelian and $G/H$ is cyclic. Then for each $x \in G$ such that $G = H\langle x\rangle$, and each $a \in \mathbb{Z}$ prime to $|G|$ such that $a \equiv 1$ (mod $|G/H|$), there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(h) = h^a$ for each $h \in H \cup \{x\}$.*

(b) *Let $G$ be a group, let $Z \le Z(G)$ be a central subgroup, and let $\psi \in \mathrm{Hom}(G, Z)$ be a homomorphism such that $Z \le \mathrm{Ker}(\psi)$. Then there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(g) = g\psi(g)$ for each $g \in G$.*

*Proof.* **(a)** Define $\alpha\colon G \longrightarrow G$ by setting, for each $h \in H$ and each $i \in \mathbb{Z}$, $\alpha(hx^i) = h^a x^{ai}$. By assumption, every element in $G$ can be written in this form. If $hx^i = kx^j$ for $h, k \in H$ and $i, j \in \mathbb{Z}$, then $k^{-1}h = x^{j-i}$, so $k^{-a}h^a = (k^{-1}h)^a = x^{aj-ai}$, and hence $h^a x^{ai} = k^a x^{aj}$. So $\alpha$

is well defined as a map of sets. Finally, if $h,k\in H$ and $i,j\in\mathbb{Z}$ are arbitrary, then

$$\begin{aligned}\alpha((hx^i)(kx^j)) &= \alpha(h(x^ikx^{-i})x^{i+j}) = h^a(x^ik^ax^{-i})x^{ai+aj} = h^a(x^{ai}k^ax^{-ai})x^{ai+aj}\\ &= (h^ax^{ai})(k^ax^{aj}) = \alpha(hx^i)\alpha(kx^j),\end{aligned}$$

where the third equality holds since $x^{(a-1)i}\in H$ and hence commutes with $k^a$. So $\alpha$ is an automorphism.

**(b)** One easily checks that $\alpha$ is a homomorphism with inverse $(g\mapsto g\psi(g)^{-1})$. □

As a first, very simple, application of Lemma 3.1(a), we have:

**Example 3.2.** *If a finite group $G$ contains a normal abelian subgroup of index at most 3, then $\mathscr{N}_G=0$.*

*Proof.* Assume $H\trianglelefteq G$ is abelian of index at most 3. Fix $g\in G$; then either $g\in H$ or $G=H\langle g\rangle$. By Lemma 3.1(a), for each $a\in\mathbb{Z}$ such that $\gcd(a,|G|)=1$, and such that $a\equiv 1 \pmod 3$ if $|G/H|=3$, there is $\alpha\in\mathrm{Aut}(G)$ such that $\alpha(g)=g^a$. Thus all generators of $\langle g\rangle$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate to $g$. Since $g\in G$ was arbitrary, Theorem 2.6 now implies

$$\mathscr{N}_G = \#\big\{\mathbb{R}\text{-}\mathrm{Aut}(G)\text{-conjugacy classes}\big\} - \#\big\{\mathbb{Q}\text{-}\mathrm{Aut}(G)\text{-conjugacy classes}\big\} = 0. \qquad \square$$

With a little more work, one can show that if $G$ is a finite group with $\mathscr{N}_G=0$, then $\mathscr{N}_{G\times H}=0$ for each finite group $H$ that contains an abelian subgroup of index at most 2. However, if we let $G\cong C_5\rtimes C_4$ (induced by an injection $C_4\longrightarrow \mathrm{Aut}(C_5)$), and let $H$ be a nonabelian group of order 21, then $\mathscr{N}_G=\mathscr{N}_H=0$, but $\mathscr{N}_{G\times H}>0$.

We next look at some more small groups $G$ for which $\mathscr{N}_G$ vanishes.

**Example 3.3.** *If $G$ is a group of order at most 54, then $\mathscr{N}_G=0$.*

*Proof.* If each element of $G$ has order dividing 4 or 6, then all generators of each cyclic subgroup of $G$ are $\mathbb{R}$-$G$-conjugate, so $\mathscr{N}_G=0$. If $G$ is abelian, or contains a normal abelian subgroup of index 2 or 3, then $\mathscr{N}_G=0$ by Example 3.2. From now on, we use these without repeating the references each time.

