

# INHERITANCE PROPERTIES OF THE FARRELL-JONES CONJECTURE FOR TOTALLY DISCONNECTED GROUPS

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ABSTRACT. In this paper we formulate and lay the foundations for the  $K$ -theoretic Farrell-Jones Conjecture for the Hecke algebra of totally disconnected groups. The main result of this paper is the proof that it passes to closed subgroups. Moreover, we carry out some constructions such as the diagonal tensor product and prove some results that will be used in the actual proof of the Farrell-Jones Conjecture for reductive  $p$ -adic groups, which will appear in a different paper.

## 1. INTRODUCTION

1.A. **The Cop-Farrell-Jones Conjecture for Hecke algebras of td-groups.** Let  $R$  be a (not necessarily commutative) associative unital ring with  $\mathbb{Q} \subseteq R$ . Let  $G$  be a td-group i.e., locally compact second countable totally disconnected topological Hausdorff group. Let  $\mathcal{H}(G; R)$  be the associated *Hecke algebra*. We are interested in the algebraic  $K$ -groups  $K_n(\mathcal{H}(G; R))$ . In particular the projective class group  $K_0(\mathcal{H}(G; R))$  is important for the theory of smooth representations of  $G$  with coefficients in  $R$ .

The following Conjecture 1.1 was stated in [11, Conjecture 119 on page 773] for  $R = \mathbb{C}$ . A ring is called *uniformly regular*, if it is Noetherian and there exists a natural number  $l$  such that any finitely generated projective  $R$ -module admits a resolution by projective  $R$ -modules of length at most  $l$ . We write  $\mathbb{Q} \subseteq R$ , if for any integer  $n$  the element  $n \cdot 1_R$  is a unit in  $R$ . Examples for uniformly regular rings  $R$  with  $\mathbb{Q} \subseteq R$  are fields of characteristic zero.

**Conjecture 1.1** (Cop-Farrell-Jones Conjecture for Hecke algebras). *A td-group  $G$  satisfies the Cop-Farrell-Jones Conjecture for Hecke algebras if for every uniformly regular ring  $R$  with  $\mathbb{Q} \subseteq R$  the map induced by the projection  $E_{\text{Cop}}(G) \rightarrow G/G$  induces for every  $n \in \mathbb{Z}$  an isomorphism*

$$H_n^G(E_{\text{Cop}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R)).$$

Here  $H_n^G(-; \mathbf{K}_R)$  is a *smooth  $G$ -homology theory* satisfying  $H_n^G(G/U; \mathbf{K}_R) \cong K_n(\mathcal{H}(U; R))$  for every open subgroup  $U$  of  $G$  and  $n \in \mathbb{Z}$ , see Subsection 2.C, and  $E_{\text{Cop}}(G)$  is the *classifying space for proper smooth  $G$ -actions*, see Section 3.

The isomorphism above yields a computation of the  $K$ -theory of  $\mathcal{H}(G; R)$  in terms of the  $K$ -theory of the compact open subgroups of  $G$ . In particular it implies that the canonical map induced by the various inclusions  $K \subseteq G$  for the set  $\text{Cop}$  of compact open subgroups of  $G$

$$(1.2) \quad \bigoplus_{K \in \text{Cop}} K_0(\mathcal{H}(K; R)) \rightarrow K_0(\mathcal{H}(G; R))$$

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is surjective. This and further consequences of Conjecture 1.1 will be discussed in Theorem 6.11.

We will show in [3, Cor. 1.8] that Conjecture 1.1 is true if  $G$  is a reductive  $p$ -adic group.

Dat [6] has shown that the map (1.2) is rationally surjective for  $G$  a reductive  $p$ -adic group and  $R = \mathbb{C}$ . In particular, the cokernel of it is a torsion group. Dat [5, Conj. 1.11] conjectured that this cokernel is  $\tilde{w}_G$ -torsion. Here  $\tilde{w}_G$  is a certain multiple of the order of the Weyl group of  $G$ . Dat proved this conjecture for  $G = \mathrm{GL}_n(F)$  [5, Prop. 1.13] and asked about the integral version, see the comment following [5, Prop. 1.10]. A consequence of the proof of Conjecture 1.1 is that the integral version is true.

### 1.B. The Cop-Farrell-Jones Conjecture for Hecke categories with $G$ -support.

The Farrell-Jones Conjecture 1.1 for Hecke algebras does not pass to subgroups. Note that subgroups are always understood to be closed. It is interesting to have this inheritance to subgroups, since important subgroups of a reductive  $p$ -adic group such that the Borel subgroup are not necessarily reductive  $p$ -adic groups again. Therefore we develop in this paper a more general version of the Farrell-Jones Conjecture 1.1, which has the inheritance to closed subgroups more or less built in and for which the proof of the Farrell-Jones Conjecture for reductive  $p$ -adic groups in [3] carries over. The idea is to allow more general coefficients than just a ring  $R$ .

We will introduce in Definition 2.1 the notion of a *category with  $G$ -support*  $\mathcal{B}$  and associate to it a  $G$ -homology theory  $H_n^G(-; \mathbf{K}_{\mathcal{B}}^\infty)$  in Section 2. Then one can consider the assembly map

$$H_n^G(E_{\mathrm{Cop}}(G); \mathbf{K}_{\mathcal{B}}^\infty) \rightarrow H_n^G(G/G; \mathbf{K}_{\mathcal{B}}^\infty) = \pi_n(\mathbf{K}^\infty(\mathcal{B}_\oplus))$$

and ask whether it is bijective for all  $n \in \mathbb{Z}$ . In this generality this is not true. However, if one uses the stronger notion of a *Hecke category with  $G$ -support* of Definition 5.1 and requires a regularity assumption, then the following version is realistic.

**Conjecture 1.3** (The Cop-Farrell-Jones Conjecture). *A  $td$ -group  $G$  satisfies the Cop-Farrell-Jones Conjecture if for every Hecke category with  $G$ -support  $\mathcal{B}$  satisfying condition (Reg), see Definition 3.2, the Cop-assembly map induced by the projection  $E_{\mathrm{Cop}}(G) \rightarrow G/G$*

$$H_n^G(E_{\mathrm{Cop}}(G); \mathbf{K}_{\mathcal{B}}^\infty) \rightarrow H_n^G(G/G; \mathbf{K}_{\mathcal{B}}^\infty) = \pi_n(\mathbf{K}^\infty(\mathcal{B}_\oplus))$$

*is bijective for all  $n \in \mathbb{Z}$ .*

We will show in [3, Thm. 1.11].

**Theorem 1.4.** *Every reductive  $p$ -adic group satisfies the Cop-Farrell-Jones Conjecture 1.3.*

Given a uniformly regular ring  $R$  with  $\mathbb{Q} \subseteq R$ , one can construct a Hecke category with  $G$ -support  $\mathcal{B}$  satisfying the condition (Reg), see Definition 3.2, such that the assembly map appearing in Conjecture 1.3 is the assembly map appearing in Conjecture 1.1. Hence Conjecture 1.3 implies Conjecture 1.1. All this is explained in Section 6.c.

**1.c. The main theorem about inheritance to subgroups.** The main theorem of this papers is

**Theorem 1.5.** *Suppose that the Cop-Farrell-Jones Conjecture 1.3 holds for the  $td$ -group  $G$ .*

Then it also holds for every *td*-group  $G'$  that contains a (not necessarily open) normal compact subgroup  $K' \subseteq G'$  such that  $G'/K'$  is isomorphic to some subgroup of  $G$ .

Theorem 1.5 will follow from Theorem 4.1 and Lemma 5.3.

**1.D. Some input for the proof of the Cop-Farrell-Jones Conjecture for Hecke algebras of *td*-groups.** In Section 7 we present some constructions and results which will be needed in the proof of Conjecture 1.3 for every reductive  $p$ -adic group in [3]. There we mainly deal with the construction and the main properties of the so called *diagonal tensor product*.

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## 2. THE SMOOTH $K$ -THEORY SPECTRUM AND THE ASSOCIATED SMOOTH $G$ -HOMOLOGY THEORY

**2.A. The definition of a category with  $G$ -support.** A  $\mathbb{Z}$ -category is a small category  $\mathcal{A}$  enriched over the category of  $\mathbb{Z}$ -modules, i.e., for every two objects  $A$  and  $A'$  in  $\mathcal{A}$  the set of morphisms  $\text{mor}_{\mathcal{A}}(A, A')$  has the structure of a  $\mathbb{Z}$ -module such that composition is a  $\mathbb{Z}$ -bilinear map.

**Definition 2.1.** Let  $G$  be a td-group. A *category with  $G$ -support* is a  $\mathbb{Z}$ -category  $\mathcal{B}$  together with a map that assigns to every morphism  $\varphi$  in  $\mathcal{B}$  a compact subset  $\text{supp}_{\mathcal{B}}(\varphi)$  of  $G$ , often denoted by  $\text{supp}(\varphi)$  for short. For  $B \in \mathcal{B}$  we set  $\text{supp}(B) := \text{supp}(\text{id}_B)$ .

We require the following axioms for any object  $B$ , and any morphisms  $\varphi, \varphi': B \rightarrow B'$ :

- (i)  $\text{supp}(\varphi) = \emptyset \iff \varphi = 0$ ;
- (ii)  $\text{supp}(\varphi' \circ \varphi) \subseteq \text{supp}(\varphi') \cdot \text{supp}(\varphi)$ ;
- (iii)  $\text{supp}(\varphi + \varphi') \subseteq \text{supp}(\varphi) \cup \text{supp}(\varphi')$  and  $\text{supp}(-\text{id}_B) = \text{supp}(B)$ .

We are *not* requiring that  $\text{supp}(s) \subseteq G$  is open, since later we want to allow not necessarily open group homomorphism  $\alpha: G \rightarrow G'$  in Theorem 4.1.

**2.B. The smooth  $K$ -theory spectrum associated to a category with  $G$ -support.** Let  $\mathcal{B}$  be a category with  $G$ -support in the sense of Definition 2.1. Let  $S$  be a *smooth  $G$ -set*, i.e., a  $G$ -set such that the isotropy group of each point in  $S$  is an open subgroup of  $G$ . Define the  $\mathbb{Z}$ -category  $\mathcal{B}[S]$  as follows. Objects are pairs  $(x, B)$  consisting of an element  $x \in S$  and an object  $B \in \mathcal{B}$  such that  $\text{supp}(B) \subseteq G_x$ . A morphism  $\varphi: (x, B) \rightarrow (x', B')$  is a morphism  $\varphi: B \rightarrow B'$  in  $\mathcal{B}$  satisfying  $\text{supp}(\varphi) \subseteq G_{x, x'} := \{g \in G \mid x' = gx\}$ . Composition is given by the composition in  $\mathcal{B}$ . The identity morphism  $\text{id}_B$  yields the identity morphism  $(x, B) \rightarrow (x, B)$  in  $\mathcal{B}[S]$ . The structure of a  $\mathbb{Z}$ -category on  $\mathcal{B}$  induces the structure of a  $\mathbb{Z}$ -category on  $\mathcal{B}[S]$ .

Let  $f: S \rightarrow S'$  be a  $G$ -map. It induces a functor of  $\mathbb{Z}$ -categories  $\mathcal{B}[f]: \mathcal{B}[S] \rightarrow \mathcal{B}[S']$  by sending  $(x, B)$  to  $(f(x), B)$  and a morphism  $\varphi: (x, B) \rightarrow (x', B')$  given by a morphism  $\varphi: B \rightarrow B'$  in  $\mathcal{B}$  to the morphism  $\varphi: (f(x), B) \rightarrow (f(x'), B')$  given by  $\varphi: B \rightarrow B'$  in  $\mathcal{B}$  again.

Given a  $\mathbb{Z}$ -category  $\mathcal{A}$ , one can associate to it an additive  $\mathbb{Z}$  category  $\mathcal{A}_{\oplus}$  with functorial finite sums. One can assign to any additive category  $\mathcal{A}$  its non-connective  $K$ -theory spectrum  $\mathbf{K}^{\infty}(\mathcal{A})$ . All these classical notions are summarized with references to the relevant papers in [1, Section 2 and 3].

**Definition 2.2** (The smooth  $K$ -theory spectrum). We obtain a functor called the *smooth  $K$ -theory spectrum associated to the category with  $G$ -support  $\mathcal{B}$*  with the category of smooth  $G$ -sets as source

$$\mathbf{K}_{\mathcal{B}}^{\infty}: G\text{-SETS}_{\text{sm}} \rightarrow \text{Spectra}, \quad S \mapsto \mathbf{K}^{\infty}(\mathcal{B}[S]_{\oplus}).$$

**Lemma 2.3.** *Let  $S$  be a smooth  $G$ -set. For an orbit  $O \subseteq G \backslash S$ , let  $j_O: O \rightarrow S$  be the inclusion of  $G$ -sets.*

*Then the induced map*

$$\bigvee_{O \in G \backslash S} \mathbf{K}_{\mathcal{B}}^{\infty}(j_O): \bigvee_{O \in G \backslash S} \mathbf{K}_{\mathcal{B}}^{\infty}(O) \rightarrow \mathbf{K}_{\mathcal{B}}^{\infty}(S)$$

*is a weak homotopy equivalence of spectra.*

*Proof.* Let  $(x, B)$  and  $(x', B')$  be objects in  $\mathcal{B}[S]$ . Then  $\text{mor}_{\mathcal{B}[S]}((x, B), (x', B)) \neq \{0\}$  holds only, if  $x$  and  $x'$  belong to the same  $G$ -orbit in  $S$ . Therefore we obtain an equivalence of additive categories

$$(2.4) \quad \bigoplus_{O \in G \backslash S} \mathcal{B}[j_O]_{\oplus} : \bigoplus_{O \in G \backslash S} \mathcal{B}[O]_{\oplus} \rightarrow \mathcal{B}[S]_{\oplus}.$$

Now the claim follows from the fact that algebraic  $K$ -theory of additive categories is compatible with direct sums over arbitrary index sets, see [12, Corollary 7.2].  $\square$

**Remark 2.5.** All the definitions and results of this Subsection 2.B do make sense, if one drops the condition smooth. However, then the resulting spectrum over the orbit category  $\text{Or}(G)$  will turn out not to be the appropriate one, when we will give proofs of the Farrell-Jones Conjecture and will have to consider homogeneous spaces, which are not necessarily smooth, in forthcoming papers, e.g. [3]. This will actually be one of the main technical difficulties. To avoid such problems, we will consider in this paper only smooth spaces and the smooth orbit category. This will be sufficient to state the Farrell-Jones Conjecture and prove some inheritance properties.

**2.c. Smooth  $G$ -homology theories.** Let  $\text{Or}_{\text{sm}}(G)$  be the *smooth orbit category*. Objects are homogeneous spaces  $G/H$  for  $H \subseteq G$  open and morphisms are  $G$ -maps.

Note that  $G/H'$  is for any open subgroup  $H' \subseteq G$  a discrete space and hence  $\text{map}_G(G/H, G/H')$  carries the discrete topology for any subgroup  $H \subseteq G$ . Hence we can view  $\text{Or}_{\text{sm}}(G)$  just as a category without taking any topology on the set of objects or set of morphisms between two objects into account. So in particular all the material of Davis-Lück [7] applies, if we take the category  $\mathcal{C}$  to be  $\text{Or}_{\text{sm}}(G)$ .

Let  $X$  be a smooth  $G$ -CW-complex, i.e., a  $G$ -CW-complex, all whose isotropy groups are open. For an introduction to  $G$ -CW-complexes we refer for instance to [8, Chapter 1 and 2]. We can assign to  $X$  a contravariant  $\text{Or}_{\text{sm}}(G)$ -space

$$(2.6) \quad O^G(X) : \text{Or}_{\text{sm}}(G) \rightarrow \text{Spaces}, \quad G/H \mapsto \text{map}_G(G/H, X).$$

**Remark 2.7** ( $O^G(X)$  is a  $\text{Or}_{\text{sm}}(G)$ -CW-complex). If  $X$  a smooth  $G$ -CW-complex, then  $O^G(X)$  is a free  $\text{Or}_{\text{sm}}(G)$ -CW-complex in the sense of [7, Definition 3.8]. In the sequel we will just talk about a  $\text{Or}_{\text{sm}}(G)$ -CW-complex instead of a free  $\text{Or}_{\text{sm}}(G)$ -CW-complex. Note that in (2.6) we consider  $\text{map}_G(G/H, X)$  as a topological space in order to ensure that the canonical bijection  $\text{map}_G(G/H, X) \xrightarrow{\cong} X^H$  sending  $f$  to  $f(eH)$  is a homeomorphism. If  $G/L$  is an object in  $\text{Or}_{\text{sm}}(G)$ , then  $G/L$  is a discrete space and the topology on  $\text{map}_G(G/H, G/L)$  is the discrete one. Hence the topological space  $\text{map}_G(G/H, G/L)$  agrees with the set of morphisms  $\text{mor}_{\text{Or}_{\text{sm}}(G)}(G/H, G/L)$  equipped with the discrete topology. This is one key ingredient in the proof that  $O^G(X)$  is a  $\text{Or}_{\text{sm}}(G)$ -CW-complex. The other two ingredients are that for a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I} G/L_i \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} G/L_i \times D^n & \longrightarrow & X_{n-1} \end{array}$$

and a subgroup  $H \subseteq G$ , we obtain after applying  $\text{map}_G(G/H, -)$  the pushout

$$\begin{array}{ccc} \coprod_{i \in I} \text{map}_G(G/H, G/L_i) \times S^{n-1} & \longrightarrow & \text{map}_G(G/H, X_{n-1}) \\ \downarrow & & \downarrow \\ \coprod_{i \in I} \text{map}_G(G/H, G/L_i) \times D^n & \longrightarrow & \text{map}_G(G/H, X_n) \end{array}$$

and that  $\text{map}_G(G/H; X)$  carries the weak topology with respect to the filtration by the subspaces  $\text{map}_G(G/H, X_n)$ .

Let  $\mathbf{E}: \text{Or}_{\text{sm}}(G) \rightarrow \text{Spectra}$  be any covariant  $\text{Or}_{\text{sm}}(G)$ -spectrum. Given a smooth  $G$ -CW-complex  $X$ , we obtain a spectrum  $\mathbf{E}(X) := O^G(X)_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{E}$ . The smash product  $\wedge_{\text{Or}_{\text{sm}}(G)}$  is defined for instance in [7, Section 1] and denoted by  $\otimes_{\text{Or}_{\text{sm}}(G)}$  there. Given a pair  $(X, A)$  of smooth  $G$ -CW-complexes, let  $\mathbf{E}(X, A)$  be the cofiber of the maps of spectra  $\mathbf{E}(A) \rightarrow \mathbf{E}(X)$  induced by the inclusion  $A \rightarrow X$ . Define

$$(2.8) \quad H_n^G(X, A; \mathbf{E}) := \pi_n(\mathbf{E}(X, A)).$$

Then we obtain a  $G$ -homology theory  $H_*^G(-, \mathbf{E})$  on the category of smooth  $G$ -CW-complexes, i.e., we obtain a covariant functor from the category of pairs of smooth  $G$ -CW-complexes to the category of  $\mathbb{Z}$ -graded abelian groups sending  $(X, A)$  to  $H_*(X, A; \mathbf{E})$  satisfying the obvious axioms, namely,  $G$ -homotopy invariance, the long exact sequence of a pair, excision, and the disjoint union axiom. We get for every object  $G/H$  in  $\text{Or}_{\text{sm}}(G)$  and every  $n \in \mathbb{Z}$  an isomorphism

$$(2.9) \quad \pi_n(\mathbf{E}(G/H)) \xrightarrow{\cong} H_n(G/H; \mathbf{E}),$$

which is natural in  $G/H$  and  $\mathbf{E}$ . We leave it to the reader to figure out the straightforward proof that all these claims follow from [7, Sections 4 and 7].

