Survey on L^2-torsion and its (future) applications

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Bonn, May 2024

Motivation

- L^2-Betti numbers have many applications to algebra, geometry, and group theory, often as obstructions, for instance against fibering.
- If they all vanish, a secondary invariant, the L^2-torsion is defined. It is much more sophisticated and richer than the notion of an L^2-Betti number, but also harder to access.
- We want to discuss some open problems and potential applications of L^2-torsion without going into technical details.
- Hopefully these will be picked up as interesting research projects by some mathematicians.
- We are not planning to go over all the slides in the talk.
- The slides can be downloaded from my homepage.

Topics

- Basics definitions
- Basic properties of L^2-torsion
- Knots
- An invariant of group automorphisms
- Twisting with finite dimensional representations
- The Thurston norm and the degree of the ϕ-twisted L^2-torsion function
- Homological growth and L^2-torsion
- Simplicial volume and L^2-invariants
- L^2-torsion and measure equivalence
- (Generalized) Lehmer's problem
- References

Basic definitions

- Let G be a group and Y be a G-space. Define

$$b_n^{(2)}(Y;\mathcal{N}(G)) = \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(Y))\big) \quad \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

  for the dimension function $\dim_{\mathcal{N}(G)}$ defined for arbitrary $\mathcal{N}(G)$-modules.

- For a group G define

$$b_n^{(2)}(G) = b_n^{(2)}(EG;\mathcal{N}(G)) = b_n^{(2)}(\underline{E}G;\mathcal{N}(G)) \quad \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

- Let X be a connected finite CW-complex with universal covering $\widetilde{X} \to X$ and fundamental group π. Define

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{X};\mathcal{N}(\pi)) = \dim_{\mathcal{N}(G)}\big(\ker(\mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{Z}\pi} \widetilde{\Delta}_n)\big) \quad \in \mathbb{R}^{\geq 0}$$

  for $\widetilde{\Delta}_n \colon C_n(\widetilde{X}) \to C_n(\widetilde{X})$ the combinatorial Laplacian over $\mathbb{Z}\pi$.

- If M is a closed Riemannian manifold, the L^2-Betti numbers can be defined analytically in terms of the heat kernel on $\widetilde{M}$

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, d\mathrm{vol}_{\widetilde{M}}\,.$$

- A connected finite CW-complex X is called L^2-acyclic if $b_n^{(2)}(\widetilde{X}) = 0$ holds for all $n \geq 0$.

- In this case we can define a secondary invariant as follows, where we will ignore in the sequel discussions about $\det \geq 1$ class since this is satisfied in all cases of interest.

- Let X be connected finite CW-complex X which is L^2-acyclic. Define its L^2-torsion

$$\rho^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln\bigl(\det_{\mathcal{N}(\pi)}(\mathrm{id}_{\mathcal{N}(\pi)} \otimes_{\mathbb{Z}\pi} \widetilde{\Delta}_n)\bigr) \quad \in \mathbb{R}$$

where $\det_{\mathcal{N}(\pi)}$ is the Fuglede-Kadison determinant.

- If M is a closed Riemannian manifold which is L^2-acyclic, it has an analytic expression in terms of the heat kernel on $\widetilde{M}$, namely for any choice of $\epsilon > 0$ we have

$$\begin{aligned} \rho^{(2)}(\widetilde{M}) \quad := \quad & \frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \Biggl(\frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^{\epsilon} t^{s-1} \cdot \theta_n(t) dt \, \Bigg|_{s=0} \\ & + \int_{\epsilon}^{\infty} t^{-1} \cdot \theta_n(t) dt \Biggr) \end{aligned}$$

for $\theta_n(t) = \int_{\mathcal{F}} \mathrm{tr}(e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, d\mathrm{vol}_{\widetilde{M}}$.

