ALMOST EQUIVARIANT MAPS FOR TD-GROUPS

BARTELS, A. AND LÜCK, W.

Abstract. We construct certain maps from buildings associated to td-groups to a space closely related to the classifying numerable G-space for the family $\mathcal{C}$vcy of covirtually cyclic subgroups. These maps are used elsewhere to study the K-theory of Hecke algebras in the spirit of the Farrell–Jones conjecture.

1. Introduction

The Farrell–Jones conjecture [13] originated in the surgery theory and has applications to the classification of manifolds, notably it implies (in dimension ≥ 5) Borel's conjecture on the topological rigidity of aspherical manifolds. The conjecture concerns the K- and L-groups of group rings and expresses these in terms of an equivariant homology theory. It can be viewed as reducing computations to the case of group rings for virtually cyclic groups. Further information on the conjecture can be found for instance in [19].

The main result from this paper are used in [5] to obtain computations for the K-theory of Hecke algebras that are in spirit similar to the Farrell–Jones conjecture. It can viewed as extending results from [3] which are a central ingredient to the proof of the Farrell–Jones conjecture for CAT(0)-groups [4] from the setting of discrete groups to td-groups.

1.a. Discrete case.

Let Γ be a discrete groups. Typically a Γ-space cannot be both compact and Γ-CW-complex with small isotropy groups. Compromises between these two properties are central to axiomatic results for the Farrell–Jones conjecture, see for example [2, Sec. 2]. For CAT(0)-groups such a compromise was established in [3, Main Thm] and [23, Thm 3.4].

For a collection $\mathcal{F}$ of subgroups and $N \in \mathbb{N}$ we consider the n+1-fold join $E^N_{\mathcal{F}}(\Gamma) := *_{i=0}^N(\coprod_{V \in \mathcal{F}} \Gamma/V)^1$. As $E^N_{\mathcal{F}}(\Gamma)$ is a simplicial complex we can equip it with the ℓ^∞-metric d_E. We note that this metric is Γ-invariant. Let X be a CAT(0)-space of finite covering dimension with a cocompact proper isometric Γ-action. We fix a basepoint in X and write B_R for the closed ball of radius R around the basepoint. Let $\pi_R \colon X \to B_R$ be the radial projection. Let $\mathcal{C}$vcy be the family of subgroups of Γ that admit a map to a cyclic group with finite kernel.

Theorem 1.1. *There is $N \in \mathbb{N}$ such that for all finite $M \subseteq \Gamma$ and $\epsilon > 0$ there is $\mathcal{V} \subseteq \mathcal{C}$vcy finite such that for all $L > 0$ we find $R > 0$ and a (continuous) map $f \colon X \to E^N_{\mathcal{V}}(G)$ satisfying:*

(i) *for $x \in B_{R+L}, g \in M$ we have $d_E(f(gx), gf(x)) < \epsilon$;*
(ii) *for $x \in B_{R+L}$, $R' \geq R$ we have $d_E(f(x), f(\pi_{R'}(x))) < \epsilon$.*

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[1] The infinite join $*_{i=0}^\infty(\coprod_{V \in \mathcal{F}} \Gamma/V)$ is a model for the classifying Γ-CW-complex for the family consisting of all subgroups of Γ that are subconjugated to one of the $V \in \mathcal{F}$, compare [8, App. A1].

Theorem 1.1 is the discrete precursor to our main result in the totally disconnected case, see Theorem 1.2. We discuss in Remark 1.7 how it is implied by the main result. Theorem 1.1 has not been stated before, but it can be viewed as a reformulation of the results from [3, 23] cited above.

There is a homotopy Γ-action on B_R where $g \in \Gamma$ acts as $x \mapsto \pi_R(gx)$. Roughly speaking (i) says that f is almost G-equivariant and (ii) ensures that the tracks of the homotopies of the homotopy action on B_R have small images in $E^N_{\mathcal{V}}(G)$.

1.B. **The setup.** Throughout this paper we fix a td-group G, i.e., a locally compact second countable topological Hausdorff group. We also fix an action of G on a locally compact CAT(0)-space X of finite covering dimension. We assume that the action is by isometries, continuous, cocompact, smooth and proper. In particular, all isotropy groups G_x for the action are compact open in G. The main result will also depend on a further more technical assumption. Informally, the assumption is that isotropy groups of geodesics in the space of geodesics with bounded period for the G-action get only smaller in small neighborhoods, at least on a suitable fundamental domain. Technically this is formulated as Assumption 2.7 using the flow space for X. The main example where Assumption 2.7 is satisfied is the action of a reductive p-adic group on its associated extended Bruhat-Tits building, see Appendix A. This is the main example we are interested in.

1.C. **The space $J^N_{\mathcal{F}}(G)$.** Let $\mathcal{F}$ be a collection of closed subgroups of G. As in the discrete case we can consider for $N \in \mathbb{N}$ the $N{+}1$-fold join

$$J^N_{\mathcal{F}}(G) := *_{i=0}^N \Big(\coprod_{V \in \mathcal{F}} G/V \Big).$$

For closed subgroups V, V' of G the product $G/V \times G/V'$ can as a G-space not necessarily be written as a coproduct of orbits. For this reason $J^N_{\mathcal{F}}(G)$ is not G-CW-complex. But it is still a numerable G-space in the sense of [18, Def. 2.1][2]. In contrast to $E^N_{\mathcal{F}}(\Gamma)$, there is in general no G-invariant metric on $J^N_{\mathcal{F}}(G)$. In fact, for a closed (but neither open nor compact) subgroup V of G there may be no G-invariant metric on the orbit G/V that generates the topology. This is a more substantial difficulty to formulating Theorem 1.1 for td-groups. We will explain our solution to this difficulty next.

1.D. **V-foliated distance in G.** We can equip G with a left invariant proper metric d_G that generates the topology of G, see [14, Thm. 4.5] or [1, Thm. 1.1]. Let V be a closed subgroup of G. As a replacement for the in general not existing G-invariant metric on G/V we will use the following V-foliated distance in G. For $g, g' \in G$, $\beta, \eta > 0$ we write

$$d_{V\text{-fol}}(g, g') < (\beta, \eta)$$

if there is $v \in V$ with $d_G(e, v) = d_G(g, gv) < \beta$ and $d_G(gv, g') < \eta$. Note that $d_{V\text{-fol}}$ is left G-invariant in the sense that

$$d_{V\text{-fol}}(g, g') < (\beta, \eta) \Longleftrightarrow d_{V\text{-fol}}(g''g, g''g') < (\beta, \eta)$$

holds for all $g, g', g'' \in G$.

Two elements g, g' in G satisfy $gV = g'V$ if and only if there exists $\beta > 0$ such that for every $\eta > 0$ we have $d_{V\text{-fol}}(g, g') < (\beta, \eta)$. One might be tempted to consider

$$d_{G/V}(gV, g'V) := \inf\{\eta \mid d_{V\text{-fol}}(g, g') < (\beta, \eta) \text{ for some } \beta > 0\},$$

[2] We note that $\operatorname{colim}_{N\to\infty} J^N_{\mathcal{F}}(G)$ is a model for the classifying numerable G-space for the family $\mathcal{F}$, see [18, Def. 2.3] and [8, App. A1]. We will not need this fact, but it motivates the definition.

but this infimum can be 0 for $gV \neq g'V$. This happens for example for V the subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ consisting of diagonal matrices $g = e$ and g' unipotent.

1.E. **The space $J^N_{\mathcal{F}}(G)^\wedge$.** Let $\mathcal{F}$ be a collection of closed subgroups of G. For $N \in \mathbb{N}$ let

$$J^N_{\mathcal{F}}(G)^\wedge := *_{i=0}^N \Big(\coprod_{V \in \mathcal{F}} G \Big).$$

The projections $G \to G/V$ induce a G-equivariant map $J^N_{\mathcal{F}}(G)^\wedge \to J^N_{\mathcal{F}}(G)$.

As $\coprod_{V \in \mathcal{F}} G = G \times \mathcal{F}$ we can write elements in $J^N_{\mathcal{F}}(G)^\wedge$ as $[t_0(g_0, V_0), \ldots, t_n(g_n, V_n)]$ with $t_i \in [0,1]$, $g_i \in G$, $V_i \in \mathcal{F}$ such that $\sum t_i = 1$. In this notation we have $[t_0(g_0, V_0), \ldots, t_n(g_n, V_n)] = [t'_0(g'_0, V'_0), \ldots, t'_n(g'_n, V'_n)]$ if and only if $t_i = t'_i$ for all i and $(g_i, V_i) = (g'_i, V'_i)$ for all i with $t_i \neq 0 \neq t'_i$.

1.F. ***J*-foliated distance.** As discussed in Subsection 1.C there is in general no G-invariant metric on $J^N_{\mathcal{F}}(G)$. As a replacement we work with the following notion of foliated distance on $J^N_{\mathcal{F}}(G)^\wedge$. For

$$y = [t_0(g_0, V_0), \ldots, t_N(g_N, V_N)],\ y' = [t'_0(g'_0, V'_0), \ldots, t'_N(g'_N, V'_N)] \in J^N_{\mathcal{F}}(G)^\wedge$$

and $\beta, \eta, \epsilon > 0$ we write

$$d_{J\text{-fol}}(y, y') < (\beta, \eta, \epsilon)$$

if $|t_i - t'_i| < \epsilon$ for all i and, in addition, for all i with $\max\{t_i, t'_i\} \geq \epsilon$ we have

$$V_i = V'_i, \quad \text{and} \quad d_{V_i\text{-fol}}(g_i, g'_i) < (\beta, \eta).$$

There is a map $J^N_{\mathcal{F}}(G)^\wedge \to *_{i=0}^N \mathcal{V}$, $[t_0(g_0, V_0), \ldots, t_N(g_N, V_N)] \mapsto [t_0 V_0, \ldots, t_N V_N]$. The first requirement implies that the images of y and y' in the join $*_{i=0}^N \mathcal{V}$ are of distance $< \epsilon$ with respect to the ℓ^∞-metric. Two points y and y' in $J^N_{\mathcal{F}}(G)^\wedge$ have the same image under the projection $J^N_{\mathcal{F}}(G)^\wedge \to J^N_{\mathcal{F}}(G)$ if and only if there exists $\beta > 0$ such that for any $\epsilon > 0$ and any $\eta > 0$ we have $d_{J\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$.

1.G. **Statement of main result.** We write now $\mathcal{C}\mathrm{vcy}$ for the family of subgroups V of G which are *covirtually cyclic*, i.e., V is compact or there exists an exact sequence of topological groups $1 \to K \xrightarrow{i} V \to \mathbb{Z} \xrightarrow{p} 1$, where i is the inclusion of a compact open subgroup K of V and $\mathbb{Z}$ is equipped with the discrete topology.

As before we fix a base point $b \in X$ and write B_R for the closed ball of radius R around b in X. We write $\pi_R \colon X \to B_R$ for the radial projection.

Theorem 1.2 (Main Theorem)**.** *Suppose that Assumption 2.7 holds. There is $N \in \mathbb{N}$ such that for all $M \subseteq G$ compact and $\epsilon > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite with the following property: For all $\eta > 0$ and all $L > 0$ we find $R > 0$ and a (not necessarily continuous) map $f \colon X \to J^N_{\mathcal{V}}(G)^\wedge$ satisfying:*

- *(i) for $x \in B_{R+L}$, $g \in M$ we have $d_{J\text{-fol}}(f(gx), gf(x)) < (\beta, \eta, \epsilon)$;*
- *(ii) for $x \in B_{R+L}$, $R' \geq R$ we have $d_{J\text{-fol}}(f(x), f(\pi_{R'}(x))) < (\beta, \eta, \epsilon)$;*
- *(iii) there is $\rho > 0$ such that for all $x, x' \in X$ with $d_X(x, x') < \rho$ we have $d_{J\text{-fol}}(f(x), f(x')) < (\beta, \eta, \epsilon)$.*

Remark 1.3 (Quantifiers)**.** Using quantifiers the beginning of Theorem 1.2 reads as

$$\exists N\ \forall M, \epsilon\ \exists \beta, \mathcal{V}\ \forall \eta, L\ \exists R, f \text{ such that} \ldots$$

Remark 1.4 (Failure of continuity)**.** The map f appearing in Theorem 1.2 is not necessarily continuous, but this should not be viewed as a serious problem; (iii) is a sufficient replacement for continuity.

This issue arise in Proposition 6.8. We discuss in Remark 6.10 how it might be circumvented with more careful bookkeeping.

Remark 1.5 (Strategy of the proof of Theorem 1.2)**.** The proof of Theorem 1.2 will use the flow space FS from [3] that mimics the geodesic flow on non-positively curved manifolds. More precisely the map f from Theorem 1.2 will be constructed as a composition

$$X \xrightarrow{f_0} FS \xrightarrow{f_1} J^N_{\mathcal{V}}(G).$$

The map f_0 is constructed in Theorem 4.1. This uses the dynamic properties of the flow on FS coming from the CAT(0)-geometry of X. The map f_1 is constructed in Theorem 4.3 and uses an adaptation of the long and thin covers for flow spaces from [6, 15] to the case of td-groups.

Remark 1.6 (About Assumption 2.7)**.** We do not know whether our main theorem fails in the absence of Assumption 2.7. We do not know whether or whether not Assumption 2.7 is always satisfied. It is not difficult to check that Assumption 2.7 holds automatically if G is discrete. It is not difficult to check that Assumption 2.7 implies that for $\ell > 0$ the collection of all V_c with $\tau_c \leq \ell$ contains up to conjugation only finitely many subgroups. We do not know whether or not the converse holds.

Remark 1.7 (Back to the discrete case)**.** As any discrete group Γ is also a td-group we can apply Theorem 1.2 in the situation of Subsection 1.A. Write $p\colon J^N_{\mathcal{V}}(\Gamma)^\wedge \to J^N_{\mathcal{V}}(\Gamma) = E^N_{\mathcal{V}}(\Gamma)$ for the canonical projection. As Γ is discrete there is $\eta > 0$ such that $d_\Gamma(g, g') < \eta$ implies $g = g'$. For such η and all $\beta, \epsilon > 0$, $y, y' \in J^N_{\mathcal{V}}(\Gamma)^\wedge$ we then have

$$d_{J\text{-fol}}(y, y') < (\beta, \eta, \epsilon) \implies d_E(p(y), p(y')) < \epsilon.$$

Therefore we can simply compose f from Theorem 1.2 with p to obtain Theorem 1.1. Note that (iii) from Theorem 1.2 implies that $p \circ f$ is continuous (in fact uniformly).

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The paper is organized as follows:

Contents

2. The flow space FS

2.A. **Construction of the flow space.** Given a metric space Z, denote by $FS = FS(Z)$ the associated flow space defined in [3, Section 1]. It consists of all generalized geodesics. A *generalized geodesic* is a continuous map $c\colon \mathbb{R} \to Z$ whose restriction to some interval[3] is an isometric embedding and is locally constant on the complement of this interval. We do allow that c is constant. The metric on FS is given by

$$d_{FS}(c, c') := \int_{\mathbb{R}} \frac{d_Z(c(t), c'(t))}{2e^{|t|}}\, dt.$$

We recall from [3, Prop. 1.7] that this metric generates the topology of uniform convergence on compact subsets. The flow Φ on FS is defined by

$$(\Phi_\tau c)(t) := c(t+\tau).$$

We also write $FS_\infty = FS_\infty(Z)$ for the subspace of FS consisting of all generalized geodesics that are bi-infinite geodesics, i.e., are nowhere locally constant.