### 3. THE $\text{Cop}$ -ASSEMBLY MAP FOR CATEGORIES WITH $G$ -SUPPORT

Let  $G$  be a td-group. We denote by  $E_{\text{Cop}}(G)$  its classifying  $G$ -CW-complex for the family  $\text{Cop}$  of compact open subgroups. This is a proper smooth  $G$ -CW-complex such that the  $H$ -fixed point set  $E_{\text{Cop}}(G)^H$  is weakly contractible for every compact open subgroup  $H \subseteq G$ . A  $G$ -CW-complex  $X$  is proper and smooth if and only if each of its isotropy group is compact and open, see [8, Theorem 1.23 on page 18]. Two models for  $E_{\text{Cop}}(G)$  are  $G$ -homotopy equivalent. This follows from the universal property that for any proper smooth  $G$ -CW-complex  $X$  there is up to  $G$ -homotopy precisely one  $G$ -map from  $X$  to  $E_{\text{Cop}}(G)$ . We mention that the canonical  $G$ -map  $E_{\text{Cop}}(G) \rightarrow J_{\text{Cop}}(G)$  is a  $G$ -homotopy equivalence, if  $J_{\text{Cop}}(G)$  denotes the numerable version of the classifying space for the family  $\text{Cop}$ , see [9, Lemma 3.5]. For more information about classifying spaces for families, we refer for instance to [9].

**Problem 3.1.** For which categories  $\mathcal{B}$  with  $G$ -support is the  $\text{Cop}$ -assembly map induced by the projection  $E_{\text{Cop}}(G) \rightarrow G/G$

$$H_n^G(E_{\text{Cop}}(G); \mathbf{K}_{\mathcal{B}}^\infty) \rightarrow H_n^G(G/G; \mathbf{K}_{\mathcal{B}}^\infty) = \pi_n(\mathbf{K}^\infty(\mathcal{B}_\oplus))$$

bijection for all  $n \in \mathbb{Z}$ ?

Given an additive category  $\mathcal{A}$ , we have defined in [1, Definition 6.2 (iii)] the notion  $l$ -uniformly regular coherent for a natural number  $l$ . The additive category  $\mathcal{A}$  is  $l$ -uniformly regular coherent, if and only if its idempotent completion  $\text{Idem}(\mathcal{A})$  is  $l$ -uniformly regular coherent, see [1, Lemma 6.4 (vi)]. Intrinsic equivalent definitions of the notion  $l$ -uniformly regular coherent for idempotent categories are presented in [1, Lemma 6.6]. For instance, if  $l \geq 2$  and  $\mathcal{A}$  is idempotent complete,  $\mathcal{A}$  is  $l$ -uniformly regular coherent, if and only if for every morphism  $f_1: A_1 \rightarrow A_0$  we can find a sequence of length  $l$  in  $\mathcal{A}$

$$0 \rightarrow A_l \xrightarrow{f_l} A_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0,$$

which is exact at  $A_i$  for  $i = 1, 2, \dots, n$  in the sense that for any object  $B$  in  $\mathcal{A}$  the induced sequence  $\text{hom}_{\mathcal{A}}(B, A_{i+1}) \xrightarrow{(f_{i+1})^*} \text{hom}_{\mathcal{A}}(B, A_i) \xrightarrow{(f_i)^*} \text{hom}_{\mathcal{A}}(B, A_{i-1})$  is exact.

Given an additive category  $\mathcal{A}$ , we define by  $\mathcal{A}[\mathbb{Z}]$  the associated additive category of finite Laurent series over  $\mathcal{A}$  as follows. It has the same objects as  $\mathcal{A}$ . Given two objects  $A$  and  $B$ , a morphism  $f: A \rightarrow B$  in  $\mathcal{A}[\mathbb{Z}]$  is a formal sum  $f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i$ , where  $f_i: A \rightarrow B$  is a morphism in  $\mathcal{A}$  from  $A$  to  $B$  and only finitely many of the morphisms  $f_i$  are non-trivial. If  $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j$  is a morphism in  $\mathcal{A}[\mathbb{Z}]$  from  $B$  to  $C$ , we define the composite  $g \circ f: A \rightarrow C$  by

$$g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{\substack{i, j \in \mathbb{Z}, \\ i+j=k}} g_j \circ f_i \right) \cdot t^k.$$

For a natural number  $d$  we define inductively  $\mathcal{A}[\mathbb{Z}^d] = (\mathcal{A}[\mathbb{Z}^{d-1}])[\mathbb{Z}]$ .

Without some regularity assumptions the answer to Problem 3.1 is in general not positive, as the example  $G = \mathbb{Z}$  and  $R = \mathbb{Z}[t]/t^2$  together with the Bass-Heller-Swan decomposition shows, see [13, Theorem 3.2.22 on page 149 and Exercise 3.2.23 on page 151]

**Definition 3.2** (Reg). A category  $\mathcal{B}$  be with  $G$ -support in the sense of Definition 2.1 satisfies the condition (Reg), if for every natural number  $d$  there is a natural number  $l(d)$  such that for every compact open subgroup  $K \subseteq G$  the additive category  $\mathcal{B}[G/K]_{\oplus}[\mathbb{Z}^d]$  is  $l(d)$ -uniformly regular coherent.

**Remark 3.3.** The Farrell-Jones Conjecture formulated for categories with  $G$ -support would predict that the answer to Problem 3.1 is positive for every td-group  $G$  and every category  $\mathcal{B}$  with  $G$ -support that satisfies condition (Reg) of Definition (3.2).

However, this is already for discrete groups  $G$  a far too optimistic statement, since the notion of a category with  $G$ -support is very general and the actual proofs of the Farrell-Jones Conjecture for certain classes of discrete groups have no chance to go through in this general setting, the problem is the construction of certain transfer, see [3, Section 13]. The adequate formulation of the Farrell-Jones Conjecture has been given in Conjecture 1.3.

#### 4. INHERITANCE TO SUBGROUPS MODULO NORMAL COMPACT SUBGROUPS

The main results of this section is

**Theorem 4.1** (Inheritance to closed subgroups modulo normal compact groups). *Let  $\alpha: G \rightarrow G'$  be a (not necessarily open) group homomorphism with compact kernel. Let  $\mathcal{B}$  be a category with  $G$ -support.*

*Then there exists a category with  $G'$ -support  $\text{ind}_{\alpha} \mathcal{B}$  with the following properties:*

- (i) *If  $\mathcal{B}$  satisfies condition (Reg), see Definition 3.2, then  $\text{ind}_{\alpha} \mathcal{B}$  also satisfies condition (Reg);*
- (ii) *There is a commutative diagram*

$$\begin{array}{ccc} H_n^G(E_{\text{Cop}}(G); \mathbf{K}_{\mathcal{B}}) & \longrightarrow & H_n^G(G/G; \mathbf{K}_{\mathcal{B}}) \\ \cong \downarrow & & \downarrow \cong \\ H_n^{G'}(E_{\text{Cop}}(G'); \mathbf{K}_{\text{ind}_{\alpha} \mathcal{B}}) & \longrightarrow & H_n^{G'}(G'/G'; \mathbf{K}_{\text{ind}_{\alpha} \mathcal{B}}) \end{array}$$

*whose vertical arrows are bijective.*

Its proof needs some preparation. The notion of categories with  $G$ -support has been designed to with Theorem 4.1 in mind. Its proof needs some preparation.

**4.A. The definition of the induced category with  $G'$ -support.** Fix a (not necessarily open) group homomorphism of td-groups  $\alpha: G \rightarrow G'$ . We always require for  $\alpha$  that  $\text{im}(\alpha) \subseteq G'$  is closed and that the induced group homomorphism  $G \rightarrow \text{im}(\alpha)$  is an identification, or equivalently, is open. We want to assign to a category with  $G$ -support  $\mathcal{B}$  a category with  $G'$ -support  $\text{ind}_\alpha \mathcal{B}$  as follows.

We first define a  $\mathbb{Z}$ -category  $\text{ind}_\alpha \mathcal{B}$ . An object  $(B_0, g'_0)$  in  $\text{ind}_\alpha \mathcal{B}$  is a pair consisting of elements  $B_0 \in \text{ob}(\mathcal{B})$  and  $g'_0 \in G'$ . Given two objects  $(B_0, g'_0)$  and  $(B_1, g'_1)$  in  $\text{ind}_\alpha \mathcal{B}$ , define the  $\mathbb{Z}$ -module of morphisms between them by

$$(4.2) \quad \text{mor}_{\text{ind}_\alpha \mathcal{B}}((B_0, g'_0), (B_1, g'_1)) = \text{mor}_{\mathcal{B}}(B_0, B_1).$$

Composition and the identity elements in  $\text{ind}_\alpha \mathcal{B}$  are given by the corresponding ones in  $\mathcal{B}$ .

The support function for  $\text{ind}_\alpha \mathcal{B}$  assigns to a morphism  $\varphi: (B_0, g'_0) \rightarrow (B_1, g'_1)$  the compact subset of  $G'$  given by

$$(4.3) \quad \text{supp}_{\text{ind}_\alpha \mathcal{B}}(\varphi) := g'_1 \alpha(\text{supp}_{\mathcal{B}}(\varphi)) g'_0^{-1}.$$

In particular we get

$$(4.4) \quad \text{supp}_{\text{ind}_\alpha \mathcal{B}}((B_0, g'_0)) = g'_0 \alpha(\text{supp}_{\mathcal{B}}(B_0)) g'_0^{-1}$$

for an object  $(B_0, g'_0)$ . This finishes the definition of the category with  $G'$ -support  $\text{ind}_\alpha \mathcal{B}$ .

**4.B. The smooth  $K$ -theory spectrum is compatible with induction.** The smooth  $K$ -theory spectrum  $\mathbf{K}_{\mathcal{B}}^\infty$  of Definition 2.2 induces a covariant  $\text{Or}_{\text{sm}}(G)$ -spectrum denoted in the same way

$$\mathbf{K}_{\mathcal{B}}^\infty: \text{Or}_{\text{sm}}(G) \rightarrow \text{Spectra}, \quad G/H \mapsto \mathbf{K}^\infty(\mathcal{B}[G/H]_{\oplus}).$$

Given any covariant  $\text{Or}_{\text{sm}}(G)$ -spectrum  $\mathbf{E}: \text{Or}_{\text{sm}}(G) \rightarrow \text{Spectra}$ , define the covariant  $\text{Or}_{\text{sm}}(G')$ -spectrum  $\alpha_* \mathbf{E}$ ,

$$(4.5) \quad \alpha_* \mathbf{E}: \text{Or}_{\text{sm}}(G') \rightarrow \text{Spectra},$$

$$G'/H' \mapsto \text{map}_{G'}(\alpha_* G/?, G'/H')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{E}(G/?).$$

Note that for an open subgroup  $H \subseteq G$  the subgroup  $\alpha(H)$  of  $G'$  is automatically closed, since  $\text{im}(\alpha) \subseteq G'$  is closed and  $\alpha(H) \subseteq \text{im}(\alpha)$  is an open and hence a closed subgroup of  $\text{im}(\alpha)$  as  $\alpha: G \rightarrow \text{im}(\alpha)$  is an identification. It does not matter that  $\alpha_*(G/H) = G'/\alpha(H)$  is not necessarily a smooth  $G'$ -space, since  $G'/H'$  is discrete and hence  $\text{map}_{G'}(\alpha_* G/H, G'/H') \cong (G'/H')^{\alpha(H)}$  carries the discrete topology. In particular we get the covariant  $\text{Or}_{\text{sm}}(G')$ -spectrum  $\alpha_* \mathbf{K}_{\mathcal{B}}^\infty$ . The construction above applied to  $\text{ind}_\alpha \mathcal{B}$  instead of  $\mathcal{B}$  yields another covariant  $\text{Or}_{\text{sm}}(G')$ -spectrum  $\mathbf{K}_{\text{ind}_\alpha \mathcal{B}}^\infty$ .

**Proposition 4.6.** *There is a weak homotopy equivalence of covariant  $\text{Or}_{\text{sm}}(G')$ -spectra, natural in  $\mathcal{B}$ ,*

$$\mathbf{U}: \alpha_* \mathbf{K}_{\mathcal{B}}^\infty \xrightarrow{\simeq} \mathbf{K}_{\text{ind}_\alpha \mathcal{B}}^\infty.$$

Its proof needs some preparation.

Given a smooth  $G'$ -set  $S'$ , we construct a functor of  $\mathbb{Z}$ -categories

$$(4.7) \quad W: \mathcal{B}[\alpha^* S'] \rightarrow \text{ind}_\alpha \mathcal{B}[S']$$

as follows. It sends an object  $(x, B)$  in  $\mathcal{B}[\alpha^* S']$ , which consists of an element  $x \in S'$  and an object  $B$  in  $\mathcal{B}$  to the object  $(x, (B, e'))$  in  $\text{ind}_\alpha \mathcal{B}[S']$  given by  $x \in S'$  and the object  $(B, e')$  in  $\text{ind}_\alpha \mathcal{B}$  for  $e' \in G'$  the unit. This makes sense, since the object  $(x, B)$  satisfies  $\text{supp}_{\mathcal{B}}(B) \subseteq G_x$ , we have  $G_x = \alpha^{-1}(G'_x)$  and we compute

$$\text{supp}_{\text{ind}_\alpha \mathcal{B}[S']}((x, e')) \stackrel{(4.4)}{=} e' \alpha(\text{supp}_{\mathcal{B}}(B)) e'^{-1} = \alpha(\text{supp}_{\mathcal{B}}(B)) \subseteq \alpha(G_x) \subseteq G'_x.$$



Consider two objects  $(x_0, B_0)$  and  $(x_1, B_1)$ . From the definitions we get identifications

$$\mathrm{mor}_{\mathcal{B}[\alpha^* S']}((x_0, B_0), (x_1, B_1)) = \{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid \mathrm{supp}_{\mathcal{B}}(\varphi) \subseteq G_{x_0, x_1}\},$$

and

$$\begin{aligned} & \mathrm{mor}_{\mathrm{ind}_{\alpha} \mathcal{B}[S']} (W(x_0, B_0), W(x_1, B_1)) \\ &= \mathrm{mor}_{\mathrm{ind}_{\alpha} \mathcal{B}[S']} ((x_0, (B_0, e')), (x_1, (B_1, e'))) \\ &= \{\varphi \in \mathrm{mor}_{\mathrm{ind}_{\alpha} \mathcal{B}}((B_0, e'), (B_1, e')) \mid \mathrm{supp}_{\mathrm{ind}_{\alpha} \mathcal{B}}(\varphi) \subseteq G'_{x_0, x_1}\} \\ &\stackrel{(4.2), (4.3)}{=} \{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid e' \alpha(\mathrm{supp}_{\mathcal{B}}(\varphi)) e'^{-1} \subseteq G'_{x_0, x_1}\} \\ &= \{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid \alpha(\mathrm{supp}_{\mathcal{B}}(\varphi)) \subseteq G'_{x_0, x_1}\} \\ &= \{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid \mathrm{supp}_{\mathcal{B}}(\varphi) \subseteq \alpha^{-1}(G'_{x_0, x_1})\} \\ &= \{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid \mathrm{supp}_{\mathcal{B}}(\varphi) \subseteq G_{x_0, x_1}\}. \end{aligned}$$

Under these identification we define

$$W: \mathrm{mor}_{\mathcal{B}[\alpha^* S']}((x_0, B_0), (x_1, B_1)) \rightarrow \mathrm{mor}_{\mathrm{ind}_{\alpha} \mathcal{B}[S']} (W(x_0, B_0), W(x_1, B_1))$$

by the identity on  $\{\varphi \in \mathrm{mor}_{\mathcal{B}}(B_0, B_1) \mid \mathrm{supp}_{\mathcal{B}}(\varphi) \subseteq G_{x_0, x_1}\}$ . One easily checks that  $W$  is a well-defined functor of  $\mathbb{Z}$ -categories.

**Lemma 4.8.** *The functor  $W: \mathcal{B}[\alpha^* S'] \rightarrow \mathrm{ind}_{\alpha} \mathcal{B}[S']$  of (4.7) is an equivalence of  $\mathbb{Z}$ -categories and is natural in  $S'$  and  $\mathcal{B}$ .*

*Proof.* The naturality statements are obvious. In view of the definition of  $W$  on morphisms, it remains to show that for any object  $(x, (B, g'))$  in  $\mathrm{ind}_{\alpha} \mathcal{B}[S']$  there is an object of the shape  $(x', (B', e'))$  in  $\mathrm{ind}_{\alpha} \mathcal{B}[S']$  such that  $(x, (B, g'))$  and  $(x', (B', e'))$  are isomorphic. Since  $(x, (B, g'))$  belongs to  $\mathrm{ind}_{\alpha} \mathcal{B}[S']$ , we have

$$(4.9) \quad g' \alpha(\mathrm{supp}_{\mathcal{B}}(B)) g'^{-1} \stackrel{(4.3)}{=} \mathrm{supp}_{\mathrm{ind}_{\alpha} \mathcal{B}}(g', B) \subseteq G'_x.$$

The identity  $\mathrm{id}_{\mathcal{B}}: B \rightarrow B$  in  $\mathcal{B}$  determines an isomorphism  $\varphi: (B, e' \alpha(\mathrm{supp}_{\mathcal{B}}(B))) \xrightarrow{\cong} (B, g' \alpha(\mathrm{supp}_{\mathcal{B}}(B)))$  in  $\mathrm{ind}_{\alpha} \mathcal{B}$ . We compute

$$\mathrm{supp}_{\mathrm{ind}_{\alpha} \mathcal{B}}(\varphi) \stackrel{(4.3)}{=} g' \alpha(\mathrm{supp}_{\mathcal{B}}(B)) e'^{-1} = g' \alpha(\mathrm{supp}_{\mathcal{B}}(B)) g'^{-1} g' \stackrel{(4.9)}{\subseteq} G'_x g' = G'_{g'^{-1}x, x}.$$

Analogously we get  $\mathrm{supp}_{\mathrm{ind}_{\alpha} \mathcal{B}}(\varphi^{-1}) \subseteq G'_{x, g'^{-1}x}$ . Hence we obtain an isomorphism

$$\varphi: (g'^{-1}x, (B, e' \alpha(\mathrm{supp}_{\mathcal{B}}(B)))) \xrightarrow{\cong} (x, (B, g' \alpha(\mathrm{supp}_{\mathcal{B}}(B))))$$

in  $\mathrm{ind}_{\alpha} \mathcal{B}[S']$ . This finishes the proof of Lemma 4.8.  $\square$

Given a smooth  $G$ -set  $S$ , we define a map of spectra

$$(4.10) \quad V(S): \mathrm{map}_G(G/?, S)_+ \wedge_{\mathrm{Or}_{\mathrm{sm}}(G)} \mathbf{K}_{\mathcal{B}}^{\infty}(G/?) \rightarrow \mathbf{K}_{\mathcal{B}}(S)$$

by sending  $f \otimes z$  for  $f \in \mathrm{map}_G(G/H, S)_+$ ,  $z \in \mathbf{K}_{\mathcal{B}}^{\infty}(G/H)$  and an open subgroup  $H \subseteq G$  to  $\mathbf{K}_{\mathcal{B}}^{\infty}(f)(x)$ .

**Lemma 4.11.** *The map of spectra  $V(S)$  is a weak homotopy equivalence*

*Proof.* This follows from Lemma 2.3, since under the obvious identifications

$$\begin{aligned}
& \text{map}_G(G/?, S)_+ \wedge_{\text{Or}(G)} \mathbf{K}_B^\infty(G/?) \\
&= \text{map}_G\left(G/?, \coprod_{O \in G \setminus S} O\right)_+ \wedge_{\text{Or}(G)} \mathbf{K}_B^\infty(G/?) \\
&= \left( \coprod_{O \in G \setminus S} \text{map}_G(G/?, O) \right)_+ \wedge_{\text{Or}(G)} \mathbf{K}_B^\infty(G/?) \\
&= \left( \bigvee_{O \in G \setminus S} \text{map}_G(G/?, O)_+ \right) \wedge_{\text{Or}(G)} \mathbf{K}_B^\infty(G/?) \\
&= \bigvee_{O \in G \setminus S} \left( \text{map}_G(G/?, O)_+ \wedge_{\text{Or}(G)} \mathbf{K}_B^\infty(G/?) \right) \\
&= \bigvee_{O \in G \setminus S} \mathbf{K}_B^\infty(O),
\end{aligned}$$

the map  $V(S)$  becomes the map appearing in Lemma 2.3.  $\square$

Now we are ready to give the proof of Proposition 4.6.