- The upshot of the discussion above is that whenever a connected finite CW-complex X is L^2-acyclic, then a secondary invariant, its L^2-torsion $\rho(\widetilde{X}) \in \mathbb{R}$, can be considered.
- The relation of the L^2-torsion to L^2-Betti numbers can be viewed as the L^2-analogue of the relation of the classical Reidemeister torsion to classical Betti numbers.
- The L^2-torsion is the L^2-analogue of the classical Ray-Singer torsion which is the analytic counterpart of Reidemeister torsion.
- In the sequel we tacitly assume at a few places that the K-theoretic Farrell-Jones Conjecture holds which is known to be true in all cases of interest.

Basic properties of L^2-torsion

- Homotopy invariance

  If X and Y are homotopy equivalent and X is L^2-acyclic, then Y is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{Y}).$$

- Sum formula

  If $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$, X_i is L^2-acyclic for $i = 0, 1, 2$, and the inclusions $X_i \to X$ are π_1-injective, then X is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

- Product formula

  If X is L^2-acyclic, then $X \times Y$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\widetilde{Y}).$$

- Fibration formula

  Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that F is L^2-acyclic and the inclusion $F \to E$ is π_1-injective.

  Then E is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- Poincaré duality

  If M is a closed manifold which is L^2-acyclic and of even dimension, then

$$\rho^{(2)}(\widetilde{M}) = 0.$$

- Multiplicativity

  Let $Y \to X$ be a finite covering with d-sheets. Suppose that X or Y is L^2-acyclic. Then both are L^2-acyclic and

$$\rho^{(2)}(\widetilde{Y}) = d \cdot \rho^{(2)}(\widetilde{X}).$$

- **Hyperbolic manifolds**

  If M is a closed hyperbolic manifold of odd dimension $2k+1$, then M is L^2-acyclic and there is a rational number $r_k > 0$ (depending only on k) satisfying

$$\rho^{(2)}(\widetilde{M}) = (-1)^k \cdot \pi^{-k} \cdot r_k \cdot \mathrm{Vol}(M).$$

  (There are similar formulas for locally symmetric spaces of non-compact type.)

- **3-manifolds**

  Let M be a compact connected irreducible 3-manifold with infinite π whose boundary is empty or a union of incompressible tori. Let M_1, M_2, ..., M_r be the hyperbolic pieces in its JSJ-decomposition. Define $\mathrm{Vol}(M)$ to be $\sum_{i=1}^{r} \mathrm{Vol}(M_i)$.

  Then M is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = \frac{-1}{6\pi} \cdot \mathrm{Vol}(M).$$

Knots

Theorem

Let $K \subseteq S^3$ be a knot and $M(K)$ be its knot complement which is the complement of an open regular neighborhood of K.

1. *$M(K)$ is L^2-acyclic and we can define the L^2-torsion $\rho^{(2)}(K) := \rho^{(2)}(\widetilde{M(K)})$.*
2. *We have $\rho^{(2)}(K) = 0$ if and only if K is obtained from the trivial knot by applying a finite number of times the operation "connected sum" and "cabling".*
3. *A knot is trivial if and only if both its L^2-torsion $\rho^{(2)}(K)$ and its Alexander polynomial $\Delta(K)$ are trivial.*

An invariant of group automorphisms

Definition

Let G be a group with a finite model for BG. Let $f\colon G \xrightarrow{\cong} G$ be a group automorphism. Let T_{Bf} be the mapping torus of $Bf\colon BG \to BG$. Then T_{Bf} is L^2-acyclic and we can define the L^2-torsion of f

$$\rho^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}) \quad \in \mathbb{R}$$

- One can generalize the construction above to the case where there is a finite model for $\underline{E}G$.
- Next we collect the main properties of $\rho^{(2)}(f)$.

- $\rho^{(2)}(f)$ depends only on the class of f in $\mathrm{Out}(G)$.
- Amalgamation formula

$$\rho^{(2)}(f_1 *_{f_0} f_2) = \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0).$$

- Trace property

  Let $u\colon G \xrightarrow{\cong} H$ and $v\colon H \xrightarrow{\cong} G$ group automorphisms. Then

$$\rho^{(2)}(u \circ v) = \rho^{(2)}(v \circ u).$$

  In particular $\rho^{(2)}(f)$ depends only on the conjugacy class of f in $\mathrm{Out}(G)$.

