arXiv:1501.07446v1 [math.GT] 29 Jan 2015

SURVEY ON APPROXIMATING L^2-INVARIANTS BY THEIR CLASSICAL COUNTERPARTS: BETTI NUMBERS, TORSION INVARIANTS AND HOMOLOGICAL GROWTH

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Abstract. In this paper we discuss open problems concerning L^2-invariants focusing on approximation by towers of finite coverings.

0. Introduction

We want to study in this paper the following general situation:

Setup 0.1. Let G be a (discrete) group together with a descending chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \tag{0.2}$$

such that G_i is normal in G, the index $[G : G_i]$ is finite and $\bigcap_{i \geq 0} G_i = \{1\}$.

Let $p\colon \overline{X} \to X$ be a G-covering. Put $X[i] := G_i \backslash \overline{X}$.

We obtain a $[G : G_i]$-sheeted covering $p[i]\colon X[i] \to X$. Its total space $X[i]$ inherits the structure of a finite CW-complex, a closed manifold or a closed Riemannian manifold respectively if X has the structure of a finite CW-complex, a closed manifold or a closed Riemannian manifold respectively.

Let α be a classical topological invariant such as the Euler characteristic, the signature, the nth Betti number with coefficients in the field $\mathbb{Q}$ or $\mathbb{F}_p$, torsion in the sense of Reidemeister or Ray-Singer, the minimal number of generators of the fundamental group, the minimal number of generators of the nth homology group with integral coefficients, or the logarithm of the cardinality of the torsion subgroup of the nth homology group with integral coefficients. We want to study the sequence

$$\left(\frac{\alpha(X[i])}{[G : G_i]}\right)_{i \geq 0}.$$

Problem 0.3 (Approximation Problem)**.**

1. *Does the sequence converge?*
2. *If yes, is the limit independent of the chain?*
3. *If yes, what is the limit?*

The hope is that the answer to the first two questions is yes and the limit turns out to be an L^2-analogue $\alpha^{(2)}$ of α applied to the G-space $\overline{X}$, i.e., one can prove an equation of the type

$$\lim_{i \to \infty} \frac{\alpha(X[i])}{[G : G_i]} = \alpha^{(2)}(\overline{X}; \mathcal{N}(G)). \tag{0.4}$$

Here $\mathcal{N}(G)$ stands for the group von Neumann algebra and is a reminiscence of the fact that the G-action on $\overline{X}$ plays a role. Equation (0.4) is often used to compute

Date: January, 2015 **Comment 1**: Has to be updated.

2000 *Mathematics Subject Classification.* 59Q99,22D25,46L99, 58J52.

Key words and phrases. L^2-invariants, approximation, homological growth.

the L^2-invariant $\alpha^{(2)}(\overline{X};\mathcal{N}(G))$ by its finite-dimensional analogues $\alpha(X[i])$. On the other hand, it implies the existence of finite coverings with large $\alpha(X[i])$, if $\alpha^{(2)}(\overline{X};\mathcal{N}(G))$ is known to be positive.

For some important invariants α one can prove (0.4), for instance for α the Euler characteristic, the signature or the n-th Betti number with rational coefficients. In other very interesting cases Problem 0.3 and the equality (0.4) are open, and hence there is the intriguing and hard challenge to find a proof. Here we are thinking of α to be one of the following invariants:

- the n-th Betti number $b_n(X[i];\mathbb{F}_p)$ of $X[i]$ with coefficients in the field $\mathbb{F}_p$ for a prime p;
- the minimal number of generators $d(G_i)$ or the deficiency $\operatorname{def}(G_i)$ of $G_i = \pi_1(X[i])$, if $\overline{X}$ is contractible;
- Reidemeister or Ray-Singer torsion $\rho_{\mathrm{an}}(X[i])$ if X is a closed Riemannian manifold;
- the logarithm of the cardinality of the torsion in the n-th singular homology with integer coefficients $\ln\big(\big|\operatorname{tors}\big(H_n(X[i])\big)\big|\big)$, if X is an aspherical closed manifold and $\overline{X}$ its universal covering.

Here are two highlights of open problems which will be treated in more detail later in the manuscript.

Question 3.4 (Rank gradient, cost, first L^2-Betti number and approximation)
Let G be a finitely presented residually finite group. Let (G_i) be a descending chain of normal subgroups of finite index of G with $\bigcap_{i\ge 0} G_i = \{1\}$. Let F be any field.

When do we have

$$\lim_{i\to\infty} \frac{b_1(G_i;F)-1}{[G:G_i]} = b_1^{(2)}(G) - b_0^{(2)}(G) = \operatorname{cost}(G) - 1 = \operatorname{RG}(G;(G_i)_{i\ge 0}),$$

where $\operatorname{cost}(G)$ denotes the cost and $\operatorname{RG}(G;(G_i)_{i\ge 0})$ the rank gradient?

Conjecture 10.1 (Homological growth and L^2-torsion for aspherical closed manifolds)
Let M be an aspherical closed manifold of dimension $d \ge 1$ and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then

(1) For any natural number n with $2n \ne d$ we get
$$b_n^{(2)}(\widetilde{M}) = 0.$$
If $d = 2n$, we have
$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \ge 0.$$
If $d = 2n$ and M carries a Riemannian metric of negative sectional curvature, then
$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) > 0;$$

(2) Let $(G_i)_{i\ge 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G:G_i]$ and trivial intersection $\bigcap_{i\ge 0} G_i = \{1\}$. Put $M[i] = G_i\backslash\widetilde{M}$.
Then we get for any natural number n and any field F
$$b_n^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_n(M[i];F)}{[G:G_i]} = \lim_{i\to\infty} \frac{d\big(H_n(M[i];\mathbb{Z})\big)}{[G:G_i]},$$

where $d\bigl(H_n(M[i];\mathbb{Z})\bigr)$ is the minimal numbers of generators of $H_n(M[i];\mathbb{Z})$, and for $n = 1$

$$b_1^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_1(M[i];F)}{[G:G_i]} = \lim_{i\to\infty} \frac{d\bigl(G_i/[G_i,G_i]\bigr)}{[G:G_i]}$$
$$= RG(G,(G_i)_{i\geq 0}) = \begin{cases} 0 & d \neq 2; \\ -\chi(M) & d = 2; \end{cases}$$

(3) We get for the truncated Euler characteristic in dimension m

$$\lim_{i\to\infty} \frac{\chi_m^{\mathrm{trun}}(M[i])}{[G:G_i]} = \begin{cases} \chi(M) & \text{if } d \text{ is even and } 2m \geq d; \\ 0 & \text{otherwise;} \end{cases}$$

(4) If $d = 2n+1$ is odd, we get for the L^2-torsion

$$(-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\bigl(\widetilde{M}\bigr) \geq 0;$$

If $d = 2n+1$ is odd and M carries a Riemannian metric with negative sectional curvature, we have

$$(-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\bigl(\widetilde{M}\bigr) > 0;$$

(5) Let $(G_i)_{i\geq 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G:G_i]$ and trivial intersection $\bigcap_{i\geq 0} G_i = \{1\}$. Put $M[i] = G_i\backslash\widetilde{M}$.

Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i\to\infty} \frac{\ln\Bigl(\bigl|\mathrm{tors}\bigl(H_n(M[i])\bigr)\bigr|\Bigr)}{[G:G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i\to\infty} \frac{\ln\Bigl(\bigl|\mathrm{tors}\bigl(H_n(M[i])\bigr)\bigr|\Bigr)}{[G:G_i]} = (-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\bigl(\widetilde{M}\bigr) \geq 0.$$

The earliest reference, where a version of Problem 0.3 appears, is to our knowledge Kazhdan [48], where the inequality $\limsup_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} \leq b_n^{(2)}(\overline{X};\mathcal{N}(G))$ for X a closed manifold is discussed, see also Gromov [46, pages 13, 153].

Commencing with Section 12 we will drop the condition that $[G:G_i]$ is finite.

We assume that the reader is familiar with basic concepts concerning L^2-Betti numbers and L^2-torsion. More information about these notions can be found for instance in [66, 67].

Most of the article consists of surveys of open problems and known results. There are a few new aspects in this manuscript:

- In Section 4.1 we introduce the truncated Euler characteristic which leads to a high-dimensional version of the rank gradient and to Question 4.2 about asymptotic Morse inequalities;
- In Theorem 16.16 we discuss a strategy to prove the Approximation Conjecture for Fuglede-Kadison determinants under a uniform logarithmic estimate;
- The vanishing of the regulators on the homology comparing the inner product coming from a Riemannian metric with the one coming from a triangulation in the L^2-acyclic case, see Theorem 7.7, or more generally Theorem 14.10.
- The question whether L^2-torsion can be approximated by integral torsion depends only on the $\mathbb{Q}G$-chain homotopy type of a finite based free L^2-acyclic $\mathbb{Z}G$-chain complex, see Lemma 18.4.

Acknowledgments. This paper is financially supported by the Leibniz-Preis of the author, granted by the Deutsche Forschungsgemeinschaft DFG. The author thanks Nicolas Bergeron, Florian Funke, Holger Kammeyer, Clara Löh, Roman Sauer, and Andreas Thom for their useful comments.

The paper is organized as follows:

Contents

1. Euler characteristic and signature

1.1. **Euler characteristic.** Let us begin with one of the oldest and most famous invariants, the *Euler characteristic* $\chi(X)$ for a finite CW-complex. It is defined as $\sum_{n\geq 0}(-1)^n \cdot c_n$, where c_n is the number of n-cells. It is easy to see that it is multiplicative under finite coverings. Since this implies $\chi(X) = \frac{\chi(X[i])}{[G:G_i]}$, the answer in this case is yes for all three questions appearing in Problem 0.3 and the limit is

$$\lim_{i\to\infty} \frac{\chi(X[i])}{[G:G_i]} \quad = \quad \chi(X). \tag{1.1}$$

1.2. **Signature of closed oriented manifolds.** Next we consider the *signature* of a closed oriented topological $4k$-dimensional manifold M. It is defined as the signature of the non-degenerate symmetric bilinear $\mathbb{R}$-pairing given by the intersection form

$$H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to \mathbb{R}, \quad (x,y) \mapsto \langle x \cup y, [M]_{\mathbb{R}}\rangle.$$

It is known that it is multiplicative under finite coverings, but the proof is however more involved than the one for the Euler characteristic. It follows for instance from Hirzebruch's Signature Theorem, see [47], or Atiyah's L^2-index theorem [6, (1.1)] in the smooth case, for closed topological manifolds see Schafer [88, Theorem 8]. Since this implies $\operatorname{sign}(X) = \frac{\operatorname{sign}(X[i])}{[G:G_i]}$, the answer in this case is yes for all three questions appearing in Problem 0.3 and the limit is

$$\lim_{i\to\infty} \frac{\operatorname{sign}(X[i])}{[G:G_i]} \quad = \quad \operatorname{sign}(X). \tag{1.2}$$

1.3. **Signature of finite Poincaré complexes.** The next level of generality is to pass from a topological manifold to a finite Poincaré complex whose definition is due to Wall [97]. For them the signature is still defined if the dimension is divisible by 4. There are Poincaré complexes X for which the signature is not multiplicative under finite coverings, see [85, Example 22.28], [97, Corollary 5.4.1]. Hence the situation is more complicated here. Nevertheless, it turns out that the answer in this case is yes for all three questions appearing in Problem 0.3 and the limit is

$$\lim_{i\to\infty} \frac{\operatorname{sign}(X[i])}{[G:G_i]} \quad = \quad \operatorname{sign}^{(2)}(X;\mathcal{N}(G)), \tag{1.3}$$

where $\operatorname{sign}^{(2)}(\overline{X};\mathcal{N}(G))$ denotes the L^2-signature which is in general different from $\operatorname{sign}(X)$ for a finite Poincaré complex X.

Actually, for any closed oriented $4k$-dimensional topological manifold M one has $\operatorname{sign}(M) = \operatorname{sign}^{(2)}(\overline{M};\mathcal{N}(G))$. Equation (1.3) for finite Poincaré complexes extends to finite Poincaré pairs. For a detailed discussion of these notions and results we refer to [72, 73].

2. Betti numbers

2.1. **Characteristic zero.** Fix a field F of characteristic zero. We consider the *n-th Betti number with F-coefficients* $b_n(X;F) := \dim_F(H_n(X;F))$. Notice that $b_n(X;F) = b_n(X;\mathbb{Q}) = \operatorname{rk}_{\mathbb{Z}}(H_n(X;\mathbb{Z}))$ holds, where $\operatorname{rk}_{\mathbb{Z}}$ denotes the rank of a finitely generated abelian group. In this case the answer is yes for all three questions appearing in Problem 0.3 by the main result of Lück [62].