*Proof.* Given a smooth  $G'$ -set  $S'$ , we have the natural adjunction isomorphism of discrete  $\text{Or}_{\text{sm}}(G)$ -sets

$$(4.12) \quad a: \text{map}_{G'}(\alpha_* G/?, S') \xrightarrow{\cong} \text{map}_G(G/?, \alpha^* S').$$

If we precompose  $V(\alpha^* S')$  defined in (4.10) with the induced isomorphism

$$\begin{aligned}
a_+ \wedge_{\text{Or}_{\text{sm}}(G)} \text{id}: \text{map}_{G'}(\alpha_* G/?, S')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{K}_B^\infty(G/?) \\
\xrightarrow{\cong} \text{map}_G(G/?, \alpha^* S')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{K}_B^\infty(G/?),
\end{aligned}$$

we obtain a weak homotopy equivalence of spectra, natural in  $S'$ ,

$$V'(S'): \text{map}_{G'}(\alpha_* G/?, S')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{K}_B^\infty(G/?) \xrightarrow{\cong} \mathbf{K}_B^\infty[\alpha^* S'].$$

If we compose  $V'(S')$  with the weak homotopy equivalence  $\mathbf{K}^\infty(W_\oplus)$  induced on the  $K$ -theory spectrum by the equivalence of  $\mathbb{Z}$ -categories  $W$ , see (4.7) and Lemma 4.8, we obtain a weak homotopy equivalence of spectra, natural in  $S'$ ,

$$\mathbf{U}(S'): \text{map}_{G'}(\alpha_* G/?, S')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{K}_B^\infty(G/?) \xrightarrow{\cong} \mathbf{K}_{\text{ind}_\alpha B}^\infty(S').$$

If we let  $S'$  run through the objects of  $\text{Or}_{\text{sm}}(G')$ , we get from the collection of the  $\mathbf{U}(S')$ -s the desired functor  $\mathbf{U}$ . This finishes the proof of Proposition 4.6.  $\square$

**4.C. The Adjunction Theorem for spectra and categories with  $G$ -support.** Let  $\alpha: G \rightarrow G'$  be a (not necessarily open) group homomorphism. Let  $\mathbf{E}: \text{Or}_{\text{sm}}(G) \rightarrow \text{Spectra}$  be a covariant  $\text{Or}_{\text{sm}}(G)$ -spectrum. We have defined the covariant  $\text{Or}_{\text{sm}}(G')$  spectrum  $\alpha_* \mathbf{E}$  in (4.5).

**Theorem 4.13** (Adjunction Theorem for spectra). *Let  $(X', A')$  be a pair of smooth  $G'$ -CW-complexes. Then:*

- (i) *The  $G'$ -pair  $\alpha^*(X', A')$  obtained from  $(X', A')$  by restriction with  $\alpha$  is a pair of smooth  $G$ -CW-complexes;*
- (ii) *We obtain an isomorphism of  $\mathbb{Z}$ -graded abelian groups*

$$\alpha_*^{\text{sm}}(X', A'): H_n^G(\alpha^*(X', A'); \mathbf{E}) \xrightarrow{\cong} H_n^{G'}(X', A'; \alpha_* \mathbf{E}),$$

*which is natural in  $(X', A')$  and  $\mathbf{E}$ ;*

- (iii) *The collection of the isomorphisms  $\alpha_*^{\text{sm}}(X', A')$  yield an isomorphism of smooth  $G'$ -homology theories.*

*Proof.* For simplicity we consider only the case  $A' = \emptyset$ .

(i) The functor sending a  $G'$ -space  $X'$  to the  $G$ -space  $\alpha^* X'$  obtained from  $X'$  by restriction with  $\alpha$  is compatible with pushouts and directed colimits. If  $H' \subseteq G'$  is an open subgroup, then  $\alpha^* G'/H'$  is  $G$ -homeomorphic to the disjoint union of its  $G$ -orbits and each of these  $G$ -orbit is  $G$ -homeomorphic to  $G/H$  for some open subgroup  $H \subseteq G$ . This implies that  $\alpha^* X'$  is a smooth  $G$ -CW-complex, if  $X'$  is a smooth  $G'$ -CW-complex, the  $n$ -skeleton  $(\alpha^* X)_n$  of  $\alpha^* X$  is defined to be  $\alpha^*(X_n)$ .

(ii) and (iii) Given a smooth  $G'$ -CW-complex  $X'$ , we construct a map of spectra

$$\mathbf{a}(X'): (\alpha_* \mathbf{E})(X') \rightarrow \mathbf{E}(\alpha^* X').$$

We get from the definitions, the adjunction  $(\alpha_*, \alpha^*)$ , and the associativity of the smash products over the smooth orbit categories identifications

$$(\alpha_* \mathbf{E})(X') = (\text{map}_{G'}(G'/?!, X') \times_{\text{Or}_{\text{sm}}(G')} \text{map}_{G'}(\alpha_* G/?!, G'/?!))_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{E}(G/?),$$

and

$$\mathbf{E}(\alpha^* X') = \text{map}_{G'}(\alpha_* G/?!, X')_+ \wedge_{\text{Or}_{\text{sm}}(G)} \mathbf{E}(G/?).$$

Hence it suffices to construct for every object  $G/H$  in  $\text{Or}_{\text{sm}}(G)$  a map of (unpointed) spaces

$$\text{map}_{G'}(G'/?!, X') \times_{\text{Or}_{\text{sm}}(G')} \text{map}_{G'}(\alpha_* G/H, G'/?! ) \rightarrow \text{map}_{G'}(\alpha_* G/H, X'),$$

which is natural in  $G/H$ . It is given by  $(u, v) \mapsto u \circ v$ .

The collection of the maps of spectra  $\mathbf{a}(X')$  defines a transformation of smooth  $G'$ -homology theories

$$\alpha_*^{\text{sm}}(-): H_*^G(\alpha^*(-); \mathbf{E}) \rightarrow H_n^{G'}(-; \alpha_* \mathbf{E}),$$

where the left hand side is indeed a smooth  $G'$ -homology theory because of assertion (i).

It remains to show that  $\alpha_*^{\text{sm}}(X')$  is an isomorphism for every smooth  $G'$ -CW-complex  $X'$ . Since the  $G'$ -homology theories satisfy the disjoint union axiom, the canonical maps

$$\begin{aligned} \text{colim}_{n \rightarrow \infty} H_*^G(\alpha^* X'_n; \mathbf{E}) &\xrightarrow{\cong} H_*^G(\alpha^* X'; \mathbf{E}); \\ \text{colim}_{n \rightarrow \infty} H_n^{G'}(X'_n; \alpha_* \mathbf{E}) &\xrightarrow{\cong} H_n^{G'}(X'; \alpha_* \mathbf{E}), \end{aligned}$$

are isomorphisms, since the non-equivariant proof in [14, Proposition 7.53 on page 121] carries directly over to the equivariant setting. Hence we can assume without loss of generality that  $X'$  is  $n$ -dimensional. Now using the Mayer-Vietoris sequences, the disjoint union axiom,  $G$ -homotopy invariance and the Five-Lemma, one reduces the proof to the special case  $X' = G'/H'$  for  $H' \subseteq G'$  an open subgroup. This special case follows from the definition (4.5) and the adjunction (4.12). This finishes the proof of Theorem 4.13.  $\square$

We conclude from Proposition 4.6 and Theorem 4.13 using the obvious version of [7, Theorem 3.11].

**Theorem 4.14** (Adjunction Theorem for categories with  $G$ -support). *Let  $(X', A')$  be a pair of smooth  $G'$ -CW-complexes. Let  $\mathcal{B}$  be a category with  $G$ -support. Then:*

(i) *We obtain an isomorphism of  $\mathbb{Z}$ -graded abelian groups*

$$\alpha_*^{\text{sm}}(X', A'): H_n^G(\alpha^*(X', A'); \mathbf{K}_{\mathcal{B}}^\infty) \xrightarrow{\cong} H_n^{G'}(X', A'; \mathbf{K}_{\text{ind}_\alpha \mathcal{B}}^\infty),$$

*which is natural in  $(X', A')$  and  $\mathcal{B}$ ;*

(ii) *The collection of the isomorphisms  $\alpha_*^{\text{sm}}(X', A')$  yields an isomorphism of smooth  $G'$ -homology theories.*

4.D. **Proof of Theorem 4.1.** We begin with assertion (i). Consider a compact open subgroup  $K' \subseteq G'$ . We obtain from (2.4) and Lemma 4.8 an equivalence of additive categories

$$\bigoplus_{O' \in G \backslash \alpha^*(G'/K')} \mathcal{B}[O']_{\oplus}[\mathbb{Z}^d] \xrightarrow{\cong} \text{ind}_{\alpha} \mathcal{B}[G'/K']_{\oplus}[\mathbb{Z}^d].$$

Since the kernel of  $\alpha$  is compact, each  $O'$  is a proper smooth  $G$ -orbit. We conclude from condition (Reg) that  $\mathcal{B}(O')$  is  $l(d)$ -uniformly regular coherent. Since the direct sum (over an arbitrary index set) of  $l(d)$ -uniformly regular coherent categories is again  $l(d)$ -uniformly regular coherent, see [1, Lemma 11.3 (ii)],  $\text{ind}_{\alpha} \mathcal{B}[G'/K']_{\oplus}[\mathbb{Z}^d]$  is  $l(d)$ -uniformly regular coherent.

Finally we prove assertion (ii). If  $\mathcal{C}_{\text{om}}$  is the family of compact subgroups, then the canonical map  $E_{\mathcal{C}_{\text{op}}}(G) \rightarrow E_{\mathcal{C}_{\text{om}}}(G)$  is a  $G$ -homotopy equivalence and the canonical map  $E_{\mathcal{C}_{\text{op}}}(G') \rightarrow E_{\mathcal{C}_{\text{om}}}(G')$  is a  $G'$ -homotopy equivalence, see [9, Lemma 3.5]. Since the kernel of  $\alpha$  is compact,  $\alpha^* E_{\mathcal{C}_{\text{om}}}(G')$  is a model for  $E_{\mathcal{C}_{\text{om}}}(G)$ . We conclude that  $\alpha^* E_{\mathcal{C}_{\text{op}}}(G')$  is a model for  $E_{\mathcal{C}_{\text{op}}}(G)$ . Obviously  $\alpha^* G'/G' = G/G$  holds. Hence we get from the Adjunction Theorem 4.14 for categories with  $G$ -support a commutative diagram

$$\begin{array}{ccc} H_n^G(E_{\mathcal{C}_{\text{op}}}(G); \mathbf{K}_{\mathcal{B}}) & \longrightarrow & H_n^G(G/G; \mathbf{K}_{\mathcal{B}}) \\ \cong \downarrow & & \downarrow \cong \\ H_n^{G'}(E_{\mathcal{C}_{\text{op}}}(G'); \mathbf{K}_{\text{ind}_{\alpha} \mathcal{B}}) & \longrightarrow & H_n^{G'}(G'/G'; \mathbf{K}_{\text{ind}_{\alpha} \mathcal{B}}) \end{array}$$

whose vertical arrows are bijective. This finishes the proof of Theorem 4.1.

## 5. HECKE CATEGORIES WITH $G$ -SUPPORT

Next we enrich the notion of category with  $G$ -support in the sense of Definition 2.1 so that with this new notion one can hope that the answer to Problem 3.1 has a chance to be positive.

Recall that for two subsets  $A, B \subseteq G$  we put  $A \cdot B = \{a \cdot b \mid a \in A, b \in B\} \subseteq G$ .

**Definition 5.1** (Hecke categories with  $G$ -support). A *Hecke category with  $G$ -support* is a category  $\mathcal{B}$  with  $G$ -support such that the following holds.

- (i) *Subgroups*  
 $\text{supp } B$  is a compact subgroup of  $G$  for all objects  $B$ . For any morphism  $\varphi: B \rightarrow B'$  we have  $\text{supp } \varphi = \text{supp } B' \cdot \text{supp } \varphi \cdot \text{supp } B$ . The sets  $\text{supp } B' \setminus \text{supp } \varphi$  and  $\text{supp } \varphi / \text{supp } B$  are both finite;
- (ii) *Translation*  
 For every object  $B$  in  $\mathcal{B}$  and element  $g \in G$  there exists an object  $B'$  together with an isomorphism  $\psi: B \xrightarrow{\cong} B'$  in  $\mathcal{B}$  such that  $\text{supp}(B') = g \text{supp}(B) g^{-1}$ ,  $\text{supp}(\psi) = g \text{supp}(B)$ , and  $\text{supp}(\psi^{-1}) \subseteq g^{-1} \text{supp}(B')$  holds;
- (iii) *Morphism Additivity*  
 For any finite disjoint decomposition

$$\text{supp}(\varphi) = \coprod_{i=1}^m L_i$$

for closed subsets  $L_i \subseteq \text{supp}(\varphi)$  satisfying  $\text{supp}(B') \cdot L_i \cdot \text{supp}(B) = L_i$  for  $i = 1, 2, \dots, m$ , there is a collection of morphisms  $\varphi_i: B \rightarrow B'$  for  $i = 1, 2, \dots, m$  such that  $\varphi = \sum_{i=1}^m \varphi_i$  and  $\text{supp}(\varphi_i) = L_i$  hold;

(iv) *Support cofinality*

For any object  $B$  and any subgroup  $L \subseteq \text{supp}(B)$  of finite index, there is an object  $B|_L$  and morphisms  $i_{B,L}: B \rightarrow B|_L$  and  $r_{B,L}: B|_L \rightarrow B$  such that  $\text{supp}(B|_L) = L$ ,  $\text{supp}(i_{B,L}) = \text{supp}(r_{B,L}) = \text{supp}(B)$ , and  $r_{B,L} \circ i_{B,L} = \text{id}_B$  hold.

Moreover, for any object  $B$  and any subgroups  $L' \subseteq L \subseteq \text{supp}(B)$  of finite index we require  $(B|_L)_{L'} = B|_{L'}$ ,  $i_{B,L'} = i_{B|_L,L'} \circ i_{B,L}$ , and  $r_{B,L'} = r_{B,L} \circ r_{B|_L,L'}$  and for  $L = \text{supp}(B)$  we require  $B|_L = B$  and  $i_{B,L} = r_{B,L} = \text{id}_B$ .

One can view condition *Morphism Additivity* as a kind of sheaf-condition. Note that *Translation* and *Support cofinality* are not just conditions, an additional datum is required.

**Lemma 5.2.**

- (i) Let  $\varphi_i: B \rightarrow B'$  be a collection of morphisms for  $i = 1, \dots, r$  such that  $\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset$  holds for  $i \neq j$  and we have  $\sum_{i=1}^r \varphi_i = 0$ . Then  $\varphi_i = 0$  for  $i = 1, \dots, r$ ;
- (ii) The collection of morphisms  $\varphi_i$  appearing in the axiom *Morphism Additivity* is unique.

*Proof.* (i) We use induction over  $r$ . The induction  $r = 1$  beginning is trivial, the induction step from  $r$  to  $r+1$  done as follows. Put  $\varphi' = \sum_{i=1}^r \varphi_i$ . Then  $\text{supp}(\varphi') \subseteq \bigcup_{i=1}^r \text{supp}(\varphi_i)$  and hence  $\text{supp}(\varphi') \cap \text{supp}(\varphi_{r+1}) = \emptyset$ . Since  $0 = \sum_{i=1}^{r+1} \varphi_i = 0$ , we have  $\varphi_{r+1} = -\varphi'$ . Since

$$\begin{aligned} \text{supp}(-\varphi') &= \text{supp}((- \text{id}_{B'}) \circ \varphi') \subseteq \text{supp}(- \text{id}_{B'}) \cdot \text{supp}(\varphi') \\ &= \text{supp}(B') \cdot \text{supp}(\varphi') = \text{supp}(\varphi') \end{aligned}$$

holds, we get  $\text{supp}(\varphi_{r+1}) = \text{supp}(-\varphi') = \text{supp}(\varphi')$ . We conclude  $\text{supp}(\varphi_{r+1}) = \text{supp}(\varphi') = \emptyset$  which implies  $\varphi_{r+1} = 0$  and  $\varphi' = 0$ . We get  $\varphi_i = 0$  for  $i = 1, 2, \dots, r$  from the induction hypothesis applied to  $\varphi' = \sum_{i=1}^r \varphi_i$ .

(ii) This follows directly from assertion (i) □

One easily checks

**Lemma 5.3.** Let  $\alpha: G \rightarrow G'$  be a group homomorphism of *td*-groups. Consider a Hecke category  $\mathcal{B}$  with  $G$ -support in the sense of Definition 5.1

Then the category with  $G'$ -support  $\text{ind}_\alpha \mathcal{B}$  associated to the underlying category with  $G$ -support  $\mathcal{B}$  defined in Subsection 4.A inherits the structure of a Hecke category with  $G'$ -support.

*Proof.* We leave the elementary proof that all the axioms appearing in Definition 2.1 are satisfied for  $\text{ind}_\alpha \mathcal{B}$  to the reader except for *Translation*. Given an object  $(B, g')$  in  $\text{ind}_\alpha \mathcal{B}$  and an element  $g'_0 \in G'$ , we consider the object  $(B, g'_0 g')$ . Its support satisfies

$$\begin{aligned} \text{supp}_{\text{ind}_\alpha \mathcal{B}}(B, g'_0 g') &\stackrel{(4.4)}{=} g'_0 g' \alpha(\text{supp}_{\mathcal{B}}(B))(g'_0 g')^{-1} \\ &= g'_0 (g' \alpha(\text{supp}_{\mathcal{B}}(B) g^{-1})) g_0'^{-1} \stackrel{(4.4)}{=} g'_0 \text{supp}_{\text{ind}_\alpha \mathcal{B}}(B, g') g_0'^{-1}. \end{aligned}$$

Let  $\psi: (B, g') \rightarrow (B, g'_0 g')$  be the morphism in  $\text{ind}_\alpha \mathcal{B}$  given by  $\text{id}_B$  in  $\mathcal{B}$ . Its support satisfies

$$\begin{aligned} \text{supp}_{\text{ind}_\alpha \mathcal{B}}(\psi) &\stackrel{(4.3)}{=} g'_0 g' \alpha(\text{supp}_{\mathcal{B}}(\text{id}_{\mathcal{B}})) g'^{-1} = g'_0 g' \alpha(\text{supp}_{\mathcal{B}}(B)) g'^{-1} \\ &\stackrel{(4.4)}{=} g'_0 \text{supp}_{\text{ind}_\alpha \mathcal{B}}(B, g'). \end{aligned}$$

Let  $\psi': (B, g'_0 g') \rightarrow (B, g')$  be the morphism in  $\text{ind}_\alpha \mathcal{B}$  given by  $\text{id}_B$  in  $\mathcal{B}$ . Its support satisfies

$$\begin{aligned} \text{supp}_{\text{ind}_\alpha \mathcal{B}}(\psi') &\stackrel{(4.3)}{=} g' \alpha(\text{supp}_{\mathcal{B}}(\text{id}_{\mathcal{B}}))(g'_0 g')^{-1} = g' \alpha(\text{supp}_{\mathcal{B}}(B)) g'^{-1} g'_0 \\ &= g'^{-1} ((g'_0 g') \alpha(\text{supp}_{\mathcal{B}}(B)) (g'_0 g')^{-1}) \stackrel{(4.4)}{=} g'^{-1} \text{supp}_{\text{ind}_\alpha \mathcal{B}}(B, g_0 g'). \end{aligned}$$

The morphisms  $\psi$  and  $\psi'$  in  $\text{ind}_\alpha \mathcal{B}$  are inverse to one another.  $\square$

**Notation 5.4** ( $\mathcal{B}|_H$ ). For a subgroup  $H \subseteq G$ , define  $\mathcal{B}|_H$  to be  $\mathbb{Z}$ -subcategory of  $\mathcal{B}$  consisting of objects  $B$  and morphisms  $\varphi: B \rightarrow B'$  in  $\mathcal{B}$  for which  $\text{supp}_{\mathcal{B}}(B)$  and  $\text{supp}_{\mathcal{B}}(\varphi)$  are contained in  $H$ .