- **Additivity**

  If the following diagram commutes and has exact sequences as rows and automorphisms as vertical arrows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_0 & \xrightarrow{i} & G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G_0 & \xrightarrow{i} & G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 1 \end{array}$$

  then

$$\rho^{(2)}(f_1) = \chi(BG_2) \cdot \rho^{(2)}(f_0).$$

- Multiplicativity under finite index subgroups

  If $f\colon G \xrightarrow{\cong} G$ is an automorphism of G and $H \subseteq G$ is a subgroup of finite index with $f(H) = H$, then

$$\rho^{(2)}(f|_H) = [G : H] \cdot \rho^{(2)}(f).$$

- Multiplicativity under composition

  For $m \geq 1$ we get

$$\rho^{(2)}(f^m) = m \cdot \rho^{(2)}(f)$$

  and we have

$$\rho^{(2)}(f^{-1}) = \rho^{(2)}(f).$$

- If BG is L^2-acyclic, then $\rho^{(2)}(f) = 0$.
- If there is an automorphism $a\colon S \to S$ of a compact orientable surface different from S^2 and D^2, then its mapping torus T_f is a connected compact irreducible manifold of dimension 3 whose boundary is empty or a union of incompressible tori, and we get

$$\rho^{(2)}(\pi_1(a)) = -\frac{1}{6\pi} \cdot \mathrm{Vol}(T_a).$$

- One should investigate $\rho^{(2)}(f)$ in particular for elements $f \in \mathtt{Out}(F_r)$ for the free group F_r of rank r.
- It is an interesting question whether $\rho(f)$ determines the conjugacy class of f in $\mathtt{Out}(F_r)$ up to finite ambiguity provided that f has exponential growth
- Next we describe a recipe how to compute $\rho^{(2)}(f)$ for $f \in \mathtt{Out}(F_r)$.

- Write $G = F_r \rtimes_f \mathbb{Z}$ for the semi-direct product associated to f. Let $t \in \mathbb{Z}$ be a generator and denote the corresponding element in G also by t.

- Define a (r, r)-matrix A over $\mathbb{Z}[F_r]$ by

$$A = \left(\frac{\partial}{\partial s_j} f(s_i)\right)_{1 \leq i,j \leq r}$$

where $\frac{\partial}{\partial s_j}$ denotes the Fox derivative.

- Choose a large enough real number $K > 0$.

- Denote by

$$\mathrm{tr}_{\mathbb{Z}G} \colon \mathbb{Z}G \to \mathbb{Z}, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \lambda_e$$

the standard trace on $\mathbb{Z}G$.

- Define the so called characteristic sequence for $p \geq 0$

$$c(A,K)_p = \operatorname{tr}_{\mathbb{Z}G}\left(\left(1 - K^{-2} \cdot (1 - tA)(1 - A^* t^{-1})\right)^p\right).$$

- In the setting above the sequence $c(A,K)_p$ is a monotone decreasing sequence of non-negative real numbers, and the L^2-torsion of f satisfies

$$\rho^{(2)}(f) = -r \cdot \ln(K) + \frac{1}{2} \cdot \sum_{p=1}^{\infty} \frac{1}{p} \cdot c(A,K)_p \leq 0.$$

- The convergence of the infinite sum above is exponential.

- The complexity of the computation of $\rho^{(2)}(f)$ has been analyzed by Löh-Uschold [20].

Twisting with finite dimensional representations

- One can twist L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ with a finite-dimensional real representation V and obtains the V-twisted L^2-Betti numbers $b_n^{(2)}(\widetilde{X}; V)$.
- If V is orthogonal, then it is easy to check

$$b_n^{(2)}(\widetilde{X}; V) = \dim_{\mathbb{R}}(V) \cdot b_n^{(2)}(\widetilde{X}).$$

- There is the conjecture formulated as a question in Lück [25, Question 0.1] that this holds for all finite-dimensional real representations V.
- Boschheidgen-Jaikin-Zapirain [3, Theorem 1.1] have proved it if π is sofic.
- Therefore we will tacitly assume this conjecture to be true in the sequel.
- In particular $b_n^{(2)}(\widetilde{X}; V)$ vanishes for all $n \geq 0$ if X is L^2-acyclic.