2.B. **Basic facts about the flow space.** For later reference we recall some facts about FS from [3].

Lemma 2.1. *Let (Z, d_Z) be a metric space.*

(i) The map Φ is a continuous flow and we have for $c, d \in FS(Z)$ and $\tau, \sigma \in \mathbb{R}$

$$d_{FS}\big(\Phi_\tau(c), \Phi_\sigma(d)\big) \le e^{|\tau|} \cdot d_{FS}(c, d) + |\sigma - \tau|;$$

(ii) For fixed α the map $FS \times [-\alpha, \alpha] \to FS$, $(c, t) \mapsto \Phi_t(c)$ is uniformly continuous.

Proof. Assertion (i) is proved in [3, Lemma 1.3] and implies assertion (ii). □

Lemma 2.2. *Suppose that Z is a proper metric space. Then $FS(Z)$ is a proper metric space, and for any $t \in \mathbb{R}$ the evaluation map $FS \to Z$, $c \mapsto c(t)$ is proper and uniformly continuous.*

[3] By an interval we mean a set of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$ or $(-\infty, +\infty)$

Proof. See [3, Prop. 1.9 and Lem. 1.10]. □

Lemma 2.3. *If G acts cocompactly, isometrically, or properly respectively on the proper metric space Z, then the G-action on FS is cocompact, isometric or proper respectively.*

Proof. Obviously the G-actions on FS is isometric if G acts isometrically on Z. We conclude from Lemma B.1 (vii) and (xi) and Lemma 2.2 that the G-action on FS is proper or cocompact respectively if G acts proper or cocompact respectively on Z. □

2.C. **Foliated distance in FS.** For $c, c' \in FS$, $\alpha, \delta > 0$ we write

$$d_{FS\text{-fol}}(c, c') < (\alpha, \delta)$$

to mean that there is $t \in [-\alpha, \alpha]$ with $d_{FS}(\Phi_t(c), c') < \delta$. We set

$$U^{\mathrm{fol}}_{\alpha,\delta}(c) := \{c' \in FS \mid d_{FS\text{-fol}}(c, c') < (\alpha, \delta)\}.$$

Lemma 2.4 (Basics about foliated distance)**.**

(i) For $c, c' \in FS$, we get

$$d_{FS\text{-fol}}(c, c') < (\alpha, \delta) \implies d_{FS}(c', c) < \alpha + \delta;$$

(ii) For $c, c' \in FS$ and $g \in G$, we have

$$d_{FS\text{-fol}}(c, c') < (\alpha, \delta) \iff d_{FS\text{-fol}}(gc, gc') < (\alpha, \delta).$$

Proof. (i) This follows from the triangle inequality, since Φ has at most unit speed, i.e., $d_{FS}(\Phi_t(c), c) \le |t|$ for all $c \in FS$ and $t \in \mathbb{R}$, by Lemma 2.1 (i).

(ii) Recall that d_{FS} left G-invariant and Φ is compatible with the G-action and hence $d_{FS}(\Phi_t(gc), gc') = d_{FS}(\Phi_t(c), c')$ for all $c, c' \in FS$, $g \in G$ and $t \in \mathbb{R}$. □

Lemma 2.5 (Symmetry and triangle inequality for the foliated distance)**.**

(i) For $\alpha > 0$, $\delta > 0$ there is $\epsilon > 0$ such that for all $c, c' \in FS$

$$d_{FS\text{-fol}}(c, c') < (\alpha, \epsilon) \implies d_{FS\text{-fol}}(c', c) < (\alpha, \delta);$$

(ii) For $\alpha > 0$, $\delta > 0$ there is $\epsilon > 0$ such that for all $c, c', c'' \in FS$

$$d_{FS\text{-fol}}(c, c'), d_{FS\text{-fol}}(c', c'') < (\alpha, \epsilon) \implies d_{FS\text{-fol}}(c, c'') < (2\alpha, \delta).$$

Proof. (i) Given $\alpha > 0$, $\delta > 0$, we conclude from Lemma 2.1 (i), that there is $\epsilon > 0$ such that $d_{FS}(\Phi_t(c), \Phi_t(c')) < \delta$, whenever $t \in [-\alpha, \alpha]$ and $d_{FS}(c, c') < \epsilon$. For (i), suppose now $d_{FS\text{-fol}}(c, c') < (\alpha, \epsilon)$. Then there is $t \in [-\alpha, \alpha]$ with $d_{FS}(\Phi_t(c), c') < \epsilon$. This implies $d_{FS}(c, \Phi_{-t}(c')) = d_{FS}(\Phi_{-t}(\Phi_t(c)), \Phi_{-t}(c')) < \delta$ and therefore $d_{FS\text{-fol}}(c', c) < (\alpha, \delta)$.

(ii) Given $\alpha > 0$, $\delta > 0$, we find again $\epsilon > 0$ as in (i). If $d_{FS\text{-fol}}(c, c'), d_{FS\text{-fol}}(c', c'') < (\alpha, \epsilon)$, then there exists $t, t' \in [-\alpha, \alpha]$ satisfying $d_{FS}(\Phi_t(c), c'), d_{FS}(\Phi_{t'}(c), c'') < \epsilon$. This yields $d_{FS}(\Phi_{t+t'}(c), c'') \le d_{FS}(\Phi_{t'}(\Phi_t(c)), \Phi_{t'}(c')) + d_{FS}(\Phi_{t'}(c'), c'') < \delta + \epsilon$. Hence $d_{FS\text{-fol}}(c, c'') < (2\alpha, \delta + \epsilon)$. Using $\delta' := \delta/2$ in place of δ and asking in addition for $\epsilon < \delta/2$ gives (ii). □

2.D. **The groups K_c and V_c.** For $c \in FS(X)$ we set

(2.6a) $K_c := G_c = \{g \in G \mid gc = c\} = \{g \in G \mid gc(t) = c(t) \text{ for all } t \in \mathbb{R}\}$;

(2.6b) $V_c := \{g \in G \mid \exists t \in \mathbb{R} : gc = \Phi_t(c)\}$;

(2.6c) $\tau_c := \inf\{t > 0 \mid \exists v \in V_c \setminus K_c, \text{with } \Phi_t(c) = vc\}$.

We use $\inf \emptyset = \infty$. If $\tau_c < \infty$ then we say that c is *periodic*. We have $K_c \subseteq V_c$ as the flow is G-equivariant.

Assumption 2.7. *There exists $FS_0 \subseteq FS$ compact such that*

(2.7a) $G \cdot FS_0 = FS$;

(2.7b) *for $\ell > 0$ and $c_0 \in FS_0$ there exists an open neighborhood U of c_0 in FS_0 such that for all $c \in U$ with $\tau_c \leq \ell$ we have $V_c \subseteq V_{c_0}$.*

Lemma 2.8. *Let $c \in FS$ be periodic. Then $c \in FS_\infty$. Moreover, there is $v \in V_c$ with $vc = \Phi_{\tau_c}(c)$ and any such v together with K_c generates V_c.*

Proof. Consider $c \notin FS_\infty$. Then c is constant on some interval $(-\infty, a_-)$ or (a_+, ∞). This implies that for all $v \in V_c$ there is t with $vc(t) = c(t)$. In particular, v fixes an endpoint of the image of c which is a finite geodesic or a geodesic ray. Since v also fixes the image of c as a set, it must in fact fix c. Hence $K_c = V_c$. If c is periodic we have $K_c \subsetneq V_c$ and so we must have $c \in FS_\infty$.

In particular, $t \mapsto \Phi_t(c)$ is injective. It follows that for $v \in V_c$ there is a unique $t_v \in \mathbb{R}$ with $vc = \Phi_{t_v}(c)$. We obtain a group homomorphism $V_c \to \mathbb{R}$, $v \mapsto t_v$ whose kernel is K_c. Denote the image of this homomorphism by Γ_c. We claim that Γ_c is discrete.

To prove this claim, suppose that there are $(v_n)_{n\in\mathbb{N}}$ with $t_{v_n} \to 0$ as $n \to \infty$. As $c \in FS_\infty$, $c\colon \mathbb{R} \to X$ is injective. Now for any $s \in \mathbb{R}$ we have $v_n(c(s)) = (v_n c)(s) = (\Phi_{t_n}(c)) = c(s + t_n) \to c(s)$, so $c(s)$ is an accumulation point of its G-orbit. But $G \curvearrowright X$ is smooth and proper, so orbits are discrete. Thus Γ_c is discrete.

Thus $\tau_c = \min \Gamma_c \cap \mathbb{R}_{>0}$ and there is $v \in V_c$ with $t_v = \tau_c$. Moreover, Γ_c is infinite cyclic and generated by τ_c. Thus any $v \in V_c$ with $t_v = \tau_c$ will together with K_c generate V_c. □

3. Triangle inequalities for $d_{V\text{-fol}}$ and $d_{J\text{-fol}}$

We have already proven a version of the triangle inequality for $d_{FS\text{-fol}}$ on the flow space FS in Lemma 2.5. We will need versions of the triangle inequality for $d_{V\text{-fol}}$ and $d_{J\text{-fol}}$ as well.

Lemma 3.1.

(3.1a) *Let $M \subseteq G$ be compact. For any $\epsilon > 0$ there is $\delta > 0$ such that for all $g, g' \in G$, $v \in M$ we have*

$$d_G(g, g') < \delta \implies d_G(gv, g'v) < \epsilon;$$

(3.1b) *Let $\alpha \geq 0$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that for any closed subgroup V of G and $g, g', g'' \in G$ we have*

$$d_{V\text{-fol}}(g, g'), d_{V\text{-fol}}(g', g'') \leq (\alpha, \delta) \quad \implies \quad d_{V\text{-fol}}(g, g'') \leq (2\alpha, \epsilon).$$

Proof. Assume the first statement fails for given $M \subseteq G$ compact and $\epsilon > 0$. Then there are sequences $g_n, g'_n \in G$, $v_n \in M$ with $d_G(g_n, g'_n) < 1/n$ and $d_G(g_n v_n, g'_n v_n) \geq \epsilon$. Let $x_n := g_n^{-1} g'_n$. Then $d_G(e, x_n) = d_G(g_n, g'_n) < 1/n$ and thus $\lim_{n\to\infty} x_n = e$. By passing to a subsequence, we can arrange $\lim_{n\to\infty} v_n = v$ for some $v \in M$. This implies $\lim_{n\to\infty} v_n^{-1} x_n v_n = v^{-1} e v = e$. Hence $\lim_{n\to\infty} d_G(g_n v_n, g'_n v_n) = \lim_{n\to\infty} d_G(e, v_n^{-1} x_n v_n) = 0$, a contradiction.

For the second statement, let $\alpha > 0$ and $\epsilon > 0$ be given. Using the compactness of the closed α-ball in G we find, using (3.1a), $\delta > 0$ such that

$$d_G(g, g') < \delta, d_G(v, e) < \alpha \implies d_G(gv, g'v) < \epsilon/2.$$

After decreasing δ we may assume $\delta < \epsilon/2$. Now if $d_{V\text{-fol}}(g, g'), d_{V\text{-fol}}(g', g'') \leq (\alpha, \delta)$ then there are $v, v' \in G$ with $d_G(gv, g'), d_G(g'v', g'') < \delta$, $d_G(v, e), d_G(v', e) < \alpha$. Our choice of δ implies $d_G(gvv', g'v') < \epsilon/2$. Now $d_G(vv', e) \leq d_G(vv', v) + d_G(v, e) \leq 2\alpha$ and hence $d_G(gvv', g'') \leq d_G(gvv', g'v') + d_G(g'v', g'') < \epsilon/2 + \delta < \epsilon$. Thus $d_{V\text{-fol}}(g, g'') < (2\alpha, \epsilon)$. □

Lemma 3.2. *Let $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ be finite. Fix $\beta > 0$. Then for all $\eta' > 0$ there is $\eta > 0$ with the property that for every $\epsilon > 0$ and every y, y', y'' in $J^N_{\mathcal{V}}(G)^\wedge$ we have*

$$d_{J\text{-fol}}(y, y'), d_{J\text{-fol}}(y', y'') < (\beta, \eta, \epsilon) \quad \implies \quad d_{J\text{-fol}}(y, y'') < (2\beta, \eta', 2\epsilon).$$

Proof. We use $\eta' := \delta'$ from (3.1b) with $\delta := \eta$. Write

$$\begin{aligned} y &= [t_0(g_0, V_0), \dots, t_N(g_N, V_N)]; \\ y' &= [t'_0(g'_0, V'_0), \dots, t'_N(g'_N, V'_N)]; \\ y'' &= [t''_0(g''_0, V''_0), \dots, t''_N(g''_N, V''_N)]. \end{aligned}$$

As $|t_i - t'_i|, |t'_i - t''_i| < \epsilon$ for all i, we have $|t_i - t''_i| < 2\epsilon$ for all i.

Suppose that $\max\{t_i, t''_i\} \geq 2\epsilon$. As $|t_i - t'_i|, |t'_i - t''_i| < \epsilon$, this implies both $\max\{t_i, t'_i\} \geq \epsilon$ and $\max\{t'_i, t''_i\} \geq \epsilon$. Thus $V_i = V'_i = V''_i$ and $d_{V_i\text{-fol}}(g_i, g'_i) < (\beta, \eta)$ and $d_{V_i\text{-fol}}(g'_i, g''_i) < (\beta, \eta)$. Now (3.1b) gives $d_{V_i\text{-fol}}(g_i, g''_i) < (2\beta, \eta')$. □

4. Factorization over the flow space

In this section we prove our main Theorem 1.2 modulo two results about the flow space FS.

4.A. From X to FS.

Theorem 4.1. *For all $M \subseteq G$ compact there is $\alpha > 0$ with the following property. For all $\delta > 0$, $L > 0$ there exists $R > 0$ and a uniformly continuous map $f_0 \colon X \to FS$ such that*

- *(i) for $x \in B_{R+L}$, $g \in M$ we have $d_{FS\text{-fol}}(f_0(gx), gf_0(x)) < (\alpha, \delta)$;*
- *(ii) for $x \in B_{R+L}$, $R' \geq R$ we have $d_{FS\text{-fol}}(f_0(x), f_0(\pi_{R'}(x))) < (\alpha, \delta)$, where $\pi_{R'}$ denotes the radial projection onto B_{R+L};*

Remark 4.2. Using quantifiers the beginning of Theorem 4.1 reads as

$$\forall M \ \exists \alpha \ \forall \delta, L \ \exists R, f_0 \text{ such that} \dots$$

The proof of Theorem 4.1 is given at the end of Section 5.

4.B. From FS to $|J^N_{\mathcal{V}}(G)|^\wedge$.