Theorem 2.1. *Let F be a field of characteristic zero and let X be a finite CW-complex. Then we have*

$$\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} = b_n^{(2)}(\overline{X};\mathcal{N}(G)),$$

where $b^{(2)}(\overline{X};\mathcal{N}(G))$ denotes the n-th L^2-Betti number.

2.2. **Prime characteristic.** Fix a prime p. Let F be a field of characteristic p. We consider the *n-th Betti number with F-coefficients* $b_n(X;F) := \dim_F(H_n(X;F))$. Notice that $b_n(X;F) = b_n(X;\mathbb{F}_p)$ holds, where $\mathbb{F}_p$ is the field of p-elements. In this setting a general answer to Problem 0.3 is only known in special cases. The main problem is that one does not have an analogue of the von Neumann algebra in characteristic p and the construction of an appropriate extended dimension function, see [65], is not known in general.

If G is torsionfree elementary amenable, one gets the full positive answer by Linnell-Lück-Sauer [58, Theorem 0.2], where more explanations, e.g., about Ore localizations are given and actually virtually torsionfree elementary amenable groups are considered.

Theorem 2.2. *Let F be a field (of arbitrary characteristic) and X be a connected finite CW-complex. Let G be a torsionfree elementary amenable group. Then:*

$$\dim_{FG}^{\mathrm{Ore}}\bigl(H_n(\overline{X};F)\bigr) = \lim_{n\to\infty} \frac{b_n(X[i];F)}{[G:G_n]}.$$

For a brief survey on elementary amenable groups we refer for instance to [66, Section 6.4.1 on page 256ff]. Solvable groups are examples of elementary amenable groups. Every elementary amenable group is amenable, the converse is not true in general.

Notice that Theorem 2.2 is consistent with Theorem 2.1 since for a field F of characteristic zero and a torsionfree elementary amenable group G we have $b_n^{(2)}(\overline{X};\mathcal{N}(G)) = \dim_{FG}^{\mathrm{Ore}}\bigl(H_n(\overline{X};F)\bigr)$. The latter equality follows from [66, Theorem 6.37 on page 259, Theorem 8.29 on page 330, Lemma 10.16 on page 376, and Lemma 10.39 on page 388].

Here is another special case taken from Bergeron-Lück-Linnell-Sauer [9], see also Calegari-Emerton [17, 18], where we know the answer only for special chains. Let p be a prime, let n be a positive integer, and let $\phi\colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ be a homomorphism, where $\mathbb{Z}_p$ denotes the p-adic integers. The closure of the image of ϕ, which is denoted by Λ, is a p-adic analytic group admitting an exhausting filtration by open normal subgroups $\Lambda_i = \ker\bigl(\Lambda \to \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z})\bigr)$. Put $G_i = \phi^{-1}(\Lambda_i)$.

Theorem 2.3. *Let F be a field (of arbitrary characteristic). Put $d = \dim(\Lambda)$. Let X be a finite CW-complex. Then for any integer n and as i tends to infinity, we have:*

$$b_n(X[i];F) = b_n^{(2)}(\overline{X};F)\cdot[G:G_i] + O\left([G:G_i]^{1-1/d}\right).$$

where $b_n^{(2)}(\overline{X};F)$ is the nth mod p L^2-Betti numbers occurring in [9, Definition 1.3]. *In particular*

$$\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} = b_n^{(2)}(\overline{X};F).$$

By the universal coefficient theorem we have $b_n(X[i];\mathbb{Q}) \le b_n(X[i];F)$ for any field F and hence by Theorem 2.1 the inequality

$$\liminf_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} \ge b_n^{(2)}(\overline{X};\mathcal{N}(G)).$$

If p is a prime and we additionally assume that each index $[G:G_i]$ is a p-power, then the sequence $\frac{b_n(X[i];F)}{[G:G_i]}$ is monotone decreasing and in particular $\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]}$ exists, see [9, Theorem 1.6].

Conjecture 2.4 (Approximation in zero and prime characteristic)**.** *We get*

$$\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} = b_n^{(2)}(\overline{X};\mathcal{N}(G))$$

for all fields F and $n \ge 0$, provided that $\overline{X}$ is contractible, or, equivalently, that X is aspherical, $G = \pi_1(X)$ and $\overline{X}$ is the universal covering $\widetilde{X}$.

The assumption that $\overline{X}$ is contractible is necessary in Conjecture 2.4, see [58, Example 6.2]. An obvious modification of [58, Example 6.2] applied to $G = \mathbb{Z}$ and $X = S^1 \vee Y$ for a finite aspherical CW-complex Y with $H_n(Y;\mathbb{Q}) \ne 0$ and $H_n(Y;\mathbb{F}_p) \ne 0$ yields a counterexample, where X is aspherical, (but $\overline{X}$ is not the universal covering).

Estimates of the growth of Betti-numbers in terms of the volume of the underlying manifold and examples of aspherical manifolds, where this growth is sublinear, are given in [87].

2.3. **Minimal number of the generators of the homology.** Recall the standard notation that $d(G)$ denotes the minimal number of generators of a finitely generated group G. The Universal Coefficient Theorem implies $d\big(H_n(X[i];\mathbb{Z})\big) \ge b_n(X[i];F)$ if F has characteristic zero, but this inequality is not necessarily true in prime characteristic One can make the following version of Conjecture 2.4. which is in some sense stronger, see the discussion in [68, Remark 1.3 and Lemma 2.13].

Conjecture 2.5 (Growth of number of generators of the homology)**.** *We get*

$$\lim_{i\to\infty} \frac{d\big(H_n(X[i];\mathbb{Z})\big)}{[G:G_i]} = b_n^{(2)}(\overline{X};\mathcal{N}(G)),$$

provided that $\overline{X}$ is contractible.

3. Rank gradient and cost

Let G be a finitely generated group. Let $(G_i)_{i\ge 0}$ be a descending chain of subgroups of finite index of G. The *rank gradient of* G (with respect to (G_i)) is defined by

$$\text{RG}(G;(G_i)_{i\ge 0}) = \lim_{i\to\infty} \frac{d(G_i)-1}{[G:G_i]}. \tag{3.1}$$

The above limit always exists since for any finite index subgroup H of G one has $\frac{d(H)-1}{[G:H]} \le d(G)-1$ by the Schreier index formula and hence the sequence $\frac{d(G_i)-1}{[G:G_i]}$ is a monotone decreasing sequence of non-negative rational numbers.