The main benefit of the axiom *Translation* appearing in Definition 5.1 is the following lemma

**Lemma 5.5.** *There is an equivalence of  $\mathbb{Z}$ -categories*

$$F: \mathcal{B}|_H \xrightarrow{\cong} \mathcal{B}[G/H].$$

*Proof.* The functor  $F$  sends an object  $B$  to the object  $(B, eH)$  and a morphism  $\varphi: B \rightarrow B'$  to the morphism  $(B, eH) \rightarrow (B', eH)$  given by  $\varphi$  again. Obviously  $F$  is faithful and full. In order to show that it is an equivalence of  $\mathbb{Z}$ -categories, it suffices to show that any object  $(B, gH)$  in  $\mathcal{B}[G/H]$  is isomorphic to an object in the image of  $F$ . This follows from the fact that we obtain an isomorphism  $\psi: (B, gH) \xrightarrow{\cong} (B', eH)$  in  $\mathcal{B}[G/H]$ , if  $B'$  is an object and  $\psi': B \xrightarrow{\cong} B'$  is an isomorphism in  $\mathcal{B}$  with  $\text{supp}(B') = g^{-1} \text{supp}(B)g$ ,  $\text{supp}(\psi) = g^{-1} \text{supp}(B)$ , and  $\text{supp}(\psi^{-1}) = g \text{supp}(B)$ . The existence of the pair  $(B', \psi)$  is guaranteed by *Translation*.  $\square$

In particular we get for every subgroup  $H \subseteq G$  and  $n \in \mathbb{Z}$  an isomorphism

$$(5.6) \quad K_n((\mathcal{B}|_H)_\oplus) \cong \pi_n(\mathbf{K}_{\mathcal{B}}(G/H)).$$

Our main example of a Hecke category with  $G$ -support coming from Hecke algebras will be discussed in Section 6.

**Remark 5.7** (Discrete group  $G$ ). Suppose that the td-group  $G$  is discrete.

Then the Cop-Farrell-Jones Conjecture for Hecke algebras 1.1 is the same as the  $K$ -theoretic Farrell-Jones Conjecture with coefficients in the ring  $R$  and the family  $\mathcal{F}$  of finite subgroups for a uniformly regular ring  $R$ , see [10, Conjecture 12.1 and Theorem 12.39]. Moreover, the Cvcy-Farrell-Jones Conjecture of [3, Conjecture 5.12] agrees with  $K$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories, see [10, Conjecture 12.11]. This follows from the following considerations concerning Hecke categories with  $G$ -support and additive  $G$ -categories.

Let  $\mathcal{A}$  be a  $G$ - $\mathbb{Z}$ -category, i.e. a  $\mathbb{Z}$ -category with  $G$ -action by automorphisms of  $\mathbb{Z}$ -categories. Then we can consider the  $\mathbb{Z}$ -category  $\mathcal{A}[G]$ . It has the same set of objects as  $\mathcal{A}$ . A morphism  $\sum_{g \in G} f_g \cdot g: A \rightarrow A'$  is a finite sum of morphisms in  $\mathcal{A}$ , where  $f_g$  has  $gA$  as source and  $A'$  as target. The composition is given by

$$\left( \sum_{g'' \in G} f'_{g''} \cdot g'' \right) \circ \left( \sum_{g' \in G} f_{g'} \cdot g' \right) := \sum_{g \in G} \left( \sum_{\substack{g', g'' \in G \\ g' g'' = g}} f'_{g''} \circ g'' f_{g'} \right).$$

It becomes a Hecke category with  $G$ -support in the sense of Definition 5.1, if we define the support of every object  $A$  to be  $\{e\}$  and the support of a morphism  $\sum_{g \in G} f_g \cdot g$  to be  $\{g \in G \mid f_g \neq 0\}$ .

Now let  $\mathcal{B}$  be a Hecke category with  $G$  support. Let  $\mathcal{B}_e$  be the subcategory of  $\mathcal{B}$  consisting of all morphisms and objects with support  $\{e\}$ . Thanks to the axiom *Translation* we can choose for  $g_0 \in G$  and  $A_0 \in \text{ob}(\mathcal{B}_e)$  an isomorphism

$\psi(g_0, A_0): A_0 \xrightarrow{\cong} B(g_0, A_0)$  in  $\mathcal{B}$  with  $B(g_0, A_0) \in \text{ob}(\mathcal{B}_e)$  and  $\text{supp}(\psi(g_0, A_0)) = \{g_0\}$ . We require  $B(e, A_0) = A_0$  and  $\psi(e, A_0) = \text{id}_{A_0}$ .

Let  $\mathcal{A}$  be the following  $G$ - $\mathbb{Z}$ -category. Objects are pairs  $(g_0, A_0)$  with  $g_0 \in G$  and  $A_0 \in \text{ob}(\mathcal{B}_e)$ . Morphisms  $(g_0, A_0) \rightarrow (g_1, A_1)$  are morphisms  $\varphi: B(g_0, A_0) \rightarrow B(g_1, A_1)$  in  $\mathcal{B}_e$ . We define a  $G$ -action on  $\mathcal{A}$  as follows. For  $g \in G$  and  $(g_0, A_0) \in \text{ob}(\mathcal{A})$  we set  $g \cdot (g_0, A_0) := (gg_0, A_0)$ . To define the  $G$ -action on morphisms, let  $\varphi: (g_0, A_0) \rightarrow (g_1, A_1)$  be a morphism in  $\mathcal{A}$ , i.e.,  $\varphi: B(g_0, A_0) \rightarrow B(g_1, A_1)$  is a morphism in  $\mathcal{B}_e$ . Then  $g \cdot \varphi: (gg_0, A_0) \rightarrow (gg_1, A_1)$  is the morphism  $B(gg_0, A_0) \rightarrow B(gg_1, A_1)$  in  $\mathcal{B}_e$  defined by requiring that the following diagram in  $\mathcal{B}$  commutes

$$\begin{array}{ccc} B(gg_0, A_0) & \xrightarrow{g \cdot \varphi} & B(gg_1, A_1) \\ \Psi(gg_0, A)^{-1} \downarrow & & \uparrow \psi(gg_1, A_1) \\ A & & A' \\ \psi(g_0, A_0) \downarrow & & \uparrow \psi(g_1, A_1)^{-1} \\ B(g_0, A_0) & \xrightarrow{\varphi} & B(g_1, A_1) \end{array}$$

Now consider  $\mathcal{A}[G]$ . It is a Hecke category with  $G$ -support as described above. The point is that there is a functor of  $\mathbb{Z}$ -categories

$$(5.8) \quad F: \mathcal{A}[G] \rightarrow \mathcal{B}$$

such that  $F$  respects the supports and induces for every  $G$ -set  $S$  an equivalence of additive categories

$$\text{Idem}(F[S]_{\oplus}): \text{Idem}((\mathcal{A}[G])[S]_{\oplus}) \rightarrow \text{Idem}(\mathcal{B}[S]_{\oplus}),$$

where  $(\mathcal{A}[G])[S]$  and  $\mathcal{B}[S]$  have been defined in Subsection 2.B.

The construction of  $F$  is as follows. Set  $F(g_0, A_0) := B(g_0, A_0)$ . Recall that a morphism  $\varphi: (g_0, A_0) \rightarrow (g_1, A_1)$  in  $\mathcal{A}[G]$  is of the form  $\varphi = \sum_{g_0 \in G} \varphi_g \cdot g$ , where  $\varphi_g: B(gg_1, A_1) \rightarrow B(g_1, A_1)$  is a morphism in  $\mathcal{B}_e$ . Now we put  $F(\varphi) = \sum_g F(\varphi_g \cdot g): B(g_0, A_0) \rightarrow B(g_1, A_1)$ , where  $F(\varphi_g \cdot g)$  is the composite

$$B(g_0, A_0) \xrightarrow{\Psi(g_0, A_0)^{-1}} A_0 \xrightarrow{\psi(gg_0, A_0)} B(gg_0, A_0) \xrightarrow{\varphi_g} B(g_1, A_1).$$

One easily checks that  $F$  respects the support of objects and morphisms and the  $\mathbb{Z}$ -structures. Moreover,  $F$  is full and faithful by the following consideration. Consider two objects  $(g_0, A_0)$  and  $(g_1, A_1)$  in  $\mathcal{A}[G]$ . Let  $\mu: F(g_0, A_0) = B(g_0, A_0) \rightarrow F(g_1, A_1) = B(g_1, A_1)$  be any morphisms in  $\mathcal{B}$  from  $F(g_0, A_0)$  to  $F(g_1, A_1)$ . Because of the axiom *Morphism Additivity* and Lemma 5.2 (ii) there is precisely one collection of morphisms  $\{\mu_g: B(g_0, A_0) \rightarrow B(g_1, A_1) \mid g \in \text{supp}(\mu)\}$  such that  $\text{supp}(\mu_g) = \{g\}$  holds for  $g \in \text{supp}(\mu)$  and we have  $\mu = \sum_{g \in G} \mu_g$ . Define a morphism  $\varphi_g$  in  $\mathcal{B}_e$  by the composite

$$\varphi_g: B(gg_0, A_0) \xrightarrow{\psi(gg_0, A_0)^{-1}} A_0 \xrightarrow{\psi(g_0, A_0)} B(g_0, A_0) \xrightarrow{\varphi_g} B(g_1, A_1).$$

Define a morphism in  $\mathcal{A}[G]$  by  $\phi = \sum_{g \in \text{supp}(\mu)} \varphi_g: (g_0, A_0) \rightarrow (g_1, A_1)$ . One easily checks that  $F(\phi) = \mu$  and any other morphism  $\phi': (g_0, A_0) \rightarrow (g_1, A_1)$  in  $\mathcal{A}[G]$  with  $F(\phi') = \mu$  satisfies  $\phi = \phi'$ .

Consider any object  $B$  in  $\mathcal{B}$ . By the axiom *Support Cofinality* we can find an object  $B_0 \in \mathcal{B}_e$  and morphisms  $B \xrightarrow{i} B_0 \xrightarrow{r} B$  in  $\mathcal{B}$  such that  $\text{supp}(i) = \text{supp}(r) = \text{supp}(B)$  and  $r \circ i = \text{id}_B$  holds. Now one easily checks that  $\text{Idem}(F[S]_{\oplus})$  is an equivalence of additive categories for every  $G$ -set  $S$ .

## 6. THE EXAMPLE COMING FROM HECKE ALGEBRAS

**6.A. The basic setup for Hecke algebras.** We briefly recall the basic setup of [2, Section 2.A].

Let  $R$  be a (not necessarily commutative) associative unital ring with  $\mathbb{Q} \subseteq R$ . Let  $G$  be a td-group with a normal (not necessarily open or central) subgroup  $N \subseteq G$ . Put  $Q = G/N$ . Then we obtain an extension of td-groups  $1 \rightarrow N \rightarrow G \xrightarrow{\text{pr}} Q \rightarrow 1$ .

Consider a group homomorphism  $\rho: G \rightarrow \text{aut}(R)$ , where  $\text{aut}(R)$  is the group of automorphism of the unital ring  $R$ . We will assume that the kernel of  $\rho$  is open, in other words,  $G$  acts smoothly on  $R$ .

A *normal character* is a locally constant group homomorphism

$$\omega: N \rightarrow \text{cent}(R)^\times$$

to the multiplicative group of central units of  $R$  satisfying  $\omega(gng^{-1}) = \omega(n)$  for all  $n \in N$  and  $g \in G$ . We will need the following compatibility condition between the normal character and the  $G$ -action  $\rho$  on  $R$ , namely for  $n \in N$ ,  $g \in G$ , and  $r \in R$  we require  $\rho(g)(\omega(n)r) = \omega(n)\rho(g)r$  and  $\rho(n)(r) = r$ .

Let  $\mu$  be a  $\mathbb{Q}$ -valued Haar measure on  $Q$ , i.e., a Haar measure  $\mu$  on  $Q$  such that for every compact open subgroup  $K \subseteq Q$  we have  $\mu(K) \in \mathbb{Q}^{>0}$ . Given any Haar measure  $\mu$  on  $Q$ , we can normalize it to a  $\mathbb{Q}$ -valued Haar measure by choosing a compact open subgroup  $L_0 \subseteq Q$  and defining  $\mu' = \frac{1}{\mu(L_0)} \cdot \mu$ .

**6.B. The construction of the Hecke algebra.** An element  $s$  in the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)$  is given by a map  $s: G \rightarrow R$  with the following properties

- The map  $s: G \rightarrow R$  is locally constant;
- The image of its support  $\text{supp}(s) := \{g \in G \mid s(g) \neq 0\} \subseteq G$  under  $\text{pr}: G \rightarrow Q$  is a compact subset of  $Q$ ;
- For  $n \in N$  and  $g \in G$  we have  $s(ng) = \omega(n) \cdot s(g)$  and  $s(gn) = s(g) \cdot \omega(n)$ .

Let  $P_{\rho, \omega}$  the set of compact open subgroups  $K \subseteq G$  satisfying  $\rho(k)(r) = r$  for  $k \in K$ ,  $r \in R$  and  $\omega(n) = 1$  for  $n \in N \cap K$ . We call an element  $K \in P_{\rho, \omega}$  *admissible* for  $s: G \rightarrow R$ , if for all  $g \in G$  and  $k \in K$  we have  $s(kg) = s(g)$  and  $s(gk) = s(g)$ . Note that the existence of an admissible element  $K \in P_{\rho, \omega}$  is equivalent to the condition that  $s$  is locally constant.

For two elements  $s, s'$  in  $\mathcal{H}(G; R, \rho, \omega)$ , define  $(s + s')(g) = s(g) + s'(g)$  and  $(-s)(g) = -s(g)$  for  $g \in G$ . In order to define the product, choose  $K \in P_{\rho, \omega}$  which is admissible for  $s$  and admissible for  $s'$ , and a transversal  $T$  for the projection  $p: G \rightarrow G/NK$ , where  $NK$  is the subgroup of  $G$  given by  $\{nk \mid n \in N, k \in K\}$ . Define the product  $s \cdot s'$  by

$$(6.1) \quad (s \cdot s')(g) := \mu(\text{pr}(K)) \cdot \sum_{g' \in T} s(gg') \cdot \rho(gg')(s'(g'^{-1})).$$

It is not hard to check that this definition is independent of the choice of  $K$  and  $T$ . One may think of this as an integral  $(s \cdot s')(g) = \int_G s(gx) \cdot \rho(gx)(s'(x^{-1})) d\mu(x)$ , where  $\mu$  is a left invariant Haar measure. More information and details can be found in [2, Section 2.B].

If  $\rho$  is trivial and  $N = \{1\}$  and hence  $G = Q$ , then we write

$$(6.2) \quad \mathcal{H}(G; R) = \mathcal{H}(G; R, \rho, \omega).$$

**6.C. The Hecke category with  $Q$ -support associated to Hecke algebras.** Next we define a Hecke category  $\mathcal{B} = \mathcal{B}(G; R, \rho, \omega)$  with  $Q$ -support.

The set of objects in  $\mathcal{B}$  is the set  $P_{\rho, \omega}$  defined in Subsection 6.B. A morphism  $s: K \rightarrow K'$  is a function  $s: G \rightarrow R$  satisfying

- the image of  $\{g \in G \mid s(g) \neq 0\}$  under  $\text{pr}: G \rightarrow Q$  is compact;



- $s(gk) = s(g)$  for  $g \in G$  and  $k \in K$ ;
- $s(k'g) = s(g)$  for  $g \in G$  and  $k' \in K'$ ;
- $s(ng) = \omega(n) \cdot s(g)$  for  $n \in N$  and  $g \in G$ ;
- $s(gn) = s(g) \cdot \omega(n)$  for  $n \in N$  and  $g \in G$ .

Note that  $s$  defines an element in  $\mathcal{H}(G; R, \rho, \omega)$ . The composition in  $\mathcal{B}$  is given by the multiplication in  $\mathcal{H}(G; R, \rho, \omega)$ , see (6.1). The identity  $\text{id}_K$  of an object  $K$  is defined by

$$\text{id}_K(g) = \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) & \text{if } g = nk \text{ for } n \in N, k \in K; \\ 0 & \text{otherwise.} \end{cases}$$

The support of a morphism  $s$  is defined by  $\text{supp}_{\mathcal{B}}(s) = \text{pr}(\{g \in G \mid s(g) \neq 0\})$ . In particular the support of an object  $K$  is  $\text{pr}(K)$ .

Next we define a  $G$ -action on  $\mathcal{B}$ . For an object  $K$  and an element  $g \in G$  we define  $g \cdot K := gKg^{-1}$ . For a morphism  $s: K \rightarrow K'$  define  $g \cdot s: gKg^{-1} \rightarrow gK'g^{-1}$  by

$$(g \cdot s)(g') := \frac{\mu(\text{pr}(K))}{\mu(\text{pr}(gKg^{-1}))} \cdot s(g^{-1}g'g) \quad \text{for } g' \in G.$$

Define for an object  $K$  and  $g \in G$  an isomorphism  $\Omega_g(K): K \xrightarrow{\cong} gKg^{-1}$  by

$$\Omega_g(K)(g') = \begin{cases} \frac{1}{\mu(\text{pr}(gKg^{-1}))} \omega(n) & \text{if } g' = gnk \text{ for } n \in N, k \in K; \\ 0 & \text{otherwise.} \end{cases}$$

Its inverse is given by  $\Omega_{g^{-1}}(gKg^{-1}): gKg^{-1} \rightarrow K$ .

We leave the elementary proof to the reader that  $\mathcal{B}$  satisfies all the axioms appearing in Definition 5.1 except for the axioms *Translation* and *Support cofinality*. *Translation* follows from the  $G$ -action and the isomorphisms  $\Omega_{g^{-1}}(gKg^{-1})$  constructed above, since  $\text{pr}: G \rightarrow Q$  is surjective. For *Support cofinality*, consider an object  $K$  of  $\mathcal{B}$  and a compact open subgroup  $L \subseteq \text{supp}_{\mathcal{B}}(K) = \text{pr}(K)$ . We define the object  $K|_L$  in  $\mathcal{B}$  to be  $K \cap \text{pr}^{-1}(L)$ . We have to define morphisms  $i: K \rightarrow K|_L$  and  $r: K|_L \rightarrow K$  in  $\mathcal{B}$  such that  $r \circ i = \text{id}_K$  holds and both  $\text{supp}_{\mathcal{B}}(i)$  and  $\text{supp}_{\mathcal{B}}(r)$  agree with  $\text{supp}_{\mathcal{B}}(K) := \text{pr}(K)$ . This is done by putting for  $g' \in G$

$$(6.3) \quad i(g') = r(g') = \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot 1_R & \text{if } g' = nk \text{ for } n \in N, k \in K; \\ 0 & \text{otherwise,} \end{cases}$$

Let  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$  be the group ring of  $\mathbb{Z}^d$  with coefficients in  $\mathcal{H}(G; R, \rho, \omega)$ . Denote by  $\underline{\mathcal{H}}(G; R, \rho, \omega)[\mathbb{Z}^d]$  the  $\mathbb{Z}$ -category which has precisely one object whose endomorphisms are given by elements in  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$ . The  $\mathbb{Z}$ -structure comes from the additive structure of  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$ , while composition comes from the multiplicative structure of  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$ . Since  $\mathcal{H}(G; R, \rho, \omega)$  is a ring without unit in general,  $\underline{\mathcal{H}}(G; R, \rho, \omega)[\mathbb{Z}^d]$  is in general non-unital in the sense that there may be no identity morphisms for objects.

For a (not necessarily unital)  $\mathbb{Z}$ -category  $\mathcal{B}$ , let  $\mathcal{B}_{\oplus}$  be the  $\mathbb{Z}$ -category whose objects  $\underline{B}$  are  $n$ -tuples  $(B_1, \dots, B_n)$  consisting of objects  $B_1, \dots, B_n$  in  $\mathcal{B}$  for  $n \geq 1$  or the object  $0$ , which will be an initial and terminal object in  $\mathcal{B}$ . A morphism  $\underline{\varphi}: \underline{B} = (B_1, \dots, B_n) \rightarrow \underline{B}' = (B'_1, \dots, B'_{n'})$  is a collection of morphisms  $\varphi: B_i \rightarrow B'_{i'}$  in  $\mathcal{B}$  for  $i \in \{1, \dots, n\}$  and  $i' \in \{1, \dots, n'\}$ . Composition is given by matrix multiplication. The direct sum is defined by concatenation. For any objects  $\underline{B}$  and  $\underline{B}'$  and  $\underline{C}$  there is a natural isomorphism of  $\mathbb{Z}$ -modules

$$\text{mor}_{\mathcal{B}_{\oplus}}(\underline{B}, \underline{C}) \oplus \text{mor}_{\mathcal{B}_{\oplus}}(\underline{B}', \underline{C}) \xrightarrow{\cong} \text{mor}_{\mathcal{B}_{\oplus}}(\underline{B} \oplus \underline{B}', \underline{C}).$$

If  $\mathcal{B}$  is unital, this agrees with the earlier definition introduced before Definition 2.2.