Theorem 4.3. *Suppose that Assumption 2.7 holds.*
There is $N \in \mathbb{N}$ such that for any $\alpha > 0$ and any $\epsilon > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite such that for any $\eta > 0$ there are $\delta > 0$, $f_1 \colon FS \to J^N_{\mathcal{V}}(G)^\wedge$, satisfying the following properties.

- *(i) For $c, c' \in FS$ with $d_{FS\text{-fol}}(c, c') < (\alpha, \delta)$ we have $d_{J\text{-fol}}(f_1(c), f_1(c')) < (\beta, \eta, \epsilon)$;*
- *(ii) For $c \in FS$, $g \in G$ we have $d_{J\text{-fol}}(f_1(gc), gf_1(c)) < (\beta, \eta, \epsilon)$.*

Remark 4.4. Using quantifiers the beginning of Theorem 4.3 reads as

$$\exists N \ \forall \alpha, \epsilon \ \exists \beta, \mathcal{V} \ \forall \eta \ \exists \delta, f_1 \text{ such that} \dots$$

The proof of Theorem 4.3 (modulo three results proven later) is given at the end of Section 6.

4.C. Proof of Main Theorem using Theorems 4.1 and 4.3.

We restate our Main Theorem 1.2 from the introduction.

Theorem (Main Theorem)**.** *Suppose that Assumption 2.7 holds.*
There is $N \in \mathbb{N}$ such that for all $M \subseteq G$ compact and $\epsilon > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite with the following property: For all $\eta > 0$ and all $L > 0$ we find $R > 0$ and a (not necessarily continuous) map $f \colon X \to J^N_{\mathcal{V}}(G)^\wedge$ satisfying:

- *(i) for $x \in B_{R+L}$, $g \in M$ we have $d_{J\text{-fol}}(f(gx), gf(x)) < (\beta, \eta, \epsilon)$;*

(ii) *for* $x \in B_{R+L}$*,* $R' \geq R$ *we have* $d_{J\text{-fol}}(f(x), f(\pi_{R'}(x))) < (\beta, \eta, \epsilon)$*;*
(iii) *there is* $\rho > 0$ *such that for all* $x, x' \in X$ *with* $d_X(x, x') < \rho$ *we have* $d_{J\text{-fol}}(f(x), f(x')) < (\beta, \eta, \epsilon)$*.*

Proof of Main Theorem. Let N be the number from Theorem 4.3. Let $M \subseteq G$ be compact and $\epsilon > 0$. Theorem 4.1 gives us a number $\alpha > 0$. Theorem 4.3 gives us $\beta/2 > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite. Let $\eta > 0$ be given. Because of Lemma 3.2 we can find $0 < \eta_0 \leq \eta$ with the property that every y, y', y'' in $J^N_{\mathcal{F}}(G)^\wedge$ we have

(4.5a) $d_{J\text{-fol}}(y, y'), d_{J\text{-fol}}(y', y'') < (\beta/2, \eta_0, \epsilon/2) \implies d_{J\text{-fol}}(y, y'') < (\beta, \eta, \epsilon)$.

Theorem 4.3 gives us $\delta > 0$ and $f_1 \colon FS \to J^N_{\mathcal{V}}(G)^\wedge$ satisfying

(4.5b) if $c, c' \in FS$ fullfils $d_{FS\text{-fol}}(c, c') < (\alpha, \delta)$, then we have $d_{J\text{-fol}}(f_1(c), f_1(c')) < (\beta/2, \eta_0, \epsilon/2)$;
(4.5c) for $c \in FS$, $g \in G$ we have $d_{J\text{-fol}}(f_1(gc), gf_1(c)) < (\beta/2, \eta_0, \epsilon/2)$.

Let $L > 0$ be given. Theorem 4.1 gives us $R > 0$ and $f_0 \colon X \to FS$ uniformly continuous satisfying

(4.5d) for $x \in B_{R+L}$, $g \in M$ we have $d_{FS\text{-fol}}(f_0(gx), gf_0(x)) < (\alpha, \delta)$;
(4.5e) for $x \in B_{R+L}$, $R \leq t$ we have $d_{FS\text{-fol}}(f_0(x), f_0(\pi_t(x))) < (\alpha, \delta)$.

We set now $f := f_1 \circ f_0$. It remains to verify the three assertions from Theorem 1.2.

(i) Let $x \in B_{R+L}$, $g \in M$. Then $d_{FS\text{-fol}}(f_0(gx), gf_0(x)) < (\alpha, \delta)$ by (4.5d). Thus (4.5b) implies $d_{J\text{-fol}}(f_1(f_0(gx)), f_1(gf_0(x))) < (\beta/2, \eta_0, \epsilon/2)$. On the other hand (4.5c) implies $d_{J\text{-fol}}\big(f_1(gf_0(x)), gf_1(f_0(x))\big) < (\beta/2, \eta_0, \epsilon/2)$. Now we conclude

$$d_{J\text{-fol}}(f(gx), gf(x)) = d_{J\text{-fol}}\big(f_1(f_0(gx)), gf_1(f_0(x))\big) < (\beta, \eta, \epsilon)$$

from (4.5a).

(ii) Let $x \in B_{R+L}$, $R' \geq R$. Then $d_{FS\text{-fol}}(f_0(x), f_0(\pi_{R'}(x))) < (\alpha, \delta)$ by (4.5e). We conclude

$$\begin{aligned} d_{J\text{-fol}}(f(x), f(\pi_{R'}(x))) &= d_{J\text{-fol}}(f_1(f_0(x)), f_1(f_0(\pi_{R'}(x)))) \\ &< (\beta/2, \eta_0, \epsilon/2) \leq (\beta, \eta, \epsilon) \end{aligned}$$

from (4.5b).

(iii) Since f_0 is uniformly continuous there is $\rho > 0$ such that $d_{FS}(f_0(x), f_0(x')) < \delta$ (and in particular $d_{FS\text{-fol}}(f_0(x), f_0(x')) < (\alpha, \delta)$) for all $x, x' \in X$ with $d_X(x, x') < \rho$. Using (4.5b) we obtain

$$d_{J\text{-fol}}(f(x), f'(x)) < (\beta/2, \eta_0, \epsilon/2) \leq (\beta, \eta, \epsilon)$$

for all $x, x' \in X$ with $d_X(x, x') < \rho$. □

5. The map to the flow space

The proof of Theorem 4.1, given in this section, will follow closely arguments from similar results for actions of discrete groups in [3].

For $x, x' \in X$ we write $c_{x,x'} \in FS$ for the generalized geodesic from x to x', i.e., for the generalized geodesic characterized by

$$\begin{aligned} c_{x,x'}(t) &= x \quad t \in (-\infty, 0], \\ c_{x,x'}(t) &= x \quad t \in [d(x, x'), +\infty). \end{aligned}$$

Recall that we fixed a base point b and write B_R for the closed ball of radius R around b in X. Recall also that $\pi_R \colon X \to B_R$ denotes the radial projection.

Lemma 5.1. *The map* $X \to FS$*,* $x \mapsto c_{b,x}$ *is uniformly continuous.*

Proof. The map is continuous by [9, II.1.4 (1) on page 160] and [3, Proposition 1.7]. As it is equivariant for the cocompact actions of G, it is also uniformly continuous. (Alternatively, uniform continuity can also be checked directly without using the G-action.) □

Lemma 5.2. *For all $\delta > 0$ there is $\Delta > 0$ such that for all R', T with $R' \geq T + \Delta$, $x \in X$ we have*

$$d_{FS}(\Phi_T(c_{b,x}), \Phi_T(c_{b,\pi_{R'}(x)})) < \delta.$$

Proof. Choose $\Delta > 0$ such that $\int_\Delta^\infty \frac{s}{2e^{|s|}} ds < \delta$ holds. Let $x \in X$. For $s + T \leq R'$ we have

$$\big(\Phi_T(c_{b,x})\big)(s) = c_{b,x}(s+T) = c_{b,\pi_{R'}(x)}(s+T) = \big(\Phi_T(c_{b,\pi_{R'}(x)})\big)(s),$$

while for $s + T \geq R'$ we have

$$\begin{aligned} d_X\big(\big(\Phi_T(c_{b,x})\big)(s), \big(\Phi_T(c_{b,\pi_{R'}(x)})\big)(s)\big) \\ = d_X\big(c_{b,x}(s+T), c_{b,\pi_{R'}(x)}(s+T)\big) = s + T - R' = s - (R' - T). \end{aligned}$$

Thus, for $R' \geq T + \Delta$,

$$\begin{aligned} d_{FS}(\Phi_T(c_{b,x}), &\Phi_T(c_{b,\pi_{R'}(x)})) \\ &= \int_{-\infty}^{\infty} \frac{d_X\big(\big(\Phi_T(c_{b,x})\big)(s), \big(\Phi_T(c_{b,\pi_{R'}(x)})\big)(s)\big)}{2e^{|s|}} ds \\ &= \int_{R'-T}^{\infty} \frac{s - (R' - T)}{2e^{|s|}} ds \\ &\leq \int_{\Delta}^{\infty} \frac{s}{2e^{|s|}} ds \\ &\leq \delta. \end{aligned}$$

□

Lemma 5.3. *Let r', L, $\alpha > 0$, $r'' > \alpha$. Let $T := r' + r''$, $R := r'' + 2r' + \alpha$. Let $x, x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \alpha$, $d_X(x_1, x) \leq R + L$. Set $\tau := d_X(x, x_1) - d_X(x, x_2)$. Then for all $t \in [-r', r']$*

$$d_X(c_{x_1,x}(T + \tau + t), c_{x_2,x}(T + t)) \leq \frac{2\alpha(L + 2r' + 2\alpha)}{r''}.$$

Proof. This is an application of the CAT(0)-condition, see [3, Lemma 3.6(i)][4]. □

Lemma 5.4. *For all $\alpha > 0$, $\Delta > 0$, $L > 0$ and $\delta > 0$ there are $R > 0$, $0 \leq T \leq R - \Delta$ such that for all $x, x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \alpha$, $d_X(x_1, x) \leq R + L$ we have*

$$d_{FS\text{-fol}}(\Phi_T(c_{x_1,x}), \Phi_T(c_{x_2,x})) < (\alpha, \delta).$$

Proof. Pick $1 > \delta' > 0, r' > \Delta, r'' > \alpha$ such that

$$\int_{-\infty}^{-r'} \frac{2|t| + 1}{2e^{|t|}} dt < \frac{\delta}{3}, \quad \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt < \frac{\delta}{3} \quad \text{and} \quad \frac{2\alpha(L + 2r' + 2\alpha)}{r''} < \delta'.$$

Set $R := 2r' + r'' + \alpha$ and $T := r' + r''$. Then $0 \leq T = R - r' - \alpha \leq R - \Delta$. Set $\tau := d_X(x, x_1) - d_X(x, x_2)$. Then $\tau \in [-\alpha, \alpha]$ as $d_X(x_1, x_2) \leq \alpha$. By Lemma 5.3 we have for all $t \in [-r', r']$

$$d_X(c_{x_1,x}(T + \tau + t), c_{x_2,x}(T + t)) < \delta'.$$

[4]Strictly speaking, this reference gives $d_X(c_{x_1,x}(T + t), c_{x_2,x}(T + t + \tau)) \leq \frac{2\alpha(L+2r'+\alpha)}{r''}$. The statement is not quite symmetric in x_1 and x_2 as we only know $d_X(x, x_2) \leq d_X(x, x_1) + d_X(x_1, x_2) \leq R + L + \alpha$. Thus in applying [3, Lemma 3.6(i)] we need to use $L' := L + \alpha$. In any event, the exact estimate is not important.

For $t \in (-\infty, -r']$ we have

$$\begin{aligned} & d_X(c_{x_1,x}(T+\tau+t), c_{x_2,x}(T+t)) \\ & \qquad \leq \quad d_X(c_{x_1,x}(T+\tau+t), c_{x_1,x}(T+\tau-r')) \\ & \qquad \quad + \quad d_X(c_{x_1,x}(T+\tau-r'), c_{x_2,x}(T-r')) \\ & \qquad \quad + \quad d_X(c_{x_2,x}(T-r'), c_{x_2,x}(T+t)) \\ & \qquad < \quad |t+r'| + \delta' + |t+r'| \\ & \qquad = \quad 2|t+r'| + \delta' \leq 2|t| + \delta' < 2|t| + 1. \end{aligned}$$

Similarly, for $t \in [r', \infty)$ we have

$$d_X(c_{x_1,x}(T+t), c_{x_2,x}(T+\tau+t)) < 2|t| + 1.$$

We can now estimate

$$\begin{aligned} & d_{FS}\big(\Phi_{T+\tau}(c_{x_1,x}), \Phi_T(c_{x_2,x})\big) \\ & \quad = \int_{-\infty}^{\infty} \frac{d_X(c_{x_1,x}(T+\tau+t), c_{x_2,x}(T+t))}{2e^{|t|}} dt \\ & \quad < \int_{-\infty}^{-r'} \frac{2|t|+1}{2e^{|t|}} dt + \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt + \int_{r'}^{+\infty} \frac{2|t|+1}{2e^{|t|}} dt \\ & \quad < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} \quad = \quad \delta. \end{aligned}$$

Therefore $d_{FS\text{-fol}}(\Phi_T(c_{x_1,x}), \Phi_T(c_{x_2,x})) < (\alpha, \delta)$. □

Proof of Theorem 4.1. Let $M \subseteq G$ be compact. By compactness of $M \cdot b$ there is $\alpha > 0$ such that $d_X(b, gb) \leq \alpha$ for all $g \in M$. Let $\delta > 0$, $L > 0$ be given. Let Δ be the number from Lemma 5.2. Let $R > 0$ and $0 \leq T \leq R - \Delta$ be the numbers from Lemma 5.4. Let $f_0 \colon X \to FS$ be the map $x \mapsto \Phi_T(c_{b,x})$. It is uniformly continuous by Lemma 2.1 and 5.1. We can now verify the two assertions from Theorem 4.1.

(i) Let $g \in M$ and $x \in B_{R+L}$. Then $f_0(gx) = \Phi_T(c_{gb,gx})$ and $gf_0(x) = \Phi_T(c_{b,gx})$. As $g \in M$, we have $d_X(b, gb) < \alpha$. Also $d_X(gb, gx) = d_X(b, x) \leq R + L$. We can therefore apply Lemma 5.4 and conclude

$$d_{FS\text{-fol}}(f_0(gx), gf_0(x)) = d_{FS\text{-fol}}(\Phi_T(c_{gb,gx}), \Phi_T(c_{b,gx})) < (\alpha, \delta).$$

(ii) Let $x \in B_{R+L}$ and $R' \geq R$. Then $f_0(x) = \Phi_T(c_{b,x})$ and $f_0(\pi_{R'}(x)) = \Phi_T(c_{b,\pi_{R'}(x)})$. As $T \leq R - \Delta$ we have $R' \geq R \geq T + \Delta$. We can therefore apply Lemma 5.2 and conclude $d_{FS}\big(f_0(x), f_0(\pi_{R'}(x))\big) = d_{FS}(\Phi_T(c_{b,x}), \Phi_T(c_{b,\pi_{R'}(x)})) < \delta$. In particular we get

$$d_{FS\text{-fol}}\big(f_0(x), f_0(\pi_{R'}(x))\big) < (\alpha, \delta).$$

□

6. Three properties of the flow space

Recall from Subsection 4.c and Section 5 that to prove our main Theorem 1.2 it remains to prove Theorem 4.3 about maps $FS \to J^N_{\mathcal{V}}(G)$. In this section we formulate three propositions about the flow space and use them to prove Theorem 4.3. These three propositions will be proved in the forthcoming sections.