When $q=p^k$ for a prime $p$ and $k\ge 1$, we let $E_q$ denote an elementary abelian $p$-group of order $q$ and rank $k$. When $P$ is a $p$-group for a prime $p$, we let $\Phi(P)$ denote its Frattini subgroup: the intersection of the maximal proper subgroups of $P$, and the smallest normal subgroup such that $P/\Phi(P)$ is elementary abelian.

**Case 1: $|G|=n$ where $n\le 53$ is odd.** If $n$ is prime or the square of a prime, or $n=35$ or 45, then $G$ is abelian, so $\mathscr{N}_G=0$. Otherwise, $n=3m$ where $m>3$ is prime or $m=9$, so $G$ contains a normal abelian subgroup of index 3, and $\mathscr{N}_G=0$.

**Case 2: $|G|=2n$ where $n\le 27$ is odd.** In these cases, $G$ always contains a normal subgroup $H\trianglelefteq G$ of order $n$ and index 2. If $H$ is abelian, then $\mathscr{N}_G=0$. If $H$ is nonabelian, then $n=21$ or 27.

If $n=21$ and $H$ is nonabelian, then $G$ is a semidirect product of the form $C_{14}\rtimes C_3$ or $C_7\rtimes C_6$. In the first case, $\mathscr{N}_G=0$. In the second case, all elements of $G$ have order 7 or a divisor of 6, and the elements of order 7 are permuted transitively by $\mathrm{Aut}(G)$ by Lemma 3.1(a). So $\mathscr{N}_G=0$ also in this case.

This leaves the case where $|H|=n=27$ and $H$ is nonabelian. If $H$ has exponent 3, then the order of each element of $G$ divides 6, and so $\mathscr{N}_G=0$. So assume $H$ is nonabelian of exponent 9. There are three subgroups of order 9 in $H$, at least one of which must be invariant under the conjugation action of $G/H\cong C_2$. So there is a normal subgroup $K\trianglelefteq G$

with $K \cong C_9$. If $C_G(K) > K$, then $G$ has a normal abelian subgroup of index 3, and so $\mathscr{N}_G = 0$. Otherwise, $G/K \cong C_6$ (since $\mathrm{Aut}(K) \cong C_6$), and $G$ is a semidirect product $K \rtimes C_6$ where $C_G(K) = K$.

In this last case, $G$ has presentation $\langle a, b \,|\, a^9 = 1 = b^6,\ bab^{-1} = a^2 \rangle$. So by Lemma 3.1(a), there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = a^7$ and $\alpha(b) = b$. Then $\alpha$ induces the identity on $G/\langle a^3 \rangle$, so for each cyclic subgroup $C \le G$ of order 9, $\alpha(C) = C$ and hence all generators of $C$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate. Since all other elements have order dividing 6, this shows that $\mathscr{N}_G = 0$.

**Case 3: $|G| = 4n$ where $n \le 13$ is odd.** If $n = 1$ or $n \equiv 3 \pmod 4$, then $G$ contains an abelian subgroup of index 2. If $n = 5$ or 13, then either $G$ has an abelian subgroup of index 2, or $G \cong C_n \rtimes C_4$, and each element of $G$ has order 1, 2, 4, or $n$. All elements in $G$ of order $n$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate by Lemma 3.1(a), so $\mathscr{N}_G = 0$.

Assume $n = 9$, so $|G| = 36$. If there is $H \trianglelefteq G$ of order 9, then either $G$ has an abelian subgroup of index 2, or $H \cong E_9$, $C_G(H) = H$, and all elements of $G$ have order $1, 2, 3, 4, 6$. So $\mathscr{N}_G = 0$ in all such cases. Otherwise, $|\mathrm{Syl}_3(G)| = 4$, the conjugation action on this set induces a homomorphism $\chi\colon G \longrightarrow \Sigma_4$ with image $A_4$, and so $G$ has a normal abelian subgroup of index 3.

**Case 4: $|G| = 8$, 16, or 32.** If $G/Z(G)$ has an element of order 8, generated by the class of $g \in G$, then $Z(G)\langle g \rangle$ is abelian of index at most 2 in $G$. Also, if $G$ has an element of order 16, then it generates a cyclic subgroup of index at most 2. So $\mathscr{N}_G = 0$ in all such cases.

Assume from now on that $G/Z(G)$ has exponent at most 4 and $G$ has exponent at most 8. So if $g \in G$ has order 8, then $g^4 \in Z(G)$. If $g \notin \Phi(G)$, then by Lemma 3.1(b), applied with $Z = \langle g^4 \rangle$, there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(g) = g^5$. So all generators of $\langle g \rangle$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate in this case.

