ON THE STABLE CANNON CONJECTURE

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Abstract. The Cannon Conjecture for a torsion-free hyperbolic group G with boundary homeomorphic to S^2 says that G is the fundamental group of an aspherical closed 3-manifold M. It is known that then M is a hyperbolic 3-manifold. We prove the stable version that for any closed manifold N of dimension greater or equal to 2 there exists a closed manifold M together with a simple homotopy equivalence $M \to N \times BG$. If N is aspherical and $\pi_1(N)$ satisfies the Farrell-Jones Conjecture, then M is unique up to homeomorphism.

0. Introduction

0.1. **The motivating conjectures by Wall and Cannon.** This paper is motivated by the following two conjectures which will be reviewed in Sections 1 and 2.

Conjecture 0.1 (A Conjecture on Poincaré duality groups and closed aspherical 3-manifolds by Wall)**.** *Every Poincaré duality group of dimension* 3 *is the fundamental group of an closed aspherical* 3*-manifold.*

Conjecture 0.2 (Cannon Conjecture in the torsion-free case)**.** *Let G be a torsion-free hyperbolic group. Suppose that its boundary is homeomorphic to S^2.*

Then G is the fundamental group of a closed hyperbolic 3*-manifold.*

We will investigate whether these conjectures are true stably. More precisely, we ask whether for any closed smooth manifold N of dimension ≥ 2 the product $BG \times N$ is simple homotopy equivalent to a closed smooth manifold. Notice that for a torsionfree hyperbolic group G there is a finite CW-complex model for BG by the Rips complex. The Whitehead group of G is known to be trivial, so the simple homotopy type of BG is well-defined. We will also consider the analogous questions in the in the PL and topological categories.

0.2. **The main results.** In the sequel $\underline{\mathbb{R}^a}$ denotes the trivial a-dimensional vector bundle.

Theorem 0.3 (Vanishing of the surgery obstruction)**.** *Let G be a hyperbolic* 3*-dimensional Poincaré duality group.*

Then there exist a closed smooth 3-manifold M and a normal map of degree one (in the sense of surgery theory)

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "f"] & BG \end{tikzcd}$$

satisfying

(1) *The space BG is a finite* 3*-dimensional CW-complex;*

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(2) *The map* $H_n(f;\mathbb{Z})\colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(BG;\mathbb{Z})$ *is bijective for all* $n \geq 0$*;*
(3) *The simple algebraic surgery obstruction* $\sigma(f,\overline{f}) \in L_3^s(\mathbb{Z}G)$ *vanishes.*

Notice that the vanishing of the surgery obstruction does not imply that we can arrange by surgery that f is a simple homotopy equivalence since this works only in dimensions ≥ 5. In dimension 3 we can achieve at least a $\mathbb{Z}G$-homology equivalence. See [29, Theorem 11.3A].

However, if we cross the normal map with a closed manifold N of dimension ≥ 2, the resulting normal map has also vanishing surgery obstruction by the product formula and hence can be transformed by surgery into a simple homotopy equivalence. Thus Theorem 0.3 implies assertion (1) of Theorem 0.4 below; the proof of assertion (2) of Theorem 0.4 below will require more work.

Theorem 0.4 (Stable Cannon Conjecture)**.** *Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2.*

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

satisfying

(1) *The map f is a simple homotopy equivalence;*
(2) *Let $\widehat{M} \to M$ be the G-covering associated to the composite of the isomorphism $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ with the projection $G \times \pi_1(N) \to G$. Suppose additionally that N is aspherical,* $\dim(N) \geq 3$*, and $\pi_1(N)$ is a Farrell-Jones group.*

   Then $\widehat{M}$ is homeomorphic to $\mathbb{R}^3 \times N$. Moreover, there is a compact topological manifold $\overline{\widehat{M}}$ whose interior is homeomorphic to $\widehat{M}$ and for which there exists a homeomorphism of pairs $(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$.

We call a group G a *Farrell-Jones-group* if it satisfies the Full Farrell-Jones Conjecture. We will review what is known about the class of Farrell-Jones groups in Theorem 4.1. For now, we mention that hyperbolic groups, CAT(0)-groups, and the fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary) are Farrell-Jones groups.

We have the following uniqueness statement.

Theorem 0.5 (Borel Conjecture)**.** *Let M_0 and M_1 be two closed aspherical manifolds of dimension n satisfying $\pi_1(M_0) \cong \pi_1(M_1)$. Suppose one of the following conditions hold:*

- *We have $n \leq 3$;*
- *We have $n = 4$ and $\pi_1(M_0)$ is a Farrell-Jones group which is good in the sense of Freedman* [28]*;*
- *We have $n \geq 5$ and $\pi_1(M_0)$ is a Farrell-Jones group.*

Then any map $f\colon M_0 \to M_1$ inducing an isomorphism of fundamental groups is homotopic to a homeomorphism.

Proof. The Borel Conjecture is true obviously in dimension $n \leq 1$. The Borel Conjecture is true in dimension 2 by the classification of closed manifolds of dimension 2. It is true in dimension 3 since Thurston's Geometrization Conjecture holds. This follows from results of Waldhausen (see Hempel [34, Lemma 10.1 and

Corollary 13.7]) and Turaev, see [60], as explained for instance in [40, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [39, 47] following ideas of Perelman. The Borel Conjecture follows from surgery theory in dimension ≥ 4, see for instance [4, Proposition 0.3]. □

One cannot replace homeomorphism by diffeomorphism in Theorem 0.5. The torus T^n for $n \geq 5$ is a counterexample, see [63, 15A]. Other counterexamples involving negatively curved manifolds are constructed by Farrell-Jones [25, Theorem 0.1].

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The paper is organized as follows:

Contents

1. Short review of Poincaré duality groups

Definition 1.1 (Poincaré duality group)**.** A *Poincaré duality group G of dimension* n is a group satisfying:

- G is of type FP, i.e. $\mathbb{Z}$ admits a finite resolution by finitely generated projective $\mathbb{Z}G$-modules;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

1.1. Basic facts about Poincaré duality groups.

- A Poincaré duality group is finitely generated and torsion free;
- For $n \geq 4$ there exist n-dimensional Poincaré duality groups which are not finitely presented, see [19, Theorem C];
- If G is a Poincaré duality group of dimension $n \geq 3$ then BG is a finitely dominated n-dimensional Poincaré complex in the sense of Wall [62] if and only if G is finitely presented, see [35, Theorem 1]. If $\widetilde{K}_0(\mathbb{Z}G)$ vanishes, then BG is homotopy equivalent to finite n-dimensional CW-complex, see [61, Theorem F];
- If G is the fundamental group of a closed aspherical manifold of dimension n, then BG is homotopy equivalent to a finite n-dimensional CW-complex and in particular G is finitely presented. In fact, every compact ENR of dimension $n > 2$ is homotopy equivalent to a finite n-dimensional polyhedron, see West [68];
- To our knowledge there exists in the literature no example of a 3-dimensional Poincaré duality group which is not homotopy equivalent to a finite 3-dimensional CW-complex;
- Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface. This result is due to Bieri, Eckmann and Linnell, see for instance, [23].

1.2. Some prominent conjectures and results about Poincaré duality groups.

Conjecture 1.2 (Poincaré duality groups and closed aspherical manifolds)**.** *Every finitely presented Poincaré duality group is the fundamental group of a closed aspherical topological manifold.*

A weaker version is

Conjecture 1.3 (Poincaré duality groups and closed aspherical ENR homology manifolds)**.** *Every finitely presented Poincaré duality group is the fundamental group of a closed aspherical* ENR *homology manifold.*

Michel Boileau has informed us about the following two facts:

Theorem 1.4. *A Poincaré duality group G of dimension* 3 *is the fundamental group of a closed aspherical* 3*-manifold if and only if G contains a subgroup H, which is the fundamental group of a closed aspherical* 3*-manifold.*

Proof. Let H be a subgroup of G which is the fundamental group of an irreducible closed 3-manifold. Suppose that the index of H in G is infinite. Then the cohomological dimension of H is smaller than the cohomological dimension of G by [59]. Since the cohomological dimension of both H and G is three, we get a contradiction. Hence the index of H in G is finite. The solution of Thurston's Geometrization Conjecture by Perelman, see [47], implies that G is the fundamental group of an irreducible closed 3-manifold, see for instance [30, Theorem 5.1]. Since a closed 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group, Lemma 1.4 follows. □

Moreover, Theorem 1.4 and the works of Cannon-Cooper [14], Eskin-Fisher-Whyte [24], Kapovich-Leeb [38], and Rieffel [55] imply

Theorem 1.5. *A Poincaré duality group G of dimension 3 is the fundamental group of a closed aspherical 3-manifold if and only if it is quasiisometric to the fundamental group of a closed aspherical 3-manifold.*

The next result is due to Bowditch [11, Corollary 0.5].

Theorem 1.6. *If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.*

The following result follows from the algebraic torus theorem of Dunwoody-Swenson [22].

Theorem 1.7. *Let G be a 3-dimensional Poincaré duality group. Then precisely one of the following statements are true:*

1. *It is the fundamental group of a closed Seifert 3-manifold;*
2. *It splits as an amalgam or HNN extension over a subgroup $\mathbb{Z} \oplus \mathbb{Z}$;*
3. *It is atoroidal, i.e., it contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

Conjecture 1.8 (Weak hyperbolization Conjecture)**.** *An atoroidal 3-dimensional Poincaré duality group is hyperbolic.*

The next result is due to Kapovich-Kleiner [37, Theorem 2].

Theorem 1.9. *A 3-dimensional Poincaré duality group which is a CAT(0)-group and atoroidal is hyperbolic.*

We conclude from [9, Theorem 2.8 and Remark 2.9].

Theorem 1.10. *Let G be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to S^2.*

1.3. **High-dimensions.**

Theorem 1.11 (Poincaré duality groups and ENR homology manifolds)**.** *Let G be a finitely presented torsion-free group which is a Farrell-Jones group.*

1. *Then for $n \geq 6$ the following are equivalent:*
   - (a) *G is a Poincaré duality group of dimension n;*
   - (b) *There exists a closed* ENR *homology manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$;*
2. *If the statements in assertion* (1) *hold, then the closed* ENR *homology manifold M appearing there can be arranged to have the DDP, see Definition 6.2;*
3. *If the statements in assertion* (1) *hold, then the closed* ENR *homology manifold M appearing there is unique up to s-cobordism of* ENR *homology manifolds;*

Proof. See Bartels-Lück-Weinberger [8, Theorem 1.2]. It relies strongly on the surgery theory for ENR homology manifolds, see for instance [13, 27, 49]. □

The question whether a closed ENR homology manifold, which has dimension ≥ 5 and has the DDP, is a topological manifold is decided by Quinn's obstruction, see Section 8.

More information about Poincaré duality groups can be found for instance [20] and [64].

2. Short review of the Cannon Conjecture

The following conjecture is taken from [15, Conjecture 5.1].

Conjecture 2.1 (Cannon Conjecture)**.** *Let G be a hyperbolic group. Suppose that its boundary is homeomorphic to S^2.*

Then G acts properly cocompactly and isometrically on the 3*-dimensional hyperbolic space.*

If G is torsion free, then the Cannon Conjecture 2.1 reduces to the Cannon Conjecture for torsion free groups 0.2.

Remark 2.2. We mention that Conjecture 0.2 is open and does not follow from Thurston's Geometrization Conjecture which is known to be true by the work of Perelman, see Morgan-Tian [47].

The next result is due to Bestvina-Mess [10, Theorem 4.1] and says that for the Cannon Conjecture one just has to find some closed aspherical 3-manifold with G as fundamental group.

Theorem 2.3. *Let G be a hyperbolic group which is the fundamental group of a closed aspherical* 3*-manifold M.*

Then the universal covering $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ and its compactification by ∂G is homeomorphic to D^3, and the Geometrization Conjecture of Thurston implies that M is hyperbolic and G satisfies the Cannon Conjecture 0.2.

Ursula Hamenstädt informed us that she has a proof for the following result.

Theorem 2.4 (Hamenstädt)**.** *Let G be a hyperbolic group G whose boundary is homeomorphic to S^{n-1}.*

Then G acts properly and cocompactly on $S^{n-1} \times \mathbb{R}^n$.

Hamenstädt's result is proved by completely different methods and does not need the assumption that G is torsion free. It aims for $n = 3$ at construction of the sphere tangent bundle of the universal covering of the conjectured hyperbolic 3-manifold M appearing in the Cannon Conjecture 2.1, where we aim at constructing M for $BG \times N$ for any closed manifold N with $\dim(N) \geq 2$.

2.1. The high-dimensional analogue of the Cannon Conjecture.

The following result is taken from [8, Theorem A].

Theorem 2.5 (High-dimensional Cannon Conjecture)**.** *Let G be a torsion free hyperbolic group and let n be an integer ≥ 6. The following statements are equivalent:*

1. *The boundary ∂G is homeomorphic to S^{n-1};*
2. *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n;*

Moreover, the aspherical manifold M appearing in assertion (2) *is unique up to homeomorphism.*

In high dimensions there are exotic examples of hyperbolic n-dimensional Poincaré duality groups G, see [8, Section 5]. For instance, for any integer $k \geq 2$ there are examples satisfying $\partial G = S^{4k+1}$ such that G is the fundamental group of a closed aspherical topological manifold, but not of an closed aspherical smooth manifold. For $n \geq 6$ there exists a closed aspherical topological manifold whose fundamental group is hyperbolic but which cannot be triangulated, see [21, page 800].

We mention without giving the details that using the method of this paper one can prove Theorem 2.5 also in the case $n = 5$.

2.2. **The Cannon Conjecture 0.2 in the torsion free case implies Theorem 0.3 and Theorem 0.4.** Let G be hyperbolic 3-dimensional Poincaré duality group. We want to show that then all claims in Theorem 0.3 and Theorem 0.4 are obviously true, provided that the Cannon Conjecture 0.2 in the torsion free case holds for G.

We know already that there is a 3-dimensional finite model[1] for BG and ∂G is S^2. By the Cannon Conjecture 0.2 we can find a closed hyperbolic 3-manifold M together with a homotopy equivalence $f\colon M \to BG$. Since G is a Farrell-Jones group, f is a simple homotopy equivalence. We obviously can cover f by a bundle map $\overline{f}\colon TM \to \xi$ if we take ξ to be $(f^{-1})^*TM$ for some homotopy inverse $f^{-1}\colon BG \to M$ of f. Hence we get Theorem 0.3 and assertion (1) of Theorem 0.4. It remains to prove assertion (2) of Theorem 0.4.

The universal covering $\widetilde{M}$ is the hyperbolic 3-space. Hence it is homeomorphic to $\mathbb{R}^3$ and the compactification $\overline{\widetilde{M}} = \widetilde{M} \cup \partial G$ is homeomorphic to D^3. In particular $\overline{\widetilde{M}}$ is a compact manifold whose interior is $\widetilde{M}$ and whose boundary is S^2. Hence $\overline{\widetilde{M}} \times N$ is a compact manifold and there is a homeomorphism $(\overline{\widetilde{M}} \times N, \partial(\overline{\widetilde{M}} \times N)) \xrightarrow{\cong} (D^3 \times N, S^2 \times N)$.

2.3. **When does the Cannon Conjecture 0.2 in the torsion free case follow from Theorem 0.4.** Next we discuss what would be needed to conclude the Cannon Conjecture 0.2 in the torsion free case from Theorem 0.4.

Let G be a hyperbolic group such that ∂G is S^2. Then G is a 3-dimensional Poincaré duality group by Bestvina-Mess [10, Corollary 1.3]. Fix any closed aspherical manifold N of dimension ≥ 2 such that $\pi_1(N)$ is a Farrell-Jones group.

We get from Theorem 0.4 a closed aspherical $(3+\dim(N))$-dimensional manifold M together with a homotopy equivalence $f\colon M \to BG \times N$. Let $\alpha\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ be the isomorphism $\pi_1(f)$. If M' is any other closed aspherical manifold together with an isomorphism $\alpha'\colon \pi_1(M') \xrightarrow{\cong} G \times \pi_1(N)$, then we conclude from Theorem 4.1 (1a) and (2b) that $\pi_1(M) \cong G \times \pi_1(N)$ is a Farrell-Jones group and from Theorem 0.5 that there exists a homeomorphism $u\colon M \to M'$ such that $\alpha' \circ \pi_1(u)$ and α agree (up to inner automorphisms). Hence the pair (M, α) is unique and thus an invariant depending on G and N only.