Note that in the idempotent completion  $\text{Idem}(\mathcal{B}_\oplus)$  every object has an identity and the direct sum in  $\mathcal{B}_\oplus$  induces the structure of an additive category on  $\text{Idem}(\mathcal{B}_\oplus)$ .

**Remark 6.4.** The algebraic  $K$ -groups of the (non-unital) ring  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$  are defined by

$$(6.5) \quad K_n(\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]) := K_n(\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)).$$

This agrees with the usual definition  $K_n(R) := \text{cok}(K_n(\mathbb{Z}) \rightarrow K_n(R_+))$  for a non-unital ring  $R$ , where  $R_+$  is the unitalization of  $R$ , if  $R$  has an approximate unit, which is the case for  $\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]$ .

**Lemma 6.6.** *Consider any natural number  $d$ . Then there exists an equivalence of additive categories*

$$\text{Idem}(F): \text{Idem}(\mathcal{B}(G; R, \rho, \omega)_\oplus[\mathbb{Z}^d]) \xrightarrow{\cong} \text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus).$$

*Proof.* We begin with defining a functor

$$F: \mathcal{B}(G; R, \rho, \omega)_\oplus[\mathbb{Z}^d] \rightarrow \text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus),$$

It assigns to an object  $\underline{K} = (K_1, \dots, K_n)$  in  $\mathcal{B}(G; R, \rho, \omega)_\oplus[\mathbb{Z}^d]$  the object in  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)$  given by  $\text{id}_{K_1} \oplus \dots \oplus \text{id}_{K_n}$ . Consider a morphism  $\underline{s} = (s_{i,i'}) : (K_1, \dots, K_n) \rightarrow (K'_1, \dots, K'_{n'})$  in  $\mathcal{B}(G; R, \rho, \omega)_\oplus$ . It is sent to the morphism  $\underline{s}' = (s'_{i,i'}) : \text{id}_{K_1} \oplus \dots \oplus \text{id}_{K_n} \rightarrow \text{id}_{K'_1} \oplus \dots \oplus \text{id}_{K'_{n'}}$  given by the same collection  $(s_{i,i'})$  having in mind that each  $s_{i,i'}$  is an element in  $\mathcal{H}(G; R, \rho, \omega)$  satisfying  $\text{id}_{K'_{i'}} \circ s_{i,i'} \circ \text{id}_{K_i} = s_{i,i'}$ .

Next we show that

$$\text{Idem}(F): \text{Idem}(\mathcal{B}(G; R, \rho, \omega)_\oplus[\mathbb{Z}^d]) \xrightarrow{\cong} \text{Idem}(\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus))$$

is an equivalence of additive categories. One easily checks that  $F$  and hence  $\text{Idem}(F)$  is faithful and full. Hence it suffices to show that the image of  $F$  is cofinal in  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)$ . Consider an object  $\underline{p} = (p_{i,i'}) : *^n \rightarrow *^{n'}$  in  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)$ , where  $*^n$  is the  $n$ -tuple  $(*, \dots, *)$ . For each  $p_{i,i'}$  there exists elements  $\bar{K}_i$  and  $\bar{K}'_{i'}$  in  $P_{\rho, \omega}$  such that  $\text{id}_{\bar{K}'_{i'}} \circ p_{i,i'} \circ \text{id}_{\bar{K}_i} = p_{i,i'}$  holds. Put

$$K = \bigcap_{i=1}^n \bar{K}_i \cap \bigcap_{i'=1}^{n'} \bar{K}'_{i'}.$$

Then  $\text{id}_K \circ p_{i,i'} \circ \text{id}_K = p_{i,i'}$  holds for every  $i$  and  $i'$ . Consider the object  $\text{id}_K^n : *^n \rightarrow *^n$  in  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)$  which is given by the  $n$ -fold direct sums of copies of  $\text{id}_K : * \rightarrow *$ . Let  $\underline{i} : \underline{p} \rightarrow \text{id}_K^n$  and  $\underline{r} : \text{id}_K^n \rightarrow \underline{p}$  be the morphisms in  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus)$  that are in both cases given by the morphism  $\underline{p}$  in  $\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus$ . One easily checks  $\underline{r} \circ \underline{i} = \text{id}_{\underline{p}}$ . Since  $\text{id}_K^n$  is in the image of  $F$ , the image of  $F$  is cofinal. Hence  $\text{Idem}(F)$  is an equivalence of additive categories.

We obtain an equivalences of additive categories,

$$\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus) \xrightarrow{\cong} \text{Idem}(\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus))$$

from [2, Lemma 5.6]. Since there is an obvious isomorphism

$$\underline{\mathcal{H}(G; R, \rho, \omega)[\mathbb{Z}^d]}_\oplus \xrightarrow{\cong} \underline{\mathcal{H}(G; R, \rho, \omega)}_\oplus[\mathbb{Z}^d],$$

Lemma 6.6 follows.  $\square$

**Remark 6.7.** Let  $U \subseteq Q$  be an open subgroup of  $Q$ . Then we get the equality

$$\mathcal{B}(G; R, \rho, \omega)|_U = \mathcal{B}(\text{pr}^{-1}(U); R, \rho|_{\text{pr}^{-1}(U)}, \omega),$$

where the source has been defined in Notation 5.4.

**6.D. Consequences of the Cop-Farrell-Jones Conjecture for Hecke algebras.** Let  $\mathbf{K}_{\mathcal{B}(G;R,\rho,\omega)}: \text{Or}_{\text{sm}}(G) \rightarrow \text{Spectra}$  be the covariant  $\text{Or}_{\text{sm}}(G)$ -spectrum of Definition 2.2 associated to the Hecke category with  $Q$ -support  $\mathcal{B}(G;R,\rho,\omega)$ , see Subsection 6.C. We conclude from (5.6) and Remark 6.7 that for every open subgroup  $U \subseteq Q$  and  $n \in \mathbb{Z}$  there is an isomorphism

$$(6.8) \quad \pi_n(\mathbf{K}_{\mathcal{B}(G;R,\rho,\omega)}(G/U)) = K_n(\mathcal{H}(\text{pr}^{-1}(U); R, \rho|_{\text{pr}^{-1}(U)}, \omega)).$$

The *smooth subgroup category*  $\text{Sub}_{\text{sm}}(G)$  has as objects the open subgroups  $H$  of  $G$ . For subgroups  $H$  and  $K$  of  $G$ , denote by  $\text{conhom}_G(H, K)$  the set of group homomorphisms  $f: H \rightarrow K$ , for which there exists an element  $g \in G$  with  $gHg^{-1} \subseteq K$  such that  $f$  is given by conjugation with  $g$ , i.e.  $f = c(g): H \rightarrow K$ ,  $h \mapsto ghg^{-1}$ . Note that  $c(g) = c(g')$  holds for two elements  $g, g' \in G$  with  $gHg^{-1} \subseteq K$  and  $g'Hg'^{-1} \subseteq K$ , if and only if  $g^{-1}g'$  lies in the centralizer  $C_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$  of  $H$  in  $G$ . The group of inner automorphisms  $\text{Inn}(K)$  of  $K$  acts on  $\text{conhom}_G(H, K)$  from the left by composition. Define the set of morphisms

$$\text{mor}_{\text{Sub}_{\text{cop}}(G)}(H, K) := \text{Inn}(K) \backslash \text{conhom}_G(H, K).$$

There is an obvious bijection

$$(6.9) \quad K \backslash \{g \in G \mid gHg^{-1} \subseteq K\} / C_G H \xrightarrow{\cong} \text{Inn}(K) \backslash \text{conhom}_G(H, K), \\ KgC_G H \mapsto [c(g)],$$

where  $[c(g)] \in \text{Inn}(K) \backslash \text{conhom}_G(H, K)$  is the class represented by the element  $c(g): H \rightarrow K$ ,  $h \mapsto ghg^{-1}$  in  $\text{conhom}_G(H, K)$  and  $K$  acts from the left and  $C_G H$  from the right on  $\{g \in G \mid gHg^{-1} \subseteq K\}$  by the multiplication in  $G$ .

**Lemma 6.10.** *The (Hecke) category with  $Q$ -support  $\mathcal{B}(G;R,\rho,\omega)$  satisfies condition (Reg), see Definition 3.2, if  $R$  is uniformly regular.*

*Proof.* This follows from [2, Theorem 7.2] and Lemma 6.6.  $\square$

**Theorem 6.11.** *Suppose that the td-group  $Q$  satisfies the Cop-Farrell-Jones Conjecture 1.3, e.g.,  $Q$  is modulo a normal compact subgroup a subgroup of some reductive  $p$ -adic group. Let  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ . Suppose that  $N \subseteq G$  is locally central, i.e., its centralizer  $C_G N$  in  $G$  is an open subgroup of  $G$ . Then:*

(i) *The assembly map induced by the projection  $E_{\text{Cop}}(Q) \rightarrow Q/Q$*

$$H_n^G(E_{\text{Cop}}(G); \mathbf{K}_{\mathcal{B}(G;R,\rho,\omega)}) \rightarrow H_n^G(G/G; \mathbf{K}_{\mathcal{B}(G;R,\rho,\omega)}) = K_n(\mathcal{H}(G; R, \rho, \omega))$$

*is an isomorphism for  $n \in \mathbb{Z}$ ;*

(ii) *The canonical map induced by the various inclusions  $U \subseteq Q$*

$$\text{colim}_{U \in \text{Sub}_{\text{cop}}(G)} K_0(\mathcal{H}(U; R; \rho|_U, \omega)) \rightarrow K_0(\mathcal{H}(G; R, \rho, \omega))$$

*can be identified with the assembly map of assertion (i) in degree  $n = 0$  and hence is bijective;*

(iii) *We have  $K_n(\mathcal{H}(G; R, \rho, \omega)) = 0$  for  $n \leq -1$ .*

*Proof.* (i) This follows from Theorem 1.4, Theorem 1.5, and Lemma 6.10.

(ii) See [4, Theorem 1.1 (iii)].

(iii) See [4, Theorem 1.1 (iv)].  $\square$

## 7. SOME INPUT FOR THE PROOF OF THE FARRELL-JONES CONJECTURE

In this section we provided some technical input for the proof of the Cop-Farrell-Jones Conjecture 1.3 for reductive  $p$ -adic groups, which we will present in [3].

7.A. **The category  $\mathcal{S}^G(\Omega)$ .** Throughout this section we fix a  $G$ -set  $\Omega$ .

**Definition 7.1.** We define the additive category  $\mathcal{S}^G(\Omega)$  as follows. Objects are pairs  $\mathbf{V} = (\Sigma, c)$  where  $\Sigma$  is a smooth  $G$ -set and  $c: \Sigma \rightarrow \Omega$  is a  $G$ -map. A morphism  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  is an  $\Sigma \times \Sigma'$ -matrix  $(\rho_\sigma^{\sigma'})_{\sigma \in \Sigma, \sigma' \in \Sigma'}$  over  $\mathbb{Z}$  satisfying the following two conditions

(7.1a) for all  $\sigma \in \Sigma$  the set  $\{\sigma' \in \Sigma' \mid \rho_\sigma^{\sigma'} \neq 0\}$  is finite;

(7.1b) for all  $g \in G, \sigma \in \Sigma, \sigma' \in \Sigma'$  we have  $\rho_{g\sigma}^{g\sigma'} = \rho_\sigma^{\sigma'}$ .

The *support* of  $\rho$  is

$$\text{supp}_2(\rho) := \left\{ \left( \begin{array}{c} c'(\sigma') \\ c(\sigma) \end{array} \right) \mid \rho_\sigma^{\sigma'} \neq 0 \right\} \subseteq \Omega \times \Omega.$$

Composition is matrix multiplication

$$(\rho' \circ \rho)_\sigma^{\sigma''} := \sum_{\sigma'} \rho'_{\sigma'}^{\sigma''} \circ \rho_\sigma^{\sigma'}.$$

The identity of  $\mathbf{V} = (\Sigma, c)$  is given by the morphism  $\rho$  with  $\rho_\sigma^{\sigma'} = 1$  for  $\sigma = \sigma'$  and  $\rho_\sigma^{\sigma'} = 0$  for  $\sigma \neq \sigma'$ .

7.B. **The category  $\mathcal{B}_G(\Omega)$ .**

**Definition 7.2.** Let  $\mathcal{B}$  be a category with  $G$ -support. We define the additive category  $\mathcal{B}_G(\Omega)$  as follows. Objects are triples  $\mathbf{B} = (S, \pi, B)$ , where

(7.2a)  $S$  is a set,

(7.2b)  $\pi: S \rightarrow \Omega$  is a map,

(7.2c)  $B: S \rightarrow \text{ob}(\mathcal{B})$  is a map.

Morphisms  $\mathbf{B} = (S, \pi, B) \rightarrow \mathbf{B}' = (S', \pi', B')$  in  $\mathcal{B}_G(\Omega)$  are matrices  $\varphi = (\varphi_s^{s'}: B(s) \rightarrow B'(s'))_{s \in S, s' \in S'}$  of morphisms in  $\mathcal{B}$ . Morphisms are required to be column finite: for each  $s \in S$  there are only finitely many  $s' \in S'$  with  $\varphi_s^{s'} \neq 0$ . Composition is matrix multiplication (using the composition in  $\mathcal{B}$ )

$$(\varphi' \circ \varphi)_s^{s''} := \sum_{s'} \varphi'_{s'}^{s''} \circ \varphi_s^{s'}.$$

The identity of an object  $\mathbf{B} = (S, \pi, B)$  is given by the morphisms  $\varphi$  with  $\varphi_\sigma^{\sigma'} = \text{id}_{B(\sigma)}$  for  $\sigma' = \sigma$  and  $\varphi_\sigma^{\sigma'} = 0$  for  $\sigma' \neq \sigma$ . The direct sum in  $\mathcal{B}_G(\Omega)$  comes from disjoint unions, i.e.,

$$(S, \pi, B) \oplus (S', \pi', B') \cong (S \sqcup S', \pi \sqcup \pi', B \sqcup B').$$

**Definition 7.3** (Support for  $\mathcal{B}_G(\Omega)$ ). The *support* of an object  $\mathbf{B} = (S, \pi, B)$  in  $\mathcal{B}_G(\Omega)$  is defined to be

$$\text{supp}_1(\mathbf{B}) := \pi(S) \subseteq \Omega$$

The *support* of a morphism  $\varphi: (S, \pi, B) \rightarrow (S', \pi', B')$  in  $\mathcal{B}_G(\Omega)$  is defined to be

$$\text{supp}_2(\varphi) := \left\{ \left( \begin{array}{c} \pi'(s') \\ g\pi(s) \end{array} \right) \mid s \in S, s' \in S', g \in G, g \in \text{supp}_G(\varphi_s^{s'}) \right\} \subseteq \Omega \times \Omega.$$

We set  $\text{supp}_2(\mathbf{B}) := \text{supp}_2(\text{id}_B) = \{(\pi(s), g\pi(s)) \mid s \in S, g \in \text{supp}_B(\pi(s))\}$ . The  $G$ -*support* of a morphism  $\varphi$  in  $\mathcal{B}_G(\Omega)$  is

$$\text{supp}_G(\varphi) = \bigcup_{s \in S, s' \in S'} \text{supp}_B(\varphi_s^{s'}).$$

We set  $\text{supp}_G(B) := \text{supp}_G(\text{id}_B)$ .

### 7.C. The diagonal tensor product for Hecke categories with $G$ -support.

In this subsection we want to define a bilinear pairing

$$(7.4) \quad - \otimes - : \mathcal{S}^G(\Omega) \times \mathcal{B} \rightarrow \mathcal{B}_G(\Omega),$$

where  $\mathcal{B}$  is a Hecke category with  $G$ -support in the sense of Definition 5.1.

Given  $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$  and  $B \in \mathcal{B}$ , we define  $\mathbf{V} \otimes_0 B$  in  $\mathcal{B}_G(\Omega)$  by

$$(7.5) \quad \mathbf{V} \otimes_0 B := (\Sigma, c, \sigma \mapsto B|_{\text{supp}(B)_\sigma}),$$

where  $\text{supp}(B)_\sigma = \text{supp}(B) \cap G_\sigma$  denotes the isotropy group of  $\sigma \in \Sigma$  for the action of  $\text{supp}(B) \subseteq G$  on  $\Sigma$ .

For morphisms  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  in  $\mathcal{S}^G(\Omega)$  and  $\varphi: B \rightarrow B'$  in  $\mathcal{B}$  we define

$$(7.6) \quad \rho \otimes_0 \varphi: \mathbf{V} \otimes_0 B \rightarrow \mathbf{V}' \otimes_0 B'$$

as follows. Thanks to the axiom *Support cofinality*, we can define a morphism  $\varphi_\sigma^{\sigma'}: B|_{\text{supp}(B)_\sigma} \rightarrow B'|_{\text{supp}(B')_{\sigma'}}$  by the composite

$$(7.7) \quad \varphi_\sigma^{\sigma'}: B|_{\text{supp}(B)_\sigma} \xrightarrow{r_{B, \text{supp}(B)_\sigma}} B \xrightarrow{\varphi} B' \xrightarrow{i_{B', \text{supp}(B')_{\sigma'}}} B'|_{\text{supp}(B')_{\sigma'}}.$$

By the property *Morphism Additivity* and Lemma 5.2 (ii) one can write

$$\varphi_\sigma^{\sigma'} = \sum_{x \in \text{supp}(B')_{\sigma'} \setminus G / \text{supp}(B)_\sigma} \varphi_\sigma^{\sigma'}[x]$$

for morphisms  $\varphi_\sigma^{\sigma'}[x]: B|_{\text{supp}(B)_\sigma} \rightarrow B'|_{\text{supp}(B')_{\sigma'}}$  that are uniquely determined by  $\text{supp}(\varphi_\sigma^{\sigma'}[x]) = \text{supp}(\varphi_\sigma^{\sigma'}) \cap x$ . For an element  $x \in G_{\sigma'} \setminus G / G_\sigma$  define the integer

$$(7.8) \quad \rho_{x\sigma}^{\sigma'} := \rho_{g\sigma}^{\sigma'}$$

for any  $g \in x$ . This definition is indeed independent of the choice of  $g$ , since any other choice is of the form  $g_1 g g_0$  for  $g_0 \in G_\sigma$  and  $g_1 \in G_{\sigma'}$  and we get  $\rho_{g_1 g g_0 \sigma}^{\sigma'} = \rho_{g_1 g \sigma}^{\sigma'} = \rho_{g\sigma}^{\sigma'}$ . For an element  $x \in \text{supp}(B')_{\sigma'} \setminus G / \text{supp}(B)_\sigma$  we abuse the notation and put  $\rho_{x\sigma}^{\sigma'} := \rho_{g\sigma}^{\sigma'}$  the integer  $\rho_{\text{pr}(x)\sigma}^{\sigma'} := \rho_{g\sigma}^{\sigma'}$  for the projection  $\text{pr}: \text{supp}(B')_{\sigma'} \setminus G / \text{supp}(B)_\sigma \rightarrow G_{\sigma'} \setminus G / G_\sigma$ .

We define

$$(7.9) \quad (\rho \otimes_0 \varphi)_\sigma^{\sigma'} = \sum_{x \in \text{supp}(B')_{\sigma'} \setminus G / \text{supp}(B)_\sigma} \rho_{x\sigma}^{\sigma'} \cdot \varphi_\sigma^{\sigma'}[x].$$

This definition makes sense, since  $\{x \in \text{supp}(B')_{\sigma'} \setminus G / \text{supp}(B)_\sigma \mid \varphi_\sigma^{\sigma'}[x] \neq 0\}$  is the finite set  $\text{supp}(B')_{\sigma'} \setminus \text{supp}(\varphi_\sigma^{\sigma'}) / \text{supp}(B)_\sigma$ .