If $g \in \Phi(G)$ has order 8, then $\Phi(G) = \langle g \rangle$ and $G/\Phi(G) \cong C_2 \times C_2$, which is impossible unless $G$ has a cyclic subgroup of index 2.

**Case 5: $|G| = 24$.** If $|\mathrm{Syl}_3(G)| = 1$, then $G$ contains a normal abelian subgroup of index 2, so $\mathscr{N}_G = 0$. Otherwise, $|\mathrm{Syl}_3(G)| = 4$, and the conjugation action on this set defines a homomorphism $\chi\colon G \longrightarrow \Sigma_4$ whose image contains $A_4$. So either $G \cong \Sigma_4$, or $G/Z(G) \cong A_4$ and $|Z(G)| = 2$, and in all such cases, all elements of $G$ have order dividing 4 or 6. So $\mathscr{N}_G = 0$.

**Case 6: $|G| = 40$.** In all cases, $|\mathrm{Syl}_5(G)| = 1$. Then either $G$ has an abelian subgroup of index 2 (and $\mathscr{N}_G = 0$), or there is a normal subgroup $H \trianglelefteq G$ such that $H \cong C_{10}$, $G/H \cong C_4$, and $C_G(H) = H$.

Assume we are in this last case. By Lemma 3.1(a), the generators of $H$ are permuted transitively by $\mathrm{Aut}(G)$, and similarly for the elements of order 5. All other elements have order dividing 8. Let $x \in H$ be the element of order 2, and let $\alpha \in \mathrm{Aut}(G)$ be the automorphism that is the identity on the subgroup of index 2 that contains $H$, and sends $g$ to $gx$ otherwise. If $g \in G$ has order 8, then $x = g^4$ and $\alpha(g) = gx = g^5$, so all generators of $\langle g \rangle$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate. Thus $\mathscr{N}_G = 0$.

**Case 7: $|G| = 48$.** By the Sylow theorems, $|\mathrm{Syl}_3(G)| = 1$, 4, or 16.

**Case 7A:** If $|\mathrm{Syl}_3(G)| = 16$, then the Sylow 2-subgroup $S \le G$ is normal and contains all elements not of order 3. So $S$ has an automorphism of order 3 that fixes only the identity element, hence $[S, S]$ is trivial or noncyclic, which implies $S \cong E_{16}$ or $C_4 \times C_4$. Thus all elements of $G$ have order at most 4, and so $\mathscr{N}_G = 0$.

**Case 7B:** Assume $|\mathrm{Syl}_3(G)| = 4$, and let $\chi\colon G \longrightarrow \Sigma_4$ be the homomorphism defined by the conjugation action. If $\chi$ is onto, then $\mathrm{Ker}(\chi) = Z(G) = \langle x\rangle$ where $|x| = 2$, so by Lemma 3.1(b), there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha|_{\chi^{-1}(A_4)} = \mathrm{Id}$ and $\alpha(g) = gx$ for $g \in G$ with $\chi(g) \notin A_4$. So if $g \in G$ has order 8, then $g^4 = x$ and $\alpha(g) = gx = g^5$. Thus all generators of each cyclic subgroup of order 8 are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate. Since all elements of $G$ have order dividing 6 or 8, this proves that $\mathscr{N}_G = 0$.

If $\chi(G) = A_4$, then either $\mathrm{Ker}(\chi) \cong E_4$, or $\mathrm{Ker}(\chi) = Z(G) \cong C_4$ and $G \cong C_4 \times A_4$ or $C_4 \circ SL_2(3)$ (central product). In all such cases, all elements of $G$ have order dividing 4 or 6, so $\mathscr{N}_G = 0$.

**Case 7C:** Now assume $|\mathrm{Syl}_3(G)| = 1$, let $H \trianglelefteq G$ be the normal subgroup of order 3, and choose $S \in \mathrm{Syl}_2(G)$. Then $C_G(H) = HT$ for some $T \leq S$ of index at most 2, and $\mathscr{N}_G = 0$ if $T$ is abelian. So assume $S \cong G/H$ is nonabelian of order 16, with a nonabelian subgroup $T \leq S$ of index at most 2.

If $T = S$, then $G \cong H \times S$. If $S/Z(S)$ has a cyclic subgroup of index 2, then $S$ has an abelian subgroup of index 2. Otherwise, $|Z(S)| = 2$ and $S/Z(S) \cong E_8$, in which case some pair of commuting elements in $S/Z(S)$ lift to commuting elements in $S$. So in all cases, $S$, and hence $G$, have abelian subgroups of index 2, and so $\mathscr{N}_G = 0$.