The rank gradient was originally introduced by Lackenby [53] as a tool for studying 3-manifold groups, but is also interesting from a purely group-theoretic point of view, see, e.g., [3, 4, 81, 90].

In the sequel let $(G_i)_{i\ge 0}$ be a descending chain of normal subgroups of finite index of G with $\bigcap_{i\ge 0} G_i = \{1\}$. The following inequalities are known to hold:

$$b_1^{(2)}(G) - b_0^{(2)}(G) \le \text{cost}(G) - 1 \le \text{RG}(G;(G_i)_{i\ge 0}). \tag{3.2}$$

The first inequality is due to Gaboriau [38, Corollaire 3.16, 3.23] and the second was proved by Abért and Nikolov [4, Theorem 1]. See [37, 38, 39] for the definition and some key results about the cost of a group.

It is not known if either inequality in (3.2) can be strict. It is strict if G is finite since then all values are $|G|^{-1}$. The following questions remain open:

Question 3.3 (Rank gradient, cost, and first L^2 Betti number)**.** *Let G be an infinite finitely generated residually finite group. Let $(G_i)_{i\geq 0}$ be a descending chain of normal subgroups of finite index of G with $\bigcap_{i\geq 0} G_i = \{1\}$.*

Do we have

$$b_1^{(2)}(G) = \operatorname{cost}(G) - 1 = \operatorname{RG}(G;(G_i)_{i\geq 0})?$$

Question 3.4 (Rank gradient, cost, first L^2-Betti number and approximation)**.** *Let G be a finitely presented residually finite group. Let (G_i) be a descending chain of normal subgroups of finite index of G with $\bigcap_{i\geq 0} G_i = \{1\}$. Let F be any field.*

Do we have

$$\lim_{i\to\infty} \frac{b_1(G_i;F) - 1}{[G:G_i]} = b_1^{(2)}(G) - b_0^{(2)}(G) = \operatorname{cost}(G) - 1 = \operatorname{RG}(G;(G_i)_{i\geq 0})?$$

Notice that a positive answer to the questions above also includes the statement, that $\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]}$ and $\operatorname{RG}(G;(G_i)_{i\geq 0})$ are independent of the chain.

One can ask for any finitely generated group G (without assuming that it is residually finite) whether $b_1^{(2)}(G) = \operatorname{cost}(G) - 1$ is true, and whether the Fixed Prize Conjecture is true which predicts that the cost of every standard action of G, i.e., an essentially free G-action on a standard Borel space with G-invariant probability measure, is equal to the cost of G.

Remark 3.5 (Minimal number of generators versus rank of the abelianization)**.** A positive answer to Question 3.4 is equivalent to the assertion that

$$\lim_{i\to\infty} \frac{d(G_i) - \operatorname{rk}_{\mathbb{Z}}(H_1(G))}{[G:G_i]} = 0.$$

This is surprising since in general one would not expect that for a finitely generated group H the minimal number of generators $d(H)$ agrees with the rank of its abelianization $\operatorname{rk}_{\mathbb{Z}}(H_1(G))$. So a positive answer to Question 3.4 would imply that this equality is true asymptotically. This raises the question whether this equality holds for a "random group" in the sense of Gromov.

Remark 3.6 (Known cases)**.** The answer to Question 3.3 and 3.4 is known to be positive if G contains a normal infinite amenable subgroup. Namely, in this case all values are 0 since $\operatorname{RG}(G;(G_i)_{i\geq 0}) = 0$ for all descending chains $(G_i)_{i\geq 0}$ of normal subgroups of finite index of G with trivial intersection, as proved by Lackenby [53, Theorem 1.2] when G is finitely presented, and by Abért and Nikolov [4, Theorem 3] in general, where actually more general chains are considered.

It is also positive for limit groups by Bridson-Kochlukova [15, Theorem A and Corollary C], where all values are $-\chi(G)$.

Remark 3.7 (All conditions are necessary)**.** One cannot drop in Question 3.4 the assumption that the intersection $\bigcap_{i\geq 0} G_i$ is trivial. Namely, there exists a finitely presented group G together with a descending chain $(G_i)_{i\geq 0}$ of normal subgroups G_i of finite index of G, but with non-trivial intersection $\bigcap_{i\geq 0} G_i$ such that

$$\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{Q})}{[G:G_i]} < \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]} < \lim_{i\to\infty} \frac{d(H_1(G_i;\mathbb{Z}))}{[G:G_i]} < RG(G;(G_i))$$

holds, see Ershof-Lück [33, Section 4].

The condition that each subgroup G_i is normal in G cannot be discarded in Question 3.4. Namely one can conclude from Abért and Nikolov [4, Proposition 14], see [36, Proposition 3.14] for details, that there exists for every prime p a finitely presented group together with a descending chain $(G_i)_{i\geq 0}$ of (not normal) subgroups of finite index of G with $\bigcap_{i\geq 0} G_i$ satisfying

$$\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{Q})}{[G:G_i]} < b_1^{(2)}(G) < \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]} < \lim_{i\to\infty} \frac{d(H_1(G_i;\mathbb{Z}))}{[G:G_i]} < RG(G;(G_i)).$$

One can find also examples where G is the fundamental group of an oriented hyperbolic 3-manifold of finite volume and the rank gradient is positive for a descending chain $(G_i)_{i\geq 0}$ of (not normal) subgroups of finite index of G with $\bigcap_{i\geq 0} G_i$, whereas the first L^2-Betti number of G is zero, see Girão [40, 41].

Also the condition that G is finitely presented has to appear in Question 3.4. (Notice that in Question 3.3 we demand G only to be finitely generated.) For finitely generated G and F of characteristic zero one knows at least $\limsup_{i\to\infty} \frac{b_1(G_i;F))}{[G:G_i]} \leq b_1^{(2)}(G)$, see Lück-Osin [71, Theorem 1.1]. However, for every prime p there exists an infinite finitely generated residually p-finite p-torsion group G such that for any descending chain of normal subgroups $(G_i)_{i\geq 0}$, for which $[G:G_i]$ is finite and a power of p and $\bigcap_{i\geq 0} G_i$ is trivial,

$$0 = \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{Q})}{[G:G_i]} < b_1^{(2)}(G) \leq \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{[G:G_i]}$$

holds. This follows from Ershof-Lück [33, Theorem 1.6] and Lück-Osin [71, Theorem 1.2].