6.A. Long thin covers.

Definition 6.1 (α-long cover)**.** A cover $\mathcal{U}$ of the flow space FS is said to be *α-long* if for any $c_0 \in FS$ there exists $U \in \mathcal{U}$ such that $\Phi_{[-\alpha,\alpha]}(c_0) \subseteq U$.

Typically such covers are thin in directions transversal to the flow and are often referred to as long thin covers.

Remark 6.2 (Discrete versus totally disconnected)**.** For a discrete group Γ it is possible to construct G-equivariant maps from the flow space FS to G-simplicial complexes (such as $E^N_{\mathcal{V}}(\Gamma)$) via the α-long $\mathcal{C}$vcy-covers of FS from [6, Sec. 5] or [15]. Here a $\mathcal{C}$vcy-cover consists of open subsets U of FS for which there is $V \in \mathcal{C}$vcy such that: $gU \cap U \neq \emptyset$ iff $g \in V$ and then $gU = U$. For a td-group G such U will not exist; elements of $g \in G \setminus V$ that are close to V will typically produce $gU \cap U \neq \emptyset$. The way out is to consider $G \cdot U$. In the discrete case there is a map $G \cdot U \to G/V$ and this generalizes to the totally disconnected case. In fact we will even construct $U \to G$, but this will typically not be equivariant and create some (β,η)-errors. Some equalities from the case of discrete groups, we will here be replaced with foliated bounds for the distance. This applies for instance to the map $FS \to J^N_{\mathcal{V}}(G)^\wedge$.

Since G-action commutes with the flow, we get also a flow on $G\backslash FS$.

Lemma 6.3. *(i) Let $\mathcal{U}$ be a α-long cover of the flow space FS such that each element $U \in \mathcal{U}$ is G-invariant. Then $\mathcal{U}' := \{p(U) \mid U \in \mathcal{U}\}$ is a α-long covering of $G\backslash FS$;*

(ii) Let $\mathcal{V}$ be a α-long covering of $G\backslash FS$. Then $\mathcal{V}' = \{p^{-1}(V) \mid V \in \mathcal{V}$ is a α-long cover of the flow space FS such that each element $V \in \mathcal{U}$ is G-invariant.

Proof. This is obvious from the definitions. □

6.B. Partitions of unity for long thin covers.

Proposition 6.4 (Partition of unity)**.** *For all $\alpha > 0$, $\varepsilon > 0$, $N \in \mathbb{N}$ there is $\alpha' > 0$ such that the following holds. Let $\mathcal{U}$ be a α'-long cover of dimension $\leq N$ by G-invariant open subsets of FS. Then there exists a partition of unity $\{t_U \colon FS \to [0,1] \mid U \in \mathcal{U}\}$ subordinate to $\mathcal{U}$ and $\delta > 0$ such that*

(i) for $U \in \mathcal{U}$, $c, c' \in FS$ with $d_{FS\text{-fol}}(c,c') < (\alpha,\delta)$ we have

$$|t_U(c) - t_U(c')| < \varepsilon;$$

(ii) the t_U are G-invariant.

Proof. This follows from Lemma 6.3 and from Proposition 7.1 applied to $G\backslash FS$. □

Remark 6.5. Using quantifiers the beginning of Proposition reads as

$$\forall \alpha, \epsilon, N\ \exists \alpha'\ \forall \mathcal{U}\ \exists \{t_U\}, \delta \text{ such that} \ldots$$

6.C. Dimension of long thin covers.

Proposition 6.6 (Dimension of long thin covers)**.** *There is $N \in \mathbb{N}$ such that for any $\alpha' > 0$ there is α'' such that the following is true. Let $\mathcal{W}$ be an α''-long cover of FS by G-invariant open subsets. Then there exist collections $\mathcal{U}_0, \ldots, \mathcal{U}_N$ of open G-invariant subsets of FS such that*

(i) $\mathcal{U} := \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ is an α'-long cover of FS, in particular $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for $i \neq j$;

(ii) for each i the open sets in $\mathcal{U}_i$ are pairwise disjoint;

(iii) for each $U \in \mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ there is $W \in \mathcal{W}$ with $U \subseteq W$.

Proof. This follows from Lemma 6.3 and Proposition 8.1 applied to $G\backslash FS$. □

Remark 6.7. Using quantifiers the beginning of Proposition 6.6 reads as

$$\exists N\ \forall \alpha'\ \exists \alpha''\ \forall \mathcal{W}\ \exists \mathcal{U}_0, \ldots, \mathcal{U}_N \text{ such that} \ldots$$

6.D. **The local structure of FS.**

Proposition 6.8 (Local structure)**.** *Suppose that Assumption 2.7 holds.*
For all $\alpha'' > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite with the following property. For all $\eta > 0$ and all $c_0 \in FS$ there exist $U \subseteq FS$ open, $h\colon U \to G$, $V \in \mathcal{V}$ and $\delta'' > 0$ such that

(i) *for some neighborhood U_0 of the orbit Gc_0 we have $\Phi_{[-\alpha'',\alpha'']}(U_0) \subseteq U$;*
(ii) *U is G-invariant;*
(iii) *for $c, c' \in U$ we have*
$$d_{FS\text{-fol}}(c, c') < (\alpha'', \delta'') \implies d_{V\text{-fol}}(h(c), h(c')) < (\beta, \eta);$$
(iv) *for $c \in U$, $g \in G$ we have;*
$$d_{V\text{-fol}}(h(gc), gh(c)) < (\beta, \eta)$$

Proposition 6.8 is proven in Section 9.

Remark 6.9. Using quantifiers the beginning of Proposition 6.8 reads as

$$\forall \alpha''\ \exists \beta, \mathcal{V}\ \forall \eta, c_0\ \exists U, h, \delta'' \text{ such that} \ldots$$

Remark 6.10 (Failure of continuity)**.** The failure of continuity in Theorem 1.2 comes from the failure of continuity of h in Proposition 6.8. The action of G on the flow space FS is not free and so maps $h\colon U \to G$ defined on open subsets of the flow space are necessarily a bit pathological. The action of G on the flow space is proper and it seems possible to construct maps $U \to G/K(V)$ that are continuous where $K(V)$ is the maximal compact subgroup of V. It should then be possible to obtain continuous maps in Theorem 1.2 after one replaces $J^N_{\mathcal{V}}(G)^\wedge$ with $*_{i=0}^{n}\big(\coprod_{V \in \mathcal{V}} G/K(V)\big)$. We will not work this out in detail here.

6.E. **Proof of Theorem 4.3 using Propositions 6.4, 6.6 and 6.8.**

Proof of Theorem 4.3. We will use for N the number appearing in Proposition 6.6. Given $\alpha > 0$ and $\epsilon > 0$ Proposition 6.4 gives us a number α' and Proposition 6.6 gives us then a number $\alpha'' > 0$. We can assume $\alpha'' \geq \alpha$. Next Proposition 6.8 gives us a number $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite. Let now $\eta > 0$ be given. From Proposition 6.8 we get for every $c_0 \in FS$ an open subset $U(c_0) \subseteq FS$, an open neighborhood $U_0(c_0)$ of the orbit Gc_0, a map $h(c_0)\colon U(c_0) \to G$, an element $V(c_0) \in \mathcal{V}$ and $\delta''(c_0) > 0$ such that the assertions (i), (ii), (iii) and (iv) hold. Since G acts cocompactly on FS, we can find a finite subset $I \subseteq FS$ such that $FS = \bigcup_{c_0 \in I} U_0(c_0)$ holds. Define $\delta'' := \min\{\delta''(c_0) \mid c_0 \in I\}$ and $\mathcal{W} = \{U(c_0) \mid c_0 \in I\}$. Then $\mathcal{W}$ an α''-long cover of FS by G-invariant open subsets and comes for every $W \in \mathcal{W}$ with maps $h_W\colon W \to G$ and an element $V_W \in \mathcal{V}$ satisfying

(6.10a) for $W \in \mathcal{W}$ and $c, c' \in W$ we have
$$d_{FS\text{-fol}}(c, c') < (\alpha'', \delta'') \implies d_{V_W-\text{fol}}(h_W(c), h_W(c')) < (\beta, \eta);$$
(6.10b) for $W \in \mathcal{W}$, $c \in W$ and $g \in G$ we have
$$d_{V_W-\text{fol}}(h_W(gc), gh_W(c)) < (\beta, \eta).$$

We apply Proposition 6.6 to the cover $\mathcal{W}$ and obtain collections $\mathcal{U}_0, \ldots, \mathcal{U}_N$ of open G-invariant subsets of FS such that

(6.10c) $\mathcal{U} := \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ is an α'-long cover of FS;
(6.10d) for each i the open sets in $\mathcal{U}_i$ are pairwise disjoint;

(6.10e) for each $U \in \mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$, there is $W_U \in \mathcal{W}$ with $U \subseteq W_U$; we pick such a W_U for each U and set $V_U := V_{W_U}$, $h_U := h_{W_U}$.

Proposition 6.4 yields a G-invariant partition of unity $\{t_U \colon FS \to [0,1] \mid U \in \mathcal{U}\}$ subordinate to $\mathcal{U}$ and $\delta > 0$, such that

(6.10f) for $U \in \mathcal{U}$, $c, c' \in FS$ with $d_{FS\text{-fol}}(c,c') < (\alpha,\delta)$ we have

$$|t_U(c) - t_U(c')| < \epsilon.$$

We can assume $\delta \le \delta''$. We now define $f_1 \colon FS \to J^N_{\mathcal{V}}(G)^\wedge$ by

$$c \mapsto [t_{U_0}(c)(h_{U_0}(c), V_{U_0}), \ldots, t_{U_N}(c)(h_{U_N}(c), V_{U_N})],$$

where for $i = 0, \ldots, N$, $U_i \in \mathcal{U}_i$ is determined by $c \in U_i$. There is at most one such U_i by (6.10d). If there is no $U_i \in \mathcal{U}_i$ containing c, then $t_U(c) = 0$ for all $U \in \mathcal{U}_i$ and the choice of U_i is without effect. It remains to verify the two assertions of Theorem 4.3.

(i) Let $c, c' \in FS$ with $d_{FS\text{-fol}}(c,c') < (\alpha,\delta)$. We write

$$\begin{aligned} f_1(c) &= [t_{U_0}(c)(h_{U_0}(c), V_{U_0}), \ldots, t_{U_N}(c)(h_{U_N}(c'), V_{U_N})]; \\ f_1(c') &= [t_{U_0'}(c')(h_{U_0'}(c'), V_{U_0'}), \ldots, t_{U_N'}(c')(h_{U_N'}(c'), V_{U_N'})]. \end{aligned}$$

If $\max\{t_{U_i}(c), t_{U_i'}(c')\} \ge \epsilon$ then, by (6.10f), we necessarily have $U_i = U_i'$ and $|t_{U_i}(c) - t_{U_i'}(c')| < \epsilon$. Moreover, by (6.10a) $d_{V_{U_i}-\text{fol}}(h_{U_i}(c), h_{U_i'}(c')) < (\beta,\eta)$. If $\max\{t_{U_i}(c), t_{U_i'}(c')\} < \epsilon$, then $|t_{U_i}(c) - t_{U_i'}(c')| < \epsilon$, as $t_{U_i}(c), t_{U_i'}(c') \in [0,1]$. Thus $d_{J\text{-fol}}(f_1(c), f_1(c')) < (\beta,\eta,\epsilon)$.

(ii) Let $c \in FS$ and $g \in G$. We write

$$\begin{aligned} f_1(c) &= [t_{U_0}(c)(h_{U_0}(c), V_{U_0}), \ldots, t_{U_N}(c)(h_{U_N}(c'), V_{U_N})]; \\ f_1(gc) &= [t_{U_0'}(gc)(h_{U_0'}, (gc)V_{U_0'}), \ldots, t_{U_N'}(gc)(h_{U_N'}(gc), V_{U_N'})]. \end{aligned}$$

Then

$$g f_1(c) = [t_{U_0}(c)(g h_{U_0}(c), V_{U_0}), \ldots, t_{U_N}(c)(g h_{U_N}(c), V_{U_N'})].$$

As the U_i and the t_{U_i} are all G-invariant we have $U_i = U_i'$, $t_{U_i}(c) = t_{U_i'}(gc)$ for all i. Moreover, by (6.10b), $d_{V_{U_i}\text{-fol}}(h_{U_i}(gc), g h_{U_i}(c)) < (\beta,\eta)$. In particular, $d_{J\text{-fol}}(f_1(gc), g f_1(c)) < (\beta,\eta,\epsilon)$. ☐

In order to prove our main Theorem 1.2, it suffices to prove Propositions 6.4, 6.6 and 6.8. This we will do in the forthcoming sections.

7. Partition of unity

In this section we finish the proof Proposition 6.4 by proving Proposition 7.1 below.

Proposition 7.1 (Partition of unity). *Let Z be a compact metric space with a flow Φ. For all $\alpha > 0$, $\epsilon > 0$, $N \in \mathbb{N}$ there is $\widehat{\alpha} > 0$ such that the following holds. Let $\mathcal{U}$ be a $\widehat{\alpha}$-long cover of Z of dimension $\le N$ by open subsets. Then there exists a partition of unity $\{t_U \colon Z \to [0,1] \mid U \in \mathcal{U}\}$ subordinate to $\mathcal{U}$ and $\delta > 0$ such that for $U \in \mathcal{U}$, $z, z' \in Z$ with $d_{FS\text{-fol}}(z,z') < (\alpha,\delta)$ we have*

$$|t_U(z) - t_U(z')| < \varepsilon.$$

Remark 7.2. Using quantifiers the beginning of Proposition 7.1 reads as

$$\forall \alpha, \epsilon, N \ \exists \widehat{\alpha} \ \forall \mathcal{U} \ \exists \{t_U\}, \delta > 0 \ \forall U, z, z' \text{ we have} \ldots$$

Lemma 7.3. *Let $\widehat{\alpha} > 0$. Let $K \subseteq Z$ be compact and U be an open neighborhood of K. Then there exists a (continuous) map $f \colon Z \to [0,1]$ satisfying*

(i) $f|_K \equiv 1$;

(ii) $f|_{Z\setminus\Phi_{[-\widehat{\alpha},\widehat{\alpha}]}U} \equiv 0$;

(iii) *for* $s \in \mathbb{R}$, $z \in Z$ *we have* $|f(\Phi_s(z)) - f(z)| \leq \frac{|s|}{\widehat{\alpha}}$.