What does the Cannon Conjecture 0.2 tell us about (M, α) and what do we need to know about (M, α) in order to prove the Cannon Conjecture 0.2? This is answered by the next result.

Lemma 2.6. *Assume Theorem 0.4 for a given (G, N) and consider the above unique (M, α). The following statements are equivalent*

1. *The Cannon Conjecture 0.2 holds for G;*
2. *There is a closed 3-manifold M' and a homeomorphism $h\colon M \xrightarrow{\cong} M' \times N$ such that for the projection $p\colon M' \times N \to N$ the map $\pi_1(p \circ h)$ agrees with the composite $\pi_1(M) \xrightarrow{\alpha} G \times \pi_1(N) \xrightarrow{\mathrm{pr}} \pi_1(N)$ for pr the projection;*
3. *There is a closed 3-manifold M' and a map $p\colon M \to N$ with homotopy fiber M' such that $\pi_1(p)$ agrees with the composite $\pi_1(M) \xrightarrow{\alpha} G \times \pi_1(N) \xrightarrow{\mathrm{pr}} \pi_1(N)$ for pr the projection.*

Proof. (1) $\implies$ (2). By the Cannon Conjecture 0.2 there exists a closed hyperbolic 3-manifold M' with $\pi(M') = G$. Since M' models BG, we can find a homotopy equivalence $h\colon M \to M' \times N$ with $\pi_1(h) = \alpha$. By Theorem 0.5 we can assume that h is a homeomorphism.

(2) $\implies$ (3) This is obvious.

[1]Since G is hyperbolic, $\widetilde{K}_0(\mathbb{Z}G)$ vanishes. The rest follows from Subsection1.1 above.

(3) $\implies$ (1) The long exact homotopy sequence associated to p implies that $\pi_1(M') \cong G$ and M' is aspherical. We conclude from Theorem 2.3 that M' is a closed hyperbolic 3-manifold. Hence G satisfies the Cannon Conjecture 0.2. $\square$

2.4. **The special case $N = T^k$.** Now suppose that in the situation of Subsection 2.3 we take $N = T^k$ for some $k \geq 2$. Then we get a criterion, where α does not appear anymore.

Lemma 2.7. *Fix an integer $k \geq 2$. Let M be a closed aspherical $(3+k)$-dimensional manifold with fundamental group $G\times\mathbb{Z}^k$, where G is hyperbolic with $\partial G = S^2$. Then the following statements are equivalent:*

1. *The Cannon Conjecture 0.2 holds for G;*
2. *There is closed 3-manifold M' together with a homeomorphism $h\colon M \xrightarrow{\cong} M' \times T^k$;*
3. *There is a closed 3-manifold M' and a map $p\colon M \to T^k$ with homotopy fiber M'.*

Proof. (1) $\implies$ (2) This follows from Theorem 2.6.

(2) $\implies$ (3) This is obvious.

(3) $\implies$ (1) First we explain that we can assume that $\pi_1(p)\colon \pi_1(M) \to \pi_1(T^k)$ is surjective. Since M' is compact and has only finitely many path components, we conclude from the exact long homotopy sequence that the image of $\pi_1(p)\colon \pi_1(M) \to \pi_1(T^k)$ has finite index. Let $q\colon T^k \to T^k$ be a finite covering such that the image of $\pi_1(p)$ and $\pi_1(q)$ agree. Then we can lift $p\colon M \to T^k$ to a map $p'\colon M \to T^k$ such that $q \circ p' = p$. One easily checks that that $\pi_1(p')$ is surjective and the homotopy fiber of p' fiber is a finite covering of M' and in particular a closed 3-manifold. Hence we assume without loss of generality that $\pi_1(p)$ is surjective, otherwise replace p by p'.

Let K be the kernel of the map $\pi_1(p)\colon \pi_1(M) \cong G \times \mathbb{Z}^k \to \pi_1(T^k) \cong \mathbb{Z}^k$. Since M and T^k are aspherical, the homotopy fiber of p is homotopy equivalent to BK. Hence K is the fundamental group of the closed aspherical 3-manifold M'. Define $K' := K \cap \{1\} \times \mathbb{Z}^k$. This is a normal subgroup of both K and $\mathbb{Z}^k$ if we identify $\{1\} \times \mathbb{Z}^k = \mathbb{Z}^k$.

We begin with the case, where K' is trivial. Then the projection $\mathrm{pr}\colon G \times \mathbb{Z}^k \to G$ induces an isomorphism $K \xrightarrow{\cong} L$ for $L = \mathrm{pr}(K) \subseteq G$. We conclude from Theorem 1.4 that G is the fundamental group of a closed 3-manifold. Theorem 2.3 implies that G is the fundamental group of a closed hyperbolic 3-manifold.

Next we consider the case where K' is non-trivial. Consider the following commutative diagram

$$\begin{tikzcd}
& \{1\} \arrow[d] & \{1\} \arrow[d] & \{1\} \arrow[d] & \\
\{1\} \arrow[r] & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/K' \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathbb{Z}^k \arrow[r] \arrow[d] & G \times \mathbb{Z}^k \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathbb{Z}^k/K' \arrow[r] \arrow[d] & \mathbb{Z}^k \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \{1\} \\
& \{1\} & \{1\} & \{1\} &
\end{tikzcd}$$

where the upper and the middle rows and the left and the middle columns are the obvious exact sequences, and the map $\mathbb{Z}^k/K' \to \mathbb{Z}^k$ is the map making the diagram commutative. The group Q is defined to be the cokernel of the map $\mathbb{Z}^k/K' \to \mathbb{Z}^k$, and all other arrows are uniquely determined by the property that the diagram commutes. The so called nine-lemma, which can be proved by an easy diagram chase, shows that all rows and columns are exact.

Since $K' \subseteq \mathbb{Z}^k \times \{1\} \subseteq \mathbb{Z}^k \times G$, we have $K' \subseteq \operatorname{cent}(G \times \mathbb{Z}^k)$. Since $K' \subseteq K \subseteq G \times \mathbb{Z}^k$, we conclude $K' \subseteq \operatorname{cent}(K)$. Since K is torsionfree and K' is non-trivial, the center of K contains a copy of $\mathbb{Z}$. We conclude from Theorem 1.6 that there is a closed aspherical Seifert 3-manifold S such that $K = \pi_1(N)$. There exists a finite covering $\overline{S} \to S$ such that $\overline{S}$ is orientable, there is a principal S^1-fiber bundle $S^1 \to \overline{S} \to F_g$ for a closed orientable surface of genus $g \geq 1$, see [58, page 436 and Theorem 2.3]. We obtain a short exact sequence $\{1\} \to \pi_1(S^1) \to \pi_1(\overline{S}) \to \pi_1(F_g) \to \{1\}$. The center of $\pi_1(\overline{S})$ contains the image of $\pi_1(S^1) \to \pi_1(\overline{S^1})$ since we are considering a principal S^1-fiber bundle $S^1 \to \overline{S} \to F_g$ and the fiber transport is by self-homotopy equivalences of S^1 which are all homotopic to the identity. The center cannot be larger if $g \geq 2$ since $\operatorname{cent}(\pi_1(F_g))$ is trivial for $g \geq 2$. If the center is larger and $g = 1$, the extension has to be trivial, after possibly passing to a finite covering of $\overline{S}$. Hence we can arrange that there is a subgroup $\overline{K} \subseteq K$ of finite index such that $\operatorname{cent}(\overline{K}) \cong \mathbb{Z}$ and $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ holds for some $g \geq 1$, or we have $\overline{K} \cong \mathbb{Z}^3$; just take $\overline{K} = \pi_1(\overline{S})$.

Next we show that $\operatorname{cent}(\overline{K})$ must be infinite cyclic. If $\operatorname{cent}(\overline{K})$ is not infinite cyclic, then $\overline{K}$ has to be $\mathbb{Z}^3$. We conclude that K and hence also K/K' are virtually finitely generated abelian. Since Q is abelian, we have the exact sequence $1 \to K/K' \to G \to Q \to 1$ and G has cohomological dimension 3, the group G cannot be hyperbolic, a contradiction. Hence $\operatorname{cent}(\overline{K})$ must be infinite cyclic and and $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ for some $g \geq 1$.

We have $\{0\} \neq K' \subseteq \operatorname{cent}(K)$ and $\operatorname{cent}(K) \cap \overline{K} \subseteq \operatorname{cent}(\overline{K}) \cong \mathbb{Z}$. Since K' is torsion free and $[K : \overline{K}]$ is finite, $\operatorname{cent}(K)$ is a non-trivial torsion-free virtually cyclic group and hence $\operatorname{cent}(K)$ is infinite cyclic. Since $\operatorname{cent}(K)/K'$ is a finite subgroup of K/K' and K/K' is isomorphic to a subgroup of the torsion free group G, we have $K' = \operatorname{cent}(K)$. The group $\overline{K}/(\overline{K} \cap \operatorname{cent}(K))$ is a subgroup of $K/K' = K/\operatorname{cent}(K)$ of finite index and admits an epimorphism onto $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ whose kernel $\operatorname{cent}(\overline{K})/(\overline{K} \cap \operatorname{cent}(K))$ is finite. Since $\overline{K}/(\overline{K} \cap \operatorname{cent}(K))$ is isomorphic to a subgroup of the torsion free group G, this kernel is trivial and hence $\overline{K}/(\overline{K} \cap \operatorname{cent}(K)) \cong \pi_1(F_g)$.

Since K' is infinite cyclic, Q contains a copy of $\mathbb{Z}$ of finite index. Hence we can find a subgroup G' of G of finite index together with a short exact sequence $\{1\} \to K/K' \to G' \to \mathbb{Z} \to \{1\}$. So there exists an automorphism $\phi\colon K/K' \to K/K'$ such that G' is isomorphic to the semi-direct product $K/K' \rtimes_\phi \mathbb{Z}$. If we put $L := \overline{K}/(\overline{K} \cap \operatorname{cent}(K))$, then $L \cong \pi_1(F_g)$ and L is a subgroup of finite index of the finitely generated group K'/K. Then $L' = \bigcap_{n \in \mathbb{Z}} \phi^n(L)$ is a subgroup of K/K' of finite index again which satisfies $L' \subseteq L$ and $\phi(L') = L'$ and for which there is an isomorphism $u\colon L' \xrightarrow{\cong} \pi_1(F_{g'})$ for some $g' \geq 1$. Let $\phi'\colon L' \to L'$ be the automorphism induced by ϕ. Then $G'' := L' \rtimes_{\phi'} \mathbb{Z}$ is isomorphic to a subgroup of G of finite index in G. Choose a homeomorphism $h'\colon F_{g'} \to F_{g'}$ satisfying $\pi_1(h') = u \circ \phi' \circ u^{-1}$. The mapping torus $T_{h'}$ is a closed aspherical 3-manifold with $\pi_1(T_{h'}) \cong G''$. Theorem 1.4 shows that G is the fundamental group of a closed 3-manifold. Theorem 2.3 implies that G is the fundamental group of a closed hyperbolic 3-manifold. □

Remark 2.8 (manifold approximate fibration)**.** Some evidence for Lemma 2.7 comes from the conclusion of [26, Theorem 1.8] that one can find for any epimorphism $\alpha\colon \pi_1(M) \to \pi_1(T^k)$ at least a manifold approximate fibration $p\colon M \to T^k$ such that $\pi_1(p) = \alpha$.

3. The existence of a normal map of degree one

We call a connected finite Poincaré complex X *oriented* if we have chosen a generator $[X]$ of the infinite cyclic group $H_n^{\pi_1(X)}(\widetilde{X}; \mathbb{Z}^{w_1(X)})$. Notice that we do allow non-trivial $w_1(X)$. In this section we show

Theorem 3.1 (Existence of a normal map)**.** *Let X be a connected finite 3-dimensional Poincaré complex. Then there exist an integer $a \geq 0$ and a vector bundle ξ over X and a normal map of degree one*

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Proof. Any element $c \in H^k(BO; \mathbb{Z}/2)$ determines up to homotopy a unique map $\widehat{c}\colon BSG \to K(\mathbb{Z}/2, k)$. It is characterized by the property that $c = H^k(\widehat{c}; \mathbb{Z}/2)(\iota_k)$ for the canonical element $\iota_k \in H^k(K(\mathbb{Z}/2, k); \mathbb{Z}/2)$ which corresponds to $\operatorname{id}_{\mathbb{Z}/2}$ under the isomorphism

$$\begin{aligned} H^k(K(\mathbb{Z}/2, k); \mathbb{Z}/2) &\cong \hom_{\mathbb{Z}}(H_k(K(\mathbb{Z}/2, k); \mathbb{Z}), \mathbb{Z}/2) \\ &\cong \hom_{\mathbb{Z}}(\pi_k(K(\mathbb{Z}/2, k)), \mathbb{Z}/2) \cong \hom_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2). \end{aligned}$$

Next, we claim that the product of the maps given by the universal first and second Stiefel-Whitney classes $w_1 \in H^1(BO; \mathbb{Z}/2)$ and $w_2 \in H^2(BO; \mathbb{Z}/2)$

$$\widehat{w_1} \times \widehat{w_2}\colon BO \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \tag{3.2}$$

is 4-connected. Since BO is connected, $\pi_1(BO) \cong \pi_2(BSO) = \mathbb{Z}/2$ and $\pi_3(BSO) = 0$, it suffices to show that $\pi_k(\widehat{w_k})\colon \pi_k(BO) \to \pi_k(K(\mathbb{Z}/2, k))$ is non-trivial for $k = 1, 2$. This is easily proved using the fact the Hopf fibration $S^1 \to S^3 \to S^2$ has non-trivial second Stiefel-Whitney class. Hence for any 3-dimensional complex X

stable vector bundles over X are classified by w_1 and w_2. The map induced by composition with the map (3.2)

$$[X, BSO] \to [X, K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)] = H^1(X; \mathbb{Z}/2) \times H^2(X; \mathbb{Z}/2),$$

is bijective. For a vector bundle ξ with classifying map f_ξ the class $[f_\xi]$ goes to $(w_1(\xi), w_2(\xi))$.

We conclude from [33, page 44] that there is a closed manifold M together with a map $f \colon M \to X$ such that $w_1(M) = f^*w_1(X)$ and the induced map $H_3^{\pi_1(M)}(\widetilde{M}; \mathbb{Z}^{w_1(M)}) \xrightarrow{\cong} H_3^{\pi_1(X)}(\widetilde{X}; \mathbb{Z}^{w_1(X)})$ is an isomorphism of infinite cyclic groups. The proof in the general case is a variation of the one for trivial $w_1(X)$ which we sketch next. The Atiyah-Hirzebruch spectral sequence applied to the homology theory Ω_* given by oriented bordism yields an epimorphism

$$\Omega_3(X) \xrightarrow{\cong} H_3(X; \mathbb{Z}), \quad [f \colon M \to BG] \mapsto f_*([M])$$

since the projection $X \to \{\bullet\}$ induces an epimorphism $\Omega_3(X) \to \Omega_3(\{\bullet\})$ and there is a map of degree one from X to S^3. We can choose the fundamental class $[M] \in H_3(M; \mathbb{Z}^{w_1(M)})$ so that it is mapped to $[X] \in H_3(M; \mathbb{Z}^{w_1(X)})$ under the isomorphism $H_3^{\pi_1(M)}(\widetilde{M}; \mathbb{Z}^{w_1(M)}) \xrightarrow{\cong} H_3^{\pi_1(X)}(\widetilde{X}; \mathbb{Z}^{w_1(X)})$.

Choose a vector bundle ξ over X with $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(X) \cup w_1(X)$. Its pull back $f^*\xi$ satisfies $w_2(f^*\xi) = w_1(M) \cup w_1(M)$ and $w_1(f^*\xi) = w_1(M)$. The Wu formula, see for instance [46, Theorem 11.14 on page 132], implies $w_2(TM) = w_1(TM) \cup w_1(TM)$ and hence $w_2(f^*\xi) = w_2(TM)$ and $w_1(f^*\xi) = w_1(TM)$. Therefore TM and $f^*\xi$ are stably isomorphic. Hence we can cover $f \colon M \to X$ by a bundle map $\overline{f} \colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ after possibly replacing ξ by $\xi \oplus \underline{\mathbb{R}^b}$. □

Notice that the sphere bundle of ξ is necessarily the Spivak normal bundle of X. Hence we see that the Spivak normal fibration of X has a vector bundle reduction.