4. A high dimensional version of the rank gradient

One may speculate about the following higher dimensional analogue of Question 3.4.

4.1. **Truncated Euler characteristics.** Let d be a natural number and let X be a space. Denote by $\mathcal{CW}_d(X)$ the set of CW-complexes Y which have a finite d-skeleton Y_d and are homotopy equivalent to X.

Provided that $\mathcal{CW}_d(X)$ is not empty, define the *d-th truncated Euler characteristic of X* by

(4.1)
$$\chi_d^{\mathrm{trun}}(X) \quad := \quad \begin{cases} \min\{\chi(Y_d) \mid Y \in \mathcal{CW}_d(X)\} & \text{if } d \text{ is even;} \\ \max\{\chi(Y_d) \mid Y \in \mathcal{CW}_d(X)\} & \text{if } d \text{ is odd,} \end{cases}$$

where $\chi(Y_d)$ is the Euler characteristic of the d-skeleton Y_d of Y.

If X is a finite CW-complex, then $\chi_d^{\mathrm{trun}}(X) = \chi(X)$ if $d \geq \dim(X)$.

Fix a G-covering $\overline{X} \to X$. Consider $Y \in \mathcal{CW}_d(X)$. Choose a homotopy equivalence $h\colon Y \to X$. We obtain a G-covering $\overline{Y} \to Y$ by applying the pullback construction to $\overline{X} \to X$ and $h\colon Y \to X$. We get using [66, Theorem 6.80 (1) on page 277]

$$\begin{aligned}
\chi(Y_d) &= \chi^{(2)}(\overline{Y_d};\mathcal{N}(G)) \\
&= \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{Y_d};\mathcal{N}(G)) \\
&= (-1)^d \cdot b_d^{(2)}(\overline{Y_d};\mathcal{N}(G)) + \sum_{n=0}^{d-1}(-1)^n \cdot b_n^{(2)}(\overline{Y_d};\mathcal{N}(G)) \\
&= (-1)^d \cdot b_d^{(2)}(\overline{Y_d};\mathcal{N}(G)) + \sum_{n=0}^{d-1}(-1)^n \cdot b_n^{(2)}(\overline{Y};\mathcal{N}(G)) \\
&= (-1)^d \cdot \left(b_d^{(2)}(\overline{Y_d};\mathcal{N}(G)) - b_d^{(2)}(\overline{Y};\mathcal{N}(G))\right) + \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{Y};\mathcal{N}(G)) \\
&= (-1)^d \cdot \left(b_d^{(2)}(\overline{Y_d};\mathcal{N}(G)) - b_d^{(2)}(\overline{Y};\mathcal{N}(G))\right) + \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X},\mathcal{N}(G)).
\end{aligned}$$

Since $b_d^{(2)}(\overline{Y_d};\mathcal{N}(G)) \geq b_d^{(2)}(\overline{Y};\mathcal{N}(G))$ holds, we always have the inequality

$$\chi(Y_d) \begin{cases} \geq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X},\mathcal{N}(G)) & \text{if } d \text{ is even;} \\ \leq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X},\mathcal{N}(G)) & \text{if } d \text{ is odd.} \end{cases}$$

This implies that $\chi_d^{\mathrm{trun}}(X)$ is a well-defined integer satisfying

$$\chi_d^{\mathrm{trun}}(X) \begin{cases} \geq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X},\mathcal{N}(G)) & \text{if } d \text{ is even;} \\ \leq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X},\mathcal{N}(G)) & \text{if } d \text{ is odd.} \end{cases}$$

Next we show that the limit $\lim_{i\to\infty} \frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}$ always exists. Consider a natural number i. Choose an element $Y[i] \in \mathcal{CW}_d(X[i])$ such that $\chi_d^{\mathrm{trun}}(X[i]) = \chi(Y[i]_d)$ holds. We can find a $[G_i : G_{i+1}]$-sheeted covering $Y[i+1] \to Y[i]$ such that $Y[i+1]$ belongs to $\mathcal{CW}_d[X[i+1])$. Obviously $\chi(Y[i+1]_d) = \chi(Y[i]_d) \cdot [G_i : G_{i+1}]$. Suppose that d is even. We conclude

$$\begin{aligned}
\frac{\chi_d^{\mathrm{trun}}(X[i+1])}{[G:G_{i+1}]} &= \frac{\chi_d^{\mathrm{trun}}(X[i+1])}{[G:G_i]\cdot[G_i:G_{i+1}]} \\
&\leq \frac{\chi(Y[i+1]_d)}{[G:G_i]\cdot[G_i:G_{i+1}]} \\
&= \frac{\chi(Y[i]_d)}{[G:G_i]} \\
&= \frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}.
\end{aligned}$$

Hence the sequence $\left(\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}\right)_{i\geq 0}$ is monotone decreasing. Since we get by an argument similar to the one above

$$\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]} \geq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(X[i];\mathcal{N}(G/G_i)) = \sum_{n=0}^{d}(-1)^n \cdot \frac{b_n(X[i];\mathbb{Q})}{[G:G_i]}$$

for all i, we conclude from Theorem 2.1 that the sequence $\left(\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}\right)_{i\geq 0}$ is bounded from below by $\sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X};\mathcal{N}(G))$. Hence its limit exists and satisfies

$$\lim_{i\to\infty} \frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]} \geq \sum_{n=0}^{d}(-1)^n \cdot b_n^{(2)}(\overline{X};\mathcal{N}(G)),$$

if d is even. Provided that d is odd, one analogously shows that the sequence $\left(\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}\right)_{i\geq 0}$ is monotone increasing, bounded from above by $\sum_{n=0}^{d}(-1)^n\cdot b_n^{(2)}(\overline{X};\mathcal{N}(G))$ and hence converges with

$$\lim_{i\to\infty}\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}\leq\sum_{n=0}^{d}(-1)^n\cdot b_n^{(2)}(\overline{X};\mathcal{N}(G)).$$

This leads to

Question 4.2 (Asymptotic Morse equality)**.** *Let $\overline{X}\to X$ be a G-covering and let d be a natural number such that $\mathcal{CW}_d(X)$ is not empty. When do we have*

$$\lim_{i\to\infty}\frac{\chi_d^{\mathrm{trun}}(X[i])}{[G:G_i]}=\sum_{n=0}^{d}(-1)^n\cdot b_n^{(2)}(\overline{X};\mathcal{N}(G))?$$

In this generality, the answer to Question 4.2 is not positive in general. For instance if G is trivial and $d=1$, a positive answer to Question 4.2 would mean for a connected CW-complex X with non-empty $\mathcal{CW}_1(X)$ that $\pi_1(X)$ is finitely generated and satisfies $d(\pi_1(X))=b_1(X)$ which is not true in general.