Assume for the rest of the proof that $T$ has index 2 in $S$. Since $G$ is a semidirect product $G = H \rtimes S$, each $\alpha \in \mathrm{Aut}(S)$ with $\alpha(T) = T$ extends to $\widehat{\alpha} \in \mathrm{Aut}(G)$ such that $\widehat{\alpha}|_H = \mathrm{Id}_H$. Also, there is $\beta \in \mathrm{Aut}(G)$ such that $\beta|_S = \mathrm{Id}_S$ and $\beta(h) = h^{-1}$ for $h \in H$.

If $S$ has an element $g$ of order 8, then since $T$ is nonabelian, we have $S = \langle g, x\rangle$ and $T = \langle g^2, x\rangle$ for some $x \in T \smallsetminus \langle g\rangle$. By Lemma 3.1(a), for each odd $a \in \mathbb{Z}$, there is $\alpha \in \mathrm{Aut}(S)$ that $\alpha(g) = g^a$ and $\alpha(x) = x^a$, so $\alpha(T) = T$, and hence $\alpha$ extends to $\widehat{\alpha} \in \mathrm{Aut}(G)$. Thus $N_{\mathrm{Aut}(G)}(\langle g\rangle)$ permutes transitively the generators of $\langle g\rangle$.

If $g \in G$ has order 12, then $H = \langle g^4\rangle$ and $g^3 \in T$. Thus $\beta(g^4) = g^{-4}$ and $\beta(g^3) = g^3$, so $\beta(g) = g^5$, proving that all generators of $\langle g\rangle$ are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate. All elements of $G$ have order 12, 8, 6, or at most 4, so $\mathscr{N}_G = 0$ in all of these cases. □

By Example 3.3, the smallest example of a finite group $G$ such that $\mathscr{N}_G > 0$ has order at least 55. We now prove that there is such a group. Let $\varphi$ denote the Euler function: $\varphi(n) = |(\mathbb{Z}/n)^\times|$.

**Example 3.4.** *Fix a prime $p$, and an integer $m \geq 3$ such that $m|(p-1)$. Let $G$ be a semidirect product $G \cong C_p \rtimes C_m$, where $C_m$ acts faithfully on $C_p$. Then $\mathscr{N}_G \geq (\varphi(m)/2) - 1$, with equality if $m$ is prime. Thus $\mathscr{N}_G > 0$ if $m = 5$ or $m \geq 7$.*

*Proof.* Fix elements $a, b \in G$ with $|a| = p$ and $|b| = m$, and set $H = \langle a\rangle \trianglelefteq G$. Let $k \in \mathbb{Z}$ be such that $bab^{-1} = a^k$. For each $\alpha \in \mathrm{Aut}(G)$, $\alpha(a) = a^s$ for some $s$, and so $\alpha(b)\alpha(a)\alpha(b)^{-1} = \alpha(a)^s = b\alpha(a)b^{-1}$. Hence $\alpha(b) \in bH$. In other words, each automorphism of $G$ induces the identity on $G/H \cong C_m$.

Thus the $\varphi(m)$ generators of $\langle b\rangle$ all lie in one $\mathbb{Q}$-$\mathrm{Aut}(G)$-conjugacy class, but in separate $\mathrm{Aut}(G)$-conjugacy classes. Since each $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugacy class is the union of at most two $\mathrm{Aut}(G)$-conjugacy classes, there are at least $\varphi(m)/2$ $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugacy classes in the $\mathbb{Q}$-$\mathrm{Aut}(G)$-conjugacy class of $b$. So $\mathscr{N}_G \geq (\varphi(m)/2) - 1$.

If $m$ is prime, then every nonidentity element of $G$ has order $p$ or $m$ and hence is $\mathbb{Q}$-$G$-conjugate to $a$ or $b$. All generators of $\langle a\rangle$ are $\mathrm{Aut}(G)$-conjugate to $a$, and there are $\varphi(m)/2$ $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugacy classes of generators of $\langle b\rangle$. So $\mathscr{N}_G = (\varphi(m)/2) - 1$ in this case. □

With a little more work, one can show that $\mathcal{N}_G = \sum_{2<d|m}((\varphi(d)/2)-1)$ in the situation of Example 3.4.