- This raises the question whether, for a connected finite CW-complex X which is L^2-acyclic, we can twist L^2-torsion $\rho^{(2)}(\widetilde{X})$ with a finite-dimensional real representation V and obtain the V-twisted L^2-torsion $\rho^{(2)}(\widetilde{X};V)$.
- This is easy if V is orthogonal but the result is not interesting since it will satisfy
$$\rho^{(2)}(\widetilde{X};V) = \dim_{\mathbb{R}}(V) \cdot \rho^{(2)}(\widetilde{X}).$$
- If V is any finite-dimensional real representation V, the proof that $\rho^{(2)}(\widetilde{X};V)$ is well-defined is much harder.
- It has been carried out by Lück [25, Theorem 7.7] provided that V is a $\mathbb{Q}\pi$-module which is finitely generated as $\mathbb{Q}$-module or if the representation V considered as a homomorphisms $\rho_V \colon \pi \to GL_d(\mathbb{R})$ factorizes through $\mathbb{Z}^k$ for $k \geq 0$.

- Let X be a finite connected CW-complex with fundamental group π which is L^2-acyclic. Let $\operatorname{Rep}_{\mathbb{R}}(\pi, d)$ be the real algebraic variety of d-dimensional real representations, i.e., of group homomorphisms $\pi \to GL_d(\mathbb{R})$.

Conjecture

The function

$$\rho_X^{(2)} \colon \operatorname{Rep}_{\mathbb{R}}(\pi, d) \to \mathbb{R}$$

is well-defined, continuous, and even smooth on manifold strata.

- We expect that $\rho_X^{(2)}$ carries interesting information, in particular when X is a compact connected irreducible 3-manifold M with infinite π whose boundary is empty or a union of incompressible tori.
- Question: Can we recover the Casson invariant of an integral homology 3-sphere N from $\rho_N^{(2)}$?
- Partial results show that $\rho_X^{(2)}$ seems to carry a lot of information.

- We know already that $\rho_M^{(2)}$ evaluated at the trivial d-dimensional representation is $-\frac{d}{6\pi} \cdot \mathrm{Vol}(M)$ for such M.

- If M is above, one can calculate $\rho_M^{(2)}(V)$ in terms of characteristic sequences as indicated above for group automorphisms, where the relevant matrices A can be read off from π and the representation $\pi \to GL_d(\mathbb{R})$.

- Next we explain the relation between $\rho_M^{(2)}$ and the Thurston norm, where M is a compact connected irreducible orientable 3-manifold M with infinite π whose boundary is empty or a union of incompressible tori. See [7, 8, 9, 10, 18, 19, 25].

The Thurston norm and the degree of the ϕ-twisted L^2-torsion function

- Consider an element $\phi \in H^1(M;\mathbb{Q}) = \hom(\pi,\mathbb{Q})$.
- We obtain for every $t \in (0,\infty)$ a 1-dimensional real representation $\mathbb{R}_{\phi,t}$ whose underlying real vector space is $\mathbb{R}$ and on which $w \in \pi$ acts by multiplication with $t^{\phi(w)}$.
- We obtain the ϕ-twisted L^2-torsion function

$$\rho^{(2)}(M;\phi)\colon (0,\infty) \to \mathbb{R}, \quad t \mapsto \rho^{(2)}(\widetilde{M};\mathbb{R}_{\phi,t}).$$

- Actually this function depends on a choice of a Spin^c-structure, but we will ignore this point since a different choice changes the $\rho^{(2)}_{\phi}$ by adding a function of the shape $E \cdot \ln(t)$.
- It turns out to be well-defined and continuous.