Of particular interest is the case, where $\overline{X}$ is contractible, or, equivalently, $X=BG$ and $\overline{X}=EG$. Since $\chi_d^{\mathrm{trun}}(X)$ depends only on the homotopy type of X, we will abbreviate $\chi_d^{\mathrm{trun}}(G_i):=\chi_d^{\mathrm{trun}}(BG_i)$, provided that $\mathcal{CW}_d(BG)$ for some (and hence all) model for BG is not empty. Then Question 4.2 reduces to the following question, for which we do not know an example, where the answer is negative.

Question 4.3 (Asymptotic Morse equality for groups)**.** *Let G be a group and let d be a natural number such that $\mathcal{CW}_d(BG)$ is not empty. When do we have*

$$\lim_{i\to\infty}\frac{\chi_d^{\mathrm{trun}}(G_i)}{[G:G_i]}=\sum_{n=0}^{d}(-1)^n\cdot b_n^{(2)}(G)?$$

Example 4.4 (Morse relation in degree $d=1,2$)**.** Question 4.3 is in the case $d=1$ precisely Question 3.4, since a group H is finitely generated if and only if there is a model for BH with finite 1-skeleton and in this case $\chi_1^{\mathrm{trun}}(H)=1-d(H)$.

In the case $d=2$ Question 4.3 can be rephrased as the question when for a finitely presented group G we have

$$\lim_{i\to\infty}\frac{1-\mathrm{def}(G_i)}{[G:G_i]}=b_2^{(2)}(G)-b_1^{(2)}(G)+b_0^{(2)}(G),$$

where $\mathrm{def}(H)$ denotes for a finitely presented group H its *deficiency*, i.e, the maximum over the numbers $g-r$ for all finite presentations $H=\langle s_1,s_2,\dots,s_g\mid R_1,R_2,\dots R_r\rangle$.

Remark 4.5 (Asymptotic Morse inequalities imply Approximation for Betti numbers over any field)**.** Let G be a group with a finite model for BG. It is not hard to show that Conjecture 2.4 is true for G, provided that the answer to Question 4.3 is positive for all $d\geq 0$. The main idea of the proof is to show for every field F and CW-complex Z with non-empty $\mathcal{CW}_d(Z)$

$$\chi_d^{\mathrm{trun}}(Z)\begin{cases}\geq\ \sum_{n=0}^{d}(-1)^n\cdot b_n(Z;F) & \text{if } d \text{ is even;}\\ \leq\ \sum_{n=0}^{d}(-1)^n\cdot b_n(Z;F) & \text{if } d \text{ is odd,}\end{cases}$$

and then show in the situation of Conjecture 2.4 for $n=0,1,2,\dots$ by induction using Theorem 2.1 the equality

$$\lim_{i\to\infty}\frac{b_n(X[i];F)}{[G:G_i]}=b_n^{(2)}(\widetilde{X}).$$

More generally, Conjecture 2.5 is true for G, provided that the answer to Question 4.3 is positive for all $d \geq 0$, since for $Y \in \mathcal{CW}_d(Z)$ we get $b_d(Y_d;\mathbb{Q}) = d(H_d(Y_d;\mathbb{Z})) \geq d(H_d(Y;\mathbb{Z})) = d(H_d(Z;\mathbb{Z}))$.

4.2. **Groups with slow growth.** The answer to Question 4.3 is positive by Bridson-Kochlukova [15, Theorem A and Corollary C] if G is a limit group. Their proofs are based on various notions of groups with slow growth. It is interesting that limit groups may have non-trivial first L^2-Betti numbers.

Here is a another case, where the answer to Question 4.3 is positive. Following Bridson-Kochlukova [15, page 4] we make

Definition 4.6 (Slow growth in dimensions $\leq d$)**.** We say that a residually finite group has *slow growth in dimension* $\leq d$ if for any chain $(G_i)_{i\geq 0}$ of normal subgroups of finite index with trivial intersection there is a choice of CW-complexes $(X[i])_{i\geq 0}$ such that $X[i]$ has a finite d-skeleton and is a model for BG_i for each $i \geq 0$, and $\lim_{i\to\infty} \frac{c_k(X[i])}{[G:G_i]} = 0$ holds for every $k = 0,1,2\ldots,d$, where $c_k(X[i])$ is the number of k-cells in $X[i]$.

Lemma 4.7. *Suppose that G has* slow growth in dimension $\leq d$. *Then we get for* $k = 0,1,2,\ldots,d$

$$\begin{aligned} \lim_{i\to\infty} \frac{\chi_k^{\mathrm{trun}}(G_i)}{[G:G_i]} &= 0, \\ b_k^{(2)}(G) &= 0. \end{aligned}$$