The next example also involves metacyclic groups, and shows that there are $p$-groups $G$ (for an arbitrary prime $p$) such that $\mathcal{N}_G > 0$. Note that the smallest examples constructed in this way have order $p^3$ when $p \geq 5$, or order $3^6$ or $2^{10}$ when $p = 3$ or 2.

**Example 3.5.** *Fix integers $1 < r|q$, and let $G$ be the group of order $qr^2$ with presentation*

$$G = \langle a, b \,|\, a^{qr} = 1 = b^r,\ bab^{-1} = a^{q+1} \rangle.$$

*Then $\mathcal{N}_G > 0$ if $\varphi(r) > 4$, or if $\varphi(r) = 4$ and $q$ is odd or $2r|q$.*

*Proof.* Set $t = q+1$ for short. Note first that $t^s = (1+q)^s \equiv 1 + sq \pmod{q^2}$ for each $s \geq 1$. In particular, $t^r \equiv 1 \pmod{qr}$, so the above presentation does define a group of order $qr^2$.

Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism that normalizes the cyclic subgroup $\langle b \rangle$. Thus

$$\alpha(a) = a^i b^j \qquad \text{and} \qquad \alpha(b) = b^k$$

for some $i, j, k \in \mathbb{Z}$. Then $\gcd(k, r) = 1$ since $\alpha(b)$ has order $r$, and $\gcd(i, q) = 1$ since $\langle \alpha(a), \alpha(b) \rangle = G$. Since $\alpha$ is a homomorphism,

$$a^{it^k} b^j = b^k(a^i b^j)b^{-k} = \alpha(b)\alpha(a)\alpha(b)^{-1} = \alpha(bab^{-1}) = \alpha(a^t) = (a^i b^j)^t = a^{iN} b^{tj},$$

where $N = 1 + t^j + t^{2j} + \cdots + t^{(t-1)j}$. Since $\gcd(i, q) = 1$, this implies that $N \equiv t^k \pmod{qr}$. Hence

$$t^k \equiv N \equiv 1 + (1+jq) + (1+2jq) + \cdots + (1+(t-1)jq) = t + jq^2t/2 \pmod{qr}$$

(recall $q = t-1$). So $t^k \equiv t \pmod{qr}$ if $q$ is odd or $2r|q$, and $t^k \equiv t \pmod{qr/2}$ otherwise. Since $t = 1+q$ and $t^k \equiv 1 + kq \pmod{qr}$, we now get that $k \equiv 1 \pmod{r}$ if $q$ is odd or $2r|q$, and $k \equiv 1 \pmod{r/2}$ otherwise.

Thus if $q$ is odd or $2r|q$, then the only generators of $\langle b \rangle$ that are $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugate to $b$ are $b$ and $b^{-1}$, so $\mathcal{N}_G > 0$ if $\varphi(r) \geq 4$. Otherwise, $r$ is even, the $\mathbb{R}$-$\mathrm{Aut}(G)$-conjugacy class of $b$ in $\langle b \rangle$ contains the four elements $b^{\pm 1}$ and $b^{(r/2)\pm 1}$, and $\mathcal{N}_G > 0$ if $\varphi(r) > 4$. □

We finish by looking at a few examples of finite simple groups $G$ where $\mathcal{N}_G > 0$. These are larger groups in most cases, but working with them has the advantage that the automorphism groups of simple groups are well known, and their outer automorphism groups are in most cases quite small. Also, the properties of conjugacy classes of elements of $G$ needed to determine $\mathcal{N}_G$ are in many cases listed in the Atlas [Atl].

We start with the easiest case.

**Example 3.6.** *If $G$ is an alternating or symmetric group, then $\mathcal{N}_G = 0$.*

*Proof.* If $G \cong A_n$ or $\Sigma_n$, then two generators of the same cyclic subgroup of $G$ are always conjugate in $\Sigma_n$, and hence in $\mathrm{Aut}(G)$. So $\mathbb{Q}$-$\mathrm{Aut}(G)$-conjugate elements are also $\mathrm{Aut}(G)$-conjugate, and hence $\mathcal{N}_G = 0$. □

We next look at the 26 sporadic simple groups, where we observe that the largest groups are not necessarily the ones for which $\mathcal{N}_G$ or $\mathrm{rk}(\mathrm{Wh}(G))$ are largest. For example, the Whitehead group of $F_1$ (the monster) is finite, while $\mathcal{N}_G > 0$ when $G \cong F_2$ (the baby monster) or the Janko group $J_1$ (one of the smallest sporadic groups).