Next we want to figure out the simple surgery obstruction

$$\sigma^s(f, \overline{f}) \in L_3^s(\mathbb{Z}[\pi_1(X)], w_1(X))$$

of the normal one map of degree one appearing in Theorem 3.1. The goal is to find one $(f, \overline{f})$ such that $\sigma^s(f, \overline{f})$ vanishes. Notice that the definition of the surgery obstruction makes sense in all dimensions, in particular also in dimension 3. For this purpose we will need the Full Farrell-Jones Conjecture.

4. Short review of Farrell-Jones groups

Recall that a group G is called a Farrell-Jones group if it satisfies the Full Farrell-Jones Conjecture, which means that it satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive categories and with finite wreath products. A detailed exposition on the Farrell-Jones Conjecture will be given in [44].

The reader does not need to know any details about the Full Farrell-Jones Conjecture since this paper is written so that FJ can be used as a black box. We will mention the consequences which we need in this paper when they appear. For now, we record the following important consequences for a torsion free Farrell-Jones group G.

- The projective class group $\widetilde{K}_0(\mathbb{Z}G)$ vanishes. This implies that any finitely presented n-dimensional Poincaré duality group has a finite n-dimensional model for BG;
- The Whitehead group $\mathrm{Wh}(G)$ vanishes. Hence any homotopy equivalence of finite CW-complexes with G as fundamental group is a simple homotopy

equivalence and every h-cobordism of dimension ≥ 6 with G as fundamental group is trivial;
- The negative K-groups $K_n(\mathbb{Z}G)$ for $n \leq -1$ all vanish. Hence the decorations $L_n^\epsilon(\mathbb{Z}G)$ in the L-groups do not matter;
- The L-theoretic assembly map, see (5.2),

$$\mathrm{asmb}_n^\epsilon(G,w)\colon H_n^G(EG;\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to H_n^G(\{\bullet\};\mathbf{L}^\epsilon_{\mathbb{Z},w}) = L_n^\epsilon(\mathbb{Z}G,w)$$

is an isomorphism for $n \in \mathbb{Z}$ and all decorations ϵ;
- The Borel Conjecture holds for closed aspherical manifolds of dimension ≥ 5 whose fundamental group is G.

The reader may appreciate the following status report.

Theorem 4.1 (The class $\mathcal{FJ}$)**.** *Let class $\mathcal{FJ}$ of Farrell-Jones groups has the following properties.*

(1) *The following classes of groups belong to $\mathcal{FJ}$:*
  (a) *Hyperbolic groups;*
  (b) *Finite dimensional* CAT(0)*-groups;*
  (c) *Virtually solvable groups;*
  (d) *(Not necessarily cocompact) lattices in second countable locally compact Hausdorff groups with finitely many path components;*
  (e) *Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;*
  (f) *The groups $GL_n(\mathbb{Q})$ and $GL_n(F(t))$ for $F(t)$ the function field over a finite field F;*
  (g) *S-arithmetic groups;*
  (h) *mapping class groups;*

(2) *The class $\mathcal{FJ}$ has the following inheritance properties:*
  (a) Passing to subgroups
  Let $H \subseteq G$ be an inclusion of groups. If G belongs to $\mathcal{FJ}$, then H belongs to $\mathcal{FJ}$;
  (b) Passing to finite direct products
  If the groups G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 \times G_1$ belongs to $\mathcal{FJ}$;
  (c) Group extensions
  Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{FJ}$ and that the group Q belongs to $\mathcal{FJ}$.
  Then G belongs to $\mathcal{FJ}$;
  (d) Directed colimits
  Let $\{G_i \mid i \in I\}$ be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to $\mathcal{FJ}$.
  Then the colimit $\mathrm{colim}_{i\in I} G_i$ belongs to $\mathcal{FJ}$;
  (e) Passing to finite free products
  If the groups G_0 and G_1 belong to $\mathcal{FJ}$, then $G_0 * G_1$ belongs to $\mathcal{FJ}$;
  (f) Passing to overgroups of finite index
  Let G be an overgroup of H with finite index $[G:H]$. If H belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

Proof. See [1, 2, 3, 4, 6, 7, 36, 57, 65, 66]. □

5. The total surgery obstruction

The results of this section are inspired and motivated by Ranicki's total surgery obstruction, see for instance [41, 51, 54]. Since we consider only aspherical Poincaré complexes whose fundamental groups are Farrell-Jones groups, the exposition simplifies drastically and we get some valuable additional information. Moreover, we get a version of Quinn's resolution obstruction which does not require the structure of an ENR homology manifold on the relevant Poincaré complexes. Then the total surgery obstruction and hence Quinn's resolution obstruction are already determined by the symmetric signature of the finite Poincaré complex.

The main result of this section will be

Theorem 5.1. *Let G be a finitely presented 3-dimensional Poincaré duality group which is a Farrell-Jones group.*

Let X be a finite 3-dimensional CW complex modeling BG. The following statements are equivalent:

1. *There exists a closed aspherical topological manifold N_0 with Farrell-Jones fundamental group such that $BG \times N_0$ is homotopy equivalent to a closed topological manifold;*
2. *Let N be any closed smooth manifold, closed PL-manifold, or closed topological manifold respectively of dimension ≥ 2. Then there is exists a normal map of degree one for some vector bundle ξ over X*

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & (\xi \times TN) \oplus \underline{\mathbb{R}^b} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \times N \end{array}$$

such that M is a smooth manifold, PL-manifold, or topological manifold respectively and f is a simple homotopy equivalence.

5.1. **The quadratic total surgery obstruction.** Let G be a group together with an orientation homomorphism $w\colon G \to \{\pm 1\}$. Then there is a covariant functor

$$\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

from the orbit category to the category of spectra, where the decoration ϵ is $\langle i \rangle$ for some $i \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$, see [54, Definition 4.1 on page 145]. Notice that the decoration $\langle i \rangle$ for $i = 2, 1, 0$ is also denoted by s, h, p in the literature. From $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}$ we obtain a G-homology theory on the category of G-CW-complexes $H^G_*(-; \mathbf{L}^{\epsilon}_{\mathbb{Z},w})$ such that for every subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ we have identifications

$$H^G_n(G/H; \mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \cong \pi_n(\mathbf{L}^{\epsilon}_{\mathbb{Z},w}(G/H)) \cong L^{\epsilon}_n(\mathbb{Z}H, w|_H),$$

where $L^{\epsilon}_n(\mathbb{Z}H, w|_H)$ denotes the n-th quadratic L-group with decoration ϵ of $\mathbb{Z}G$ with the w-twisted involution, see [18, Section 4 and 7]. The projection $EG \to \{\bullet\}$ induces the so called assembly map

$$\operatorname{asmb}^{\epsilon}_n(G, w)\colon H^G_n(EG; \mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \to H^G_n(\{\bullet\}; \mathbf{L}^{\epsilon}_{\mathbb{Z},w}) = L^{\epsilon}_n(\mathbb{Z}G, w), \tag{5.2}$$

which is induced by the projection $EG \to \{\bullet\}$.

In the sequel we denote for a spectrum $\mathbf{E}$ by $\mathbf{i}(\mathbf{E})\colon \mathbf{E}\langle 1 \rangle \to \mathbf{E}$ its 1-connective cover. This is a map of spectra such that $\pi_n(\mathbf{i}(\mathbf{E}))$ is an isomorphism for $n \geq 1$ and $\pi_n(\mathbf{E}\langle 1 \rangle) = 0$ for $n \leq 0$. We claim that there is a functorial construction of the 1-connective cover so that we get from the covariant functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ another covariant functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1 \rangle\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ together with a natural transformation $\mathbf{i}\colon \mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1 \rangle \to \mathbf{L}^{\epsilon}_{\mathbb{Z},w}$ such that $\mathbf{i}(G/H)$ is a cofibration of spectra. Then we can also define a functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1 \rangle\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$

together with a natural transformation $\mathbf{pr}\colon \mathbf{L}^{\epsilon}_{\mathbb{Z},w} \to \mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle$ such that for every object G/H in $\mathsf{Or}(G)$ we obtain a cofibration sequence of spectra

$$\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H) \xrightarrow{\mathbf{i}(G/H)} \mathbf{L}^{\epsilon}_{\mathbb{Z},w}(G/H) \xrightarrow{\mathbf{pr}(G/H)} \mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H).$$

For every G-CW-complex Y this induces a long exact sequence

$$\begin{aligned}(5.3)\quad \cdots \to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) &\to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \\ &\to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \to H^G_{n-1}(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \to \cdots\end{aligned}$$

and we have

$$\pi_n(\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H)) \cong \begin{cases} L^{\epsilon}_n(H;w|_H) & n \le 0; \\ 0 & n \ge 1. \end{cases}$$

Now consider an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphism $w = w_1(X)\colon G \to \{\pm 1\}$ in the sense of [62]. We can read w from the underlying CW-complex X as follows. For any abelian group A we denote by A^w the $\mathbb{Z}G$-module whose underlying abelian group is A and on which $g \in G$ acts by multiplication with $w(g)$. Now we use the isomorphism $H_n(C^{n-*}(\widetilde{X})) \cong_{\mathbb{Z}\pi} \mathbb{Z}^w$ coming from Poincaré duality, where $C^{n-*}(\widetilde{X})$ is the (untwisted) $\mathbb{Z}\pi$-dual chain complex of the cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$ of the universal covering $\widetilde{X}$.

There is an equivariant version of the Atiyah-Hirzebruch spectral sequence, whose E^2-term is given by $E^2_{p,q} = H^G_p(\widetilde{X};\pi_q(\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle))$ and which converges to $H^G_{p+q}(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle)$, see for instance [18, 4.7]. It implies $H^G_{n+1}(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) = 0$ and yields an isomorphism

$$(5.4)\qquad H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{\cong} H^G_n(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})^w).$$

Poincaré duality yields an isomorphism

$$(5.5)\qquad H^G_n(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})^w) \xrightarrow{\cong} H^0_G(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})),$$

where G acts trivially on $L^{\epsilon}_0(\mathbb{Z})$ in $H^0_G(\widetilde{X};L^{\epsilon}_0(\mathbb{Z}))$. There is an obvious isomorphism

$$(5.6)\qquad H^0_G(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})) \xrightarrow{\cong} H^0(X;L^{\epsilon}_0(\mathbb{Z})) \cong L^{\epsilon}_0(\mathbb{Z}).$$

Notice that $L^{\epsilon}_0(\mathbb{Z})$ is independent of the decoration ϵ and hence we abbreviate $L_0(\mathbb{Z}) = L^{\epsilon}_0(\mathbb{Z})$. We obtain from (5.4), (5.5), and (5.6) an isomorphism

$$(5.7)\qquad H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{\cong} L_0(\mathbb{Z}).$$

Its composition with $H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \to H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle)$ is denoted by

$$(5.8)\qquad \lambda^{\epsilon}_n(X)\colon H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \to L_0(\mathbb{Z}).$$

From the exact sequence (5.3) we obtain a short exact sequence

$$(5.9)\qquad 0 \to H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{H^G_n(\mathrm{id}_{\widetilde{X}};\mathbf{i})} H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \xrightarrow{\lambda^{\epsilon}_n(X)} L_0(\mathbb{Z}).$$

For every ϵ there is a natural transformation

$$\mathbf{e}^{\epsilon}\colon \mathbf{L}^{\epsilon} \to \mathbf{L}^{\langle -\infty\rangle}$$

such that $e_n^\epsilon := \pi_n(\mathbf{e}^\epsilon)\colon L_n^\epsilon(\mathbb{Z}G,w) \to L_n^{\langle -\infty\rangle}(\mathbb{Z}G,w)$ is the classical change of decoration homomorphism and the following diagram

$$\begin{tikzcd} H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}_n^\epsilon(X)"] \arrow[d, "H_n^G(\operatorname{id}_{\widetilde{X}};\mathbf{e}^\epsilon)"'] & H_n^G(\{\bullet\};\mathbf{L}^\epsilon_{\mathbb{Z},w}) = L_n^\epsilon(\mathbb{Z}G,w) \arrow[d, "e_n^\epsilon"] \\ H_n^G(\widetilde{X};\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}_n^{\langle -\infty\rangle}(X)"'] & H_n^G(\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z},w}) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G,w) \end{tikzcd}$$

commutes. Note that since X is aspherical, G must be torsion free. If G is a Farrell-Jones group, then $\operatorname{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_m(\mathbb{Z}G)$ for $m \le -1$ vanish and hence all maps in the commutative diagram are isomorphisms, in particular, the choice of the decoration ϵ does not matter.

Let $\mathcal{N}(X)$ be the set of normal bordism classes of degree one normal maps with target X. Suppose that $\mathcal{N}(X)$ is not empty. Consider a normal map $(f,\overline{f})$ of degree one with target X

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

One can assign to it its simple surgery obstruction $\sigma^s(f,\overline{f}) \in L_n^s(\mathbb{Z}G,w)$. (This makes sense for all dimensions n.) Fix a normal map $(f_0,\overline{f_0})$. Then there is a commutative diagram

$$\begin{tikzcd} \mathcal{N}^{\mathrm{TOP}}(X) \arrow[r, "\sigma^s(-,-)-\sigma^s(f_0,\overline{f_0})"] \arrow[d, "s_0"', "\cong"] & L_n^s(\mathbb{Z}G,w) \\ H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\operatorname{id}_{\widetilde{X}};\mathbf{i})"'] & H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}) \arrow[u, "\cong"', "\operatorname{asmb}_n^s(X)"] \end{tikzcd} \tag{5.10}$$

whose vertical arrows are bijections. The upper arrow sends the class of $(f,\overline{f})$ to the difference $\sigma^s(f,\overline{f})-\sigma^s(f,\overline{f_0})$. This follows from the work of Ranicki [54, Proof of Theorem 17.4 on pages 191ff] using [16, Theorem B1]. A detailed and careful exposition of the proof of the existence of the diagram above can be found in [41, Proposition 14.18]. The right vertical arrow is an isomorphism, provided that G is a Farrell-Jones group.

Now consider the composition

$$\mu_n^s(X)\colon \mathcal{N}(X) \xrightarrow{\sigma^s} L_n^s(\mathbb{Z}G,w) \xrightarrow{\operatorname{asmb}_n^s(X)^{-1}} H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}) \xrightarrow{\lambda_n^\epsilon(X)} L_0(\mathbb{Z}), \tag{5.11}$$

where the map $\lambda_n^\epsilon(X)$ has been defined in (5.8). From the exact sequence (5.9) and the diagram 5.10 we conclude that there is precisely one element, called the *quadratic total surgery obstruction*,

$$s(X) \in L_0(\mathbb{Z}) \tag{5.12}$$

such that for any element $[(f,\overline{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$. Moreover, we get

Theorem 5.13 (The quadratic total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphism $w = w_1(X)\colon G \to \{\pm 1\}$. Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty. Then:*

(1) *There exists a normal map $(f,\overline{f})$ of degree one with target X whose simple surgery obstruction $\sigma^s(f,\overline{f}) \in L_n^s(\mathbb{Z}G,w)$ vanishes, if and only if $s(X) \in L_0(\mathbb{Z})$ vanishes;*
(2) *If X is homotopy equivalent to a closed topological manifold, then $s(X) \in L_0(\mathbb{Z})$ vanishes.*

Proof. 1 The "only if"-statement is obvious. The "if"- statement is proved as follows. The vanishing of $s(X) \in L_0(\mathbb{Z})$ implies that the element $-(f_0,\overline{f_0})$ in $\mathcal{N}^{\mathrm{TOP}}(X)$ is sent under $\mu_n^s(X)$ to zero. The exact sequence (5.9) implies that the composite $\mathcal{N}(X) \xrightarrow{\sigma^s} L_n^s(\mathbb{Z}G,w) \xrightarrow{\mathrm{asmb}_n^s(X)^{-1}} H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w})$ sends $-(f_0,\overline{f_0})$ to an element which is in the image of $H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})\colon H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}\langle 1\rangle) \to H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w})$. We conclude from the diagram (5.10) that $-\sigma^s(f_0,\overline{f_0})$ lies in the image of the upper horizontal arrow of the diagram 5.10. Therefore there is an element $-(f,\overline{f})$ in $\mathcal{N}^{\mathrm{TOP}}(X)$ which satisfies $\sigma^s(f,\overline{f})-\sigma^s(f_0,\overline{f_0}) = -\sigma^s(f_0,\overline{f_0})$ and hence $\sigma^s(f,\overline{f}) = 0$.
2 If X is simply homotopy equivalent to a closed topological manifold, then there exists an element in $[(f,\overline{f})]$ in $\mathcal{N}(X)$ with $\sigma^s(f,\overline{f}) = 0$. Now apply assertion 1. □

Notice that Theorem 5.13 (1) holds also in dimensions $n \leq 4$. We are *not* claiming in Theorem 5.13 (1) that that we can arrange f to be a simple homotopy equivalence. This conclusion from the vanishing of the simple surgery obstruction does require $n \geq 5$.