- There exist constants $C \geq 0$ and $D \geq 0$ such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \rho^{(2)}(M;\phi)(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho^{(2)}(M;\phi)(t) \leq C \cdot \ln(t) + D.$$

- Define the degree of $\overline{\rho}^{(2)}(M;\phi)$ to be the non-negative real number

$$\deg\bigl(\rho^{(2)}(M;\phi)\bigr) := \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)}.$$

- Recall the definition of Thurston [30] of the so-called Thurston norm of $\phi \in H^1(M;\mathbb{Z})$

  $x_M(\phi) := \min\{\chi_-(F) \mid F \subset M \text{ properly embedded surface dual to } \phi\},$

  where, given a surface F with connected components $F_1, F_2, \ldots, F_k$, we define

  $$\chi_-(F) \;:=\; \sum_{i=1}^{k} \max\{-\chi(F_i), 0\}.$$

- Thurston [30] showed that this defines a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$.
- In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(M;\mathbb{R})$

  $$x_M(r \cdot \phi) = |r| \cdot x_M(\phi).$$

- If $K \subseteq S^3$ is a knot and we take M as its knot complement, then the Thurston norm of the element ϕ_K given by the knot is $2 \cdot \mathrm{genus}(K) - 1$.

- If $p\colon \overline{M} \to M$ is a finite covering with n sheets, then Gabai [11, Corollary 6.13] showed that

$$x_{\overline{M}}(p^*\phi) = n \cdot x_M(\phi).$$

- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle for a 3-manifold M and compact surface F, and $\phi \in H^1(M;\mathbb{Z})$ is given by the homomorphism $H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$, then by Thurston [30, Section 3] we have

$$x_M(\phi) = \begin{cases} -\chi(F), & \text{if } \chi(F) \leq 0; \\ 0, & \text{if } \chi(F) \geq 0. \end{cases}$$

Theorem (The Thurston norm and the degree of the ϕ-twisted L^2-torsion function)

We have

$$x_M(\phi) = \deg\bigl(\rho^{(2)}(M;\phi)\rho^{(2)}(M;\phi)\bigr).$$

- Actually, Thurston defines the so-called Thurston polytope which is essentially the unit ball with respect to the Thurston norm and carries information about the question which ϕ in $H^1(M;\mathbb{Z})$ are fibered.

- The Thurston polytope can be read of the universal L^2-torsion defined by Friedl-Lück [7] using [18] which actually determines also $\rho^{(2)}_X$ and hence $\rho^{(2)}(M;\phi)$.

Homological growth and L^2-torsion

- A normal chain $\{G_i\}$ for the group G is a descending chain of subgroups
$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \tag{1}$$
such that G_i is normal in G and $\bigcap_{i \geq 0} G_i = \{1\}$.

- A normal chain is a finite index normal chain, if and only if $[G : G_i]$ is finite for each i.

- If $G = \pi_1(M)$, them $M[i] \to M$ is the G/G_i-covering associated to $G_i \subseteq G$.

- The following conjecture is taken from Lück [23, Conjecture 1.12 (2)]. For locally symmetric spaces it reduces to the conjecture of Bergeron and Venkatesh [2, Conjecture 1.3].

Conjecture (Homological torsion growth and L^2-torsion)

Let M be an aspherical closed manifold and

$$\pi_1(M) = G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

be any finite index normal chain.
Then we get for any natural number n with $2n+1 \neq \dim(M)$

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i];\mathbb{Z})\right)\right|\right)}{[G:G_i]} = 0.$$

If the dimension $\dim(M) = 2m+1$ is odd, then $\widetilde{M}$ is $\det$-L^2-acyclic and we get

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_m(M[i];\mathbb{Z})\right)\right|\right)}{[G:G_i]} = (-1)^m \cdot \rho^{(2)}(\widetilde{M}).$$

Theorem (Lück [23])

Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup.

Then M is L^2-acyclic and we get for all $n \geq 0$ and any finite index normal chain $(G_i)_{i\geq 0}$

$$\begin{aligned} \lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} &= 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The Singer Conjecture and the Conjecture on Homological torsion growth and L^2-torsion cannot both be true in general. Namely, if both are true, then the so called $\mathbb{F}_p$-Singer Conjecture would be true as pointed out by Avramidi-Okun-Schreve [1]. Moreover, the $\mathbb{F}_p$-Singer Conjecture is not true in general, see [1, Theorem 4].