**Example 3.7.** *Among the sporadic simple groups $G$,*

(a) $\mathrm{Wh}(G)$ *is finite, and hence* $\mathscr{N}_G = 0$*, when* $G$ *is one of the five Matthieu simple groups, one of Conway's simple groups* $Co_n$ *for* $n = 1, 2, 3$*, or one of the groups* HS, McL, $F_3$*, or* $F_1$*;*

(b) $\mathrm{rk}(\mathrm{Wh}(G)) > 0$ *but* $\mathscr{N}_G = 0$ *when* $G \cong J_2$, Suz*, or* $Fi_{22}$*; and*

(c) $\mathscr{N}_G > 0$ *when* $G$ *is one of the Janko groups* $J_n$ *for* $n = 1, 3, 4$*, or* $G \cong$ He, Ly, Ru, O'N, $Fi_{23}$*,* $Fi'_{24}$*,* $F_5$ *or* $F_2$*.*

*Proof.* From the character tables in [Atl], we see that every pair of $\mathbb{Q}$-$G$-conjugate elements is $\mathbb{R}$-$G$-conjugate, and hence $\mathrm{Wh}(G)$ is finite, whenever $G$ is one of the Matthieu groups, one of Conway's simple groups $Co_n$ for $n = 1, 2, 3$, or HS, McL, $F_3$ or $F_1$. Note that an entry "`5A B*`" at the top of the character table means that class `5B` is $\mathbb{Q}$-$G$-conjugate but not $\mathbb{R}$-$G$-conjugate to class `5A`, while "`5A B**`" means that the classes are $\mathbb{R}$-$G$-conjugate.

The computations of $\mathrm{rk}(\mathrm{Wh}(G))$ and $\mathscr{N}_G$ in the other cases are described in Table 3.1. □

| $G$ | $\mathrm{rk}(\mathrm{Wh}(G))$ | $\mathscr{N}_G$ | classes |
|---|---|---|---|
| $J_1$ | 5 | 5 | $5AB, 10AB, 15AB, 19ABC$ |
| $J_2$ | 5 | 0 | $5AB!, 5CD!, 10AB!, 10CD!, 15AB!$ |
| $J_3$ | 6 | 3 | $5AB!, 9ABC, 10AB!, 15AB!, 17AB$ |
| $J_4$ | 11 | 11 | $20AB, 24AB, 31ABC, 33AB, 37ABC, 40AB, 43ABC, 66AB$ |
| Suz | 3 | 0 | $13AB!, 15AB!, 21AB!$ |
| He | 2 | 1 | $17AB!, 21AB$ |
| Ly | 11 | 11 | $21AB, 24BC, 31ABCDE, 37AB, 40AB, 42AB, 67ABC$ |
| Ru | 7 | 7 | $14ABC, 20BC, 24AB, 26ABC, 29AB$ |
| O'N | 6 | 5 | $15AB, 16AB, 16CD, 19ABC, 28AB$ |
| $Fi_{22}$ | 1 | 0 | $13AB!$ |
| $Fi_{23}$ | 3 | 3 | $13AB, 26AB, 39AB$ |
| $Fi'_{24}$ | 8 | 2 | $21CD!, 24FG!, 29AB!, 33AB, 39AB, 39CD!, 42BC!, 45AB!$ |
| $F_5$ | 8 | 1 | $5CD!, 10DE!, 10GH!, 15BC!, 20AB, 20DE!, 25AB!, 30BC!$ |
| $F_2$ | 3 | 3 | $32AB, 34BC, 56AB$ |

TABLE 3.1. In the last column, we list families of $G$-conjugacy classes that are $\mathbb{Q}$-$G$-conjugate but not $\mathbb{R}$-$G$-conjugate to each other. For example, "$19ABC$" means that the three classes $19A$, $19B$, and $19C$ together form a $\mathbb{Q}$-$G$-conjugacy class, no two of which are $\mathbb{R}$-$G$-conjugate. An exclamation point "!" means that the classes are permuted transitively by $\mathrm{Aut}(G)$. When $G =$ O'N, the classes $16AB$ are exchanged with $16CD$ by an outer automorphism.

We finish by looking at the projective special linear groups.