5.2. **The symmetric total surgery obstruction.** There is also a symmetric version of the material of Subsection 5.1. There is a covariant functor

$$\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

from the orbit category to the category of spectra such that for every subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ we have identifications

$$H_n(G/H;\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \cong \pi_n(\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}(G/H)) \cong L^n_\epsilon(\mathbb{Z}H, w|_H),$$

where $L^n_\epsilon(\mathbb{Z}H,w|_H)$ denotes the 4-periodic n-th symmetric L-group with decoration ϵ of $\mathbb{Z}G$ with the w-twisted involution. The projection $EG \to \{\bullet\}$ induces the symmetric assembly map

$$\mathrm{asmb}_n^{\epsilon,\mathrm{sym}}(X)\colon H_n^G(EG;\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \to H_n^G(\{\bullet\};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) = L^n_\epsilon(\mathbb{Z}G,w), \tag{5.14}$$

which is induced by the projection $\widetilde{X} \to \{\bullet\}$.

There is a natural transformation called symmetrization of covariant functors $\mathsf{Or}(G) \to \mathsf{SPECTRA}$

$$\mathbf{sym}^\epsilon\colon \mathbf{L}^\epsilon_{\mathbb{Z},w} \to \mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}. \tag{5.15}$$

It induces the classical symmetrization homomorphisms on homotopy groups

$$\mathrm{sym}_n^\epsilon(G/H)\colon L_n^\epsilon(\mathbb{Z}H,w|_H) \to L^n_\epsilon(\mathbb{Z}H,w|_H), \tag{5.16}$$

which are isomorphism after inverting 2, see [52, Proposition 8.2]. We obtain a natural transformation of G-homology theories, see [18, Lemma 4.6].

$$H_*^G(-;\mathbf{sym}^\epsilon)\colon H_*^G(-;\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to H_*^G(-;\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \tag{5.17}$$

satisfying

Theorem 5.18. *For every $n \in \mathbb{Z}$ and every G-CW-complex X the maps*

$$H_*^G(-;\mathbf{sym}^\epsilon)\colon H_n^G(X;\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to H_n^G(X;\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w})$$

are isomorphisms after inverting 2.

The following diagram commutes

$$
\begin{tikzcd}
H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}^\epsilon_n(X)"] \arrow[d, "H_n^G(EG,\mathbf{sym}^\epsilon)"'] & L_n^\epsilon(\mathbb{Z}G,w) \arrow[d, "\operatorname{sym}^\epsilon_n(G/G)"] \\
H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}^{\epsilon,\mathrm{sym}}_n(X)"'] & L^n_\epsilon(\mathbb{Z}G,w).
\end{tikzcd}
\tag{5.19}
$$

There is an obvious symmetric analog of the map (5.8)

$$\lambda_n^{\epsilon,\mathrm{sym}}(X)\colon H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \to L^0(\mathbb{Z}), \tag{5.20}$$

and of the short exact sequence (5.9)

$$0 \to H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})} H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \xrightarrow{\lambda_n^{\epsilon,\mathrm{sym}}(X)} L^0(\mathbb{Z}). \tag{5.21}$$

The following diagram

(5.22)

$$
\begin{tikzcd}
0 \arrow[r] & H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})"] \arrow[d, "H_n^G(\mathrm{id};\mathbf{sym}^\epsilon\langle 1\rangle)"] & H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \arrow[r, "\lambda_n^\epsilon(X)"] \arrow[d, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{sym}^\epsilon)"] & L_0(\mathbb{Z}) \arrow[d, "\operatorname{sym}_0"] \\
0 \arrow[r] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})"] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[r, "\lambda_n^{\epsilon,\mathrm{sym}}(X)"] & L^0(\mathbb{Z})
\end{tikzcd}
$$

commutes, has exact rows, and all its vertical arrows are bijections after inverting 2 since the map (5.16) is bijective after inverting 2 and we have [18, Theorem 4.7]. Under the standard identifications

$$h_0\colon L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; \tag{5.23}$$

$$h^0\colon L^0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \tag{5.24}$$

the map $\operatorname{sym}_0\colon L_0(\mathbb{Z}) \to L^0(\mathbb{Z})$ becomes $8\cdot\mathrm{id}\colon \mathbb{Z}\to\mathbb{Z}$, see the proof of [52, Proposition 8.2], and hence is injective. Define the *symmetric total surgery obstruction*

$$s^{\mathrm{sym}}(X) \in L^0(\mathbb{Z}) \tag{5.25}$$

to be the image of $s(X)$ defined in (5.12) under the injection $\operatorname{sym}_0\colon L_0(\mathbb{Z}) \to L^0(\mathbb{Z})$. Theorem 5.13 implies

Theorem 5.26 (The symmetric total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X}\to X$, fundamental group $G=\pi_1(X)$ and orientation homomorphisms $w=w_1(X)\colon G\to\{\pm1\}$. Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty. Then*

1. *There exists a normal map of degree one $(f,\overline{f})$ with target X whose simple surgery obstruction $\sigma^s(f,\overline{f})\in L_n^s(\mathbb{Z}G,w)$ vanishes, if and only if $s^{\mathrm{sym}}(X)\in L^0(\mathbb{Z})$ vanishes;*
2. *If X is homotopy equivalent to a closed topological manifold, then $s^{\mathrm{sym}}(X)\in L^0(\mathbb{Z})$ vanishes.*

Now we study the main properties of the symmetric total surgery obstruction.

If A is an abelian group, denote by $A/2$-tors its quotient by the abelian subgroup of elements in A, whose order is finite and a power of two. For an element $a\in A$ denote by $[a]_2$ its image under the projection $A\to A/2$-tors.

Next we show that $s^{\mathrm{sym}}(X)$ and $s(X)$ are determined by the image $[\sigma_G^{s,\mathrm{sym}}(\widetilde{X})]_2$ of $\sigma_G^{s,\mathrm{sym}}(\widetilde{X})$ under $L_s^n(\mathbb{Z}G,w)\to L_s^n(\mathbb{Z}G,w)/2$-tors, where $\sigma_G^{s,\mathrm{sym}}(\widetilde{X})$ is the symmetric signature in the sense of [52, Proposition 6.3] taking into account, that G is a torsionfree Farrell-Jones group and hence the decorations do not matter.

Theorem 5.27. *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphism $w = w_1(X)\colon G \to \{\pm 1\}$. Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty.*

Then there is precisely one element $u \in H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors}$ such that the injective map

$$\mathrm{asmb}_n^{s,\mathrm{sym}}(X)/2\text{-tors}\colon H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors} \to L_s^n(\mathbb{Z}G, w)/2\text{-tors}$$

sends u to the element $[\sigma_G^{s,\mathrm{sym}}(\widetilde{X})]_2$ associated to the symmetric signature $\sigma_G^{s,\mathrm{sym}}(\widetilde{X})$, and the composite

$$H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors} \xrightarrow{\lambda^{s,\mathrm{sym}}(\widetilde{X})/2\text{-tors}} L^0(\mathbb{Z})/2\text{-tors} \xrightarrow{h_0/2\text{-tors}} \mathbb{Z}/2\text{-tors} = \mathbb{Z}$$

sends u to $1 - h^0(s^{\mathrm{sym}}(X)) = 1 - 8 \cdot h_0(s(X))$.

Proof. Consider a normal map$(f, \overline{f})$ of degree one from M to X.

Since G is a Farrell-Jones group, the assembly map asmb of (5.2) is bijective for all $n \in \mathbb{Z}$. The homomorphism $\mathrm{sym}_n^s\colon L_n^s(\mathbb{Z}G, w) \to L_s^n(\mathbb{Z}G, w)$ sends $\sigma^s(f, \overline{f})$ to $\sigma_G^{s,\mathrm{sym}}(\overline{M}) - \sigma_G^{s,\mathrm{sym}}(\widetilde{X})$, where $\overline{M} \to M$ is the pull back of the G-covering $\widetilde{X} \to BG$ by f, see [53, Section 6]. We conclude from the commutative diagram (5.19) that there is an element $u' \in H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors}$ whose image under $\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-tors}$ is $[\sigma_G^{s,\mathrm{sym}}(\overline{M})]_2 - [\sigma_G^{s,\mathrm{sym}}(\widetilde{X})]_2$.

We conclude from the commutative diagram (5.19) that the assembly map

$$\mathrm{asmb}_n^{\epsilon,\mathrm{sym}}(\widetilde{X})\colon H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}) \to H_n^G(\{\bullet\}; \mathbf{L}^{\epsilon,\mathrm{sym}}) = L_\epsilon^n(\mathbb{Z}G, w)$$

of (5.14) is an isomorphism after inverting 2 since the upper horizontal arrow is the bijective map (5.2), and the two vertical arrows are isomorphisms after inverting 2, see (5.16) and Theorem 5.18. Hence the map

$$\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-tors}\colon H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors} \to L_s^n(\mathbb{Z}G, w)/2\text{-tors}$$

is injective.

We conclude from the diagram (5.22) that the image of u' under the composite

$$H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors} \xrightarrow{\lambda^{s,\mathrm{sym}}(\widetilde{X})/2\text{-tors}} L^0(\mathbb{Z})/2\text{-tors} \xrightarrow{h_0/2\text{-tors}} \mathbb{Z}/2\text{-tors} = \mathbb{Z}$$

is $h^0(s^{\mathrm{sym}}(X))$. We have $8 \cdot h_0(s(X)) = h^0(s^{\mathrm{sym}}(X))$. Hence it suffices to show that there is an element $u'' \in H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors}$ such that its image under the map $\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-tors}$ is $[\sigma_G^{s,\mathrm{sym}}(\overline{M})]_2$ and the image of u'' under the composite

$$H_n^G(\widetilde{X}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors} \xrightarrow{\lambda_n^{s,\mathrm{sym}}(X)/2\text{-tors}} L^0(\mathbb{Z})/2\text{-tors} \xrightarrow{h^0/2\text{-tors}} \mathbb{Z}/2\text{-tors} = \mathbb{Z}$$

is 1 since then we can take $u = u'' - u'$.

For simplicity we give the proof of the existence of the element u'' only in the special case, where w is trivial. For every $n \geq 0$ and every connected CW-complex X, the symmetric signature defines a map, see [53, Proposition 6.3],

$$\sigma_n^{s,\mathrm{sym}}(X)\colon \Omega_n^{\mathrm{TOP}}(X) \to L_s^n(\mathbb{Z}[\pi_1(X)]), \quad [f\colon M \to X] \mapsto \sigma_G^{s,\mathrm{sym}}(\overline{M}).$$

Without giving the details of the proof, we claim that this natural transformation of functors from the category of CW-complexes to the category of $\mathbb{Z}$-graded abelian groups can be implemented as a functor from the category of CW-complexes to the category of spectra. We conclude from the general theory about assembly maps, see [18, Section 6] or [67], that we can lift $\sigma_n^{s,\mathrm{sym}}(X)$ over $\mathrm{asmb}_n^{s,\mathrm{sym}}(X)$ to a map $\tau_n^{s,\mathrm{sym}}(X)$

$$\begin{tikzcd} & H_n^G(\widetilde{X};\mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}}) \arrow[d, "\mathrm{asmb}_n^{s,\mathrm{sym}}(X)"] \\ \Omega_n^{\mathrm{TOP}}(X) \arrow[ur, "\tau_n^{s,\mathrm{sym}}(X)"] \arrow[r, "\sigma_n^{s,\mathrm{sym}}(X)"'] & L_h^n(\mathbb{Z}G) \end{tikzcd}$$

such that $\tau_*^{s,\mathrm{sym}}(-)$ is a transformation of homology theories. Such a construction seems also to be contained in [42]. Consider the map

$$\nu_n(X)\colon \Omega_n^{\mathrm{TOP}}(X) \xrightarrow{d_n} H_n(X;\mathbb{Z}) \xrightarrow{-\cap[X]} H^0(X;\mathbb{Z}) \to \mathbb{Z},$$

where the first map d_n sends $[f\colon M \to X]$ to $f_*([M])$. The naturality of the Atiyah-Hirzebruch spectral sequence implies that the following diagram

$$\begin{tikzcd} \Omega_n^{\mathrm{TOP}}(X) \arrow[r, "\nu_n(X)"] \arrow[d, "\tau_n^{s,\mathrm{sym}}(X)"'] & \mathbb{Z} \\ H_n^G(\widetilde{X};\mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}}) \arrow[r, "\lambda^{s,\mathrm{sym}}(X)"'] & L^0(\{\bullet\}) \arrow[u, "h^0"'] \end{tikzcd}$$

commutes. Define u'' to be the image of $f\colon M \to X$ under the composite

$$\Omega_n^{\mathrm{TOP}}(X) \xrightarrow{\tau_n^{s,\mathrm{sym}}(X)} H_n^G(\widetilde{X};\mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}}) \to H_n^G(\widetilde{X};\mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-tors}\,.$$

Since the degree of $f\colon M \to X$ is one, the image of $[f\colon M \to X]$ under $\nu_n(X)$ is 1. An easy diagram chase shows that u'' has the desired properties. This finishes the proof of Theorem 5.27. $\square$

Theorem 5.27 together with the homotopy invariance of the symmetric signature implies the homotopy invariance of the total surgery obstruction. More precisely, we have

Theorem 5.28 (Homotopy invariance of the total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex such that $\pi_1(X)$ is a Farrell-Jones group and $\mathcal{N}(X)$ is non-empty. Let Y be a finite n-dimensional CW-complex which is homotopy equivalent to X.*

Then Y is an aspherical oriented finite n-dimensional Poincaré complex such that $\pi_1(Y)$ is a Farrell-Jones group and such that $\mathcal{N}(Y)$ is non-empty. We get

$$\begin{aligned} s(X) &= s(Y); \\ s^{\mathrm{sym}}(X) &= s^{\mathrm{sym}}(Y). \end{aligned}$$

Proof. Choose a homotopy equivalence $f\colon X \to Y$. Define $w_1(Y) \in H^n(Y;\mathbb{Z}/2)$ to be $f^*w_1(X)$. Then obviously Y inherits the structure of an oriented finite n-dimensional Poincaré complex from X if we take as fundamental class $[Y]$ the image of $[X]$ under the isomorphism $H_n^{\pi_1(X)}(\widetilde{X};\mathbb{Z}^{w_1(X)}) \xrightarrow{\cong} H_n^{\pi_1(Y}(\widetilde{Y};\mathbb{Z}^{w_1(Y)})$ induced by f. A consequence of the basic features of the symmetric signature and G being a torsionfree Farrell-Jones group is that the isomorphism induced by $\pi_1(f)$

$$L_s^n(\mathbb{Z}[\pi_1(X)], w_1(X)) \xrightarrow{\cong} L_s^n(\mathbb{Z}[\pi_1(Y)], w_1(Y))$$

sends $\sigma_{\pi_1(X)}^{s,\mathrm{sym}}(\widetilde{X})$ to $\sigma_{\pi_1(Y)}^{s,\mathrm{sym}}(\widetilde{Y})$. Now apply Theorem 5.27. $\square$

Next we show a product formula.

Theorem 5.29 (Product formula)**.** *For each $i \in \{0,1\}$, let X_i be an aspherical oriented finite n_i-dimensional Poincaré complex with fundamental group $G_i = \pi_1(X_i)$ and orientation homomorphism $v_i := w_1(X_i)\colon G_i \to \{\pm 1\}$ such that G_i is a Farrell-Jones group and that $\mathcal{N}(X_i)$ is non-empty.*

Then $X_0 \times X_1$ is an aspherical oriented finite (n_0+n_1)-dimensional Poincaré complex with fundamental group $G_0 \times G_1$ and orientation homomorphisms $v := w_1(X \times N)\colon G_0 \times G_1 \to \{\pm 1\}$ sending (g_0, g_1) to $v_0(g_0) \cdot v_1(g_1)$ such that $G_0 \times G_1$ is a Farrell-Jones group and that $\mathcal{N}(X_0 \times X_1)$ is non-empty, and we get in $\mathbb{Z}$

$$(1 - 8 \cdot h^0(s(X_0 \times X_1))) = (1 - 8 \cdot h^0(s(X_0))) \cdot (1 - 8 \cdot h^0(s(X_1))).$$

Proof. The product $G_0 \times G_1$ is a Farrell-Jones group by Theorem 4.1 (2b).