**Lemma 7.10.** *Let  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  and  $\rho': \mathbf{V}' = (\Sigma', c') \rightarrow \mathbf{V}'' = (\Sigma'', c'')$  be composable morphisms in  $\mathcal{S}^G(\Omega)$  and  $\varphi: B \rightarrow B'$  and  $\varphi': B' \rightarrow B''$  be composable morphisms in  $\mathcal{B}$ .*

*Then we get in  $\mathcal{B}_G(\Omega)$*

$$(\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi) = (\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi).$$

*Proof.* For the remainder of the proof we fix  $\sigma \in \Sigma$  and  $\sigma'' \in \Sigma''$ . We have to show

$$(7.11) \quad ((\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi))_\sigma^{\sigma''} = \sum_{\sigma' \in \Sigma'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'}.$$

We introduce the following abbreviations  $S = \text{supp}(B)$ ,  $S' = \text{supp}(B')$ , and  $S'' = \text{supp}(B'')$ . For a compact subgroup  $K \subseteq G$  and  $\sigma \in \Sigma$ ,  $\sigma' \in \Sigma'$ ,  $\sigma'' \in \Sigma''$ , we

write  $K_\sigma = K \cap G_\sigma$ ,  $K_{\sigma'} = K \cap G_{\sigma'}$ , and  $K_{\sigma''} = K \cap G_{\sigma''}$ . Put

$$(7.12) \quad \widehat{\Sigma}' = \{\sigma' \in \Sigma' \mid \rho_{x\sigma}^{\sigma'} \neq 0, \varphi_\sigma^{\sigma'}[x] \neq 0 \text{ for some } x \in S'_{\sigma'} \backslash G/S_\sigma \\ \text{and } \rho_{x'\sigma'}^{\sigma''} \neq 0, \varphi_{\sigma'}^{\sigma''}[x'] \neq 0 \text{ for some } x' \in S_{\sigma''} \backslash G/S'_{\sigma'}\}.$$

The set  $\{\sigma' \in \Sigma' \mid \varphi_\sigma^{\sigma'} \neq 0\}$  is finite and for  $\sigma' \in \Sigma'$  and  $x \in S'_{\sigma'} \backslash G/S_\sigma$  we have the implication  $\varphi_\sigma^{\sigma'}[x] \neq 0 \implies \varphi_\sigma^{\sigma'} \neq 0$ . This implies that  $\widehat{\Sigma}'$  is finite.

We get from the definitions as  $\{\sigma' \in \Sigma' \mid (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \neq 0 \text{ and } (\rho \otimes_0 \varphi)_\sigma^{\sigma'} \neq 0\}$  is contained in  $\widehat{\Sigma}'$

$$(7.13) \quad \sum_{\sigma' \in \Sigma'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'} = \sum_{\sigma' \in \widehat{\Sigma}'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'}.$$

In the first step of the proof we show that we can assume without loss of generality

**Assumption 7.14.** *For every  $\sigma' \in \widehat{\Sigma}'$ , we have  $\text{supp}(B') \subseteq G_{\sigma'}$ .*

Consider any compact open subgroup  $K' \subseteq S'$ . Next we show for every  $\sigma' \in \Sigma'$

$$(7.15) \quad i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'} = (\rho \otimes_0 (i_{B', K'} \circ \varphi))_\sigma^{\sigma'};$$

$$(7.16) \quad (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ r_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} = (\rho' \otimes_0 (\varphi' \circ r_{B'|_{K', K'_{\sigma'}}}))_{\sigma'}^{\sigma''}.$$

We begin with (7.15). Let  $\text{pr}: S_{\sigma'} \backslash G/S_\sigma \rightarrow K'_{\sigma'} \backslash G/S_\sigma$  be the canonical projection. By *Morphism Additivity* we get

$$(7.17) \quad (i_{B', K'} \circ \varphi)_\sigma^{\sigma'} = \sum_{y \in K'_{\sigma'} \backslash G/S_\sigma} (i_{B', K'} \circ \varphi)_\sigma^{\sigma'}[y]$$

for morphisms  $(i_{B', K'} \circ \varphi)_\sigma^{\sigma'}[y]: B|_{S_\sigma} \rightarrow B'|_{K'_{\sigma'}}$  with  $\text{supp}((i_{B', K'} \circ \varphi)_\sigma^{\sigma'}[y]) = \text{supp}((i_{B', K'} \circ \varphi)_\sigma^{\sigma'}) \cap y$ . Analogously we get

$$(7.18) \quad \varphi_\sigma^{\sigma'} = \sum_{x \in S'_{\sigma'} \backslash G/S_\sigma} \varphi_\sigma^{\sigma'}[x]$$

for morphisms  $\varphi_\sigma^{\sigma'}[x]: B|_{S_\sigma} \rightarrow B'|_{S'_{\sigma'}}$  with  $\text{supp}(\varphi_\sigma^{\sigma'}[x]) = \text{supp}(\varphi_\sigma^{\sigma'}) \cap x$ . We have

$$\begin{aligned} \text{supp}(i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x]) &\subseteq \text{supp}(i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}}) \cdot \text{supp}(\varphi_\sigma^{\sigma'}[x]) \\ &= S'_{\sigma'} \cdot \text{supp}(\varphi_\sigma^{\sigma'}[x]) \\ &= \text{supp}(\varphi_\sigma^{\sigma'}[x]) \\ &= \text{supp}(\varphi_\sigma^{\sigma'}) \cap x. \end{aligned}$$

By *Morphism Additivity* we get a decomposition

$$(7.19) \quad i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x] = \sum_{\substack{y \in K'_{\sigma'} \backslash G/S_\sigma \\ \text{pr}(y)=x}} (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x])[y]$$

for morphisms  $(i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x])[y]: B|_{S_\sigma} \rightarrow B'|_{K'_{\sigma'}}$  with

$$\text{supp}((i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x])[y]) = \text{supp}(i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x]) \cap y \subseteq y.$$

Hence we get

(7.20)

$$\begin{aligned}
 i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'} &\stackrel{(7.18)}{=} i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \left( \sum_{x \in S_{\sigma'} \setminus G/S_{\sigma}} \varphi_{\sigma}^{\sigma'}[x] \right) \\
 &= \sum_{x \in S_{\sigma'} \setminus G/S_{\sigma}} i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x] \\
 &\stackrel{(7.19)}{=} \sum_{x \in S_{\sigma'} \setminus G/S_{\sigma}} \sum_{\substack{y \in K_{\sigma'} \setminus G/S_{\sigma} \\ \text{pr}(y)=x}} (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x])[y].
 \end{aligned}$$

We have

$$\begin{aligned}
 (7.21) \quad i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'} &\stackrel{(7.7)}{=} i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ i_{B', S'_{\sigma'}} \circ \varphi \circ r_{B, S_{\sigma}} = i_{B', K'_{\sigma'}} \circ \varphi \circ r_{B, S_{\sigma}} \\
 &= i_{B'|_{K', K'_{\sigma'}}} \circ i_{B', K'} \circ \varphi \circ r_{B, S_{\sigma}} \stackrel{(7.7)}{=} (i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}.
 \end{aligned}$$

Hence we get from (7.20) and (7.21)

$$(7.22) \quad (i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'} = \sum_{x \in S_{\sigma'} \setminus G/S_{\sigma}} \sum_{\substack{y \in K_{\sigma'} \setminus G/S_{\sigma} \\ \text{pr}(y)=x}} (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x])[y]$$

for morphisms  $(i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x])[y]: B_{\sigma} \rightarrow (B|_{S'_{\sigma'}})|_{K'_{\sigma'}} = B|_{K'_{\sigma'}}$  such that  $\text{supp}((i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x])[y]) \subseteq y$  holds. By *Morphisms additivity* we get

$$(7.23) \quad (i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'} = \sum_{y \in K_{\sigma'} \setminus G/S_{\sigma}} (i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}[y]$$

for morphisms  $(i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}[y]: B|_{\sigma} \rightarrow B'|_{K_{\sigma'}}$  with  $\text{supp}((i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}[y]) = \text{supp}((i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}) \cap y \subseteq y$ . We conclude from Lemma 5.2 (i) that for all  $x \in S_{\sigma'} \setminus G/S_{\sigma}$  and  $y \in K_{\sigma'} \setminus G/S_{\sigma}$  with  $\text{pr}(y) = x$  we have

$$(7.24) \quad (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_{\sigma}^{\sigma'}[x])[y] = (i_{B', K'} \circ \varphi)_{\sigma}^{\sigma'}[y].$$

Now we compute

$$\begin{aligned}
& (\rho \otimes_0 (i_{B',K'} \circ \varphi))_\sigma^{\sigma'} \\
& \stackrel{(7.9)}{=} \sum_{y \in K_{\sigma'} \backslash G/S_\sigma} \rho_{y\sigma}^{\sigma'} \cdot (i_{B',K'} \circ \varphi)_\sigma^{\sigma'}[y] \\
& = \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \sum_{\substack{y \in K_{\sigma'} \backslash G/S_\sigma \\ \text{pr}(y)=x}} \rho_{y\sigma}^{\sigma'} \cdot (i_{B',K'} \circ \varphi)_\sigma^{\sigma'}[y] \\
& = \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \sum_{\substack{y \in K_{\sigma'} \backslash G/S_\sigma \\ \text{pr}(y)=x}} \rho_{x\sigma}^{\sigma'} \cdot (i_{B',K'} \circ \varphi)_\sigma^{\sigma'}[y] \\
& = \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \rho_{x\sigma}^{\sigma'} \cdot \sum_{\substack{y \in K_{\sigma'} \backslash G/S_\sigma \\ \text{pr}(y)=x}} (i_{B',K'} \circ \varphi)_\sigma^{\sigma'}[y] \\
& \stackrel{(7.24)}{=} \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \rho_{x\sigma}^{\sigma'} \cdot \sum_{\substack{y \in K_{\sigma'} \backslash G/S_\sigma \\ \text{pr}(y)=x}} (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x])[y] \\
& \stackrel{(7.19)}{=} \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \rho_{x\sigma}^{\sigma'} \cdot (i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \varphi_\sigma^{\sigma'}[x]) \\
& = i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ \sum_{x \in S_{\sigma'} \backslash G/S_\sigma} \rho_{x\sigma}^{\sigma'} \cdot \varphi_\sigma^{\sigma'}[x] \\
& \stackrel{(7.9)}{=} i_{B'|_{S'_{\sigma'}, K'_{\sigma'}}} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'}.
\end{aligned}$$

This finishes the proof of (7.15). The one of (7.16) is analogous.

Now we conclude

$$\begin{aligned}
& ((\rho' \otimes_0 (\varphi' \circ r_{B',K'})) \circ (\rho \otimes_0 (i_{B',K'} \circ \varphi)))_\sigma^{\sigma''} \\
& = \sum_{\sigma'} (\rho' \otimes_0 (\varphi' \circ r_{B',K'}))_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 (i_{B',K'} \circ \varphi))_\sigma^{\sigma'} \\
& \stackrel{(7.15), (7.16)}{=} \sum_{\sigma'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ r_{B'|_{S_{\sigma'}, K'_{\sigma'}}} \circ i_{B'|_{S_{\sigma'}, K'_{\sigma'}}} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'} \\
& = \sum_{\sigma'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 \varphi)_\sigma^{\sigma'} \\
& = ((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi))_\sigma^{\sigma''}.
\end{aligned}$$

We also have

$$((\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi))_\sigma^{\sigma''} = ((\rho' \circ \rho) \otimes_0 ((\varphi' \circ r_{B',K'}) \circ (i_{B',K'} \circ \varphi)))_\sigma^{\sigma''}.$$

Hence

$$((\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi))_\sigma^{\sigma''} = ((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi))_\sigma^{\sigma''}$$

is true, if

$$((\rho' \circ \rho) \otimes_0 ((\varphi' \circ r_{B',K'}) \circ (i_{B',K'} \circ \varphi)))_\sigma^{\sigma''} = ((\rho' \otimes_0 (\varphi' \circ r_{B',K'})) \circ (\rho \otimes_0 (i_{B',K'} \circ \varphi)))_\sigma^{\sigma''}$$

holds. Now specify  $K'$  to be

$$K' = \text{supp}(B') \cap \bigcap_{\sigma' \in \widehat{\Sigma'}} G_{\sigma'}.$$

Since the set  $\widehat{\Sigma'}$  defined in (7.12) is finite and  $\Sigma'$  is smooth,  $K'$  is a compact open subgroup of  $\text{supp}(B')$  and  $K' \subseteq G_{\sigma'}$  holds for every  $\sigma' \in \widehat{\Sigma'}$ . Hence  $\varphi' \circ r_{B',K'}$



and  $i_{B',K'} \circ \varphi$  satisfy Assumption 7.14. We conclude from (7.13) that we can make without loss of generality the Assumption 7.14, when proving (7.11).

By *Morphism Additivity* we can write

$$(7.25) \quad \varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x] = \sum_{x'' \in S''_{\sigma'} \setminus G/S_{\sigma}} (\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x'']$$

for morphisms  $(\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x''] : B|_{S_{\sigma}} \rightarrow B''|_{S''_{\sigma'}}$  with  $\text{supp}((\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x'']) = \text{supp}(\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x]) \cap x'' \subseteq x''$ .

We compute

$$(7.26) \quad \begin{aligned} & ((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi))_{\sigma}^{\sigma''} \\ &= \sum_{\sigma' \in \Sigma'} (\rho' \otimes_0 \varphi')_{\sigma'}^{\sigma''} \circ (\rho \otimes_0 \varphi)_{\sigma}^{\sigma'} \\ &\stackrel{(7.9)}{=} \sum_{\sigma' \in \Sigma'} \left( \sum_{x' \in S'_{\sigma'} \setminus G/S'_{\sigma'}} \rho'_{x'\sigma'}^{\sigma''} \cdot \varphi'^{\sigma''}_{\sigma'}[x'] \right) \circ \left( \sum_{x \in S'_{\sigma'} \setminus G/S_{\sigma}} \rho_{x\sigma}^{\sigma'} \cdot \varphi^{\sigma'}_{\sigma}[x] \right) \\ &= \sum_{\sigma' \in \Sigma'} \sum_{x' \in S'_{\sigma'} \setminus G/S'_{\sigma'}} \sum_{x \in S'_{\sigma'} \setminus G/S_{\sigma}} \rho'_{x'\sigma'}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \cdot \varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x] \\ &\stackrel{(7.12)}{=} \sum_{\sigma' \in \widehat{\Sigma'}} \sum_{x' \in S'_{\sigma'} \setminus G/S'_{\sigma'}} \sum_{x \in S'_{\sigma'} \setminus G/S_{\sigma}} \rho'_{x'\sigma'}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \cdot \varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x] \\ &\stackrel{(7.25)}{=} \sum_{\sigma' \in \widehat{\Sigma'}} \sum_{x' \in S'_{\sigma'} \setminus G/S'_{\sigma'}} \sum_{x \in S'_{\sigma'} \setminus G/S_{\sigma}} \rho'_{x'\sigma'}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \cdot \left( \sum_{x'' \in S''_{\sigma'} \setminus G/S_{\sigma}} (\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x''] \right) \\ &= \sum_{\sigma' \in \widehat{\Sigma'}} \sum_{x' \in S'_{\sigma'} \setminus G/S'_{\sigma'}} \sum_{x \in S'_{\sigma'} \setminus G/S_{\sigma}} \sum_{x'' \in S''_{\sigma'} \setminus G/S_{\sigma}} \rho'_{x'\sigma'}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \cdot (\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x'']. \end{aligned}$$

Since

$$\text{supp}(\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x]) \subseteq \text{supp}(\varphi'^{\sigma''}_{\sigma'}[x']) \cdot \text{supp}(\varphi^{\sigma'}_{\sigma}[x]) \subseteq x'x$$

holds, we have

$$(7.27) \quad (\varphi'^{\sigma''}_{\sigma'}[x'] \circ \varphi^{\sigma'}_{\sigma}[x])[x''] \neq 0 \implies x'' \subseteq x'x.$$

We have  $S'_{\sigma'} = S'$  for  $\sigma' \in \widehat{\Sigma'}$  by Assumption 7.14. Moreover we get from (7.7) for  $\sigma' \in \widehat{\Sigma'}$

$$(7.28) \quad \varphi'^{\sigma''}_{\sigma'} = \varphi'^{\sigma''};$$

$$(7.29) \quad \varphi^{\sigma'}_{\sigma} = \varphi_{\sigma},$$

if we put  $\varphi'^{\sigma''} = i_{B'',S_{\sigma''}} \circ \varphi''$  and  $\varphi_{\sigma} = \varphi \circ r_{B,S_{\sigma}}$ . Note that  $\varphi'^{\sigma''}$  and  $\varphi_{\sigma}$  and the index sets  $S''_{\sigma''} \setminus G/\text{supp}(B')$  and  $\text{supp}(B') \setminus G/S_{\sigma}$  and are independent of  $\sigma'$ . Hence

we conclude from (7.26), (7.27), (7.28), and (7.29)

(7.30)

$$\begin{aligned}
& ((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi))_{\sigma}^{\sigma''} \\
&= \sum_{\sigma' \in \widehat{\Sigma}'} \sum_{x' \in S''_{\sigma'} \setminus G / \text{supp}(B')} \sum_{x \in \text{supp}(B') \setminus G / S_{\sigma}} \sum_{\substack{x'' \in S''_{\sigma''} \setminus G / S_{\sigma} \\ x'' \subseteq x'x}} \\
&\quad \rho'_{x'\sigma'}{}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \cdot (\varphi'^{\sigma''}[x'] \circ \varphi_{\sigma}[x])[x'']. \\
&= \sum_{x' \in S''_{\sigma''} \setminus G / \text{supp}(B')} \sum_{x \in \text{supp}(B') \setminus G / S_{\sigma}} \sum_{\substack{x'' \in S''_{\sigma''} \setminus G / S_{\sigma} \\ x'' \subseteq x'x}} \\
&\quad \left( \sum_{\sigma' \in \widehat{\Sigma}'} \rho'_{x'\sigma'}{}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} \right) \cdot (\varphi'^{\sigma''}[x'] \circ \varphi_{\sigma}[x])[x''].
\end{aligned}$$

Next we show that for any  $x'' \in S''_{\sigma''} \setminus G / S_{\sigma}$  we get for any choice of  $x \in \text{supp}(B') \setminus G / S_{\sigma}$  and  $x' \in S''_{\sigma''} \setminus G / \text{supp}(B')$  with  $x'' \subseteq x'x$

$$(7.31) \quad (\rho' \circ \rho)_{x''\sigma}^{\sigma''} = \sum_{\sigma' \in \Sigma'} \rho'_{x'\sigma'}{}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'}.$$

Choose elements  $g, g', g'' \in G$  with  $g \in x$ ,  $g' \in x'$  and  $g'' \in x''$ . The condition  $x'' \subseteq x'x$  says that we can find  $u \in S_{\sigma}$ ,  $u' \in S'_{\sigma'}$ , and  $u'' \in S''_{\sigma''}$ , such that  $u''g'u'gu = g''$  holds. We compute

$$\begin{aligned}
(\rho' \circ \rho)_{x''\sigma}^{\sigma''} &= (\rho' \circ \rho)_{g''\sigma}^{\sigma''} = \sum_{\sigma' \in \Sigma'} \rho'_{g'\sigma'}{}^{\sigma''} \circ \rho_{g\sigma}^{\sigma'} = \sum_{\sigma' \in \Sigma'} \rho'_{g'\sigma'}{}^{\sigma''} \circ \rho_{u''g'u'gu\sigma}^{\sigma'} \\
&= \sum_{\sigma' \in \Sigma'} \rho'_{g'\sigma'}{}^{\sigma''} \circ \rho_{u''g'u'g\sigma}^{\sigma'} = \sum_{\sigma' \in \Sigma'} \rho'_{u''g'\sigma'}{}^{\sigma''} \circ \rho_{u''g'u'g\sigma}^{\sigma'}.
\end{aligned}$$