Proof. As Z is a metric space, there is $\varphi\colon Z \to [0,1]$ with $\varphi|_K \equiv 1$ and $\varphi|_{Z\setminus U} \equiv 0$. Define $F\colon Z \times \mathbb{R} \to [0,1]$ by

$$F(z,t) := \begin{cases} (1 - \frac{|t|}{\widehat{\alpha}})\varphi(\Phi_t(z)) & t \in [-\widehat{\alpha},\widehat{\alpha}]; \\ 0 & \text{else.} \end{cases}$$

Put

$$f(z) := \sup\{F(z,t) \mid t \in \mathbb{R}\} = \max\{F(z,t) \mid t \in [-\widehat{\alpha},\widehat{\alpha}]\}.$$

We verify that f is continuous. Assume it is not. Then there exist $\epsilon > 0$, $z \in Z$, and a sequence $(z_n)_{n\geq 0}$ in Z such that $z_n \to z$ in Z and $|f(z_n) - f(z)| > \epsilon$ holds for all $n \geq 0$. Choose $\tau_n \in [-\widehat{\alpha},\widehat{\alpha}]$ with $f(z_n) = F(z_n,\tau_n)$. We can arrange after passing to subsequences that there exists $\tau \in [-\widehat{\alpha},\widehat{\alpha}]$ satisfying $\lim_{n\to\infty} \tau_n = \tau$. Then $f(z_n) = F(z_n,\tau_n) \to F(z,\tau) \leq f(z)$. Hence there is a natural number N such that $f(z_n) < f(z)+\epsilon$ holds for $n \geq N$. On the other hand, choose $\tau' \in [-\widehat{\alpha},\widehat{\alpha}]$ with $f(z) = F(z,\tau')$. Then $f(z_n) \geq F(z_n,\tau') \to f(z)$ and hence there is a natural number N' such that $f(z_n) > f(z) - \epsilon$ for $n \geq N'$. This is a contradiction.

It remains to check (i), (ii) and (iii).

(i) Let $z \in K$. Then $F(z,0) = \varphi(z) = 1$. Thus $f(z) = 1$.

(ii) Let $z \in Z \setminus \Phi_{[-\widehat{\alpha},\widehat{\alpha}]}(U)$. Then $\Phi_t(z) \notin U$ for all $t \in [-\widehat{\alpha},\widehat{\alpha}]$. Thus $F(z,t) = 0$ for all $t \in [-\widehat{\alpha},\widehat{\alpha}]$. Therefore $f(z) = 0$.

(iii) Next we show for all $s,t \in \mathbb{R}$ and $z \in Z$.

$$\Big|F(\Phi_s(z),t) - F(z,t)\Big| \leq \frac{|s|}{\widehat{\alpha}}. \tag{7.4}$$

If s and $s+t$ belong to $[-\widehat{\alpha},\widehat{\alpha}]$, this follows from

$$\begin{aligned}
\big|F(\Phi_s(z),t) - F(z,t+s)\big| &= \left|(1 - \frac{|t|}{\widehat{\alpha}})\varphi(\Phi_t(\Phi_s(z))) - (1 - \frac{|t+s|}{\widehat{\alpha}})\varphi(\Phi_{t+s}(z))\right| \\
&= \left|(1 - \frac{|t|}{\widehat{\alpha}})\varphi(\Phi_{t+s}(z)) - (1 - \frac{|t+s|}{\widehat{\alpha}})\varphi(\Phi_{t+s}(z))\right| \\
&= \left|\frac{|t| - |t+s|}{\widehat{\alpha}}\varphi(\Phi_{t+s}(z))\right| \\
&= \frac{\big||t| - |t+s|\big|}{\widehat{\alpha}}\varphi(\Phi_{t+s}(z)) \\
&\leq \frac{\big||t| - |t+s|\big|}{\widehat{\alpha}} \\
&\leq \frac{|s|}{\widehat{\alpha}}.
\end{aligned}$$

Suppose that $s \in [-\widehat{\alpha},\widehat{\alpha}]$ and $(s+t) \notin [-\widehat{\alpha},\widehat{\alpha}]$. Then $F(z,t+s) = 0$ and $\widehat{\alpha} \leq |t+s| \leq |s| + [t]$. Now(7.4) follows from

$$\big|F(\Phi_s(z),t)\big| = (1 - \frac{|t|}{\widehat{\alpha}})\varphi(\Phi_t(z)) \leq \frac{\widehat{\alpha} - |t|}{\widehat{\alpha}} \leq \frac{|s|}{\widehat{\alpha}}.$$

Suppose that $(s+t) \in [-\widehat{\alpha},\widehat{\alpha}]$ and $t \notin [-\widehat{\alpha},\widehat{\alpha}]$. Then $F(\Phi_s(z),t) = 0$ and $\widehat{\alpha} \leq |t| = |(s+t) - s| \leq |s+t| + |s|$. Now (7.4) follows from

$$\big|F((z),s+t)\big| = (1 - \frac{|s+t|}{\widehat{\alpha}})\varphi(z) \leq \frac{\widehat{\alpha} - |s+t|}{\widehat{\alpha}} \leq \frac{|s|}{\widehat{\alpha}}.$$

If $(s+t) \notin [-\widehat{\alpha},\widehat{\alpha}]$ and $t \notin [-\widehat{\alpha},\widehat{\alpha}]$, then $F(\Phi_s(z),t) = F(z,t+s) = 0$ and hence (7.4) is true. This finishes the proof of (7.4).

From the definitions we conclude that there exists t_0 and $t_1 \in \mathbb{R}$ such that for all $t \in \mathbb{R}$ we have

$$f(\Phi_s(z)) = F(\Phi_s(z), t_0); \tag{7.5}$$
$$F(\Phi_s(z), t) \leq F(\Phi_s(z), t_0); \tag{7.6}$$
$$f(z) = F(z, t_1); \tag{7.7}$$
$$F(z, t) \leq F(z, t_1). \tag{7.8}$$

We estimate

$$f(z) \overset{(7.7)}{=} F(z, t_1) \overset{(7.4)}{\leq} F(\Phi_s(z), t_1 - s) + \frac{|s|}{\widehat{\alpha}} \overset{(7.6)}{\leq} F(\Phi_s(z), t_0) + \frac{|s|}{\widehat{\alpha}} \overset{(7.5)}{=} f(\Phi_s(z)) + \frac{|s|}{\widehat{\alpha}},$$

and

$$f(\Phi_s(z)) \overset{(7.5)}{=} F(\Phi_s(z), t_0) \overset{(7.4)}{\leq} F(z, t_0 + s) + \frac{|s|}{\widehat{\alpha}} \overset{(7.8)}{\leq} F(z, t_1) + \frac{|s|}{\widehat{\alpha}} \overset{(7.7)}{\leq} f(z) + \frac{|s|}{\widehat{\alpha}}.$$

This finishes the proof of Lemma 7.3. □

Proof of Proposition 7.1. Let α, ϵ and N be given. Pick $\widehat{\alpha} > 0$ such that

$$\frac{(2N + 3)\alpha}{\widehat{\alpha}} < \epsilon/2.$$

For $U \in \mathcal{U}$ let $U^{-\widehat{\alpha}} := \{z \in Z \mid \Phi_{[-\widehat{\alpha}, \widehat{\alpha}]}(z) \subseteq U\}$. As $\mathcal{U}$ is $\widehat{\alpha}$-long, we can find for every $z \in Z$ an open neighborhood $U_0(z)$ and an element $U(z) \in \mathcal{U}$ such that $U_0(z) \subseteq U^{-\widehat{\alpha}}$. Since Z is compact, we can find a finite subset $I \subseteq Z$ such that $Z = \bigcup_{z \in Z} U_0(z)$. By replacing $\mathcal{U}$ with $\{U(z) \mid z \in I\}$, we can arrange that both $\mathcal{U}$ and $\mathcal{U}^{-\widehat{\alpha}} := \{U^{-\widehat{\alpha}} \mid U \in \mathcal{U}\}$ are N-dimensional finite coverings of Z. As Z is compact we can find a Lebesgue number $\ell > 0$ for $\mathcal{U}$. Define a compact subset $K_U \subseteq U$ by $K_U = \{z \in Z \mid d_Z(z, Z \setminus U) \geq \ell\}$. Then $\{K_U \mid U \in \mathcal{U}\}$ covers Z. For each $U \in \mathcal{U}$ we now choose f_U as in Lemma 7.3, i.e., such that

(7.9a) $f_U|_{K_U} \equiv 1$;
(7.9b) $f_U|_{Z \setminus U} \equiv 0$;
(7.9c) for $\tau \in \mathbb{R}$, $z \in Z$ we have $|f_U(z) - f_U(\Phi_\tau(z))| \leq \frac{|\tau|}{\widehat{\alpha}}$.

As $\mathcal{U}$ is finite, we can normalize the f_U to obtain $t_U \colon Z \to [0, 1]$ with

$$t_U(z) := \frac{f_U(z)}{\sum_{U' \in \mathcal{U}} f_{U'}(z)}.$$

Then $\{t_U \mid U \in \mathcal{U}\}$ is a partition of unity subordinate to $\mathcal{U}$. For $\tau \in [-\alpha, \alpha]$ and $z \in Z$, we next want to estimate $t_U(z) - t_U(\Phi_\tau(z))$. We abbreviate $x_V := f_V(z)$, $x'_V := f_V(\Phi_\tau(u))$ for $V \in \mathcal{U}$. By (7.9a) $x_V = 1$, $x'_{V'} = 1$ for at least one V, V'. In particular $\sum_V x_V, \sum_V x'_V \geq 1$. By (7.9b) and since the dimension of $\mathcal{U}$ is at most N, we have $x_V \neq 0$ for at most $N + 1$ different $V \in \mathcal{V}$, and similarly for x'_V. By (7.9c) $|x_V - x'_V| \leq \frac{\tau}{\widehat{\alpha}}$. Using all this we compute for $\tau \in [-\alpha, \alpha]$ and $z \in Z$, where V and V' run through $\mathcal{U}$:

$$\begin{aligned}
&|t_U(z) - t_U(\Phi_\tau(z))| \\
&= \left| \frac{x_U}{\sum_V x_V} - \frac{x'_U}{\sum_{V'} x'_{V'}} \right| \\
&= \left| \frac{\sum_{V'} x_U x'_{V'}}{\sum_{V,V'} x_V x'_{V'}} - \frac{\sum_V x'_U x_V}{\sum_{V,V'} x_V x'_{V'}} \right| \\
&\leq \frac{\sum_V \left| x_U x'_V - x_U x_V + x_U x_V - x'_U x_V \right|}{\sum_{V,V'} x_V x'_{V'}} \\
&\leq \frac{\sum_V x_U \left| x'_V - x_V \right| + \left| x_U - x'_U \right| x_V}{\sum_{V,V'} x_V x'_{V'}} \\
&\leq \frac{\sum_V \left| x_V - x'_V \right|}{\sum_{V,V'} x_V x'_{V'}} + \frac{\left| x_U - x'_U \right|}{\sum_{V'} x'_{V'}} \\
&\leq \sum_V \left| x_V - x'_V \right| + \left| x_U - x'_U \right| \\
&\leq \big(\left| \{ V \in \mathcal{U} \mid x_V \neq 0 \} \right| + \left| \{ V \in \mathcal{U} \mid x'_V \neq 0 \} \right| + 1 \big) \cdot \max \{ \left| x_V - x'_V \right| \mid V, V' \in \mathcal{U} \} \\
&\leq (2(N+1)+1) \frac{\tau}{\widehat{\alpha}} < \frac{\epsilon}{2}.
\end{aligned}$$

As Z is compact the t_U are uniformly continuous. Since $\mathcal{U}$ is finite, there is $\delta > 0$ such that for $U \in \mathcal{U}$, $z, z' \in Z$, we have

$$d_Z(z, z') < \delta \implies |t_U(z) - t_U(z')| < \frac{\epsilon}{2}.$$

Thus

$$d_{FS\text{-fol}}(z, z') < (\alpha, \delta) \implies |t_U(z) - t_U(z')| < \epsilon. \qquad \square$$

8. Dimension of long thin covers

In this section we finish the proof of Proposition 6.6 by proving Proposition 8.1 below.

Proposition 8.1. *There is N such that for any $\alpha > 0$ there is $\widehat{\alpha} > 0$ such that the following is true. Let $\mathcal{W}$ be an $\widehat{\alpha}$-long cover of $G \backslash FS$ by open subsets. Then there exists collections $\mathcal{U}_0, \ldots, \mathcal{U}_N$ of open subsets of $G \backslash FS$ such that*

(i) $\mathcal{U} := \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ is an α-long cover of $G \backslash FS$;
(ii) for each i the open sets in $\mathcal{U}_i$ are pairwise disjoint;
(iii) for each $U \in \mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ there is $W \in \mathcal{W}$ with $U \subseteq W$.

Remark 8.2. Using quantifiers the beginning of Proposition 8.1 reads as

$$\exists N \; \forall \alpha \; \exists \widehat{\alpha} \; \forall \mathcal{W} \; \exists \mathcal{U}_0, \ldots, \mathcal{U}_N \text{ such that} \ldots$$

Proof of Proposition 8.1. This follows by combining Proposition 8.4 and Lemma 8.5 below. $\square$

Lemma 8.3. *$G \backslash FS$ is of finite dimension, compact and metrizable.*

Proof. Recall that G acts on FS cocompactly, isometrically, and properly and that FS is a proper metric space, see Lemmas 2.2, and Lemma 2.3. Hence $G \backslash FS$ is compact and metrizable. A formula for a metric is

$$d_{G \backslash FS}([c], [c']) = \min\{ d_{FS}(gc, c') \mid g \in G \}.$$

By [3, Prop. 2.9] the dimension of $FS \setminus FS^{\mathbb{R}}$ is finite. As $FS^{\mathbb{R}} \cong X$ is also finite dimensional, the sum theorem from dimension theory [12, Cor. 1.5.5] now implies that FS is finite dimensional. Another result from dimension theory [12, Thm. 1.12.7] asserts that for open maps $A \to B$ with discrete fibers the dimension of A and B agree[5]. The quotient map $FS \to G\backslash FS$ is open, but as the action of G on FS is not smooth, the fibers of the quotient map are not discrete. For $R > 0$ let FS_R be the subspace of FS consisting of all generalized geodesics $c\colon \mathbb{R} \to X$ that are locally constant on the complement of $[-R, R]$. As FS_R is a closed subspace of FS we have $\dim FS_R \leq \dim FS$. The action of G on FS_R is smooth, so $FS_R \to G\backslash FS_R$ has discrete fibers and $\dim G\backslash FS_R = \dim FS_R \leq \dim FS$ is finite. There is a canonical retract $p_R\colon FS \to FS_R$ that sends c to the restriction of c to $[-R, R]$, more precisely to the generalized geodesic that agrees with c on $[-R, R]$ and is locally constant on the complement. It is not difficult to check that the fibers of p_R are of uniformly bounded diameter ϵ_R with $\epsilon_R \to 0$ as $R \to \infty$. The fibers of the induced map $\overline{p}_R\colon G\backslash FS \to G\backslash FS_R$ have the same property. Write $\mathcal{W}_{R,\delta}$ for the open cover of $G\backslash FS_R$ by all balls of radius δ. We can refine $\mathcal{W}_{R,\delta}$ to an open cover $\mathcal{W}'_{R,\delta}$ of dimension $\leq \dim FS$, i.e., every point of $G\backslash FS_R$ is contained in at most $\dim FS + 1$ sets in $\mathcal{W}'_{R,\delta}$. Now let $\mathcal{U}$ an open cover of $G\backslash FS$. By compactness $\mathcal{U}$ has a positive Lebesgue number. We then find $\delta > 0$ and $R > 0$ such that the pull-back $\overline{p}_R^*(\mathcal{W}_{R,\delta})$ refines $\mathcal{U}$. It follows that $\overline{p}_R^*(\mathcal{W}'_{R,\delta})$ is a refinement of $\mathcal{U}$ of dimension $\leq \dim FS$. Thus $\dim G\backslash FS \leq \dim FS < \infty$. □

Proposition 8.4. *There is N such that for any $\beta > 0$ there is $\widehat{\alpha} > 0$ such that the following is true. Let $\mathcal{W}$ be an $\widehat{\alpha}$-long cover of $G\backslash FS$ by open subsets. Then there exists an open cover $\mathcal{V}$ of $G\backslash FS$ such that*

(i) $\mathcal{V}$ is an β-long;
(ii) $\dim \mathcal{V} \leq N$;
(iii) for each $U \in \mathcal{V}$ there is $W \in \mathcal{W}$ with $U \subseteq W$.