Proof. By assumption there is a choice of CW-complexes $(X[i])_{i\geq 0}$ such that $X[i]$ has a finite d-skeleton and is a model for BG_i for each $i \geq 0$, and $\lim_{i\to\infty} \frac{c_n(X[i])}{[G:G_i]} = 0$ holds for every $n = 0,1,2\ldots,d$. Since $b_n(G_i;\mathbb{Q}) \leq c_n(X[i])$ holds, we conclude $\lim_{i\to\infty} \frac{b_n(G_i;\mathbb{Q})}{[G:G_i]} = 0$ for every $n = 0,1,2\ldots,d$. Theorem 2.1 implies

$$b_k^{(2)}(G) = 0 \quad \text{for } k = 0,1,2\ldots,d.$$

If $k \in \{0,1,2,\ldots,d\}$ is even, we get

$$\begin{aligned} 0 &= \sum_{n=0}^{k} (-1)^n \cdot b_n^{(2)}(G) \\ &\leq \lim_{i\to\infty} \frac{\chi_k^{\mathrm{trun}}(G_i)}{[G:G_i]} \\ &\leq \lim_{i\to\infty} \frac{\chi\big((X[i])_k\big)}{[G:G_i]} \\ &= \lim_{i\to\infty} \sum_{n=0}^{k} \frac{c_n(X[i])}{[G:G_i]} \\ &= \sum_{n=0}^{k} \lim_{i\to\infty} \frac{c_n(X[i])}{[G:G_i]} \\ &= 0, \end{aligned}$$

and hence

$$\lim_{i\to\infty} \frac{\chi_k^{\mathrm{trun}}(G_i)}{[G:G_i]} = 0.$$

The proof in the case where k is odd is analogous. □

Lemma 4.8. *Let $1 \to K \xrightarrow{j} G \xrightarrow{q} Q \to 1$ be an extensions of groups. Suppose that K has slow growth in dimensions $\leq d$. Suppose that there is a model for BQ with finite d-skeleton or that there is a model for BG with finite d-skeleton.*

Then G has slow growth in dimensions $\le d$.

Proof. If BG has a model with finite d-skeleton, then also BQ has a model with finite d-skeleton by [64, Lemma 7.2 (2)]. Hence it suffices to treat the case, where BQ has a model with finite d-skeleton.

Consider any chain $(G_i)_{i\ge 0}$ of normal subgroups of finite index with trivial intersection. Put $K_i = j^{-1}(G_i)$ and $Q_i = q(G_i)$. We obtain an exact sequence of groups $1 \to K_i \xrightarrow{j_i} G_i \xrightarrow{q_i} Q_i \to 1$, where j_i and q_i are obtained from j and q by restriction. The subgroups $K_i \subseteq K$, $G_i \subseteq G$ and $Q_i \subseteq Q$ are normal subgroups of finite index and $[G : G_i] = [K : K_i] \cdot [Q : Q_i]$. We have $\bigcap_{i\ge 0} K_i = \{1\}$. By assumption there is a choice of CW-complexes $(X[i])_{i\ge 0}$ such that $X[i]$ has a finite d-skeleton and is a model for BK_i for each $i \ge 0$, and $\lim_{i\to\infty} \frac{c_m(X[i])}{[K:K_i]} = 0$ holds for every $m = 0, 1, 2 \ldots, d$, where $c_m(X[i])$ is the number of m-cells in $X[i]$. Choose a CW-model Z for BQ with finite d-skeleton. Let $Z_i \to BQ$ be the $[Q : Q_i]$-sheeted finite covering associated to $Q_i \subseteq Q$. Equip Z_i with the CW-structure induced by the one of Z. Then Z_i is a model for BQ_i, has a finite d-skeleton, and we get for the number of n-cells for $n \in \{0, 1, 2, \ldots, d\}$

$$c_n(Z_i) \quad = \quad [Q : Q_i] \cdot c_n(Z).$$

There is a fibration $X[i] \to BG_i \to Z_i$ such that after taking fundamental groups we obtain the exact sequence $1 \to K_i \xrightarrow{j_i} G_i \xrightarrow{q_i} Q_i \to 1$. Then one can find a CW-complex Y_i which is a model for BG_i such that we get for the number of k-cells for $k \in \{0, 1, 2, \ldots, d\}$

$$c_k(Y_i) \quad = \quad \sum_{m+n=k} c_m(X[i]) \cdot c_n(Z_i),$$

see for instance [34, Section 3]. This implies for $k \in \{0, 1, 2, \ldots, d\}$

$$\begin{aligned}
\lim_{i\to\infty} \frac{c_k(Y_i)}{[G : G_i]} &= \lim_{i\to\infty} \sum_{m+n=k} \frac{c_m(X[i]) \cdot c_n(Z_i)}{[G : G_i]} \\
&= \lim_{i\to\infty} \sum_{m+n=k} \frac{c_m(X[i])}{[K : K_i]} \cdot \frac{c_n(Z_i)}{[Q : Q_i]} \\
&= \sum_{m+n=k} c_n(Z) \cdot \lim_{i\to\infty} \frac{c_m(X[i])}{[K : K_i]} \\
&= 0.
\end{aligned}$$

□

Example 4.9 (Examples of groups with slow growth in dimensions $\le d$)**.** A residually finite group has slow growth in dimensions ≤ 0 if and only if it is infinite.

Obviously $\mathbb{Z}$ has slow growth in dimensions $\le d$ for all natural numbers d since any non-trivial subgroup K of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ again and has S^1 as model for BK.

We conclude from Lemma 4.8 that any infinite virtually poly-$\mathbb{Z}$-group has slow growth in dimensions $\le d$.

Moreover, if G is any group which possesses a finite sequence $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = G$ of subgroups such that $K_0 \cong \mathbb{Z}$, K_i is normal in K_{i+1} and $B(K_{i+1}/K_i)$ has a model with finite d-skeleton for $i = 0, \ldots, (n-1)$, then G has slow growth in dimensions $\le d$ by Lemma 4.8.

5. Speed of convergence

The speed of convergence of

$$\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i]} = b_n^{(2)}(\widetilde{X})$$

(if it converges) and of

$$\lim_{i\to\infty} \frac{d(G_i)-1}{[G:G_i]} = \mathrm{RG}(G;(G_i)_{i\geq 0})$$

can be arbitrary slow for one chain and very fast for another chain in the following sense. Fix a prime p and two functions $F^s, F^f\colon \{i \in \mathbb{Z} \mid i \geq 1\} \to (0,\infty)$ such that

$$\begin{aligned} \lim_{i\to\infty} F^s(i) &= 0; \\ \lim_{i\to\infty} F^f(i) &= 0; \\ \lim_{i\to\infty} i\cdot F^f(i) &= \infty. \end{aligned}$$