The tensor product gives a pairing, see [52, Section 8],

$$\otimes\colon L_s^{n_0}(\mathbb{Z}G_0, v_0) \otimes L_s^{n_1}(\mathbb{Z}G_1, v_1) \to L^{n_0+n_1}(\mathbb{Z}[G_0, \times G_1], v). \tag{5.30}$$

Now we claim that there is a pairing

$$\times\colon H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{s,\mathrm{sym}}) \otimes H_{n_1}^{G_1}(\widetilde{X_1}; \mathbf{L}_{\mathbb{Z},v_1}^{s,\mathrm{sym}}) \to H_{n_0+n_1}^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},v}^{s,\mathrm{sym}})$$

such that the following diagram commutes
(5.31)

$$\begin{tikzcd}
L_s^{n_0}(\mathbb{Z}G_0, v_0) \otimes L_s^{n_1}(\mathbb{Z}G_1, v_1) \arrow[r, "\otimes"] & L^{n_0+n_1}(\mathbb{Z}[G_0, \times G_1], v) \\
H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{s,\mathrm{sym}}) \otimes H_{n_1}^{G_1}(\widetilde{X_1}; \mathbf{L}_{\mathbb{Z},v_1}^{s,\mathrm{sym}}) \arrow[r, "\times"] \arrow[u, "\mathrm{asmb}_{n_0}^{\epsilon,\mathrm{sym}}(\widetilde{X_0}) \otimes \mathrm{asmb}_{n_1}^{\epsilon,\mathrm{sym}}(\widetilde{X_1})"] \arrow[d, "\lambda_{n_0}^{s,\mathrm{sym}}(X_0) \otimes \lambda_{n_1}^{s,\mathrm{sym}}(X_1)"'] & H_{n_0+n_1}^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},v}^{s,\mathrm{sym}}) \arrow[u, "\mathrm{asmb}_{n_0+n_1}^{\epsilon,\mathrm{sym}}(\widetilde{X_0 \times X_1})"'] \arrow[d, "\lambda_{n_0+n_1}^{s,\mathrm{sym}}(X_0 \times X_1)"] \\
L^0(\mathbb{Z}) \otimes L^0(\mathbb{Z}) \arrow[r, "\otimes"] \arrow[d, "h^0 \otimes h^0"', "\cong"] & L^0(\mathbb{Z}) \arrow[d, "\cong"', "h^0"] \\
\mathbb{Z} \otimes \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

where the lowermost horizontal arrow is the multiplication on $\mathbb{Z}$. In order to get this diagram, one has firstly to promote the functor

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

to a functor

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}^{\mathrm{sym}}$$

to the category $\mathsf{SPECTRA}^{\mathrm{sym}}$ of symmetric spectra. Notice that the advantage of $\mathsf{SPECTRA}^{\mathrm{sym}}$ in comparison with $\mathsf{SPECTRA}$ is that $\mathsf{SPECTRA}^{\mathrm{sym}}$ has a functorial smash product $\wedge$. In the second step one has to construct a map of spectra

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_0) \wedge \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_1) \to \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}((G \times G)/(H_0 \times H_1)),$$

which on homotopy groups induces the map

$$\otimes\colon L_s^{n_0}(\mathbb{Z}H_0, v_0|_{H_0}) \otimes L_s^{n_1}(\mathbb{Z}H_1, v_1|_{H_1}) \to L_s^{n_0+n_1}(\mathbb{Z}[H_0 \times H_1], v|_{H_0 \times H_1})$$

under the identifications

$$\begin{aligned}
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_0)) &\cong L_s^k(\mathbb{Z}H_0, v_0|_{H_0}); \\
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_1)) &\cong L_s^k(\mathbb{Z}H_1, v_1|_{H_1}); \\
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G \times G/H_0 \times H_1)) &\cong L_s^k(\mathbb{Z}[H_0 \times H_1], v|_{H_0 \times H_1}),
\end{aligned}$$

and are natural in G/H_0 and G/H_1. We omit the details of this construction, see also Remark 5.32. Now the claim follows from Theorem 5.27 and the product

formula for the symmetric signature, see [53, Proposition 8.1 (i)], which says that the pairing (5.30) sends $\sigma^{s,\mathrm{sym}}_{G_0}(\widetilde{X_0}) \otimes \sigma^{s,\mathrm{sym}}_{G_1}(\widetilde{X_1})$ to $\sigma^{s,\mathrm{sym}}_{G_0\times G_1}(\widetilde{X_0 \times X_1})$.
[42] $\square$

Remark 5.32 (Special case of Theorem 5.29)**.** In the proof of Theorem 5.29 we have not given the details of the proof of the existence of the commutative diagram (5.31). We will need Theorem 5.29 only in the special case, where $n_0 = 3$ and X_1 is a closed n-dimensional manifold and then the desired assertion is

$$s^{\mathrm{sym}}(X_0) = s^{\mathrm{sym}}(X_0 \times X_1).$$

For the reader's convenience we give a direct complete proof in this special case. We have $L^0(\mathbb{Z}) \cong \mathbb{Z}$, $L^1(\mathbb{Z}) \cong \mathbb{Z}/2$ and $L^i(\mathbb{Z}) = 0$ for $i = 1, 2$, see [52, Proposition 7.2]. The Atiyah-Hirzebruch spectral sequence shows that the map $\lambda^{\epsilon,\mathrm{sym}}_n(X_0)$ of (5.20) induces an isomorphism

$$\lambda^{\epsilon,\mathrm{sym}}_{n_0}(X_0)/2\text{-tors}\colon H^{G_0}_{n_0}(\widetilde{X_0};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors} \to L^0(\mathbb{Z})/2\text{-tors} = L^0(\mathbb{Z})$$

since we assume $n_0 = 3$. We have already shown in Theorem 5.27 that

$$\mathrm{asmb}^{s,\mathrm{sym}}_{n_0}(X_0)/2\text{-tors}\colon H^{G_0}_{n_0}(\widetilde{X_0};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors} \to L^{n_0}_s(\mathbb{Z}G_0, v_0)/2\text{-tors}$$

is injective and that there is a unique element $u_0 \in H^{G_0}_n(\widetilde{X_0};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},v_0})$ which is mapped to $1 - h^0(s^{\mathrm{sym}}(X_0))$ and to $[\sigma^{s,\mathrm{sym}}_{G_0}(\widetilde{X_0})]_2$ under these maps. Let $(f_0, \overline{f_0})$ be a normal map from a closed 3-manifold M_0 to X_0. We have explained in the proof of Theorem 5.27 that there is an element $u_0'' \in H^{G}_{n_0}(\widetilde{X_0};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})$ whose image under $\mathrm{asmb}^{s,\mathrm{sym}}_{n_0}(X_0)/2\text{-tors}\colon H^{G_0}_{n_0}(\widetilde{X_0};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors} \to L^{n_0}_s(\mathbb{Z}G_0, v_0)/2\text{-tors}$ is $\sigma^{s,\mathrm{sym}}_{G_0}(\overline{M_0})$ for the G_0-covering $\overline{M_0} \to M_0$ given by the pullback of $\widetilde{X_0} \to X_0$ with f_0 and whose image under the isomorphism

$$h^0 \circ \lambda^{\epsilon,\mathrm{sym}}_{n_0}(X_0)/2\text{-tors}\colon H^{G_0}_n(\widetilde{X_0};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},v_0})/2\text{-tors} \xrightarrow{\cong} \mathbb{Z}$$

is 1. Hence we get

$$[\sigma^{s,\mathrm{sym}}_{G_0}(\widetilde{X_0})]_2 = (1 - h^0(s^{\mathrm{sym}}(X_0))) \cdot [\sigma^{s,\mathrm{sym}}_{G_0}(\overline{M_0})]_2.$$

We conclude from the product formula for the symmetric signature, see [53, Proposition 8.1 (i)],

$$\begin{array}{lrcl} (5.33) & \sigma^{s,\mathrm{sym}}_{G_0\times G_1}(\widetilde{X_0 \times X_1}) & = & \sigma^{s,\mathrm{sym}}_{G_0}(\widetilde{X_0}) \otimes \sigma^{s,\mathrm{sym}}_{G_1}(\widetilde{X_1}) \\ & & = & (1 - h^0(s^{\mathrm{sym}}(X_0)) \cdot \sigma^{s,\mathrm{sym}}_{G_0}(\overline{M_0})) \otimes \sigma^{s,\mathrm{sym}}_{G_1}(X_1) \\ & & = & (1 - h^0(s^{\mathrm{sym}}(X_0))) \cdot \sigma^{s,\mathrm{sym}}_{G_0\times G_1}(\overline{M_0} \times \widetilde{X_1}). \end{array}$$

As we have explained in the proof of Theorem 5.27, there exists a unique element $u'' \in H^{G_0\times G_1}_n(\widetilde{X_0} \times \widetilde{X_1};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors}$, whose image under

$$\begin{array}{l} \mathrm{asmb}^{s,\mathrm{sym}}_{n_0+n_1}(X_0 \times X_1)/2\text{-tors}\colon H^{G_0\times G_1}_{n_0+n_1}(\widetilde{X_0 \times X_1};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors} \\ \qquad\qquad\qquad\qquad \to L^{n_0+n_1}_s(\mathbb{Z}[G_0 \times G_1], v_0 \times v_1)/2\text{-tors} \end{array}$$

is $[\sigma^{s,\mathrm{sym}}_{G_0\times G_1}(\overline{M_0} \times \widetilde{X_1})]_2$ and whose image under

$$h^0 \circ \lambda^{\epsilon,\mathrm{sym}}_n(X_0 \times X_1)/2\text{-tors}\colon H^{G_0\times G_1}_n(\widetilde{X_0 \times X_1};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-tors} \to \mathbb{Z}$$

is 1. Here we use that X_1 and hence $M_0 \times X_1$ is a closed manifold. Theorem 5.27 together with (5.33) implies

$$(1 - h^0(s^{\mathrm{sym}}(X_0 \times X_1))) = 1 - h^0(s^{\mathrm{sym}}(X_0))).$$

Hence we get $s^{\mathrm{sym}}(X_0) = s^{\mathrm{sym}}(X_0 \times X_1)$.

5.3. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. Recall from Subsection 1.1 that there is a finite 3-dimensional Poincaré complex model X for BG. Also recall from Theorem 3.1 that $\mathcal{N}(BG)$ is non-empty. The implication $2 \implies 1$ is obviously true; the implication $1 \implies 2$ is proved as follows.

By assumption there are closed aspherical topological manifolds M_0 and N_0 and a homotopy equivalence $f_0 \colon M_0 \to X \times N_0$. We conclude from Theorem 5.28 that $s^{\mathrm{sym}}(M_0) = s^{\mathrm{sym}}(X \times N_0)$. Since M_0 and N_0 are closed aspherical topological manifolds, $s^{\mathrm{sym}}(M_0)$ and $s^{\mathrm{sym}}(N_0)$ vanish by Theorem 5.26 (2). We conclude from Theorem 5.29, or just from Remark 5.32, that $s^{\mathrm{sym}}(X) = 0$. From Theorem 5.26 (1) we obtain a normal map of degree one $(f, \overline{f})$ with target X and vanishing simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_3^s(\mathbb{Z}G, w_1(X))$. Let N be a closed smooth manifold, closed PL-manifold, or closed topological manifold respectively of dimension ≥ 2. By the product formula for the surgery obstruction, see [53, Proposition 8.1(ii)], the surgery obstruction of the normal map of degree one $(f \times \mathrm{id}_N, \overline{f} \times \mathrm{id}_{TN})$ obtained by crossing $(f, \overline{f})$ with N is trivial. Since the dimension of $X \times N$ is greater or equal to 5, we can do surgery in the smooth, PL, or topological category respectively to arrange that $f \times \mathrm{id}_N$ is a simple homotopy equivalence with a closed smooth manifold, closed PL-manifold, or closed topological manifold respectively as source. □

6. Short review of ENR homology manifolds

A topological space X is called a *Euclidean neighborhood retract* or briefly an ENR if it X is homeomorphic to a closed subset X' of some Euclidean space $\mathbb{R}^n$ such that X' has an open neighborhood U in $\mathbb{R}^n$ that retracts to X. Such a space is finite-dimensional, metrizable, separable, locally compact, and locally contractible. It is an illuminating exercise using the Tietze Extension Theorem to show that if such an X is embedded as a closed subset of any normal space, then X is a neighborhood retract in that space.

A theorem of Borsuk says that every finite-dimensional, metrizable, separable, locally compact, and locally contractible space X is an ENR. The one-point compactification of such an X is finite-dimensional and therefore embeds in a finite-dimensional sphere. Removing the point at infinity from both the one-point compactification and the sphere yields a closed embedding of X into a Euclidean space. In Theorem A7 of [32] a neighborhood retraction is constructed for compact X. The argument given extends easily to noncompact X. Hatcher focuses on the compact case in order to emphasize that compact ENRs have finitely presented fundamental groups and finitely generated homology groups.

Definition 6.1 (ENR homology manifold)**.** A *n-dimensional* ENR *homology manifold X (without boundary)* is an ENR such that for every $x \in X$ the i-th singular homology group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$. We call X *closed* if it is compact.

An ENR homology manifold in the sense of Definition 6.1 is the same as a generalized manifold in the sense of Daverman [17, page 191], as pointed out in [8, page 3]. Every closed n-dimensional topological manifold is a closed n-dimensional ENR homology manifold (see [17, Corollary 1A in V.26 page 191]).

Definition 6.2 (DDP)**.** An ENR homology manifold M is said to have the *disjoint disk property (DDP)*, if for one (and hence any) choice of metric on M, any $\epsilon > 0$ and any maps $f, g \colon D^2 \to M$, there are maps $f', g' \colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$.

Definition 6.3 (ENR homology manifold with ENR boundary)**.** An *n-dimensional* ENR *homology manifold X with boundary ∂X* is an ENR X which is a disjoint union $X = \operatorname{int} X \cup \partial X$, where

- $\operatorname{int} X$ is an n-dimensional ENR homology manifold, the "interior" of the homology manifold with boundary X;
- ∂X is an $(n-1)$-dimensional ENR homology manifold;
- for every $z \in \partial X$ the singular homology group $H_i(X, X \setminus \{z\})$ vanishes for all i.

This definition is rather general. It includes the "bad" closed complementary domain of an Alexander Horned sphere embedded in S^3. In our main application, however, the boundary will be a Z-set (see below) in X.

7. A stable ENR-version of the Cannon Conjecture

Theorem 7.1 (Stable ENR-version of the Cannon Conjecture)**.** *Let G be a torsion free hyperbolic group. Suppose that its boundary is homeomorphic to S^{n-1}. Let Γ be any d-dimensional Poincaré group for some natural number d satisfying $n + d \geq 6$ which is a Farrell-Jones group.*

Then there is a closed aspherical ENR *homology manifold X of dimension $n + d$ which has the DDP and satisfies $\pi_1(X) \cong G \times \Gamma$.*

Proof. We conclude that $G \times \Gamma$ is a Farrell-Jones group from Theorem 4.1 (1a) and (2b). Since G is a Poincaré duality group of dimension n by [10, Corollary 1.3], the product $G \times \Gamma$ is a Poincaré duality group of dimension $n + d$. Since by assumption $n + d \geq 6$, we can apply Theorem 1.11. □

8. Short review of Quinn's obstruction

In order to replace ENR homology manifolds by topological manifolds in the above result, we will use the following result that combines work of Edwards and Quinn, see [17, Theorems 3 and 4 on page 288], [50].