Proof. The main result from [16] almost gives this. More precisely Lemma 8.3 allows us to apply [16, Thm. 1.1] to $G\backslash FS$. Thus there exists N only depending on the dimension of $G\backslash FS$ such that for given $\beta > 0$ there exists a cover $\mathcal{V}$ of $G\backslash FS$ satisfying (i) and (ii).

We will argue below that the construction from [16] in fact also gives (iii).

We point out that we apply [16] to the quotient $G\backslash FS$ which no longer carries a group action. In [16] an equivariant situation is considered, but we use the special case of [16] where the group acting on the flow space is trivial.

Given $\beta > 0$, let $\hat{\alpha} := 20\beta$. Suppose that $\mathcal{W}$ is an $\hat{\alpha}$-long cover. Then for any $c \in G\backslash FS$ there is $W \in \mathcal{W}$ with $\Phi_{[-\hat{\alpha},\hat{\alpha}]}(c) \subseteq W$. As W is open there is $\delta > 0$ such that the δ-neighborhood of $\Phi_{[-\hat{\alpha},\hat{\alpha}]}(c)$ is still contained in W. As $G\backslash FS$ is compact we can choose $\delta > 0$ uniformly, that is for any $c \in G\backslash FS$ there is $W \in \mathcal{W}$ containing the δ-neighborhood of $\Phi_{[-\hat{\alpha},\hat{\alpha}]}(c)$.

The period of $c \in G\backslash FS$ is $\tau(c) := \inf\{t > 0 \mid \Phi_t(c) = c\}$. If $\Phi_t(c) \neq c$ for all $t > 0$, then $\tau(c) = \infty$. Let

$$\big(G\backslash FS\big)_{>\hat{\alpha}} := \{c \in G\backslash FS \mid \tau(c) \in (\hat{\alpha}, \infty)\}.$$

Let $\delta > 0$ be given. By [16, Thm. 5.3] there exists an β-long cover $\mathcal{V}_1$ of $\big(G\backslash FS\big)_{>\hat{\alpha}}$ of dimension $\leq N_1$, where N_1 depends only on the dimension of $G\backslash FS$. Moreover, as explained in the last line of the proof of [16, Thm. 5.3], every $V \in \mathcal{V}_1$ is contained in the δ-neighborhood of $\Phi_{[\hat{\alpha},\hat{\alpha}]}(c)$ for some c, and therefore in some $W \in \mathcal{W}$.

Next consider

$$\big(G\backslash FS\big)_{\leq\hat{\alpha}} := \{c \in G\backslash FS \mid \tau(c) \in [0, \hat{\alpha}]\}.$$

[5] Strictly speaking the two results cited from [12] are about inductive dimension, not covering dimension. However, it is not difficult to check that FS is separable, so there is no difference between covering dimension and inductive dimension [12, Thm. 1.7.7].

By [16, Lem. 7.6] there exists an open cover $\mathcal{V}_2$ of $\big(G\backslash FS\big)_{\leq\hat{\alpha}}$ of dimension $\leq N_2$, where N_2 again depends only on the dimension of of $G\backslash FS$. Moreover, for each c of period $\leq \hat{\alpha}$ there is $V \in \mathcal{U}_2$ with $\Phi_{\mathbb{R}}(c) \subseteq V$, and each $V \in \mathcal{V}_2$ is contained in the δ-neighborhood of $\Phi_{\mathbb{R}}(c)$. In fact, by construction, see the last line of the proof of [16, Lem. 7.6], c can here be chosen to be of period $\leq \hat{\alpha}$. In particular, V is contained in the δ-neighborhood of $\Phi_{[0,\hat{\alpha}]}(c) = \Phi_{\mathbb{R}}(c)$, and therefore in some $W \in \mathcal{W}$.

We have $G\backslash FS = \big(G\backslash FS\big)_{\leq\hat{\alpha}} \sqcup \big(G\backslash FS\big)_{>\hat{\alpha}}$, where the first set if closed by [16, Lem. 7.1] and the second consequently open. In particular, the $V \in \mathcal{V}_1$ are open in $G\backslash FS$. We can use [16, Lem. 2.7] to extend the $V \in \mathcal{V}_2$ to open subsets of $G\backslash FS$ while preserving the properties of $\mathcal{V}_2$. The union of the two covers is now the needed cover $\mathcal{V}$. □

Lemma 8.5. *Fix a number N. For any $\alpha > 0$ there is $\beta > 0$ with following property. Let $\mathcal{V}$ be an β-long cover of $G\backslash FS$. Assume that $\dim \mathcal{V} \leq N$. Then there exists collections $\mathcal{U}_0, \ldots, \mathcal{U}_N$ of open subsets of $G\backslash FS$ such that*

(i) $\mathcal{U} := \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ is an α-long cover of $G\backslash FS$;
(ii) for each i the open sets in $\mathcal{U}_i$ are pairwise disjoint;
(iii) for each $U \in \mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ there is $V \in \mathcal{V}$ with $U \subseteq V$.

The proof is not difficult. We translate between open covers and maps to simplicial complexes and for the latter we use barycentric subdivision.

Proof of Lemma 8.5. The metric on FS has the property that $d_{FS}(\Phi_t(c), c) \leq |t|$ for all $c \in FS$ and $t \in \mathbb{R}$. It is not difficult to check that there is metric $d_{G\backslash FS}$ with the same property. For $\lambda > 0$ we define a metric d_λ on $G\backslash FS$ as follows. For $c, c' \in G\backslash FS$ set

$$d_\lambda(c, c') := \inf \sum_{i=0}^{n} |t_i| + \lambda d_{G\backslash FS}(\Phi_{t_i}(c_i), c_{i+1})$$

where the infimum is taken over all finite sequences $c = c_0, \ldots, c_{n+1} = c'$, $t_0, \ldots, t_n \in \mathbb{R}$. Compactness of $G\backslash FS$ can be used to check that for an β-long cover $\mathcal{V}$ there is $\lambda > 0$ such that the Lebesgue number of $\mathcal{V}$ with respect to d_λ is $\geq \beta$. Let now Λ be the nerve of $\mathcal{V}$, i.e., the simplicial complex that has a vertex v_V for each $V \in \mathcal{V}$ and where $v_{V_0}, \ldots, v_{V_n}$ span a simplex iff $V_0 \cap \cdots \cap V_n \neq \emptyset$. The dimension of $\mathcal{V}$ is exactly the dimension of Λ. We equip $|\Lambda|$ with the l^1-metric d^1. There is now a map $f\colon G\backslash FS \to |\Lambda|$ satisfying

$$d_\lambda(c, c') \leq \frac{\beta}{4N} \implies d^1(f(c), f(c')) \leq \frac{16N^2}{\beta} d_\lambda(c, c'), \tag{8.6}$$

see [7, Prop. 5.3]. By its construction the map f has the following property: the preimage of the open star of v_U is exactly U. Let now Λ' be the barycentric subdivision of Λ. The vertices of Λ' correspond to the simplices of Λ. For $j = 0, \ldots, N$ let I_j be the set of simplices of Λ' to j-simplices of Λ and $\tilde{\mathcal{U}}_j$ be the collection of open stars around simplices in I_j. Then $\tilde{\mathcal{U}} := \tilde{\mathcal{U}}_0 \sqcup \cdots \sqcup \tilde{\mathcal{U}}_N$ is an open cover $|\Lambda| = |\Lambda'|$ of positive Lebesgue number L, where L depends only on the dimension of Λ. Moreover, for each j the open sets in $\tilde{\mathcal{U}}_j$ are pairwise disjoint. We now set $\mathcal{U}_j := f^*\tilde{\mathcal{U}}_j$ and $\mathcal{U} := f^*\mathcal{U}$. Using estimate 8.6 we see that the Lebesgue number of $\mathcal{U}$ with respect to d_Λ is at least $\min\{\frac{\beta}{4N}, \frac{L\beta}{16N^2}\}$. In particular, if we choose $\beta \geq \max\{4N\alpha, \frac{16N^2\alpha}{L}\}$, then the Lebesgue number of $\mathcal{U}$ with respect to d_λ is at least α. Thus $\mathcal{U}$ is α-long. Finally, each open star for Λ' is contained in an open star for Λ. Thus each U from $\mathcal{U}$ is contained in some V from $\mathcal{V}$. □

Remark 8.7. A more careful analysis of the arguments from [16] reveals that the constructions there also lead to coloured covers. This leads to a more direct proof of Proposition 8.1 and renders Lemma 8.5 superfluous.

9. Local structure

In section we will prove Proposition 6.8.

9.A. Neighborhoods in FS mapping to G.

Lemma 9.1. *Let $FS_0 \subseteq FS$ be compact. For all $\alpha > 0$ there is $\beta > 0$ such that for $g \in G$, $c \in FS_0$ we have*

$$d_{FS}(gc, c) < \alpha \quad \implies \quad d_G(g, e) < \beta.$$

Proof. Assume this fails for a given $\alpha > 0$. Then there are sequences $(c_n)_{n \geq 0}$ in FS_0, and $(g_n)_{n \geq 0}$ in G with $d_{FS}(g_n c_n, c_n) < \alpha$ but

$$\lim_{n \to \infty} d_G(g_n, e) = \infty.$$

After passing to a subsequence, we can assume $c_0 = \lim_{n \to \infty} c_n$ for some $c_0 \in FS_0$. We can choose a constant $C > 0$ such that $d_{FS}(c_n, c_0) = d_{FS}(g_n c_n, g_n c_0) \leq C$ for $n \geq 0$. Then the $g_n c_0$ are elements of the closed ball K of radius $\alpha + 2C$ around c_0. Since FS is a proper metric space by Lemma 2.2, K is compact. The set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is a compact subset of G by Lemma B.1 (iii) and contains the sequence $(g_n)_{n \geq 0}$. After passing to subsequences again, we can assume $\lim_{n \to \infty} g_n = g$ for some $g \in G$. Hence $\lim_{n \to \infty} d_G(g_n, e) = d_G(g, e)$. This contradicts $\lim_{n \to \infty} d_G(g_n, e) = \infty$. ☐

We have defined $U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ in Subsection 2.C and V_{c_0} in (2.6b).

Proposition 9.2. *Let $FS_0 \subseteq FS$ be compact. For all $\alpha > 0$ there is $\beta > 0$ such that the following is true: For all $\eta > 0$, $c_0 \in FS_0$, there are $\delta > 0$ and a (not necessarily continuous) map $h \colon G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0) \to G$ satisfying*

(i) for $c, c' \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ we have

$$d_{FS\text{-fol}}(c, c') < (\alpha, \delta) \quad \implies \quad d_{V_{c_0}-\mathrm{fol}}(h(c), h(c')) < (\beta, \eta);$$

(ii) for $g \in G$, $c \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ we have

$$d_{V_{c_0}-\mathrm{fol}}(h(gc), gh(c)) < (\beta, \eta).$$

Proof. Let $\alpha > 0$ be given. By Lemma 9.1 there is $\beta > 0$ such that for $g \in G$, $c \in FS_0$ we have

$$d_{FS}(gc, c) < 3\alpha \quad \implies \quad d_G(g, e) < \beta. \tag{9.3}$$

Next let $\eta > 0$ and $c_0 \in FS_0$ be given. For $n \in \mathbb{N}$ choose $h_n \colon G \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0) \to G$ such that $c \in h_n(c) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$ for all $c \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta_n}(c_0)$. We will show that for all sufficiently large n the map h_n satisfies (i) and (ii).

(i) Assume there are infinitely many n such that (i) fails. Then there is $I \subseteq \mathbb{N}$ infinite and for $n \in I$ there are $c_n, c'_n \in FS$ with $c_n \in h_n(c_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$, $c'_n \in h_n(c'_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$, $d_{FS\text{-fol}}(c_n, c'_n) < (\alpha, 1/n)$ such that

$$d_{V_{c_0}-\mathrm{fol}}(h_n(c_n), h_n(c'_n)) < (\beta, \eta)$$

fails. Lemma 2.5 implies that we can arrange by possibly replacing I by a smaller infinite subset that we have

$$d_{FS\text{-fol}}(h_n(c_n)c_0, h_n(c'_n)c_0) < (3\alpha, 1/n).$$

Then, with $a_n := h_n(c'_n)^{-1}h_n(c_n)$,

$$d_{FS\text{-fol}}(a_nc_0, c_0) < (3\alpha, 1/n).$$

We conclude from Lemma 2.4 (i) that a_nc_0 stays in some closed ball K around c_0 with respect to d_G. Since FS is a proper metric space, K is compact. Since FS is a proper G-space by Lemma 2.3, we conclude from Lemma B.1 (iii) that $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is a compact subset of G and contains the sequence $(a_n)_{n\geq 0}$. Hence we can arrange by passing to subsequences that $\lim_{n\to\infty} a_n = a$ holds in G for some $a \in G$.

We can choose $\tau_n \in [-3\alpha, 3\alpha]$ for $n \geq 0$ satisfying $d_{FS}(a_n\Phi_{\tau_n}(c_0), c_0) < 1/n$. This implies $\lim_{n\to\infty} a_n\Phi_{\tau_n}(c_0) = c_0$. By passing to subsequences again, we can arrange $\lim_{n\to\infty} \tau_n = \tau$ for some $\tau \in [-3\alpha, 3\alpha]$. We conclude $\lim_{n\to\infty} a_n\Phi_{\tau_n}(c_0) = a\Phi_\tau(c_0)$ from Lemma 2.1 (ii). This shows $a\Phi_\tau(c_0) = c_0$ and hence $a \in V_{c_0}$. As the flow is of at most unit speed, see Lemma 2.1 (i), and d_{FS} is left G-invariant, we get

$$d_{FS}(ac_0, c_0) = d_{FS}(ac_0, a\Phi_\tau(c_0)) = d_{FS}(c_0, \Phi_\tau(c_0)) \leq |\tau| \leq 3\alpha.$$

Hence $d_G(a, e) \leq \beta$ by 9.3. Since $\lim_{n\to\infty} a_n = a$ and $a \in V_{c_0}$ hold, there exists a natural number N such that $d_{V_{c_0}-\mathrm{fol}}(a_n, e) < (\beta, \eta)$ holds for $n \in I$ with $n \geq N$. Since D_G and hence $d_{V_{c_0}-\mathrm{fol}}$ are left G-invariant, we get $d_{V_{c_0}-\mathrm{fol}}(h_n(c_n), h_n(c'_n)) < (\beta, \eta)$ for $n \in I$ with $n \geq N$, a contradiction.