**Example 3.8.** *Assume* $G \cong PSL_n(q)$*, where* $n \geq 2$*,* $q$ *is a prime power, and* $G$ *is simple. Then* $\mathscr{N}_G > 0$*, except when* $G$ *is one of the groups* $PSL_2(4) \cong PSL_2(5) \cong A_5$*,* $PSL_2(7) \cong PSL_3(2)$*,* $PSL_2(8)$*,* $PSL_2(9) \cong A_6$*,* $PSL_3(4)$*, or* $PSL_4(2) \cong A_8$*.*

*Proof.* Choose an ordered $\mathbb{F}_q$-basis for $\mathbb{F}_{q^n}$, and let $\chi \colon \mathbb{F}_{q^n}^\times \longrightarrow GL_n(q)$ be the injective homomorphism that sends an element $u \in \mathbb{F}_{q^n}^\times$ to the matrix for multiplication by $u$ on $\mathbb{F}_{q^n}$.

Fix a generator $u_0 \in \mathbb{F}_{q^n}^\times$, set $\widetilde{x} = \chi(u_0^{q-1}) \in SL_n(q)$, and let $x \in G = PSL_n(q)$ be its class modulo the center. Then $\widetilde{x}$ has order $(q^n-1)/(q-1)$, and so $x$ has order

$$M \stackrel{\text{def}}{=} \frac{q^n-1}{(q-1)\cdot \gcd(q-1,n)}.$$

Note that $\frac{q^n-1}{q-1} \equiv n \pmod{q-1}$, and hence $\gcd(q-1,n) = \gcd(q-1, \frac{q^n-1}{q-1})$. Also, $\mathrm{Aut}_G(\langle x\rangle) \cong C_n$, generated by the Frobenius automorphism $(x \mapsto x^q)$.

Assume $q = p^k$ where $p$ is prime and $k \geq 1$. The automorphism group $\mathrm{Aut}(G)$ is generated by $\mathrm{Inn}(G)$, the field automorphism that sends $x$ to $x^p$, diagonal automorphisms induced by conjugation by $\chi(u_0)$ that send $x$ to itself, and if $n \geq 3$, the graph automorphism "transpose inverse" that (up to inner automorphism) sends $x$ to $x^{-1}$ (see, e.g., [Wi, § 3.3.4]). Thus $\mathrm{Aut}_{\mathrm{Aut}(G)}(\langle x\rangle)$ is generated by the automorphisms $(x \mapsto x^p)$ of order $nk$ and (if $n \geq 3$) $(x \mapsto x^{-1})$. It follows that

$$\left.\begin{array}{r} n = 2 \text{ and } \varphi(M) > 2k \\ \text{or } n \geq 3 \text{ and } \varphi(M) > 2nk \end{array}\right\} \implies \mathscr{N}_G > 0. \tag{3-1}$$

**Assume $n = 2$ and $p = 2$.** Then $M = 2^k + 1$ is odd, and $\mathrm{Aut}_{\mathrm{Aut}(G)}(\langle x\rangle)$ is cyclic of order $2k$, generated by the Frobenius automorphism $(x \mapsto x^2)$. If $M$ is divisible by two or more odd primes, then $\mathrm{Aut}(\langle x\rangle)$ is not cyclic, and hence $\mathscr{N}_G > 0$. Otherwise, $\varphi(M) \geq \frac{2}{3}\cdot M = \frac{2}{3}(2^k+1)$, and so $\mathscr{N}_G > 0$ if $2^k + 1 > 3k$. This holds whenever $k \geq 4$, and so $\mathscr{N}_G > 0$ whenever $q = 2^k \geq 16$.

**Assume $n = 2$ and $p$ is odd.** Then $M = (q+1)/2$, and again, $\mathrm{Aut}_{\mathrm{Aut}(G)}(\langle x\rangle)$ is cyclic of order $2k$. If $M$ is divisible by two or more odd primes, then $\mathscr{N}_G > 0$ since $\mathrm{Aut}(\langle x\rangle)$ is not cyclic. Otherwise, $\varphi(M) \geq \frac{1}{2}\cdot\frac{2}{3}\cdot M = (q+1)/6$, and so by (3-1), $\mathscr{N}_G > 0$ if $q + 1 > 12k$. This holds for $q \geq 13$ when $k = 1$, for $q = p^2 \geq 25$ when $k = 2$, for $q = p^3 \geq 5^3$ when $k = 3$, and for all odd primes $p$ when $k \geq 4$. Thus by (3-1), $\mathscr{N}_G > 0$ whenever $q \geq 13$ and $q \neq 27$.