Theorem 8.1 (Quinn's obstruction)**.** *Let X be a connected* ENR *homology manifold. There is an invariant $\iota(X) \in 1 + 8\mathbb{Z}$, known as the* Quinn obstruction*, with the following properties:*

1. *If $U \subset X$ is a connected non-empty open subset, then $\iota(U) = \iota(X)$;*
2. *Let X be an* ENR *homology manifold of dimension ≥ 5. Then the following are equivalent:*
   - *X has the DDP and $\iota(X) = 1$;*
   - *X is a topological manifold.*

The elementary proof of the following result can be found in [8, Corollary 1.6].

Lemma 8.2. *Let X be a connected* ENR *homology manifold with boundary ∂X. If ∂X is a manifold and* $\dim(X) \geq 5$*, then $\iota(\operatorname{int} X) = 1$.*

Although we do not need the next result in this paper, we mention that it follows from [54, Proposition 25.8 on page 293] using Theorem 5.27, since we assume aspherical.

Theorem 8.3 (Relating the total surgery obstruction and Quinn's obstruction)**.** *Let B be an aspherical finite n-dimensional Poincaré complex which is homotopy equivalent to an n-dimensional closed* ENR *homology manifold X. Suppose that $\pi_1(B)$ is a Farrell-Jones group.*

Then we get

$$i(X) = 8 \cdot h_0(s(B)) + 1 = h^0(s^{\mathrm{sym}}(B)) + 1.$$

Notice that in the situation of Theorem 8.3 the total surgery obstruction $s(B)$ is defined without the assumption that B is homotopy equivalent to an n-dimensional closed ENR homology manifold and therefore does make sense for any aspherical 3-dimensional Poincaré complex, and moreover, that $s(B)$ is a homotopy invariant, see Theorem 5.28.

Remark 8.4. There is no example in the literature of a closed spherical ENR homology manifold which is not homotopy equivalent to a closed topological manifold.

9. Z-sets

Definition 9.1 (Z-set)**.** A closed subset Z of a compact ENR X is called a *Z-set or a set of infinite deficiency* if for every open subset U of X the inclusion $U \setminus (U \cap Z) \to U$ is a homotopy equivalence.

Any closed subset of the boundary ∂M of a compact topological manifold M is a Z-set in M. According to [10, page 470] each of the following properties characterizes Z-sets. Here, X is a compact metric ENR. The noncompact case is similar, except that maps and homotopies are limited by arbitrary open covers rather than by fixed ϵ's.

(1) For every $\epsilon > 0$ there is a map $X \to X \setminus Z$ which is ϵ-close to the identity.
(2) For every closed subset $A \subseteq Z$, there exists a homotopy $H \colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$, $H_t|_A$ is the inclusion $A \to X$ and $H_t(X \setminus A) \subseteq X \setminus Z$ for all $t > 0$.
To this, we will add:
(3) There exists a homotopy $H \colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subseteq X \setminus Z$ for all $t > 0$.

This last is (2) with $A = \emptyset$. Clearly, (3) implies (1) and (2) implies (3), so (3) also suffices as a definition of Z-set. This is the definition we will use in what follows. Condition (3) implies that for every open $U \subset X$ the inclusion $U \setminus Z \to U$ is a homotopy equivalence. If $\alpha : S^k \to U$ is a map, then $H_t \circ \alpha : S^k \to X$ is a map homotopic to α and for $t > 0$ its image lies in $X \setminus Z$. For $t > 0$ sufficiently small, this homotopy takes place in U, so the inclusion-induced map $\pi_k(U \setminus Z) \to \pi_k(U)$ is surjective. A similar argument shows that $\pi_k(U \setminus Z) \to \pi_k(U)$ is a monomorphism – if α extends over a disk in U, push the disk off of Z. The homotopy equivalence follows from the Whitehead Theorem, since ENR$'s$ have the homotopy types of CW complexes. The next result is taken from [8, Proposition 2.5].

Lemma 9.2. *Let X be an* ENR *which is the disjoint union of an n-dimensional* ENR *homology manifold* $\operatorname{int} X$ *and an $(n-1)$-dimensional* ENR *homology manifold ∂X such that ∂X is a Z-set in X. Then X is an* ENR *homology manifold with boundary ∂X.*

Definition 9.3 (Compact sets become small at infinity)**.** Consider a pair $(\overline{Y}, Y)$ of G-spaces, G a discrete group. We say that *compact subsets of Y become small at infinity*, if, for every $y \in \partial Y := \overline{Y} \setminus Y$, open neighborhood $U \subseteq \overline{Y}$ of y, and compact subset $K \subseteq Y$, there exists an open neighborhood $V \subseteq U$ of y with the property that for every $g \in G$ we have the implication $g \cdot K \cap V \neq \emptyset \implies g \cdot K \subseteq U$.

In the sequel we will choose l large enough such that the following claims are true for the torsion free hyperbolic group G and its Rips complex $P_l(G)$.

(1) The projection $P_l(G) \to P_l(G)/G$ is a model for the universal principal G-bundle $EG \to BG$ and $P_l(G)/G$ is a finite CW-complex;
(2) One can construct a compact topological space ∂G and a compactification $\overline{P_l(G)}$ of $P_l(G)$ such that $\partial G = \overline{P_l(G)} \setminus P_l(G)$ holds, and $P_l(G)$ is open and dense in $\overline{P_l(G)}$;

(3) $\overline{P_l(G)}$ is a compact metrizable ENR such that $\partial G \subset \overline{P_l(G)}$ is Z-set and $\overline{P_l(G)}$ has finite topological dimension;
(4) Compact subsets of Y become small at infinity for the pair $(\overline{P_l(G)}, P_l(G))$.

The first claim is proved for instance in [45]. The second claim follows from [12, III.H.3.6 on page 429, III.H.3.7(3) and (4) on page 430, III.H.3.7(4) on page 430 and III.H.3.18(4) on page 433] and [5, 9.3.(ii)]. The third claim is due to Bestvina-Mess [10, Theorem 1.2], see also [56, Theorem 3.7]. The fourth assertion is for instance proved in [56, page 531].

10. Pulling back boundaries

We will need the following construction which may be interesting in its own right.

Let $(\overline{Y}, Y)$ be a topological pair. Put $\partial Y := \overline{Y} \setminus Y$. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Define a topological pair $(\overline{X}, X)$ and a map $\overline{f}\colon \overline{X} \to X$, which will turn out to be continuous, as follows. The underlying set of $\overline{X}$ is the disjoint union $X \amalg \partial Y$. We define the map of sets $\overline{f}\colon \overline{X} \to \overline{Y}$ to be $f \cup \mathrm{id}_{\partial Y}$. A subset W of $\overline{X}$ is declared to be open if there exist open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$ such that $W = \overline{f}^{-1}(U) \cup V$. We will see that this defines a topology. Obviously $\overline{X}$ and $\emptyset$ are open. Given a collection of open subsets $\{W_i \mid i \in I\}$, their union is again open by the following equality, if we write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$ and define open subsets $U := \bigcup_{i\in I} U_i \subseteq \overline{Y}$ and $V := \bigcup_{i\in I} V_i \subseteq X$:

$$\bigcup_{i\in I} W_i = \bigcup_{i\in I} \left(\overline{f}^{-1}(U_i) \cup V_i\right) = \bigcup_{i\in I} \overline{f}^{-1}(U_i) \cup \bigcup_{i\in I} V_i$$
$$= \overline{f}^{-1}\left(\bigcup_{i\in I} U_i\right) \cup \bigcup_{i\in I} V_i = \overline{f}^{-1}(U) \cup V.$$

Given two open subsets W_1 and W_2, their intersection is again open by the following equality, if we write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$ for $i = 1, 2$ and define open subsets $U := U_1 \cap U_2 \subseteq \overline{Y}$ and $V := \big(f^{-1}(U_1 \cap Y) \cap V_2\big) \cup \big(V_1 \cap f^{-1}(U_2 \cap Y)\big) \cup \big(V_1 \cap V_2\big) \subseteq X$:

$$\begin{aligned}
&W_1 \cap W_2\\
&= \big(\overline{f}^{-1}(U_1) \cup V_1\big) \cap \big(\overline{f}^{-1}(U_2) \cup V_2\big)\\
&= \big(\overline{f}^{-1}(U_1) \cap \overline{f}^{-1}(U_2)\big) \cup \big(\overline{f}^{-1}(U_1) \cap V_2\big) \cup \big(V_1 \cap \overline{f}^{-1}(U_2)\big) \cup \big(V_1 \cap V_2\big)\\
&= \overline{f}^{-1}(U_1 \cap U_2) \cup \Big(\big(f^{-1}(U_1 \cap Y) \cap V_2\big) \cup \big(V_1 \cap f^{-1}(U_2 \cap Y)\big) \cup \big(V_1 \cap V_2\big)\Big)\\
&= \overline{f}^{-1}(U) \cup V.
\end{aligned}$$

Definition 10.1 (Pulling back the boundary)**.** We say that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by *pulling back the boundary with f*.

Notice that this is the smallest topology on the set $\overline{X} = X \amalg \partial Y$ for which $\overline{f}$ is continuous and $X \subseteq \overline{X}$ is an open subset. This leads to the following universal property of the construction "pulling back the boundary".

Lemma 10.2. *Let $(\overline{Y}, Y)$ be a topological pair. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f. Consider any pair of spaces $(\overline{\overline{X}}, X)$ and map of pairs $(\overline{\overline{f}}, f)\colon (\overline{\overline{X}}, X) \to (\overline{Y}, Y)$ such that X is an open subset of $\overline{\overline{X}}$ and $\overline{\overline{f}}$ induces a map $\overline{\overline{X}} \setminus X \to \partial Y := \overline{Y} \setminus Y$.*

Then there is precisely one map $u\colon \overline{\overline{X}} \to \overline{X}$ which induces the identity on X and satisfies $\overline{f} \circ u = \overline{\overline{f}}$.

Proof. As a map of sets u exists and is uniquely determined by the properties that u induces the identity on X and $\overline{f} \circ u = \overline{\overline{f}}$. Namely, for $x \in X$ define $u(x) = x$ and for $x \in \overline{\overline{X}} \setminus X$ define $u(x)$ by $\overline{\overline{f}}(x) \in \partial\overline{Y} = \partial\overline{X} \subseteq \overline{X}$. We have to show that u is continuous, i.e., $u^{-1}(W) \subseteq \overline{\overline{X}}$ is open for every open subset $W \subseteq \overline{X}$. By definition there are open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$ such that $W = \overline{f}^{-1}(U) \cup V$. Then $u^{-1}(W) = \overline{\overline{f}}^{-1}(U) \cup V$. Since $\overline{\overline{f}}$ is continuous, $\overline{\overline{f}}^{-1}(U) \subseteq \overline{\overline{X}}$ is open. Since X is open in $\overline{\overline{X}}$ and the topology on X is the subspace topology of $X \subseteq \overline{\overline{X}}$, we conclude that for any open subset $V \subseteq X$ the subset $V \subseteq \overline{\overline{X}}$ is open. Hence $u^{-1}(W) \subseteq \overline{\overline{X}}$ is open. □

Lemma 10.3. *Let $(\overline{Y}, Y)$ be a topological pair. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f.*

1. *If $Y \subseteq \overline{Y}$ is dense and the closure of the image of f in $\overline{Y}$ contains ∂Y, then $X \subseteq \overline{X}$ is dense;*
2. *Suppose that $\overline{Y}$ is compact, $Y \subseteq \overline{Y}$ is open and $f\colon X \to Y$ is proper. Then $\overline{X}$ is compact;*
3. *We have for the topological dimension of $\overline{X}$*
$$\dim(\overline{X}) \leq \dim(X) + \dim(\overline{Y}) + 1;$$
4. *The map $\overline{f}\colon \overline{X} \to \overline{Y}$ given by $f \cup \mathrm{id}_{\partial Y}$ is continuous;*
5. *The induced map $\overline{f}$ induces a homeomorphism $\partial f\colon \partial X \to \partial Y$;*
6. *Let $g\colon Z \to X$ be a map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ and $(\overline{f \circ g}, f \circ g)\colon (\overline{Z}, Z) \to (\overline{Y}, Y)$ respectively are obtained by pulling back the boundary of $(\overline{Y}, Y)$ with f and $f \circ g$ respectively. Let $\overline{g}\colon (\overline{\overline{Z}}, Z) \to (\overline{X}, X)$ be obtained by pulling back the boundary of $(\overline{X}, X)$ with g.*

   Then we get an equality of topological spaces $\overline{\overline{Z}} = \overline{Z}$ and of maps $\overline{f} \circ \overline{g} = \overline{f \circ g}$.

Proof. (1) Consider $x \in \partial X$ and a neighborhood W of x in $\overline{X}$. We have to show $X \cap W \neq \emptyset$. We can write $W = \overline{f}^{-1}(U) \cup V$ for open subsets $U \subset \overline{Y}$ and $V \subseteq X$. Without loss of generality we can assume $V = \emptyset$, or, equivalently $W = \overline{f}^{-1}(U)$ for open subset $U \subset \overline{Y}$. Obviously U is an open neighborhood of $\overline{f}(x) \in \overline{Y}$. Since by assumption the closure of the image of f in $\overline{Y}$ contains ∂Y, we have $\mathrm{im}(f) \cap U \neq \emptyset$ and hence $X \cap W \neq \emptyset$.

(2) Let $\{W_i \mid i \in I\}$ be an open covering of $\overline{X}$. We can write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$. Then $\{U_i \cap \partial Y \mid i \in I\}$ is an open covering of ∂Y. Since $\partial Y \subseteq Y$ is closed and $\overline{Y}$ is compact by assumption, ∂Y is compact. Hence there is a finite subset $J \subseteq I$ with $\partial Y \subseteq \bigcup_{i \in J} U_i$. The set $\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)$ is closed in $\overline{Y}$ and hence compact. Since $\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)$ is contained in Y and $f\colon X \to Y$ is by assumption proper, the preimage $f^{-1}\left(\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)\right)$ is also compact. Hence there is a finite subset $J' \subseteq I$ such that $\{V_i \mid j \in J'\}$ covers $f^{-1}\left(\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)\right)$. Hence $\{W_i \mid i \in J \cup J'\}$ covers $\overline{X}$. This shows that $\overline{X}$ is compact.

(3) Consider any open covering $\mathcal{W} = \{W_i \mid i \in I\}$ of $\overline{X}$. By definition there are $U_i \subseteq \overline{Y}$ and $V_i \subseteq X$ such that $W_i = \overline{f}^{-1}(U_i) \cup V_i$. Now put

$$\begin{aligned} \mathcal{W}_{\partial X} &:= \{\overline{f}^{-1}(U_i) \mid i \in I\}; \\ \mathcal{W}_X &:= \{W_i \cap X \mid i \in I\}. \end{aligned}$$

Then $\mathcal{W}_{\partial X} \cup \mathcal{W}_X$ is an open covering of $\overline{X}$, which is a refinement of $\mathcal{W}$. Moreover, $\mathcal{W}_X$ is an open covering of X and the union of the elements in $\mathcal{W}_{\partial X}$ contains ∂X. We can find an open covering $\mathcal{V}_X$ whose covering dimension is less or equal to $\dim(X)$ and which refines $\mathcal{W}_X$. We obtain an open covering $\{U_i \mid i \in I\} \cup \{Y\}$ of $\overline{Y}$, since ∂Y is contained in $\bigcup_{i \in I} U_i$. We can find an open covering $\mathcal{V}_{\overline{Y}}$ of $\overline{Y}$ which is a refinement of $\{U_i \mid i \in I\} \cup \{Y\}$ and has dimension $\leq \dim(\overline{Y})$. Put

$$\mathcal{V}_{\partial Y} := \{V \in \mathcal{V}_{\overline{Y}} \mid V \cap \partial Y \neq \emptyset\}.$$

Then $\mathcal{V}_{\partial Y}$ is a refinement of $\{U_i \mid i \in I\}$, has covering dimension $\leq \dim(\overline{Y})$ and the union of the elements in $\mathcal{V}_{\partial Y}$ contains ∂Y. Define $\overline{f}^* \mathcal{V}_{\partial X}$ to be the collection of open subsets of $\overline{X}$ given by $\{\overline{f}^{-1}(V) \mid V \in \mathcal{V}_{\partial Y}\}$. Then $\overline{f}^* \mathcal{V}_{\partial X}$ is a refinement of $\mathcal{W}_{\partial X}$, has covering dimension $\leq \dim(\overline{Y})$ and the union of its elements contains $\partial X = \partial Y$. Put

$$\mathcal{V} = \mathcal{W}_X \cup \overline{f}^* \mathcal{V}_{\partial X}.$$

Then $\mathcal{V}$ is an open covering of $\overline{X}$ which refines $\mathcal{W}$. Its covering dimension satisfies

$$\dim(\mathcal{V}) \leq \dim(\mathcal{V}_X) + \dim(\overline{f}^* \mathcal{V}_{\partial X}) + 1 \leq \dim(X) + \dim(\overline{Y}) + 1.$$

(4) If $U \subseteq \overline{Y}$ is open, then by definition $\overline{f}^{-1}(U) \subseteq \overline{X}$ is open.