(ii) Assume there are infinitely many n such that (ii) fails. Then there is $I \subseteq \mathbb{N}$ infinite and for $n \in I$ there are $c_n \in FS$, $g_n \in G$ with $c_n \in h_n(c_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$, $g_nc_n \in h_n(g_nc_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$, such that

$$d_{V_{c_0}-\mathrm{fol}}(h_n(g_nc_n), g_nh_n(c_n)) < (\beta, \eta)$$

fails. Recall that $d_{FS\text{-fol}}$ is left G-invariant, see Lemma 2.4 (ii). Hence we get from $c_n \in h_n(c_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$ and $g_nc_n \in h_n(g_nc_n) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$ that $d_{FS}(c_n, h_n(c_n)c_0) \leq (\alpha, 1/n)$ and $d_{FS}(c_n, g_n^{-1}h_n(g_nc_n)c_0) \leq (\alpha, 1/n)$ holds for $n \in I$. Put $a_n := h_n(c_n)^{-1}g_n^{-1}h_n(g_nc_n)$. We conclude from Lemma 2.5 that we can arrange by replacing I by a possibly smaller infinite subset that

$$d_{FS\text{-fol}}(a_nc_0, c_0) < (2\alpha, 1/n)$$

holds for $n \in I$. We conclude from Lemma 2.1 (i), that a_nc_0 stays in the closed ball K of radius $2\alpha + 1$ around c. Since FS is a proper metric space, K is compact. Since FS is a proper G-space by Lemma 2.3, we conclude from Lemma B.1 (iii) that $g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is a compact subset of G and contains the sequence $(a_n)_{n\geq 0}$. Hence we can arrange by passing to subsequences that $\lim_{n\to\infty} a_n = a$ holds in G for some $a \in G$. There are $\tau_n \in [-2\alpha, 2\alpha]$ with $d_{FS}(a_n\Phi_{\tau_n}(c_0), c_0) < 1/n$. This implies $\lim_{n\to\infty} a_n\Phi_{\tau_n}(c_0) = c_0$. We can arrange by passing to subsequences $\lim_{n\to\infty} \tau_n = \tau$ for some $\tau \in [-2\alpha, 2\alpha]$. We conclude $\lim_{n\to\infty} a_n\Phi_{\tau_n}(c_0) = a\Phi_\tau(c_0)$ from Lemma 2.1 (ii). This shows $a\Phi_\tau(c_0) = c_0$ and hence $a \in V_{c_0}$. As the flow is of at most unit speed, see Lemma 2.1 (i), and d_{FS} is left G-invariant, we get

$$d_{FS}(ac_0, c_0) = d_{FS}(ac_0, a\Phi_\tau(c_0)) = d_{FS}(c_0, \Phi_\tau(c_0)) \leq |\tau| \leq 2\alpha.$$

Hence $d_G(a, e) \leq \beta$ by 9.3. Since $\lim_{n\to\infty} a_n = a$ and $a \in V_{c_0}$ hold, there is a natural number N such that $d_{V_{c_0}-\mathrm{fol}}(a_n, e) < (\beta, \eta)$ holds for $n \in I$ with $n \geq N$. As d_G and hence $d_{V_{c_0}-\mathrm{fol}}$ are left G-invariant, we get $d_{V_{c_0}-\mathrm{fol}}(h_n(g_nc_n), g_nh_n(c_n)) < (\beta, \eta)$ for $n \in I$ with $n \geq N$, a contradiction. ☐

The following addendum to Proposition 9.2 strengthens the conclusion in the case where the period τ_{c_0} of c_0 defined in (2.6c) is large relative to the given α. Recall that we have defined the compact subgroup $K_{c_0} \subset G$ to be isotropy group

G_{c_0} of $c_0 \in FS$ in (2.6a). The difference to Proposition 9.2 is that in the two conclusions $d_{K_{c_0}-\mathrm{fol}}$ is used, not $d_{V_{c_0}-\mathrm{fol}}$.

Addendum 9.4. Let $FS_0 \subseteq FS$ be compact. For all $\alpha > 0$ there are $\beta > 0$, $\ell > 0$ such that the following is true. For all $\eta > 0$, $c_0 \in FS_0$ with $\tau_{c_0} > \ell$ there are $\delta > 0$, $h\colon G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0) \to G$ satisfying the following.

(i) for $c, c' \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ we have

$$d_{FS\text{-fol}}(c,c') < (\alpha,\delta) \quad \Longrightarrow \quad d_{K_{c_0}-\mathrm{fol}}(h(c),h(c')) < (\beta,\eta);$$

(ii) for $g \in G$, $c \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ we have

$$d_{K_{c_0}-\mathrm{fol}}(h(gc), gh(c)) < (\beta,\eta).$$

Proof. We can argue almost exactly as in the proof of Proposition 9.2. For the element $a \in V_{c_0}$ produced in the proof of both conclusions there we also proved $d_{FS}(ac_0, c_0) \leq 3\alpha$. Thus if $\tau_{c_0} > \ell := 3\alpha$, then we must have $ac_0 = c_0$, i.e., $a \in K_{c_0}$ and the conclusions follow for $d_{K_{c_0}-\mathrm{fol}}$ in place of $d_{V_{c_0}-\mathrm{fol}}$. □

Remark 9.5 (The role of FS_0)**.** The construction of maps $h\colon U \to G$ will depend on a choice of base point for the orbit Gc_0, namely c_0. The same construction with respect to a different base point g_0c_0 would also work, but with respect to a different collection of subgroups and constant β. But the subgroups and constant β in Proposition 6.8 are required to be uniform over all orbits. Therefore the base points for different orbits have to be chosen somewhat consistently; in our argument we have done this by using only base points from a fixed compact subset FS_0 of FS.

9.B. **Proof of Proposition 6.8.**

Lemma 9.6. *Let $c_0 \in FS$. Then there exists an open neighborhood U of c_0 in FS and a compact open subgroup K of G such that $K_c \subseteq K$ for all $c \in U$.*

Proof. Recall that G acts cocompactly, isometrically, properly, and smoothly on X. There is an open neighborhood $W \subseteq X$ of $c_0(0)$ such that $G_x \subseteq G_{c_0(0)}$ for all $x \in W$, see Lemma B.1 (iv). Now $K := G_{c_0(0)}$ and $U := \{c \in FS \mid c(0) \in W\}$ satisfy the assertion. □

The following proof of Proposition 6.8 is the only place where we use Assumption 2.7.

Proof. Let FS_0 be the compact subset of FS from Assumption 2.7. Given $\alpha > 0$ Proposition 9.2 and Addendum 9.4 provide us with numbers $\beta > 0$, $\ell > 0$.

Next we use (2.7b) and 9.6 to find $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite and a finite cover $\mathcal{W}$ of FS_0 such that for any $W \in \mathcal{W}$ there are $K_W, V_W \in \mathcal{C}\mathrm{vcy}$ satisfying

(9.7a) for all $c \in W$ we have $K_c \subseteq K_W$;
(9.7b) for all $c \in W$ with $0 < \tau_c \leq \ell$ we have $V_c \subseteq V_W$.

Let now $\eta > 0$ and $c_0 \in FS$ be given. According to (2.7a) FS_0 is a fundamental domain for the G action. This allows us to choose $g_0 \in G$, $W \in \mathcal{W}$ such that $g_0c_0 \in W \subseteq FS_0$. If $\tau_{c_0} = \tau_{g_0c_0} > \ell$, then we set $V := K_W$ and note that by (9.7a) $K_{g_0c_0} \subseteq V$. If $\tau_{c_0} = \tau_{g_0c_0} \leq \ell$ then we set $V := V_W$ and note that by (9.7b) $V_{g_0c_0} \subseteq V$. Now Proposition 9.2 and Addendum 9.4 give us $\delta > 0$ and $h\colon G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0) \to G$ satisfying for $c, c' \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$, $g \in G$

$$\begin{aligned} d_{V\text{-fol}}(h(c),h(c')) &< (\beta,\eta) \quad \text{provided } d_{FS\text{-fol}}(c,c') < (\alpha,\delta); \\ d_{V\text{-fol}}(h(gc),gh(c)) &< (\beta,\eta). \end{aligned}$$

Of course $U := G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ is G-invariant. □

This finishes the proof of our main Theorem 1.2.

Appendix A. The Bruhat-tits building for reductive p-adic groups

Let K be a non-Archimedian local field, i.e., a finite extension of the field of p-adic numbers or the field of formal Laurent series $k((t))$ over a finite field k. Consider an algebraic group G over K whose component of the identity is reductive. Let $G(K)$ be its group of K-points. We will simply say that $G(K)$ is a reductive p-adic group. We will need the action of $G(K)$ on the associated (extended) Bruhat-Tits building. The original reference for the Bruhat-Tits building is [10, 11]. Summaries of the construction can be found in [21] and in [20, Sec. I.1].

To set up notation we briefly review aspects of the construction. Let A be the real affine space constructed in [21, 1.2, p.31,32]. It comes equipped with an action of a subgroup $N(K)$ of $G(K)$. The affine space is finite dimensional and the action of $N(K)$ is cocompact[6]. There is also a collection Φ_{af} of affine linear function $\alpha\colon A \to \mathbb{R}$, these are the affine roots [21, 1.6, p.33]. This set is symmetric, i.e., if $\alpha \in \Phi_{\mathrm{af}}$ then $-\alpha \in \Phi_{\mathrm{af}}$. After identifying A with the associated linear space V the affine roots can be described as follows: there are finitely many linear function $a\colon V \to \mathbb{R}$ (the roots) and for each a there is a discrete set $\Gamma_a \subseteq \mathbb{R}$ such that the affine roots are the maps $v \mapsto a(v)+l$ where $l \in \Gamma_a$. (This follows from the discussion in [21, 1.6, p.33], see also [20, p. 103]). Associated to $\alpha \in \Phi_{\mathrm{af}}$ is the half-apartment $A_\alpha = \{x \in A \mid \alpha(x) \geq 0\}$ and the wall $\partial A_\alpha = \{x \in A \mid \alpha(a) = 0\}$. Chambers of A are the connected components of the complements of the walls [7]. The facets of the chambers are called the facets of A. The building X is constructed as a quotient of $G \times A$ [21, 2.1, p.43]. The quotient map $G(K) \times A \to X$ is $G(K)$-equivariant and the $G(K)$-action on X extends the $N(K)$-action on A. The translates of A under $G(K)$ are the apartments of X. As $N(K)$ acts cocompactly on A and since X is the union of its apartments the action of $G(K)$ on X is cocompact as well. The apartments gA inherit an affine structure and a partition into facets from A; these structures agree on intersections of apartments; any two points (in fact any two facets) of X are contained in a common apartment [21, 2.2.1, p.44]. Given two apartments A' and A'' there is $g \in G$ with $gA' = A''$ such that g fixes $A' \cap A''$ pointwise [21, 2.2.1]. This can be used to construct a $G(K)$-invariant CAT(0)-metric d_X on X [21, 2.3, p.45][8]. Apartments are then flat subspace of X. The action of $G(K)$ on X is also proper [21, p.45]. By our assumption on K its residue field (denoted $\overline{K}$ in [21]) is finite. This assumption is used in some of the following results from [21]. The stabilizer groups of chambers (and therefore of facets) contain the Iwahori subgroups [21, p.54] and these subgroups are open [21, p.55]. In particular, all stabilizer groups for facets are open and the action of $G(K)$ on X is smooth. The chambers of X can be subdivided to give X the structure of a locally finite simplicial complex where the action of $G(K)$ is simplicial [21, 2.3.1, p.45]. Altogether X is a finite dimensional CAT(0)-space with a a proper, continuous, isometric, smooth, cocompact $G(K)$-action. Assumption 2.7 for the action of $G(K)$ on X is verified in Proposition A.7 below.

Lemma A.1. *Any generalized geodesic $c\colon \mathbb{R} \to X$ is contained in a translate of A.*

Proof. We may assume $c(0) \in A$. For $n \in \mathbb{N}$ we find an apartment A' that contains $c(\pm n)$. By the construction of the metric A' will then contain $c([-n, n])$. Now choose $g_n \in G$ such that $g_n A = A'$ and g_n fixes $A \cap A'$. Then $(g_n)_{n \in \mathbb{N}}$ is a sequence

[6] This is not explicitly mentioned [21, 1.2, p.31,32] but follows from the construction.

[7] In the quasi-simple case the facets are simplices; in the semi-simple case the facets are poly-simplices (i.e., finite products of simplices); in general the facets are products of affine spaces with poly-simplices [21, 1.7, p.33].

[8] In [21] the terminology of CAT(0)-spaces is not used, but the inequality given there is equivalent to the CAT(0)-condition, see [9, p. 163].

in the compact subgroup of $G(K)$ that fixes $c(0)$ and has an accumulation point g. As the action of $G(K)$ on X is continuous the g_nA must agree with gA on larger and larger neighborhoods of $c(0)$. It follows that the apartment gA contains the image of c. □

In the following we write FS_∞ for the subspace of FS consisting of all (bi-infinite) geodesics $c\colon \mathbb{R} \to X$ and for $Y \subseteq X$ we set $FS_\infty(Y) := FS(Y) \cap FS_\infty$.

Lemma A.2. *Let $c_0 \in FS_\infty(A)$. Then there is $\epsilon > 0$ such that for all $\alpha \in \Phi_{af}$, with $c_0 \notin FS_\infty(A_\alpha)$ we have*

$$d_{FS}\big(c_0, FS_\infty(A_\alpha)\big) \geq \epsilon.$$

Proof. As there are only finitely many roots it suffices to consider the affine roots associated to a fixed root a. The walls associated to these affine roots are then all parallel and the half-apartments A_α are linearly ordered by inclusion (because Γ_a is discrete). If c_0 is not parallel to these walls, then no half-apartment A_α contains c_0 (or any geodesic parallel to c_0) and c_0 intersects all the walls ∂A_α in the same angle. It is then not difficult to bound $d_{FS}(c_0, FS_\infty(A_\alpha))$ in terms of this angle. If c_0 is parallel to the walls ∂A_α, then among the A_α not containing c_0 there is a maximal half-apartment A_{α_0} and we can use $\epsilon := d_{FS}(c_0, FS_\infty(A_{\alpha_0}))$. □

Lemma A.3. *Let $c_0 \in FS_\infty(A)$. Then there is $\epsilon > 0$ such that for all $g \in G$ we have*

$$d_{FS}\big(c_0, FS_\infty(A \cap gA)\big) \in \{0\} \cup (\epsilon, \infty).$$

Proof. The intersection $gA \cap A$ is a union of facets of A and convex. It follows that if c_0 is not contained in $gA \cap A$, then $gA \cap A$ is contained in a half-apartment A_α that does not contain c_0. The assertion follows now from Lemma A.2. □

Lemma A.4. *Let $c_0 \in FS_\infty(A)$. Then $Gc_0 \cap FS_\infty(A)$ is discrete.*

Proof. Choose real numbers $t_- < t_+$. As geodesics in A have unique extensions in A we observe the following: if $gc_0 \in FS_\infty(A)$ and $c_0(t_\pm) = gc_0(t_\pm)$, then $c_0 = gc_0$.