Since we have

$$\rho'_{x'\sigma'}{}^{\sigma''} \cdot \rho_{x\sigma}^{\sigma'} = \rho'_{g'\sigma'}{}^{\sigma''} \cdot \rho_{g\sigma}^{\sigma'} = \rho'_{u''g'\sigma'}{}^{\sigma''} \cdot \rho_{u''g'u'g\sigma}^{\sigma'} = \rho'_{u''g'\sigma'}{}^{\sigma''} \cdot \rho_{u''g'u'g\sigma}^{\sigma'}$$

equation 7.31 follows. We conclude from (7.30) and (7.31)

(7.32)

$$\begin{aligned}
((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi))_{\sigma}^{\sigma''} &= \sum_{x' \in S''_{\sigma''} \setminus G / \text{supp}(B')} \sum_{x \in \text{supp}(B') \setminus G / S_{\sigma}} \sum_{\substack{x'' \in S''_{\sigma''} \setminus G / S_{\sigma} \\ x'' \subseteq x'x}} \\
&\quad (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \cdot (\varphi'^{\sigma''}[x'] \circ \varphi_{\sigma}[x])[x''].
\end{aligned}$$

Next we compute

$$\begin{aligned}
 (7.33) \quad & ((\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi))_\sigma^{\sigma''} \\
 & \stackrel{(7.9)}{=} \sum_{x'' \in S''_{\sigma''} \setminus G/S_\sigma} (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \cdot (\varphi' \circ \varphi)_{\sigma}^{\sigma''}[x''] \\
 & \stackrel{(7.7)}{=} \sum_{x'' \in S''_{\sigma''} \setminus G/S_\sigma} (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \cdot (\varphi'^{\sigma''} \circ \varphi_\sigma)[x''] \\
 & = \sum_{x'' \in S''_{\sigma''} \setminus G/S_\sigma} (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \\
 & \quad \cdot \left( \left( \sum_{x' \in S''_{\sigma''} \setminus G/\text{supp}(B')} \varphi'^{\sigma''}[x'] \right) \circ \left( \sum_{x \in \text{supp}(B') \setminus H/S_\sigma} \varphi_\sigma[x] \right) \right) [x''] \\
 & = \sum_{x'' \in S''_{\sigma''} \setminus G/S_\sigma} \sum_{x' \in S''_{\sigma''} \setminus G/\text{supp}(B')} \sum_{x \in \text{supp}(B') \setminus H/S_\sigma} \\
 & \quad (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \cdot (\varphi'^{\sigma''}[x'] \circ \varphi_\sigma[x])[x''] \\
 & \stackrel{(7.27)}{=} \sum_{x' \in S''_{\sigma''} \setminus G/\text{supp}(B')} \sum_{x \in \text{supp}(B') \setminus G/S_\sigma} \sum_{\substack{x'' \in S''_{\sigma''} \setminus G/S_\sigma \\ x'' \subseteq x'x}} \\
 & \quad (\rho' \circ \rho)_{x''\sigma}^{\sigma''} \cdot (\varphi'^{\sigma''}[x'] \circ \varphi_\sigma[x])[x''].
 \end{aligned}$$

Now Lemma 7.10 follows from (7.32) and (7.33).  $\square$

In general  $\text{id}_{\mathbf{V}} \otimes_0 \text{id}_B$  is not the identity on  $V \otimes_0 B$  which will force as later to pass to idempotent completions. However there is a favourite situation, where this is not necessary, which we will describe next.

**Lemma 7.34.** *Let  $\mathbf{V} = (\Sigma, c)$  be an object of  $\mathcal{S}^G(\Omega)$  and let  $B$  be an object of  $\mathcal{B}$ . Suppose that  $\Sigma$  is fixed pointwise by  $\text{supp}(B)$ .*

*Then  $\text{id}_{\mathbf{V}} \otimes_0 \text{id}_B = \text{id}_{\mathbf{V} \otimes_0 B}$ .*

*Proof.* Since  $g\sigma = \sigma$  for  $g \in \text{supp}(B)$ , we have  $\text{supp}(B) = \text{supp}(B)_\sigma$  and  $(\text{id}_B)_{\sigma}^{\sigma'} = \text{id}_B$  for every  $\sigma \in \Sigma$  and the object  $\mathbf{V} \otimes_0 B$  in  $\mathcal{B}_G(\Omega)$  is given by  $(B, \Sigma, c_B)$  for the constant function  $c_B: \Sigma \rightarrow \text{ob}(\mathcal{B})$  with value  $B$ . Recall that the identity of  $\mathbf{V} = (\Sigma, c)$  is given by the morphism  $\rho = (\rho_{\sigma}^{\sigma'})_{\sigma, \sigma' \in \Sigma}$  with  $\rho_{\sigma}^{\sigma'} = 1$  for  $\sigma = \sigma'$  and

$\rho_\sigma^{\sigma'} = 0$  for  $\sigma \neq \sigma'$ . Now we compute

$$\begin{aligned}
(\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_B)_\sigma^{\sigma'} &\stackrel{(7.9)}{=} \sum_{x \in \mathrm{supp}(B)_{\sigma'} \backslash G / \mathrm{supp}(B)_\sigma} \rho_{x\sigma}^{\sigma'} \cdot (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \sum_{\substack{x \in \mathrm{supp}(B) \backslash G / \mathrm{supp}(B) \\ \sigma' = x\sigma}} (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \sum_{\substack{x \in \mathrm{supp}(B) \backslash G / \mathrm{supp}(B) \\ \sigma' = x\sigma, \mathrm{id}_B[x] \neq 0}} (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \sum_{\substack{x \in \mathrm{supp}(B) \backslash G / \mathrm{supp}(B) \\ \sigma' = x\sigma, x \cap \mathrm{supp}(B) \neq \emptyset}} (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \sum_{\substack{x \in \mathrm{supp}(B) \backslash G / \mathrm{supp}(B) \\ \sigma' = x\sigma, x = \mathrm{supp}(B)}} (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \sum_{\substack{x \in \mathrm{supp}(B) \backslash G / \mathrm{supp}(B) \\ \sigma' = \sigma, x = \mathrm{supp}(B)}} (\mathrm{id}_B)_\sigma^{\sigma'} [x]. \\
&= \begin{cases} \mathrm{id}_B & \sigma' = \sigma; \\ 0 & \sigma' \neq \sigma. \end{cases}
\end{aligned}$$

This shows  $\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_B = \mathrm{id}_{\mathbf{V} \otimes_0 B}$ .  $\square$

**7.D. The diagonal tensor product in the case of a Hecke algebra.** It is not needed for our purposes but illuminating to figure out what the diagonal tensor product (7.4) becomes for the Hecke category with  $Q$ -support  $\mathcal{B} = \mathcal{B}(G; R, \rho, \omega)$  of Subsection 6.C. Given an object  $(\mathbf{V}, \Sigma)$  in  $\mathcal{S}^Q(\Omega)$  and an object  $K$  of  $\mathcal{B}(Q; R, \rho, \omega)$ , which is by definition just a compact open subgroup of  $G$ , we get

$$\mathbf{V} \otimes_0 K = (\Sigma, c, \sigma \mapsto K \cap \alpha^{-1}(Q_\sigma))$$

It is not hard to check that for morphisms  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  in  $\mathcal{S}^G(\Omega)$  and  $s: K \rightarrow K'$  in  $\mathcal{B}(Q; R, \rho, \omega)$  the morphism  $\rho \otimes s: \mathbf{V} \otimes_0 K \rightarrow \mathbf{V}' \otimes_0 K'$  is given by the formula

$$(7.35) \quad ((\rho \otimes_0 \varphi)_\sigma^{\sigma'})(g) = \rho_{g\sigma}^{\sigma'} \cdot s(g).$$

The following example explains the original root of the diagonal tensor product (7.4).

**Example 7.36.** Suppose that  $Q$  is discrete,  $\rho$  is trivial and no normal character is present, i.e.,  $N = \{1\}$  and  $G = Q$ . Then the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)$  is just the group ring  $RG$ . The category  $\mathcal{S}^G(G/G)$  can be viewed as a subcategory of the category  $RG\text{-Mod}_{f,R}$  of  $RG$ -modules whose underlying  $R$ -module is free by sending an object  $(\Sigma, c)$  to the permutation  $RG$ -module  $R\Sigma$ . Up to equivalence  $\mathrm{Idem}(\mathcal{B}^G(G/G))$  is the category of finitely generated projective  $RG$ -modules for  $\mathcal{B} = \mathcal{B}(G; R)$ .

The diagonal tensor product (7.4) for  $\Omega = G/G$  comes from the pairing

$$RG\text{-Mod}_{f,R} \times RG\text{-Mod}_f \rightarrow RG\text{-Mod}_f$$

sending  $(M, P)$  to  $M \otimes_R P$  equipped with the diagonal  $G$ -action.

7.E. **Construction of  $\mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \rightarrow \mathcal{B}_G(\Omega \times \Lambda)$ .** Let  $\Omega$  and  $\Lambda$  be  $G$ -sets. In this subsection we want to extend the pairing (7.4) to a bilinear pairing.

$$(7.37) \quad - \otimes_0 - : \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \rightarrow \mathcal{B}_G(\Omega \times \Lambda).$$

Let  $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$  and  $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$ . We define

$$\mathbf{V} \otimes_0 \mathbf{B} := (\Sigma \times S, c \times \pi, (\sigma, s) \mapsto B(s)|_{\text{supp}_{\mathbf{B}}(B(s))_\sigma}) \in \mathcal{B}_G(\Omega \times \Lambda),$$

where  $\text{supp}(B(s))_\sigma = G_\sigma \cap \text{supp}_{\mathbf{B}}(B(s))$ . For morphisms  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  in  $\mathcal{S}^G(\Omega)$  and  $\varphi: \mathbf{B} = (S, \pi, B) \rightarrow \mathbf{B}' = (S', \pi', B')$  in  $\mathcal{B}_G(\Lambda)$  we define  $\rho \otimes \varphi: \mathbf{V} \otimes_0 \mathbf{A} \rightarrow \mathbf{V}' \otimes_0 \mathbf{A}'$  by

$$(7.38) \quad (\rho \otimes \varphi)_{(\sigma, s)}^{(\sigma', s')} := \rho \otimes \varphi_s^{s'}$$

using the pairing of Subsection 7.C.

The proof of Lemma 7.10 can easily be extended to the pairing 7.37

**Lemma 7.39.** *Let  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  and  $\rho': \mathbf{V}' = (\Sigma', c') \rightarrow \mathbf{V}'' = (\Sigma'', c'')$  be composable morphisms in  $\mathcal{S}^G(\Omega)$  and  $\varphi: \mathbf{B} \rightarrow \mathbf{B}'$  and  $\varphi': \mathbf{B}' \rightarrow \mathbf{B}''$  be composable morphisms in  $\mathcal{B}_G(\Lambda)$ .*

*Then we get in  $\mathcal{B}_G(\Omega \times \Lambda)$*

$$(\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi) = (\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi).$$

Note that  $(\text{id}_{\mathbf{V}} \otimes \text{id}_{\mathbf{A}}) = \text{id}_{\mathbf{V} \otimes_0 \mathbf{A}}$  can fail, but this can be fixed in the idempotent completion. Lemma 7.39 implies that  $(\text{id}_{\mathbf{V}} \otimes \text{id}_{\mathbf{A}})$  is an idempotent endomorphism of  $\mathbf{V} \otimes_0 \mathbf{A}$ , and we define

$$\mathbf{V} \otimes \mathbf{A} := (\mathbf{V} \otimes_0 \mathbf{A}, \text{id}_{\mathbf{V}} \otimes_0 \text{id}_{\mathbf{A}}) \in \text{Idem}(\mathcal{B}_G(\Omega \times \Lambda)).$$

Then  $(\text{id}_{\mathbf{V}} \otimes_0 \text{id}_{\mathbf{A}}): \mathbf{V} \otimes \mathbf{A} \rightarrow \mathbf{V} \otimes \mathbf{A}$  is  $\text{id}_{\mathbf{V} \otimes \mathbf{A}}$ , and we obtain a bilinear functor

$$(7.40) \quad - \otimes - : \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \rightarrow \text{Idem}(\mathcal{B}_G(\Omega \times \Lambda)).$$

The following observation will often allow us to get rid of idempotent completions.

**Lemma 7.41.** *Let  $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$  and  $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$ . If  $\Sigma$  is fixed pointwise by all  $B(s)$ , then  $\mathbf{V} \otimes \mathbf{B} = \mathbf{V} \otimes_0 \mathbf{B}$ .*

*Proof.* One needs to show that  $\text{id}_{\mathbf{V}} \otimes_0 \text{id}_{\mathbf{B}}$  is the identity of  $\mathbf{V} \otimes_0 \mathbf{B}$ , not just an idempotent. The proof of Lemma 7.34 carries directly over this more general case.  $\square$

For  $E \subseteq \Omega \times \Omega$  and  $E' \subseteq \Lambda \times \Lambda$  we use the following convention

$$(7.42) \quad E \times E' := \left\{ \begin{pmatrix} x', \lambda' \\ x, \lambda \end{pmatrix} \mid \begin{pmatrix} x' \\ x \end{pmatrix} \in E, \begin{pmatrix} \lambda' \\ \lambda \end{pmatrix} \in E' \right\} \subseteq (\Omega \times \Lambda)^{\times 2}.$$

**Lemma 7.43.** (i) *Let  $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$  and  $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$ .*

*Then we have*

(a) *If  $\mathbf{V}$  and  $\mathbf{B}$  are finite, i.e.,  $\Sigma$  and  $S$  are finite, then  $\mathbf{V} \otimes_0 \mathbf{B}$  is finite as well;*

(b)  $\text{supp}_1(\mathbf{V} \otimes_0 \mathbf{B}) = \text{supp}_1 \mathbf{V} \times \text{supp}_1 \mathbf{B}$ .

(ii) *Let  $\rho: \mathbf{V} = (\Sigma, c) \rightarrow \mathbf{V}' = (\Sigma', c')$  in  $\mathcal{S}^G(\Omega)$ ,  $\varphi: \mathbf{B} = (S, \pi, B) \rightarrow \mathbf{B}' = (S', \pi', B')$  in  $\mathcal{B}_G(\Lambda)$ . for  $\rho \otimes_0 \varphi$  in  $\mathcal{B}_G(\Omega \times \Lambda)$  we have*

(a)  $\text{supp}_2(\rho \otimes \varphi) \subseteq \text{supp}_2 \rho \times \text{supp}_2 \varphi$ ;

(b)  $\text{supp}_G(\rho \otimes \varphi) \subseteq \text{supp}_G \varphi$ .

*Proof.* We give the proof only for assertion ((ii)a). The elementary proof for the other assertions is left to the reader. By definition we have

$$\begin{aligned} \text{supp}_2(\rho) &= \left\{ \left( \begin{array}{c} c'(\sigma') \\ c(\sigma) \end{array} \right) \middle| \rho_{\sigma'}^{\sigma'} \neq 0 \right\} \subseteq \Omega \times \Omega; \\ \text{supp}_2(\varphi) &= \left\{ \left( \begin{array}{c} \pi'(s') \\ g\pi(s) \end{array} \right) \middle| s \in S, s' \in S', g \in \text{supp}_G(\varphi_s^{s'}) \right\} \subseteq \Lambda \times \Lambda; \\ \text{supp}_2(\rho \otimes \varphi) &= \left\{ \left( \begin{array}{c} c'(\sigma'), \pi'(s') \\ g(c(\sigma), \pi(s)) \end{array} \right) \middle| \sigma \in \Sigma, \sigma' \in \Sigma', s \in S, s' \in S', g \in \text{supp}_{\mathcal{B}}((\rho \otimes \varphi)_{(\sigma, s)}^{(\sigma', s')}) \right\} \\ &\subseteq (\Omega \times \Lambda)^{\times 2}. \end{aligned}$$

We conclude from 7.9

$$\begin{aligned} \text{supp}_{\mathcal{B}}(\rho \otimes_0 \varphi_{\sigma'}^{\sigma'}) &\subseteq \{g \in G \mid \rho_{g\sigma}^{\sigma'} \neq 0, g \in \text{supp}_{\mathcal{B}}(\varphi_{\sigma'}^{\sigma'})\} \\ &= \{g \in G \mid \left( \begin{array}{c} c'(\sigma') \\ c(g\sigma) \end{array} \right) \in \text{supp}_2(\rho), g \in \text{supp}_{\mathcal{B}}(\varphi_{\sigma'}^{\sigma'})\} \\ &= \{g \in G \mid \left( \begin{array}{c} c'(\sigma') \\ c(g\sigma) \end{array} \right) \in \text{supp}_2(\rho), \left( \begin{array}{c} \pi'(s') \\ g\pi(s) \end{array} \right) \in \text{supp}_2(\varphi)\}. \end{aligned}$$

Hence we get

$$\begin{aligned} \text{supp}_2(\rho \otimes_0 \varphi) &= \left\{ \left( \begin{array}{c} c'(\sigma'), \pi'(s') \\ g(c(\sigma), \pi(s)) \end{array} \right) \middle| \sigma \in \Sigma, \sigma' \in \Sigma', s \in S, s' \in S', g \in \text{supp}_{\mathcal{B}}((\rho \otimes_0 \varphi)_{(\sigma, s)}^{(\sigma', s')}) \right\} \\ &\stackrel{(7.38)}{=} \left\{ \left( \begin{array}{c} c'(\sigma'), \pi'(s') \\ g(c(\sigma), \pi(s)) \end{array} \right) \middle| \sigma \in \Sigma, \sigma' \in \Sigma', s \in S, s' \in S', g \in \text{supp}_{\mathcal{B}}(\rho \otimes_0 \varphi_s^{s'}) \right\} \\ &= \left\{ \left( \begin{array}{c} c'(\sigma'), \pi'(s') \\ c(g\sigma), g\pi(s) \end{array} \right) \middle| \sigma \in \Sigma, \sigma' \in \Sigma', s \in S, s' \in S', g \in \text{supp}_{\mathcal{B}}((\rho \otimes_0 \varphi)_s^{s'}) \right\} \\ &\subseteq \left\{ \left( \begin{array}{c} c'(\sigma'), \pi'(s') \\ c(g\sigma), g\pi(s) \end{array} \right) \middle| \sigma \in \Sigma, \sigma' \in \Sigma', s \in S, s' \in S', \right. \\ &\quad \left. \left( \begin{array}{c} c'(\sigma') \\ c(g\sigma) \end{array} \right) \in \text{supp}_2(\rho), \left( \begin{array}{c} \pi'(s') \\ g\pi(s) \end{array} \right) \in \text{supp}_2(\varphi) \right\}. \end{aligned}$$

This finishes the proof of Lemma 7.43.  $\square$

**7.F. Compatibility of the diagonal tensor product with induction and restriction.** Let  $U$  be an open subgroup of  $G$ . Write  $\text{res}_G^U: G\text{-SETS}_{\text{sm}} \rightarrow U\text{-SETS}_{\text{sm}}$  for the restriction functor. Given a smooth  $G$ -set  $\Omega$ , it induces a *restriction functor*

$$\mathcal{S}^G(\Omega) \rightarrow \mathcal{S}^U(\text{res}_G^U \Omega), \quad (\Sigma, c) \mapsto (\text{res}_G^U \Sigma, \text{res}_G^U c),$$

that we will also denote by  $\text{res}_G^U$ .

Let  $\mathcal{B}$  be a Hecke category with  $G$ -support in the sense of Definition 5.1. We have defined the full  $\mathbb{Z}$ -linear subcategory  $\mathcal{B}|_U$  in Notation 5.4. Note that  $\mathcal{B}|_U$  inherits from  $\mathcal{B}$  the structure of a Hecke category with  $U$ -support if we define  $\text{supp}_{\mathcal{B}|_U}(\varphi) = \text{supp}_{\mathcal{B}}(\varphi)$  for any morphism  $\varphi$  in  $\mathcal{B}|_U$  and in particular  $\text{supp}_{\mathcal{B}|_U}(B) = \text{supp}_{\mathcal{B}}(B)$  for any object  $B$  in  $\mathcal{B}|_U$ . Let  $\text{ind}_U^G: \mathcal{B}|_U \rightarrow \mathcal{B}$  be the canonical inclusion. It induces an inclusion  $\text{ind}_U^G: \mathcal{B}_U(\text{res}_G^U \Omega) \rightarrow \mathcal{B}_G(\Omega)^1$  for any smooth  $G$ -set  $\Omega$ . We write

$$\begin{aligned} \otimes^G: \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Omega) &\rightarrow \text{Idem}(\mathcal{B}_G(\Omega \times \Omega)); \\ \otimes^U: \mathcal{S}^U(\text{res}_G^U \Omega) \times \mathcal{B}_U(\text{res}_G^U \Omega) &\rightarrow \text{Idem}(\mathcal{B}_U(\text{res}_G^U(\Omega \times \Omega))), \end{aligned}$$

for the diagonal tensor products introduced in (7.40).