(5) Obviously $\overline{f} \colon \overline{X} \to \overline{Y}$ induces a bijective continuous map $\partial f \colon \partial X \to \partial Y$. We have to show that it is open. An open subset of ∂X is of the form $\bigl(\overline{f}^{-1}(U) \cup V\bigr) \cap \partial X$ for some open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$. Its image under ∂f is $U \cap \partial Y$ and hence an open subset of ∂Y.

(6). Notice that as sets $\overline{\overline{Z}}$ and $\overline{Z}$ agree, both look like $Z \amalg \partial Y$. Next we show that the two topologies agree. A subset W of $\overline{\overline{Z}}$ is open if there are open subsets $U \subseteq \overline{Y}$ and $V_2 \subseteq Z$ with $W = \overline{f \circ g}^{-1}(U) \cup V_2$. A subset $W_1 \subseteq \overline{X}$ is open if there exist open subsets $U \subseteq \overline{Y}$ and $V_1 \subseteq X$ with $W_1 = \overline{f}^{-1}(U) \cup V_1$. A subset W_2 of $\overline{Z}$ is open, if there exist open subsets $W_1 \subseteq \overline{X}$ and $V_2 \subseteq Z$ such that W_2 looks like $\overline{g}^{-1}(W_1) \cup V_2$. This is equivalent to the existence of open subsets $U \subseteq \overline{Y}$, $V_1 \subseteq X$ and $V_2 \subseteq Z$ such that

$$W_2 = \overline{g}^{-1}\bigl(\overline{f}^{-1}(U) \cup V_1\bigr) \cup V_2.$$

Since

$$\overline{g}^{-1}\bigl(\overline{f}^{-1}(U) \cup V_1\bigr) \cup V_2 = \overline{f \circ g}^{-1}(U) \cup \bigl(g^{-1}(V_1) \cup V_2\bigr)$$

and $g^{-1}(V_1) \cup V_2$ is an open subset of Z, the topology on $\overline{\overline{Z}}$ is finer than the topology on $\overline{Z}$. So it remains to show that the topology on $\overline{Z}$ is finer than the topology on $\overline{\overline{Z}}$. This follows from the observation that for open subsets $U \subseteq \overline{Y}$ and $V_2 \subseteq Z$ we get

$$\overline{f \circ g}^{-1}(U) \cup V_2 = \overline{g}^{-1}(\overline{f}^{-1}(U) \cup \emptyset) \cup V_2.$$

□

Example 10.4 (One-point-compactification)**.** Let X and Y be locally compact Hausdorff spaces. Denote by X^c and Y^c their one-point-compactification. Let $f \colon X \to Y$ be a map. Denote by $(\overline{X}, X)$ the space obtained from (Y^c, Y) by pulling back the boundary with f.

Consider first the case where f is proper. Recall that a subset $W \subseteq Y^c = Y \cup \{\infty\}$ is open if it belongs to Y and is open in Y or there is a compact subset $C \subseteq Y$ such that $W = Y^c \setminus C$. This is indeed a topology, see [48, page 184]. By construction the underlying sets for $\overline{X}$ and X^c agree, namely, they are both given by $X \amalg \{\infty\}$. Next we compare the topologies.

Consider an open subset W of $\overline{X}$. We want to show that $W \subseteq X^c$ is open. We can write $W = \overline{f}^{-1}(U) \cup V$ for open subsets $U \subseteq Y^c$ and $V \subseteq X$. If ∞ does not belong to U, then U is an already open subset of Y and $\overline{f}^{-1}(U) = f^{-1}(U)$ is an open subset of X which implies that $W \subseteq X$ and hence $W \subseteq X^c$ are open. It remains to treat the case $\infty \in U$. From the definitions we conclude that we can write $W = \overline{f}^{-1}(Y^c \setminus C) \cup V$ for some compact subset $C \subseteq Y$ and an open subset V of Y. Since

$$\overline{f}^{-1}(Y^c \setminus C) = \overline{X} \setminus f^{-1}(C)$$

and by the properness of f the set $f^{-1}(C) \subseteq X$ is compact, W is open regarded as a subset of X^c.

This shows that the identity induces a continuous bijective map $X^c \to \overline{X}$. (One can also deduce this directly from Lemma 10.2.)

Since X^c is compact and $\overline{X}$ is Hausdorff, this is a homeomorphism, see [48, Theorem 5.6 in Chapter III on page 167]. Hence we get an equality of topological spaces $\overline{X} = X^c$ and of maps $\overline{f} = f^c$.

Now consider the case where f is the constant map onto some point $y_0 \in Y$. Suppose that X is not compact, or, equivalently, that the constant map f is not proper. The set $Y^c \setminus \{y_0\}$ is open in Y^c. Hence $\partial X = \{\infty\} = \overline{f}^{-1}(Y^c \setminus \{y_0\})$ is an open subset of $\overline{X}$. Since also $X \subseteq \overline{X}$ is open, $\overline{X}$ is, as a topological space, the disjoint union $X \amalg \{\infty\}$. Since X is not compact, its one-point compactification is not homeomorphic to $\overline{X}$.

Remark 10.5 (Dependency on f)**.** Example 10.4 shows that $\overline{X}$ does depend on the choice of f. So the reader should be careful when we just write $\overline{X}$ without including f in the notation.

Lemma 10.6. *Consider a pair $(\overline{Y}, Y)$ of G-spaces, G a discrete group, such that compact subsets of Y become small at infinity in the sense of Definition 9.3. Let $f\colon X \to Y$ be a G-map. Suppose that $(\overline{X}, X)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f.*

Then compact subsets of X become small at infinity.

Proof. Consider an element $x \in \partial X$, an open neighborhood $U \subseteq \overline{X}$ of x, and a compact subset $K \subseteq X$. We can find an open neighborhood $U' \subseteq Y$ of $f(x) \in \partial Y$ and an open subset $W \subseteq X$ such that $U = \overline{f}^{-1}(U') \cup W$. Put $L = f(K)$. Then $L \subseteq Y$ is compact. By assumption we can find an open neighborhood $V' \subseteq U'$ of $f(x) \in \partial Y$ such that the implication $g \cdot L \cap V' \neq \emptyset \implies g \cdot L \subseteq U'$ holds for every $g \in G$. Put $V = \overline{f}^{-1}(V')$. This is an open neighborhood of $x \in \partial X$ with $V \subseteq U$. Moreover we get for every $g \in G$

$$\begin{aligned} g \cdot K \cap V \neq \emptyset \implies & g \cdot L \cap V' \neq \emptyset \\ & \implies g \cdot L \subseteq U' \implies g \cdot f^{-1}(L) \subseteq \overline{f}^{-1}(U') \implies g \cdot K \subseteq U. \end{aligned}$$

□

Definition 10.7 (Continuously controlled over Y at ∂Y)**.** Consider a pair $(\overline{Y}, Y)$ of spaces and a homotopy equivalence $f\colon X \to Y$. We call f *continuously controlled over Y at ∂Y* if there exists a map $u\colon Y \to X$ and homotopies $h\colon f \circ u \simeq \mathrm{id}_Y$ and $k\colon u \circ f \simeq \mathrm{id}_X$ with the following property: For every $z \in \partial Y = \overline{Y} \setminus Y$ and neighborhood U of z in $\overline{Y}$ there is an open neighborhood V of z in $\overline{Y}$ with $V \subseteq U$ such that the following two implications are true:

- $y \in V \implies h(\{y\} \times [0,1]) \subseteq U$;
- $x \in f^{-1}(V) \implies f \circ k(\{x\} \times [0,1]) \subseteq U$.

Lemma 10.8. *Let $f\colon X \to Y$ be a G-map of proper free G-spaces, G a discrete group. Suppose that X is cocompact. Then f is proper.*

Proof. We have the following pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X/G \arrow[r, "f/G"] & Y/G \end{tikzcd}$$

where the vertical maps are principal G-bundles. Since X/G is compact, f/G is proper. Hence f is proper by [43, Lemma 1.16 on page 14]. □

Lemma 10.9. *Consider a pair $(\overline{Y}, Y)$ of spaces such that $\overline{Y}$ is a compact* ENR *and ∂Y is a Z-set in $\overline{Y}$. Consider a homotopy equivalence $f\colon X \to Y$ which is continuously controlled. Let $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary along f.*

Then $\overline{X}$ is an ENR *and $\partial X \subseteq \overline{X}$ is a Z-set.*

Proof. We will use the third characterization of Z-set from Definition 9.1. This characterization says that if $\overline{X} = X \cup \partial X$ with X a compact ENR and if there is a homotopy $h_t : \overline{X} \to \overline{X}$ with $h_0 = \mathrm{id}$ and $h_t(\overline{X}) \subset X$ for all $t > 0$, then $\overline{X}$ is an ENR and ∂X is a Z-set in $\overline{X}$.

The statement in [9] assumes that $\overline{X}$ is an ENR, but this is unnecessary in connection with definition (3), since Hanner's criterion, see [31, Theorem 7.2], says that a compact metric space is an ENR if it is ϵ-dominated by ENRs for every $\epsilon > 0$. The homotopy h_t above shows that the ENR X ϵ-dominates $\overline{X}$ for every $\epsilon > 0$.[2][3]

Let $c_t : \overline{Y} \to \overline{Y}$ be a homotopy so that $c_0 = \mathrm{id}_{\overline{Y}}$ and $c_t(\overline{Y}) \subset Y$ for all $t > 0$. The homotopy equivalence f has a homotopy inverse $g : Y \to X$. The continuous control condition means that f extends continuously by the identity on $\partial X = \partial Y$ to $\bar{f} : \overline{X} \to \overline{Y}$, g extends continuously by the identity to $\bar{g} : \overline{Y} \to \overline{X}$ and there are homotopies h_t from id_Y to $f \circ g$ and k_t from id_X to $g \circ f$ which extend continuously by the identity to $\bar{h}_t$ and $\bar{k}_t$. Restricted to X and Y, all of these maps and homotopies are proper.

For $x \in \overline{X}$, let $\alpha(x) = \min(\mathrm{diam}(\{\bar{k}_t(x), 0 \le t \le 1\}), \frac{1}{2})$. Set

$$\bar{e}_t = \begin{cases} \bar{k}_{t/\alpha(x)}(x) & 0 \le t \le \alpha(x),\ \alpha(x) \neq 0; \\ \bar{g} \circ c_{t-\alpha(x)} \circ \bar{f}(x) & \alpha(x) \le t \le 1 \text{ or } \alpha(x) = 0. \end{cases}$$

For $t = 0$ and $\alpha(x) \neq 0$, we have $\bar{e}_0(x) = \bar{k}_0(x) = x$. If $t = 0$ and $\alpha(x) = 0$, we have $\bar{e}_0(x) = \bar{g} \circ c_0 \circ \bar{f}(x) = x$, since $\alpha(x) = 0$ implies that $\bar{k}_t(x) = x$ for all $0 \le t \le 1$. If $t = \alpha(x) \neq 0$, $\bar{e}_t(x) = \bar{g} \circ \bar{f}(x)$ with either definition. If $t = \alpha(x)$, then both definitions give $\bar{g} \circ \bar{f}(x)$. This shows that $\bar{e}_t$ is a well-defined continuous function with $e_0 = \mathrm{id}_{\overline{X}}$. For any $x \in \partial X$, $\alpha(x) = 0$ and $\bar{e}_t(x) = \bar{g} \circ c_t \circ \bar{f}(x) = \bar{g} \circ c_t(x)$. Since $c_t(x) \in Y$, $\bar{e}_t(x) = \bar{g} \circ c_t(x) \in X$, as desired. The formula above shows that points of X have no possibility of moving back into ∂X, so the proof is complete. □

[2] A map $f : X \to Y$ between metric spaces is an ϵ-domination if there is a map $g : Y \to X$ so that the composition $f \circ g : Y \to Y$ is ϵ-homotopic to the identity.

[3] Hanner's Theorem, as stated in [31], is much more general than the version we have stated here. Hanner's theorem applies to ANR$'s$, by which he means separable metric spaces X such that whenever X is imbedded as a closed subset of another separable metric space Z, it is a retract of some neighborhood in Z. In particular, X need not be even locally compact. In order to achieve such generality, it is necessary to consider homotopy dominations limited by open covers rather than by fixed constants ϵ.

Lemma 10.10. *Consider a G-homotopy equivalence of $f\colon X \to Y$ of cocompact proper free G-spaces, G a discrete group. Suppose that Y is a subspace of the compact G-space $\overline{Y}$ such that compact subsets become small at infinity for $(\overline{Y}, Y)$.*

Then f is continuously controlled at ∂Y.

Proof. Choose a G-map $u\colon Y \to X$ and G-homotopies $h\colon f \circ u \simeq \mathrm{id}_Y$ and $k\colon u \circ f \simeq \mathrm{id}_X$. Choose a compact subset $C \subseteq Y$ such that $G \cdot C = Y$ holds.

Fix a point $z \in \partial Y = \overline{Y} \setminus Y$ and an open neighborhood U of z in $\overline{Y}$.

Since compact subsets become small at infinity for $(\overline{Y}, Y)$, we can find an open neighborhood V of z in $\overline{Y}$ with $V \subseteq U$ such that for every $g \in G$ we have the implication $g \cdot h(C \times [0,1]) \cap V \neq \emptyset \implies g \cdot h(C \times [0,1]) \subseteq U$.

Consider $y \in V$. We can find $g \in G$ with $y \in g \cdot C$. Since $y = h(y,1) \in g \cdot h(C \times [0,1])$, we get $g \cdot h(C \times [0,1]) \cap V \neq \emptyset$. This implies $g \cdot h(C \times [0,1]) \subseteq U$ and in particular $h(\{y\} \times [0,1]) \subseteq U$.

The map f is proper by Lemma 10.8. Hence $f^{-1}(C) \subseteq X$ is compact. Let $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary with f. Since compact subsets become small at infinity for $(\overline{X}, X)$ by Lemma 10.6, we can find an open neighborhood V' of $z \in \partial X = \partial Y$ in $\overline{X}$ with $V' \subseteq \overline{f}^{-1}(U)$ such that for every $g \in G$ we have the implication $g \cdot k(f^{-1}(C) \times [0,1]) \cap V' \neq \emptyset \implies g \cdot k(f^{-1}(C) \times [0,1]) \subseteq \overline{f}^{-1}(U)$. Choose an open subset $V'' \subseteq \overline{Y}$ and an open subset $W \subseteq X$ with $V' = \overline{f}^{-1}(V'') \cup W$. Since the implication above remains true if we shrink V', we can assume without loss of generality that $V' = \overline{f}^{-1}(V'')$. In particular V'' is an open neighborhood of $z \in \overline{Y}$.

Consider $x \in X$ with $f(x) \in V''$. Then $x \in \overline{f}^{-1}(V'')$. We can find $g \in G$ with $x \in g \cdot f^{-1}(C)$. Since $x = k(x,1) \in g \cdot k(f^{-1}(C) \times [0,1])$, we get $g \cdot k(f^{-1}(C) \times [0,1]) \cap \overline{f}^{-1}(V'') \neq \emptyset$. This implies $g \cdot k(f^{-1}(C) \times [0,1]) \subseteq \overline{f}^{-1}(U)$ and hence $f \circ k(\{x\} \times [0,1]) \subseteq U$. □

11. Recognizing the structure of a manifold with boundary

Recall that we have discussed the basic properties of the Rips complex $P_l(G)$ before in Section 9.