Let now $g_n \in G$ with $g_nc_0 \in FS_\infty(A)$ and $g_nc_0 \to c_1 \in FS_\infty(A)$ as $n \to \infty$. As the action of $G(K)$ on X is smooth all orbits for this action are discrete. Thus $g_nc_0(t_\pm) = c_1(t_\pm)$ for almost all n. The above observation now implies that g_nc_0 is eventually constant. Thus $Gc_0 \cap FS(A)$ is discrete, as asserted. □

Recall that an element c in FS is called periodic if there exists $g \in G$ and $t \in \mathbb{R}$ with $t > 0$, and $gc = \Phi_t(c)$. If c is periodic, then necessarily $c \in FS_\infty$.

Lemma A.5. *Let $c_0 \in FS_\infty(A)$. Let $\beta > 0$. Then there is $\epsilon > 0$ such that the following holds. Let $c \in FS_\infty(A)$ with $d_{FS}(c, c_0) < \epsilon$, $g \in G$, $t \in [-\beta, \beta]$ with $gc = \Phi_tc$. Then $g \in V_{c_0}$.*

Proof. Assume this fails. Then there are sequences $(c_n)_{n\geq 0}$ in $FS_\infty(A)$, $(t_n)_{n\geq 0}$ in $[-\beta, \beta]$, and $(g_n)_{n\geq 0}$ in $G(K)$ such that $\lim_{n\to\infty} c_n = c_0$, $g_nc_n = \Phi_{t_n}c_n$, but $g_n \notin V_{c_0}$. By passing to subsequences we can arrange $\lim_{n\to\infty} t_n = t$ for some $t \in [-\beta, \beta]$. Then we get $\lim_{n\to\infty} g_nc_n = \lim_{n\to\infty} \Phi_{t_n}(c_n) = \Phi_tc_0$ from Lemma 2.1 (ii). As $G \curvearrowright FS$ is proper, see Lemma 2.3, the g_n vary over a relatively compact set. Thus we can pass to a further subsequence and assume that $\lim_{n\to\infty} g_n = g$ for some $g \in G$. Then $\lim_{n\to\infty} g_nc_0 = gc_0$. As $G \curvearrowright FS$ is isometric we also have $\lim_{n\to\infty} g_nc_n = gc_0$. Thus $gc_0 = \Phi_t(c_0)$. We have $g_nc_n = \Phi_{t_n}c_n \in FS(A)$. Thus $c_n \in FS_\infty((g_n)^{-1}A)$. Lemma A.3 implies that $c_0 \in FS_\infty((g_n)^{-1}A)$ for almost all n. Thus $g_nc_0 \in FS(A)$ for almost all n. Now Lemma A.4 implies that $g_nc_0 = gc_0$ for almost all n. Thus $g_nc_0 = gc_0 = \Phi_t(c_0)$ for almost all n, contradicting $g_n \notin V_{c_0}$. □

Lemma A.6. *Let $c_0 \in FS_\infty(A)$. Let $\ell > 0$. Then there is $\epsilon > 0$ such that for all $c \in FS_\infty(A)$ with $d_{FS}(c, c_0) < \epsilon$ and $0 < \tau_c < \ell$ we have $V_c \subseteq V_{c_0}$.*

Recall that have defined the group K_c to be the $G(K)$-isotropy group of c in (2.6a).

Proof of Lemma A.6. Let $c \in FS(A)$ with $\tau_c > 0$, i.e., c is periodic. Lemma 2.8 tells us that there is $v \in V_c$ such that $\Phi_{\tau_c}(c) = vc$ and that v together with K_c generates V_c. The result follows therefore from Lemma A.5 □

Proposition A.7. *The action of $G(K)$ on $FS(X)$ satisfies Assumption 2.7.*

Proof. As discussed above the action of the subgroup $N(K)$ on A is cocompact. This implies that the action of $N(K)$ on $FS(A)$ is cocompact as well, see Lemma 2.3. Thus we find $FS_0 \subseteq FS(A)$ compact with $N(K) \cdot FS_0 = FS(A)$. By Lemma A.1 we have $G(K) \cdot FS(A) = FS$. So $G \cdot FS_0 = FS$, i.e., (2.7a) is satisfied.

Towards (2.7b), we first observe that $\tau_c < \infty$ implies $c \in FS_\infty$, see Lemma 2.8. Let now $c_0 \in FS_0 \subseteq FS(A)$ and $\ell > 0$ be given. We need to find an open neighborhood U of c_0 in FS_0 such that for all $c \in U$ with $\tau_c < \ell$ we have $V_c \subseteq V_{c_0}$. As $FS_\infty \subseteq FS$ is closed we can take $FS_0 \setminus FS_\infty$ if $c_0 \notin FS_\infty$. If $c_0 \in FS_\infty$, then Lemma A.6 provides a suitable ϵ-neighborhood. Thus (2.7b) is satisfied as well. □

Appendix B. Basics about group actions

A (continuous) map $f\colon X \to Y$ of (compactly generated topological) spaces is called *proper* if preimages of compact subsets are compact again. A G-space X is called *proper* if the map $\Theta_X^G\colon G \times X \to X \times X$, $(g, x) \mapsto (x, gx)$ is proper. It is called *smooth* if all isotropy group are open. It is called *cocompact* if the quotient space X/G is compact.

Lemma B.1. *Let G be a locally compact Hausdorff group and let X be a G-space.*

- *(i) The G-space X is proper if and only if for any $x \in X$ there is an open neighborhood U such that the subset $\{g \in G \mid g \cdot U \cap U \neq \emptyset\}$ of G is relatively compact, i.e., its closure in G is compact;*
- *(ii) The isotropy groups of a proper G-space are all compact. A G-CW-complex is proper if and only if all its isotropy groups are compact;*
- *(iii) If the G-space X is proper, then for every compact subset $K \subseteq X$ the subset $\{g \in G \mid gK \cap K \neq \emptyset\}$ of G is compact. The converse is true if X is locally compact;*
- *(iv) Let X be a metric space with isometric proper smooth G-action. Then for any $x \in X$ there exists $\epsilon > 0$ satisfying:*
  - *(a) $G_x = \{g \in G \mid g \cdot B_\epsilon(x) \cap B_\epsilon(x) \neq \emptyset\}$;*
  - *(b) The map*
  $$\alpha\colon G \times_{G_x} B_\epsilon(x) \xrightarrow{\cong} G \cdot B_\epsilon(x), \quad (g, y) \mapsto gy$$
  is a G-homeomorphism;
  - *(c) We have $G_y \subseteq G_x$ for every $y \in B_\epsilon(x)$.*
- *(v) Let X be a proper G-space. Let $A \subseteq X$ be a closed subspace. Let $G_A = \{g \in G \mid gA = A\}$. Then the G_A-space A is proper;*
- *(vi) Let H be a (closed) subgroup of G. If the G-space X is proper, then its restriction to an H-space is proper.*
- *(vii) Let $f\colon X \to Y$ be a proper G-map. If Y is proper, then X is proper;*
- *(viii) Let X be a metric space on which G acts isometrically and properly. Then X is smooth if and only if each orbit Gx (equipped with the subspace topology from X) is discrete;*

(ix) Suppose that X is a locally compact metric space on which G acts isometrically. Then X is proper and smooth if and only if each orbit Gx is discrete and each isotropy group G_x is compact.

(x) If the G-space X contains a compact subset C with $G \cdot C = X$, then X is cocompact.

If the G-space X is locally compact and cocompact, then there is a compact subset $C \subseteq X$ satisfying $G \cdot C = X$;

(xi) Let $f\colon X \to Y$ be a proper G-map. If Y is locally compact and cocompact, then X is cocompact.

Proof. (i) See [22, Proposition 3.21 in Chapter I on page 28].

(ii) Obviously the isotropy groups of a proper G-space are all compact. The claim about G-CW-complexes is proved in [17, Theorem 1.23 on page 18].

(iii) The set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is the image of $(\Theta_X^G)^{-1}(K \times K)$ under the projection $G \times X \to G$. Hence it is compact if X is proper. The converse follows from assertion (i).

(iv) Suppose assertion (iv)a is not true. Then there is an $x \in X$ such that for every $\epsilon > 0$ the isotropy group G_x is not equal to $\{g \in G \mid g \cdot B_\epsilon(x) \cap B_\epsilon(x) \neq \emptyset$. Hence we can choose a sequence of elements $(x_n)_{n \geq 0}$ in X and a sequence of elements $(g_n)_{n\geq 0}$ in G such that x_n and $g_n x_n$ belong to $B_{1/n}(x)$ and $g_n \notin G_x$ holds. Because of assertion (i) there is an open neighborhood U such that the subset $\overline{\{g \in G \mid g \cdot U \cap U \neq \emptyset\}}$ of G is compact. By passing to subsequences we can arrange $x_n \in U$ for $n \geq 0$. Then g_n belongs to the compact subset $\overline{\{g \in G \mid g \cdot U \cap U \neq \emptyset\}}$ of G. By passing to subsequences, we can arrange that there is an element $g \in G$ with $\lim_{n\to\infty} g_n = g$. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} g_n x_n = x$ holds and G acts isometrically, we conclude $\lim_{n\to\infty} g_n x = x$. This implies $g \in G_x$ because of $\lim_{n\to\infty} g_n = g$. Since G_x is open in G and $\lim_{n\to\infty} g_n = g$, we get for almost all g_n that $g_n = g$ and hence $g_n \in G_x$, a contradiction. This proves assertion (iv)a.

Next we show assertion (iv)b for the ϵ for which assertion (iv)a is true. Let $s\colon G/G_x \to G$ be a map of sets such that its composition with the projection $p\colon G \to G/G_x$ the identity on G/G_x. Since G/G_x is discrete, we obtain a homeomorphism

$$\beta\colon G/G_x \times B_\epsilon(x) \xrightarrow{\cong} G \times_{G_x} B_\epsilon(x), \quad (gG_x, y) \mapsto (s(gG_x), y).$$

Its inverse is given by $(g, y) \mapsto (gG_x, s(gG_x)^{-1}gy)$. The composite $\beta \circ \alpha$ is the map $G/G_x \times B_\epsilon(x) \to G \cdot B_\epsilon(x), (gG_x, y) \mapsto s(gG_x) \cdot y$ which is a homeomorphism since $G \cdot B_\epsilon(x)$ is the disjoint union of open subsets $\coprod_{gG_x \in G/G_x} s(gG_x) \cdot B_\epsilon(x)$ by assertion (iv)a. Hence assertion (iv)b is true.

Assertion (iv)c follows directly from assertion (iv)a.

(v) Let $f\colon X \to Y$ be a proper map and $B \subseteq Y$ closed. Consider any closed subspace $A \subseteq f^{-1}(B)$. Then the induced map $f|_A\colon A \to B$ is obviously proper. Now the claim follows from the commutativity of the following diagram

$$\begin{array}{ccc} G_B \times B & \xrightarrow{\Theta_A^{G_B}} & B \times B \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow[\Theta_X^G]{} & X \times X \end{array}$$

whose vertical arrows are the obvious inclusions.

(vi) This follows from the fact that the restriction of a proper map to a closed subspace is again proper.

(vii) Suppose that Y is proper. The following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\Theta_X^G} & X \times X \\ {\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f \times f} \\ G \times Y & \xrightarrow[\Theta_Y^G]{} & Y \times Y \end{array}$$

commutes. Since the lower horizontal arrow and the vertical arrows are proper, the upper arrow is proper, see [17, Lemma 1.16 on page 14], in other words, X is proper.

(viii) This follows from the fact that for a proper G-space the canonical map $G/G_x \to Gx$ is a homeomorphism for every $x \in X$, see [22, Proposition 3.19 (ii) in Chapter I on page 28] or [17, Lemma 1.19 (iii) on page 16].

(ix) Suppose that X is proper and smooth. Then each isotropy group is compact and each G-orbit is discrete by assertions (ii) and (viii).

Suppose that each each orbit Gx is discrete and each isotropy group G_x is compact. By assertion (viii) it suffices to show that the G-action is proper. Consider $x \in X$. Since Gx is discret and X locally compact, we can choose a compact neighborhood U of x in X satisfying $U \cap Gx = \{x\}$. It suffices to show that the subset $\{g \in G \mid gU \cap U \neq \emptyset\}$ of G is relative compact because of assertion (i). We can equip G with a left invariant proper metric, see [14]. It suffices to show that any sequence $(g_n)_{n\in\mathbb{N}}$ of element in G satisfying $g_n \cdot U \cap U \neq \emptyset$ has a convergent subsequence. Choose for each $n \geq 0$ elements u_n and u'_n in U such that $g_n u_n = u'_n$ holds. Since U is compact, we can arrange after passing to subsequences that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} u'_n = u'$ holds for appropriate elements $u, u' \in U$. Since G acts by isometries on X, we get $\lim_{n\to\infty} g_n u = u'$. As Gu is a discrete subspace of X, we can arrange after passing to subsequences that $g_n u = u'$ holds for $n \geq 0$. Hence $g_0^{-1} g_n u = u$ for all $n \geq 0$. Since G_u is compact, we can arrange after passing to subsequences that $g_0^{-1} g_n$ is a convergent sequence in G. This implies that g_n is a convergent sequence in G.

(x) Let $p\colon X \to X/G$ be the projection. If $C \subseteq X$ is a compact subset with $G \cdot C = X$, then $p(C) = X/G$ and hence X/G is compact.

Suppose that X is locally compact and X/G is compact. We can find for every $x \in X$ an open neighborhood $U(x)$ such that $\overline{U(x)}$ is compact. We obtain an open covering $\{p(U(x)) \mid x \in X\}$ of X/G. Since X/G is compact, we can find a finite subset $I \subseteq X$ with $\bigcup_{x\in I} p(U(x)) = X/G$. Then $C = \bigcup_{x\in I} \overline{U(x)}$ is a compact subset of X with $G \cdot C = X$.

(xi) Since Y is locally compact and cocompact, we can choose by assertion (x) a compact subset $C \subseteq Y$ with $G \cdot C = Y$. Since f is proper, $D := f^{-1}(C)$ is a compact subset of X. Since $G \cdot D = X$ holds, X/G is compact by assertion (x), in other words, X is cocompact. This finishes the proof of Lemma B.1. □

Example B.2 (Non-proper action)**.** The action of $\mathbb{Z}$ on S^1 by rotating through an irrational angle is free, isometric, and smooth, but not proper. All $\mathbb{Z}$-orbits are dense and not discrete. The canonical G-map map $G/G_x \to Gx$ is continuous and bijective, but not a homeomorphism.

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WWU Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany
Email address: a.bartels@wwu.de
URL: http://www.math.uni-muenster.de/u/bartelsa

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
Email address: wolfgang.lueck@him.uni-bonn.de
URL: http://www.him.uni-bonn.de/lueck