**Lemma 7.44.** *We get for any  $\mathbf{V} \in \text{ob}(\mathcal{S}^G(\Omega))$  and  $\mathbf{B} \in \text{ob}(\mathcal{B}_U(\text{res}_G^U \Omega))$*

$$\text{ind}_U^G(\text{res}_G^U \mathbf{V} \otimes^U \mathbf{B}) = \mathbf{V} \otimes^G \text{ind}_U^G \mathbf{B}.$$

<sup>1</sup>Strictly speaking the should be  $(\mathcal{B}|_U)_U(\text{res}_G^U \Omega)$

Similarly, for morphisms  $\rho: \mathbf{V} \rightarrow \mathbf{V}'$  in  $\mathcal{S}^G(\Omega)$  and  $\varphi: \mathbf{B} \rightarrow \mathbf{B}'$  in  $\mathcal{B}_U(\text{res}_G^U \Lambda)$  we get

$$\text{ind}_U^G(\text{res}_G^U \rho \otimes^U \varphi) = \rho \otimes^G \text{ind}_U^G \varphi.$$

*Proof.* We give the proof only in the special case where  $\Lambda$  is  $G/G$ , in other words,  $\mathcal{B}_G(\Lambda) = \mathcal{B}$  and  $\mathcal{B}_U(\text{res}_G^U \Lambda) = \mathcal{B}|_U$ . The proof for the general case is then an obvious generalization.

Consider the tensor products

$$\begin{aligned} \otimes_0^G: \mathcal{S}^G(\Omega) \times \mathcal{B} &\rightarrow \mathcal{B}_G(\Omega); \\ \otimes_0^U: \mathcal{S}^U(\text{res}_G^U \Omega) \times \mathcal{B}|_U &\rightarrow \mathcal{B}_U(\text{res}_G^U \Omega), \end{aligned}$$

introduced in (7.5) and (7.6). We obtain for  $\mathbf{V} \in \text{ob}(\mathcal{S}^G(\Omega))$  and  $B \in \text{ob}(\mathcal{B}|_U)$

$$(7.45) \quad \text{ind}_U^G(\text{res}_G^U \mathbf{V} \otimes_0^U B) = \mathbf{V} \otimes_0^G \text{ind}_U^G B,$$

as we have by definition  $\text{res}_G^U \mathbf{V} \otimes_0^U B = (\Sigma, c, \sigma \mapsto B|_{\text{supp}(B) \cap U_\sigma})$  and  $\mathbf{V} \otimes_0^G \text{ind}_U^G B = (\Sigma, c, \sigma \mapsto B|_{\text{supp}(B) \cap G_\sigma})$  and  $\text{supp}_{\mathcal{B}}(B) \subseteq U$  implies  $\text{supp}(B) \cap U_\sigma = \text{supp}(B) \cap G_\sigma$ . Given morphisms  $\rho: \mathbf{V} \rightarrow \mathbf{V}'$  in  $\mathcal{S}^G(\Omega)$  and  $\varphi: B \rightarrow B'$  in  $\mathcal{B}|_U$ , we next show

$$(7.46) \quad \text{ind}_U^G(\text{res}_G^U \rho \otimes_0^U \varphi) = \rho \otimes_0^G \text{ind}_U^G \varphi.$$

We have by (7.9)

$$(7.47) \quad (\text{res}_G^U \rho \otimes_0^U \varphi)_\sigma^{\sigma'} = \sum_{x \in (\text{supp}(B') \cap U_{\sigma'}) \setminus U / \text{supp}(B) \cap U_\sigma} \rho_{x\sigma}^{\sigma'} \cdot \varphi_\sigma^{\sigma'}[x];$$

$$(7.48) \quad (\rho \otimes_0^G \text{ind}_U^G \varphi)_\sigma^{\sigma'} = \sum_{y \in (\text{supp}(B') \cap G_{\sigma'}) \setminus G / (\text{supp}(B) \cap G_\sigma)} \rho_{y\sigma}^{\sigma'} \cdot (\text{ind}_U^G \varphi)_\sigma^{\sigma'}[y],$$

where the morphisms  $\varphi_\sigma^{\sigma'}$  in  $\mathcal{B}_U(\text{res}_G^U \Omega)$  and  $(\text{ind}_U^G \varphi)_\sigma^{\sigma'}$  in  $\mathcal{B}_G(\Omega)$  are defined by

$$(7.49) \quad \varphi_\sigma^{\sigma'}: B|_{\text{supp}(B) \cap U_\sigma} \xrightarrow{r_{B, \text{supp}(B) \cap U_\sigma}} B \xrightarrow{\varphi} B' \xrightarrow{i_{B', \text{supp}(B') \cap U_{\sigma'}}} B'|_{\text{supp}(B') \cap U_{\sigma'}}$$

and

$$(7.50) \quad (\text{ind}_U^G \varphi)_\sigma^{\sigma'}: B|_{\text{supp}(B) \cap G_\sigma} \xrightarrow{r_{B, \text{supp}(B) \cap G_\sigma}} B \xrightarrow{\text{ind}_U^G \varphi} B' \xrightarrow{i_{B', \text{supp}(B') \cap U_{\sigma'}}} B'|_{\text{supp}(B') \cap U_{\sigma'}}$$

and the morphisms  $\varphi_\sigma^{\sigma'}[x]: B|_{\text{supp}(B) \cap U_\sigma} \rightarrow B'|_{\text{supp}(B') \cap U_{\sigma'}}$  in  $\mathcal{B}_U(\text{res}_G^U \Omega)$  and  $(\text{ind}_U^G \varphi)_\sigma^{\sigma'}[y]: B|_{\text{supp}(B) \cap G_\sigma} \rightarrow B'|_{\text{supp}(B') \cap U_{\sigma'}}$  in  $\mathcal{B}_G(\Omega)$  are uniquely determined by

$$\begin{aligned} \varphi_\sigma^{\sigma'} &= \sum_{x \in (\text{supp}(B') \cap U_{\sigma'}) \setminus U / \text{supp}(B) \cap U_\sigma} \varphi_\sigma^{\sigma'}[x]; \\ \text{supp}(\varphi_\sigma^{\sigma'}[x]) &= \text{supp}(\varphi_\sigma^{\sigma'}) \cap x; \\ (\text{ind}_U^G \varphi)_\sigma^{\sigma'} &= \sum_{y \in (\text{supp}(B') \cap G_{\sigma'}) \setminus G / (\text{supp}(B) \cap G_\sigma)} (\text{ind}_U^G \varphi)_\sigma^{\sigma'}[y]; \\ \text{supp}((\text{ind}_U^G \varphi)_\sigma^{\sigma'}[y]) &= \text{supp}((\text{ind}_U^G \varphi)_\sigma^{\sigma'}) \cap y. \end{aligned}$$

As  $\text{supp}(B)$  and  $\text{supp}(B')$  are contained in  $U$ , we get  $\text{supp}(B) \cap U_\sigma = \text{supp}(B) \cap G_\sigma$  and  $\text{supp}(B') \cap U_{\sigma'} = \text{supp}(B') \cap G_{\sigma'}$ . We conclude from (7.47) and (7.48) that the morphism  $\varphi_\sigma^{\sigma'}$  and  $(\text{ind}_U^G \varphi)_\sigma^{\sigma'}$  in  $\mathcal{B}$  agree. Note that the inclusion of  $U \subseteq G$  induces an inclusion

$$(\text{supp}(B') \cap U_{\sigma'}) \setminus U / \text{supp}(B) \cap U_\sigma \subseteq (\text{supp}(B') \cap G_{\sigma'}) \setminus G / (\text{supp}(B) \cap G_\sigma).$$

Since the support of  $\varphi$  is contained in  $U$  and hence the support of  $\varphi_\sigma^{\sigma'} = (\text{ind}_U^G \varphi)_\sigma^{\sigma'}$  is contained in  $U$ , we conclude

$$(\text{ind}_U^G \varphi)_\sigma^{\sigma'}[y] = \begin{cases} \varphi_\sigma^{\sigma'}[x] & \text{if } y \in (\text{supp}(B') \cap U_{\sigma'}) \setminus U / \text{supp}(B) \cap U_\sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Now (7.46) follows from (7.47) and (7.48). Hence Lemma 7.44 follows from (7.45) and (7.46) in the special case  $\Lambda = G/G$ .  $\square$

**7.G. Flatness.** Consider a Hecke category  $\mathcal{B}$  with  $G$ -support in the sense of Definition 5.1.

We extend the notion of the support for  $\mathcal{B}$  to  $\mathcal{B}_\oplus$  as follows. The support  $\text{supp}_{\mathcal{B}_\oplus}(\underline{B})$  of an object  $\underline{B} = (B_1, B_2, \dots, B_n)$  is defined to be  $\bigcup_{i=1}^n \text{supp}_{\mathcal{B}}(B_i)$  and for  $g \in G$  we put  $g\underline{B} = (gB_1, gB_2, \dots, gB_n)$ . For a morphism  $\underline{\varphi} = (\varphi_{i,j}): \underline{B} \rightarrow \underline{B}'$  its support  $\text{supp}_{\mathcal{B}_\oplus}(\underline{\varphi})$  is defined to be  $\bigcup_{i,j} \text{supp}_G(\varphi_{i,j})$ . Note that  $\text{supp}_{\mathcal{B}_\oplus}(\underline{B})$  is not a subgroup anymore. One easily checks that the conditions appearing in Definition 5.1 are satisfied except the conditions (i) and (iii). So  $\text{supp}_{\mathcal{B}_\oplus}(\underline{B})$  is not a Hecke category with  $G$ -support. We have for any two objects  $\underline{B}$  and  $\underline{B}'$  and any two morphisms  $\varphi$  and  $\varphi'$

$$\begin{aligned} \text{supp}(\underline{B} \oplus \underline{B}') &= \text{supp}(\underline{B}) \cup \text{supp}(\underline{B}'); \\ \text{supp}(\underline{\varphi} \oplus \underline{\varphi}') &= \text{supp}(\underline{\varphi}) \cup \text{supp}(\underline{\varphi}'). \end{aligned}$$

Recall that a sequence  $A_0 \xrightarrow{u} A_1 \xrightarrow{v} A_2$  in an additive category  $\mathcal{A}$  is *exact* if for any object  $A$  in  $\mathcal{A}$  the induced sequence of abelian groups is  $\text{mor}_{\mathcal{A}}(A, A_0) \xrightarrow{u_*} \text{mor}_{\mathcal{A}}(A, A_1) \xrightarrow{v_*} \text{mor}_{\mathcal{A}}(A, A_2)$  is exact.

**Lemma 7.51.** *For any two open subgroups  $U \subseteq V$  of  $G$  and any  $n \in \mathbb{Z}$  the inclusion  $\mathcal{B}|_U \rightarrow \mathcal{B}|_V$  induces a functor of additive categories*

$$(\mathcal{B}_U)_\oplus[\mathbb{Z}^d] \rightarrow (\mathcal{B}_V)_\oplus[\mathbb{Z}^d]$$

that is exact.

*Proof.* Recall that the objects of  $(\mathcal{B}|_U)_\oplus[\mathbb{Z}^d]$  are the objects of  $(\mathcal{B}|_U)_\oplus$  and hence the support of an object  $\underline{B}$  of  $(\mathcal{B}|_U)_\oplus[\mathbb{Z}^d]$  is defined. For a morphism  $\varphi = \sum_{x \in \mathbb{Z}^d} \varphi_x \cdot x$  in  $(\mathcal{B}|_U)_\oplus[\mathbb{Z}^d]$  we set  $\text{supp}(\varphi) = \bigcup_{x \in \mathbb{Z}^d} \text{supp}(\varphi_x)$ .

Let  $\underline{A} \xrightarrow{\varphi} \underline{A}' \xrightarrow{\varphi'} \underline{A}''$  be a sequence in  $(\mathcal{B}|_U)_\oplus[\mathbb{Z}^d]$  that is exact at  $\underline{A}'$ . Let  $\underline{\psi}: \underline{B} \rightarrow \underline{A}'$  be a morphism in  $(\mathcal{B}|_V)_\oplus[\mathbb{Z}^d]$  with  $\varphi' \circ \underline{\psi} = 0$ . We need to find a lift  $\widehat{\underline{\psi}}: \underline{B} \rightarrow \underline{A}$  in  $(\mathcal{B}|_V)_\oplus[\mathbb{Z}^d]$  with  $\underline{\psi} = \varphi \circ \widehat{\underline{\psi}}$ .

$$\begin{array}{ccccc} \underline{A} & \xrightarrow{\varphi} & \underline{A}' & \xrightarrow{\varphi'} & \underline{A}'' \\ & \swarrow \widehat{\underline{\psi}} & \uparrow \underline{\psi} & \searrow 0 & \\ & & \underline{B} & & \end{array}$$

Let  $M := \text{supp}(\underline{\psi}) \cdot \text{supp}(\underline{B})$ . This is a compact subset of  $V$ . Then  $U_0 = \bigcap_{g \in M} g^{-1}Ug$  is an open subgroup of  $U$  such that  $gU_0g^{-1} \subseteq U$  holds for all  $g \in M$ .

Because of the property *Support cofinality* we find  $\underline{B} \xrightarrow{\underline{i}} \underline{B}' \xrightarrow{\underline{r}} \underline{B}$  in  $(\mathcal{B}|_V)_\oplus[\mathbb{Z}^d]$  such that  $\underline{r} \circ \underline{i} = \text{id}_{\underline{B}}$ ,  $\text{supp}(\underline{r}) = \text{supp}(\underline{i}) = \text{supp}(\underline{B})$  and  $\text{supp}(\underline{B}') \subseteq U_0$  hold. Put  $\underline{\psi}' := \underline{\psi} \circ \underline{r}$ . Then  $\text{supp}(\underline{\psi}') = \text{supp}(\underline{\psi} \circ \underline{r}) \subseteq \text{supp}(\underline{\psi}) \cdot \text{supp}(\underline{B}) = M$ . It will suffice to find  $\widehat{\underline{\psi}}': \underline{B}' \rightarrow \underline{A}$  such that  $\varphi \circ \widehat{\underline{\psi}}' = \underline{\psi}'$  because then we can set  $\widehat{\underline{\psi}} := \widehat{\underline{\psi}}' \circ \underline{i}$ .



We can write

$$\begin{aligned}\underline{\psi}' &= \sum_{UgU_0 \in U \setminus V/U_0} \underline{\psi}'_{UgU_0}; \\ \underline{\varphi}' \circ \underline{\psi}' &= \sum_{UgU_0 \in U \setminus V/U_0} (\underline{\varphi}' \circ \underline{\psi}')_{UgU_0},\end{aligned}$$

where  $\underline{\psi}'_{UgU_0}$  is a morphism  $\underline{B}' \rightarrow \underline{A}$  with  $\text{supp}(\underline{\psi}'_{UgU_0}) \subseteq UgU_0$  and  $(\underline{\varphi}' \circ \underline{\psi}')_{UgU_0}$  is a morphism  $\underline{B}' \rightarrow \underline{A}'$  with  $\text{supp}(\underline{\varphi}' \circ \underline{\psi}')_{UgU_0} \subseteq UgU_0$ . We give the argument only for  $\underline{\psi}'$ , the one for  $\underline{\varphi}' \circ \underline{\psi}'$  is analogous.

Write  $\text{supp}(\underline{B}') = (B'_1, \dots, B'_m)$  and  $\underline{A}' = (A'_1, \dots, A'_n)$ . Fix  $UgU_0 \in U \setminus V/U_0$ ,  $i \in \{1, \dots, m\}$ , and  $j \in \{1, \dots, n\}$ . Since  $\text{supp}(\underline{B}') \subseteq U_0$  and  $\text{supp}(\underline{A}') \subseteq U$  holds, we get  $\text{supp}(A'_i)UgU_0 \text{supp}(B'_j) = UgU_0$ . As  $\text{supp}(A'_j) \text{supp}(\psi'_{i,j}) \text{supp}(B'_i) = \text{supp}(\psi'_{i,j})$  holds, we conclude  $\text{supp}(A'_j)(\text{supp}(\psi'_{i,j}) \cap UgU_0) \text{supp}(B'_i) = \text{supp}(\psi'_{i,j}) \cap UgU_0$ . Obviously  $\text{supp}(\psi'_{i,j}) = \coprod_{UgU_0 \in U \setminus V/U_0} (\text{supp}(\psi'_{i,j}) \cap UgU_0)$ . *Morphism Additivity* implies that we can write  $\psi'_{i,j} = \sum_{UgU_0 \in U \setminus V/U_0} (\psi'_{i,j})_{UgU_0}$  for morphisms  $(\psi'_{i,j})_{UgU_0} : B'_i \rightarrow A'_j$  with  $\text{supp}(\psi'_{i,j})_{UgU_0} \subseteq UgU_0$ . Now define  $\psi'_{UgU_0} : \underline{B}' \rightarrow \underline{A}'$  by the collection of the morphisms  $(\psi'_{i,j})_{UgU_0}$ .

Since  $\underline{\varphi}' \circ \underline{\psi}' = 0$ , we conclude from Lemma 5.2 (i) that  $(\underline{\varphi}' \circ \underline{\psi}')_{UgU_0} = 0$  holds for all  $UgU_0 \in U \setminus V/U_0$ . Lemma 5.2 (i) implies that  $(\underline{\varphi}' \circ \underline{\psi}')_{UgU_0} = \underline{\varphi}' \circ \underline{\psi}'_{UgU_0}$ , since  $\text{supp}(\underline{\varphi}') \subseteq U$ . Hence we get  $\underline{\varphi}' \circ \underline{\psi}'_{UgU_0} = 0$  for all  $g$ . This allows us to assume without loss of generality that  $\text{supp}_G(\underline{\psi}') \subseteq UgU_0$  for some  $g \in V$ .

As  $gU_0g^{-1} \subseteq U$  we have  $UgU_0 = Ug$ . From *Translation* we obtain an object  $\underline{B}''$  and an isomorphism  $\underline{f} : \underline{B}'' \xrightarrow{\cong} \underline{B}'$  satisfying  $\text{supp}(\underline{B}'') = g \text{supp}(\underline{B}')g^{-1}$  and  $\text{supp}(\underline{f}) \subseteq g^{-1} \text{supp}(\underline{B}'')$ . Since  $\text{supp}(\underline{B}') \subseteq U_0$ , we have

$$\text{supp}(\underline{B}'') = g \text{supp}(\underline{B}')g^{-1} \subseteq gU_0g = U.$$

We have

$$\text{supp}(\underline{\psi}' \circ \underline{f}) \subseteq \text{supp}(\underline{\psi}') \cdot \text{supp}(\underline{f}) \subseteq UgU_0g^{-1} \text{supp}(\underline{B}'') \subseteq UgU_0g^{-1}U = U.$$

This implies  $(\underline{\psi}' \circ \underline{f}) \in (\mathcal{B}|_U)_{\oplus}[\mathbb{Z}^d]$  and we can apply the exactness in  $(\mathcal{B}|_U)_{\oplus}[\mathbb{Z}^d]$  to  $\underline{\psi}' \circ \underline{f}$  to obtain  $\underline{\psi} : \underline{B}'' \rightarrow \underline{A}$  with  $\underline{\psi}' \circ \underline{f} = \varphi \circ \underline{\psi}$ . Now with  $\widehat{\underline{\psi}} := \underline{\psi} \circ \underline{f}^{-1}$  we get

$$\varphi \circ \widehat{\underline{\psi}} = \varphi \circ \underline{\psi} \circ \underline{f}^{-1} = \underline{\psi}' \circ \underline{f} \circ \underline{f}^{-1} = \underline{\psi}'$$

as required.  $\square$

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