Theorem 11.1. *Let G be torsion free hyperbolic group G with boundary S^2. Consider a homotopy equivalence $f\colon M \to P_l(G)/G \times N$, where M is a closed homology* ENR*-manifold, and N is a closed topological manifold of dimension ≥ 2. Denote by $p_G\colon P_l(G) \to P_l(G)/G$ the canonical projection. Let the G-covering $\widehat{M} \to M$ be the pullback with f of the G-covering $p_G \times \mathrm{id}_N\colon P_l(G) \times N \to P_l(G)/G \times N$ and $\widehat{f}\colon \widehat{M} \to P_l(G) \times N$ be the induced G-homotopy equivalence. Let $(\overline{\widehat{f}}, \widehat{f})\colon (\overline{\widehat{M}}, \widehat{M}) \to (\overline{P_l(G)}, P_l(G))$ be obtained by pulling back the boundary along $\widehat{f}$.*

Then $\overline{\widehat{M}}$ is a compact ENR *homology manifold whose boundary $\partial\overline{\widehat{M}}$ is $S^2 \times N$ and a Z-set.*

Proof. Recall from Section 9 that $P_l(G) \to P_l(G)/G$ is a model for the universal principal G-bundle $EG \to BG$ and $P_l(G)/G$ is a finite CW-complex. Hence $P_l(G)$ is a cocompact free proper G-space. Compact subsets of $P_l(G)$ become small at infinity for the pair $(\overline{P_l(G)}, P_l(G))$. The space $\overline{P_l(G)}$ is a compact metrizable ENR and $\partial P_l(G) \subseteq \overline{P_l(G)}$ is a Z-set. We conclude from Lemma 10.9 and Lemma 10.10 that $\partial\overline{\widehat{M}} \subseteq \overline{\widehat{M}}$ is a Z-set and $\overline{\widehat{M}}$ is an ENR. We conclude that $\overline{\widehat{M}}$ is compact and has finite dimension from Lemma 10.3 (2) and (3), and Lemma 10.8. Lemma 9.2 implies that $\overline{\widehat{M}}$ is an ENR homology manifold with boundary in the sense of Definition 6.3. $\square$

12. Proof of Theorem 0.3 and Theorem 0.4

This section is entirely devoted to the proof of Theorem 0.3 and Theorem 0.4. We begin with the following considerations.

Consider a hyperbolic 3-dimensional Poincaré duality group G. Then G is torsion free and ∂G is S^2 by Theorem 1.10. Let N be a closed aspherical topological manifold of dimension $n \geq 3$ with fundamental group π. Suppose that π is a Farrell-Jones group. Then $G \times \pi$ is a finitely presented $(3+n)$-dimensional Poincaré duality group. We conclude that $G \times \pi$ is a Farrell-Jones group from Theorem 4.1 (1a) and (2b). Since $3+n \geq 6$, we conclude from Theorem 1.11 that there is a closed ENR homology manifold M having the DDP and a homotopy equivalence $M \to BG \times N$.

Denote by $p_G \colon P_l(G) \to P_l(G)/G$ the canonical projection. Let the G-covering $\widehat{M} \to M$ be the pullback with f of the G-covering $p_G \times \mathrm{id}_N \colon P_l(G) \times N \to P_l(G)/G \times N$ and $\widehat{f} \colon \widehat{M} \to P_l(G) \times N$ be the induced G-homotopy equivalence. Let $(\overline{\widehat{f}}, \widehat{f}) \colon (\overline{\widehat{M}}, \widehat{M}) \to (\overline{P_l(G)} \times N, P_l(G) \times N)$ be obtained by pulling back the boundary along $\widehat{f}$. Theorem 11.1 implies that $\overline{\widehat{M}}$ is a compact ENR homology manifold whose boundary, $\partial\overline{\widehat{M}}$ is homeomorphic to $S^2 \times N$ and is a Z-set in $\overline{\widehat{M}}$.

We conclude from Lemma 8.2 that $i(\widehat{M}) = 1$. Theorem 8.1 (1) then implies that $i(M) = 1$.

We conclude from Theorem 8.1 (2) and a collaring result due to Ferry and Seebeck, which can be found in [17, Theorem 1 in Section 40 on page 285], that $\overline{\widehat{M}}$ is a compact topological manifold with boundary $\partial\overline{\widehat{M}} = S^2 \times N$. M is a closed topological manifold since it has DDP and Quinn index 1.[4])

Since $\partial P_l(G)$ is a Z-set in $\overline{P_l(G)}$, $\partial P_l(G) \times N$ is a Z-set in $\overline{P_l(G)} \times N$. We know already that $\partial\overline{\widehat{M}}$ is a Z-set in $\overline{\widehat{M}}$. Since $P_l(G)$ is contractible, $\overline{P_l(G)}$ is contractible. Since $\widehat{f}$ is a homotopy equivalence, $\overline{\widehat{f}}$ is a homotopy equivalence. By construction $\overline{\widehat{f}}$ induces a homeomorphism $u \colon \partial\overline{\widehat{M}} = \partial P_l(G) \times N = S^2 \times N$. Since $\overline{P_l(G)}$ is contractible, we can find a homotopy equivalence $\overline{P_l(G)} \to D^3$ which is the identity on $\partial P_l(G) = S^2$. Hence there is a homotopy equivalence $(U, u) \colon (\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$ such that u is a homeomorphism. Since $\pi_1(D^3 \times N) \cong \pi$ is a Farrell-Jones group and $3+n \geq 6$, the relative Borel Conjecture[5] holds, i.e., we can change (U, u) up to homotopy relative $\partial\overline{\widehat{M}}$ such that we obtain a homeomorphism of pairs

$$(U, u) \colon (\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N).$$

Proof of Theorem 0.3. The considerations above applied in the special case $N = T^3$ show that $BG \times T^3$ is homotopy equivalent to a closed topological manifold, since $\mathbb{Z}^3$ is a Farrell-Jones group by Theorem 4.1 (1b). Hence $s^{\mathrm{sym}}(BG \times T^3)$ vanishes by Theorem 5.26 (2). We conclude from Theorem 5.29 or directly from Remark 5.32

[4] We note that a Z-set Z in a compact ENR W is automatically 1-LCC. ENRs are locally simply connected, so for every $\epsilon > 0$ there is a $\delta > 0$ so that every map $\alpha : S^1 \to W$ with diameter $< \delta$ extends to a map $\bar{\alpha} : D^2 \to W$ with diameter $< \epsilon$. The Z-set property allows us to push $\bar{\alpha}(D^2)$ off of Z by an arbitrarily small homotopy, giving the desired extension. See the discussion following Definition 9.1 for further details.

[5] This is the Borel Conjecture for compact manifolds with boundary, rel boundary. The surgery exact sequence shows that it holds in the usual dimension range whenever the assembly map is an isomorphism.

that $s^{\mathrm{sym}}(BG)$ vanishes. We conclude from Theorem 3.1 that $\mathcal{N}(BG)$ is not empty. Now Theorem 0.3 follows from Theorem 5.26 1. $\square$

Proof of Theorem 0.4. The considerations above applied in the special case $N = T^3$ show that $BG \times T^3$ is homotopy equivalent to a closed topological manifold.

Now let N be any closed smooth manifold, closed PL-manifold, or closed topological manifold respectively of dimension ≥ 2. We conclude from Theorem 5.1 applied in the case $N_0 = T^3$ that there exists a normal map of degree one for some vector bundle ξ over BG

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & (\xi \times TN) \oplus \underline{\mathbb{R}^b} \arrow[d] \\ M \arrow[r, "f"] & BG \times N \end{tikzcd}$$

such that M is a smooth manifold, PL-manifold, or topological manifold respectively and f is a simple homotopy equivalence.

The considerations above applied in the case, where N is aspherical and $n \geq 3$, imply the existence of a closed topological manifold M_0, together with a homotopy equivalence $f_0\colon M \to BG \times N$ and a homeomorphism

$$(U, u)\colon (\overline{\widehat{M_0}}, \partial\overline{\widehat{M_0}}) \to (D^3 \times N, S^2 \times N).$$

We conclude from Theorem 0.5 that M_0 and M are homeomorphic. This finishes the proof of Theorem 0.4. $\square$

References

[1] A. Bartels and M. Bestvina. The Farrell-Jones Conjecture for mapping class groups. Preprint, arXiv:1606.02844 [math.GT], 2016.

[2] A. Bartels, S. Echterhoff, and W. Lück. Inheritance of isomorphism conjectures under colimits. In Cortinaz, Cuntz, Karoubi, Nest, and Weibel, editors, *K-Theory and noncommutative geometry*, EMS-Series of Congress Reports, pages 41–70. European Mathematical Society, 2008.

[3] A. Bartels, F. T. Farrell, and W. Lück. The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups. *J. Amer. Math. Soc.*, 27(2):339–388, 2014.

[4] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)*, 175:631–689, 2012.

[5] A. Bartels, W. Lück, and H. Reich. Equivariant covers for hyperbolic groups. *Geom. Topol.*, 12(3):1799–1882, 2008.

[6] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, 172(1):29–70, 2008.

[7] A. Bartels, W. Lück, H. Reich, and H. Rüping. K- and L-theory of group rings over $\mathrm{GL}_n(\mathbf{Z})$. *Publ. Math., Inst. Hautes Étud. Sci.*, 119:97–125, 2014.

[8] A. Bartels, W. Lück, and S. Weinberger. On hyperbolic groups with spheres as boundary. *Journal of Differential Geometry*, 86(1):1–16, 2010.

[9] M. Bestvina. Local homology properties of boundaries of groups. *Michigan Math. J.*, 43(1):123–139, 1996.

[10] M. Bestvina and G. Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[11] B. H. Bowditch. Planar groups and the Seifert conjecture. *J. Reine Angew. Math.*, 576:11–62, 2004.

[12] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.

[13] J. Bryant, S. Ferry, W. Mio, and S. Weinberger. Topology of homology manifolds. *Ann. of Math. (2)*, 143(3):435–467, 1996.

[14] J. W. Cannon and D. Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. *Trans. Amer. Math. Soc.*, 330(1):419–431, 1992.

[15] J. W. Cannon and E. L. Swenson. Recognizing constant curvature discrete groups in dimension 3. *Trans. Amer. Math. Soc.*, 350(2):809–849, 1998.

[16] F. Connolly, J. F. Davis, and Q. Khan. Topological rigidity and H_1-negative involutions on tori. *Geom. Topol.*, 18(3):1719–1768, 2014.
[17] R. J. Daverman. *Decompositions of manifolds*, volume 124 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.
[18] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.
[19] M. W. Davis. The cohomology of a Coxeter group with group ring coefficients. *Duke Math. J.*, 91(2):297–314, 1998.
[20] M. W. Davis. Poincaré duality groups. In *Surveys on surgery theory, Vol. 1*, volume 145 of *Ann. of Math. Stud.*, pages 167–193. Princeton Univ. Press, Princeton, NJ, 2000.
[21] M. W. Davis, J. Fowler, and J.-F. Lafont. Aspherical manifolds that cannot be triangulated. *Algebr. Geom. Topol.*, 14(2):795–803, 2014.
[22] M. J. Dunwoody and E. L. Swenson. The algebraic torus theorem. *Invent. Math.*, 140(3):605–637, 2000.
[23] B. Eckmann. Poincaré duality groups of dimension two are surface groups. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 35–51. Princeton Univ. Press, Princeton, NJ, 1987.
[24] A. Eskin, D. Fisher, and K. Whyte. Quasi-isometries and rigidity of solvable groups. *Pure Appl. Math. Q.*, 3(4, part 1):927–947, 2007.
[25] F. T. Farrell and L. E. Jones. Negatively curved manifolds with exotic smooth structures. *J. Amer. Math. Soc.*, 2(4):899–908, 1989.
[26] T. Farrell, W. Lück, and W. Steimle. Approximately fibering a manifold over an aspherical one. *Math. Ann.*, 370(1-2):669–726, 2018.
[27] S. C. Ferry. Epsilon-delta surgery over $\mathbb{Z}$. *Geom. Dedicata*, 148:71–101, 2010.
[28] M. H. Freedman. The disk theorem for four-dimensional manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 647–663, Warsaw, 1984. PWN.
[29] M. H. Freedman and F. Quinn. Topology of 4-manifolds. Princeton University Press, Princeton, NJ, 1990.
[30] D. Groves, J. F. Manning, and H. Wilton. Recognizing geometric 3-manifold groups using the word problem. Preprint, 2012.
[31] O. Hanner. Some theorems on absolute neighborhood retracts. *Ark. Mat.*, 1:389–408, 1951.
[32] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2005.
[33] J.-C. Hausmann and P. Vogel. *Geometry on Poincaré spaces*, volume 41 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1993.
[34] J. Hempel. 3-*Manifolds*. Princeton University Press, Princeton, N. J., 1976. Ann. of Math. Studies, No. 86.
[35] F. E. A. Johnson and C. T. C. Wall. On groups satisfying Poincaré duality. *Ann. of Math. (2)*, 96:592–598, 1972.
[36] H. Kammeyer, W. Lück, and H. Rüping. The Farrell–Jones conjecture for arbitrary lattices in virtually connected Lie groups. *Geom. Topol.*, 20(3):1275–1287, 2016.
[37] M. Kapovich and B. Kleiner. The weak hyperbolization conjecture for 3-dimensional CAT(0)-groups. *Groups Geom. Dyn.*, 1(1):61–79, 2007.
[38] M. Kapovich and B. Leeb. Quasi-isometries preserve the geometric decomposition of Haken manifolds. *Invent. Math.*, 128(2):393–416, 1997.
[39] B. Kleiner and J. Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008.
[40] M. Kreck and W. Lück. Topological rigidity for non-aspherical manifolds. *Pure and Applied Mathematics Quarterly*, 5 (3):873–914, 2009. special issue in honor of Friedrich Hirzebruch.
[41] P. Kühl, T. Macko, and A. Mole. The total surgery obstruction revisited. *Münster J. Math.*, 6:181–269, 2013.
[42] G. Laures and J. E. McClure. Multiplicative properties of Quinn spectra. *Forum Math.*, 26(4):1117–1185, 2014.
[43] W. Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.
[44] W. Lück. Isomorphism Conjectures in K- and L-theory. in preparation, 2020.
[45] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.
[46] J. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.
[47] J. Morgan and G. Tian. *The geometrization conjecture*, volume 5 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014.
[48] J. R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[49] E. K. r. Pedersen, F. Quinn, and A. Ranicki. Controlled surgery with trivial local fundamental groups. In *High-dimensional manifold topology*, pages 421–426. World Sci. Publ., River Edge, NJ, 2003.
[50] F. Quinn. An obstruction to the resolution of homology manifolds. *Michigan Math. J.*, 34(2):285–291, 1987.
[51] A. A. Ranicki. The total surgery obstruction. In *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, volume 763 of *Lecture Notes in Math.*, pages 275–316. Springer, Berlin, 1979.
[52] A. A. Ranicki. The algebraic theory of surgery. I. Foundations. *Proc. London Math. Soc. (3)*, 40(1):87–192, 1980.
[53] A. A. Ranicki. The algebraic theory of surgery. II. Applications to topology. *Proc. London Math. Soc. (3)*, 40(2):193–283, 1980.
[54] A. A. Ranicki. *Algebraic L-theory and topological manifolds*. Cambridge University Press, Cambridge, 1992.
[55] E. G. Rieffel. Groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. *J. London Math. Soc. (2)*, 64(1):44–60, 2001.
[56] D. Rosenthal and D. Schütz. On the algebraic K- and L-theory of word hyperbolic groups. *Math. Ann.*, 332(3):523–532, 2005.
[57] H. Rüping. The Farrell–Jones conjecture for S-arithmetic groups. *J. Topol.*, 9(1):51–90, 2016.
[58] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.
[59] R. Strebel. A remark on subgroups of infinite index in Poincaré duality groups. *Comment. Math. Helv.*, 52(3):317–324, 1977.
[60] V. Turaev. Homeomorphisms of geometric three-dimensional manifolds. *Mat. Zametki*, 43(4):533–542, 575, 1988. translation in Math. Notes 43 (1988), no. 3-4, 307–312.
[61] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.
[62] C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.
[63] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.
[64] C. T. C. Wall. Poincaré duality in dimension 3. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 1–26 (electronic). Geom. Topol. Publ., Coventry, 2004.
[65] C. Wegner. The K-theoretic Farrell-Jones conjecture for CAT(0)-groups. *Proc. Amer. Math. Soc.*, 140(3):779–793, 2012.
[66] C. Wegner. The Farrell-Jones conjecture for virtually solvable groups. *J. Topol.*, 8(4):975–1016, 2015.
[67] M. Weiss and B. Williams. Automorphisms of manifolds and algebraic k-theory: finale. Preprint, Stockholm, 1994.
[68] J. E. West. Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk. *Ann. of Math.*, 1061:1–18, 1977.

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