**Assume $n \geq 3$.** In these cases, $\mathrm{Aut}_{\mathrm{Aut}(G)}(\langle x\rangle)$ is the product of cyclic groups of order $nk$ and 2. If $M$ is divisible by three or more odd primes, then $\mathrm{Aut}(\langle x\rangle)$ has 2-rank at least 3, hence is not equal to $\mathrm{Aut}_{\mathrm{Aut}(G)}(\langle x\rangle)$, and so $\mathscr{N}_G > 0$. If $M$ is divisible by at most two odd primes, then $\varphi(M) \geq \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot M = \frac{4}{15}M$. So by (3-1), $\mathscr{N}_G > 0$ if $2kn < \frac{4}{15}M$, and since $M \geq \frac{1}{n}\cdot\frac{q^n-1}{q-1} > \frac{1}{n}q^{n-1}$, we have

$$q^{n-1} = p^{k(n-1)} \geq \tfrac{15}{2}\cdot kn^2 \quad\implies\quad \mathscr{N}_G > 0. \tag{3-2}$$

By straightforward computation, the inequality in (3-2) holds whenever $(p,k,n)$ is equal to one of the triples

$(11,1,3)$ $(5,2,3)$ $(3,3,3)$ $(2,5,3)$ $(5,1,4)$ $(3,2,4)$ $(2,3,4)$ $(2,2,6)$ $(3,1,7)$ $(2,1,11)$

Also, if this inequality holds for a given triple $(p_0,k_0,n_0)$ (with $n_0 \geq 3$), then it holds for $(p,k,n)$ (and hence $\mathscr{N}_G > 0$) whenever $p \geq p_0$, $k \geq k_0$, and $n \geq n_0$. Among the pairs $(p^k,n)$ with $n \geq 3$ for which (3-2) does not hold, the inequality in (3-1) holds (so $\mathscr{N}_G > 0$) in the following cases:

| $q$ | 3 | 5 | 7 | 8 | 9 | 16 | 4 | 2 | 3 | 4 | 2 | 3 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 5 | 5 | 5 | 6 | 6 | 7 | 8 | 9 | 10 |
| $M$ | 13 | 31 | 19 | 73 | 91 | 91 | 85 | 31 | 121 | 341 | 63 | 182 | 127 | 255 | 511 | 1023 |
| $\varphi(M)$ | 12 | 30 | 18 | 72 | 72 | 72 | 64 | 30 | 110 | 300 | 36 | 72 | 126 | 128 | 432 | 600 |
| $2kn$ | 6 | 6 | 6 | 18 | 12 | 24 | 16 | 10 | 10 | 20 | 12 | 12 | 14 | 16 | 18 | 20 |

**Remaining cases (all $n$).** We are left with the groups $PSL_2(q)$ for $q \leq 11$ or $q = 27$, $PSL_3(q)$ for $q = 2, 4$, and $PSL_4(q)$ for $q = 2, 3$. The groups $PSL_2(4) \cong PSL_2(5) \cong A_5$, $PSL_2(9) \cong A_6$, and $PSL_4(2) \cong A_8$ were handled in Example 3.6. Also, $PSL_3(2) \cong PSL_2(7)$, so it remains to look at $PSL_2(7)$, $PSL_2(8)$, $PSL_2(11)$, $PSL_2(27)$, $PSL_3(4)$, and $PSL_4(3)$. In each of these cases, the result follows with the help of the character tables in [Atl].

For example, when $G = PSL_2(11)$, the classes $5A$ and $5B$ are not $\mathbb{R}$-Aut$(G)$-conjugate, so $\mathscr{N}_G > 0$. When $G = PSL_2(27)$, there are six conjugacy classes of elements of order 13 (all of them $\mathbb{Q}$-$G$-conjugate), permuted in two orbits by field automorphisms of order 3, hence forming two $\mathbb{R}$-Aut$(G)$-conjugacy classes. When $G = PSL_4(3)$, the subgroups of order 13 have automizers in Aut$(G)$ of order 6, so their generators form two $\mathbb{R}$-Aut$(G)$-conjugacy classes. Thus $\mathscr{N}_G > 0$ in the last two cases. When $G = PSL_2(7)$, $PSL_2(8)$, or $PSL_3(4)$, each $\mathbb{Q}$-Aut$(G)$-conjugacy class is permuted transitively by Aut$(G)$, and so $\mathscr{N}_G = 0$. $\square$

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