- There is no contradiction if we additionally assume that $\dim(M) = 3$, in which case the Singer Conjecture is known to be true, see Lott-Lück [21].
- Or one modifies the conjecture about homological torsion growth and L^2-torsion as follows.

Conjecture (Homological torsion growth and L^2-torsion, modified)

Let M be an aspherical closed manifold of odd dimension $\dim(M) = 2m + 1$ which is det-L^2-acyclic. Let $(G_i)_{i \geq 0}$ be any finite index normal chain.

Then

$$\lim_{i \to \infty} \left(\sum_{n=0}^{2m+1} (-1)^n \cdot \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i];\mathbb{Z})\right)\right|\right)}{[G : G_i]} \right) = \rho^{(2)}(\widetilde{M}).$$

- The Conjecture on Homological torsion growth and L^2-torsion is related to the following conjecture taken from Lück [24, Conjecture 14.1 on page 308].

Conjecture (Approximation Conjecture for Fuglede-Kadison determinants)

A group G satisfies the Approximation Conjecture for Fuglede-Kadison determinants if for any normal chain $\{G_i\}$ and any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant

$$\det_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr) = \lim_{i \in I} \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr).$$

- The main issue here are uniform estimates about the spectrum of the n-th Laplace operators on $M[i]$ which are independent of i.

- We are more optimistic about the conjecture above than about the conjecture on homological torsion growth and L^2-torsion since for the latter conjecture also a certain conjecture about regulators come in.

- Let M be a compact connected irreducible 3-manifold with infinite π whose boundary is empty or a union of incompressible tori. Then the conjecture above predicts for any finite index normal chain $(G_i)_{i \geq 0}$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_1(G_i)\right)\right|\right)}{[G : G_i]} = \frac{1}{6\pi} \cdot \operatorname{vol}(M).$$

  Since the volume is always positive, the equation above implies that $\left|\operatorname{tors}\left(H_1(G_i)\right)\right|$ grows exponentially in $[G : G_i]$.

- In particular this would allow to read off the volume from the profinite completion of $\pi_1(M)$, see Kammeyer [16].

Simplicial volume and L^2-invariants

- The simplicial volume of a manifold is a topological variant of the (Riemannian) volume which agrees with it for hyperbolic manifolds up to a dimension constant and was introduced by Gromov [14].

Definition (Simplicial volume)

Let M be a closed connected orientable manifold of dimension n. Define its simplicial volume to be the non-negative real number

$$||M|| := ||j([M])||_1 \quad \in \mathbb{R}^{\geq 0}$$

for any choice of fundamental class $[M] \in H_n^{\text{sing}}(M)$ and $j\colon H_n^{\text{sing}}(M) \to H_n^{\text{sing}}(M;\mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$, where $||j([M])||_1$ is the infimum over the L^1-norms of any cycle in the singular chain complex $C_*^{sing}(M;\mathbb{R})$ representing $j([M])$.

Conjecture (Simplicial volume and L^2-invariants)

Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then:

$$\begin{aligned} b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

- Gromov first asked in [15, Section 8A on page 232] whether under the conditions in the conjecture above the Euler characteristic of M vanishes, and notes that in all available examples even the L^2-Betti numbers of M vanish. The part about L^2-torsion appears in Lück [22, Conjecture 3.2].

L^2-torsion and measure equivalence

- Gaboriau [13] introduced L^2-Betti numbers of measured equivalence relations and proved that two measure equivalent countable groups have proportional L^2-Betti numbers. This notion turned out to have many important applications in recent years, most notably through the work of Popa [28].
- The notion of *measure equivalence* was introduced by Gromov [15, 0.5.E].

Definition (Measure equivalence)

Two countable groups G and H are called measure equivalent with index $c = I(G, H) > 0$ if there exists a non-trivial standard measure space (Ω, μ) on which $G \times H$ acts such that the restricted actions of $G = G \times \{1\}$ and $H = \{1\} \times H$ have measurable fundamental domains $X \subset \Omega$ and $Y \subset \Omega$, with $\mu(X) < \infty$, $\mu(Y) < \infty$, and $c = \mu(X)/\mu(Y)$. The space (Ω, μ) is called a measure coupling between G and H (of index c).

- The following conjecture is taken from Lueck-Sauer-Wegner [27, Conjecture 1.2].

Conjecture (L^2-torsion and measure equivalence)

Let G and H be two admissible groups, which are measure equivalent with index $I(G,H) > 0$. Then

$$\rho^{(2)}(G) = I(G,H) \cdot \rho^{(2)}(H).$$

- Due to Gaboriau [13], the vanishing of the nth L^2-Betti number $b_n^{(2)}(G)$ is an invariant of the measure equivalence class of a countable group G. If all L^2-Betti numbers vanish and G is an admissible group, then the vanishing of the L^2-torsion is a secondary invariant of the measure equivalence class of a countable group G provided that the conjecture above holds.

- Evidence for the conjecture above comes from Lueck-Sauer-Wegner [27, Conjecture 1.10] which says that the conjecture above is true if we replace measure equivalence by the stronger notion of uniform measure equivalence, see [27, Definition 1.3], and assume that G satisfies the Measure Theoretic Determinant Conjecture, see [27, Conjecture 1.7].

- Here is a very interesting aside concerning Fuglede-Kadison determinants and Mahler measures.

Definition (Mahler measure)

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-trivial element. Write it as $p(z) = c \cdot z^k \cdot \prod_{i=1}^{r}(z - a_i)$ for an integer $r \geq 0$, non-zero complex numbers c, a_1, ..., a_r and an integer k. Define its Mahler measure

$$M(p) = |c| \cdot \prod_{\substack{i=1,2,\ldots,r \\ |a_i|>1}} |a_i|.$$

- The following famous and open problem goes back to a question of Lehmer [17].

Problem (Lehmer's Problem)

Does there exist a constant $\Lambda > 1$ such that for all non-trivial elements $p(z) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ with $M(p) \neq 1$ we have

$$M(p) \geq \Lambda?$$

- There is even a candidate for which the minimal Mahler measure is attained, namely, Lehmer's polynomial

$$L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

- It is actual $-z^5 \cdot \Delta(z)$ for the Alexander polynomial $\Delta(z)$ of the bretzel knot given by $(2, 3, 7)$.
- It is conceivable that for any non-trivial element $p \in \mathbb{Z}[\mathbb{Z}]$ with $M(p) > 1$

$$M(p) \geq M(L) = 1.17628\ldots$$

holds.
- For a survey on Lehmer's problem we refer for instance to [4, 5, 6, 29].

Lemma

The Mahler measure $m(p)$ is the square root of the Fuglede-Kadison determinant of the operator $L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ given by multiplication with $p(z) \cdot \overline{p(z)}$.

Definition (Lehmer's constant of a group)

Define Lehmer's constant of a group G

$$\Lambda^w(G) \in [1, \infty)$$

to be the infimum of the set of Fuglede-Kadison determinants

$$\det^{(2)}_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r\bigr),$$

where A runs through all (r, r)-matrices with coefficients in $\mathbb{Z}G$ for all $r \geq 1$, for which $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r$ is a weak isomorphism and the Fuglede-Kadison determinant satisfies $\det^{(2)}_{\mathcal{N}(G)}(r_A^{(2)}) > 1$.

- We can show, see Lück [26]

$$\Lambda^w(\mathbb{Z}^n) \geq M(L)$$

for all $n \geq 1$, provided that Lehmer's problem has a positive answer.

- We know $1 \leq \Lambda^w(G) \leq M(L)$ for torsionfree G.

Problem (Generalized Lehmer's Problem)

For which torsionfree groups G do we have

$$1 < \Lambda^w(G)?$$

Example (Weeks manifold)

There is a closed hyperbolic 3-manifold W, the so called Weeks manifold, which is the unique closed hyperbolic 3-manifold with smallest volume, see Gabai-Meyerhoff-Milley [12, Corollary 1.3]. Its volume is between 0,942 and 0,943. Hence we get

$$\Lambda^w(\pi) \leq \exp\left(\frac{1}{6\pi} \cdot 0,943\right) \leq 1,06.$$

This implies $\Lambda^w(\pi) < M(L)$.

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