5.1. **Betti numbers.**

Theorem 5.1. *For every integer $n \geq 1$, there is a $(2n+1)$-dimensional Riemannian manifold X with non-positive sectional curvature and two chains $(G_i^s)_{i\geq 0}$ and $(G_i^f)_{i\geq 0}$ for $G = \pi_1(X)$ such that G_i^s and G_i^f are normal subgroups of G of finite p-power index, the intersections $\bigcap_{i\geq 0} G_i^s$ and $\bigcap_{i\geq 0} G_i^f$ are trivial, and we have for every field F*

$$\begin{aligned} \lim_{i\to\infty} \frac{b_n(X^s[i];F)}{[G:G_i^s]} &= b_n^{(2)}(\widetilde{X}) = 0; \\ \frac{b_n(X^s[i];F)}{[G^s:G_i^s]} &\geq F^s([G:G_i^s]); \\ \lim_{i\to\infty} \frac{b_n(X^f[i];F)}{[G:G_i^f]} &= b_n^{(2)}(\widetilde{X}) = 0; \\ \frac{b_n(X^f[i];F)}{[G:G_i^f]} &\leq F^f([G:G_i^f]). \end{aligned}$$

Proof. Consider any finite connected CW-complex Y with universal covering $\widetilde{Y} \to Y$ and $b_n^{(2)}(\widetilde{Y}) + b_{n-1}^{(2)}(\widetilde{Y}) > 0$ such that $K = \pi_1(Y)$ is infinite and residually p-finite. Choose any chain $(K_i)_{i\geq 0}$ of normal subgroups of K of finite index $[K:K_i]$ which is a power of p and with trivial intersection $\bigcap_{i\geq 0} K_i = \{1\}$. Because of Theorem 2.1 and [9, Theorem 1.6] the limit $\frac{b_n(Y[i];F)+b_{n-1}(Y[i];F)}{[K:K_i]}$ exists and is greater than 0. Hence there exist real numbers C_1 and C_2 (independent of i) with $0 < C_1 \leq C_2$ such that for each $i \geq 1$

$$C_1 \leq \frac{b_n(Y[i];F) + b_{n-1}(Y[i];F)}{[K:K_i]} \leq C_2.$$

Let k_i be the natural number for which $[K:K_i] = p^{k_i}$ holds. Then $(k_i)_{i\geq 0}$ is a monotone increasing sequence of natural numbers with $\lim_{i\to\infty} k_i = \infty$. Since $\lim_{k\to\infty} F^s\bigl(p^k \cdot p^m\bigr) = 0$ holds for any integer $m \geq 0$, we can construct a strictly monotone increasing sequence of natural numbers $(j_i)_{i\geq 0}$ such that we get for all $i \geq 0$

$$F^s\bigl(p^{k_{j_i}} \cdot p^i\bigr) \leq C_1 \cdot p^{-i}.$$

Since $\lim_{n\to\infty} p^n \cdot F^f\bigl(p^k \cdot p^n\bigr) = \infty$ for any natural number k, we can construct a strictly monotone increasing sequence of natural numbers $(n_i^f)_{i\geq 0}$ satisfying

$$C_2 \cdot p^{-n_i^f} \quad \leq \quad F^f\bigl(p^{k_{j_i}} \cdot p^{n_i^f}\bigr).$$

Put

$$X = Y \times S^1.$$

Then $G = \pi_1(X)$ can be identified with $K \times \mathbb{Z}$. We conclude $b_n^{(2)}(\widetilde{X}) = 0$ from [66, Theorem 1.39 on page 42]. Put

$$\begin{aligned} G_i^s &= K_{j_i} \times (p^i \cdot \mathbb{Z}); \\ G_i^f &= K_{j_i} \times (p^{n_i^f} \cdot \mathbb{Z}). \end{aligned}$$

Then $(G_i^s)_{i\geq 0}$ is a chain of normal subgroups of G with finite index $[G : G_i^s] = [K : K_{j_i}] \cdot p^i$ which is a p-power, namely $p^{k_{j_i}+i}$, and trivial intersection $\bigcap_{i\geq 0} G_i^s = \{1\}$, and analogously for $(G_i^f)_{i\geq 0}$. We estimate using then Künneth formula

$$\begin{aligned} \frac{b_n(X^s[i];F)}{[G : G_i^s]} &= \frac{b_n(Y[j_i];F) + b_{n-1}(Y[j_i];F)}{[K : K_{j_i}] \cdot p^i} \\ &\geq C_1 \cdot p^{-i} \\ &\geq F^s\bigl(p^{k_{j_i}} \cdot p^i\bigr) \\ &= F^s([G : G_i^s]), \end{aligned}$$

and

$$\frac{b_n(X^s[i];F)}{[G : G_i^s]} = \frac{b_n(Y[j_i];F) + b_{n-1}(Y[j_i];F)}{[K : K_{j_i}] \cdot p^i} \leq C_2 \cdot p^{-i}.$$

The latter implies $\lim_{i\to\infty} \frac{b_n(X^s[i];F)}{[G:G_i^s]} = 0$. We estimate

$$\begin{aligned} \frac{b_n(X^f[i];F)}{[G : G_i^f]} &= \frac{b_n(Y[j_i];F) + b_{n-1}(Y[j_i];F)}{[K : K_{j_i}] \cdot p^{n_i^f}} \\ &\leq C_2 \cdot p^{-n_i^f} \\ &\leq F^f\bigl(p^{k_{j_i}} \cdot p^{n_i^f}\bigr) \\ &= F^f([G : G_i^f]). \end{aligned}$$

Since $\lim_{i\to\infty} C_2 \cdot p^{-n_i^f} = 0$, we also get $\lim_{i\to\infty} \frac{b_n(X[i];F)}{[G:G_i^s]} = 0$.

It remains to construct the desired finite CW-complex Y. The fundamental group of an oriented closed surface of genus ≥ 2 is residually free and hence residually p-finite for any prime p by [8]. The L^2-Betti numbers of its universal covering are all zero except in dimension 1, where it is non-zero, see [66, Example 1.36 on page 40] We conclude from the Künneth formula for L^2-Betti numbers [66, Theorem 6.54 (5) on page 266] that an example for Y is the direct product of n closed oriented surfaces of genus ≥ 2. So we can arrange that X is an aspherical closed $(2n+1)$-dimensional Riemannian manifold with non-positive sectional curvature. □

Theorem 5.1 implies that one can find for any $\epsilon > 0$ two such chains $(G_i^s)_{i\geq 0}$ and $(G_i^f)_{i\geq 0}$ satisfying

$$\begin{aligned} \lim_{i\to\infty} \frac{b_n(X[i];F)}{[G : G_i^s]^{1-\epsilon}} &= \infty; \\ \lim_{i\to\infty} \frac{b_n(X[i];F)}{[G : G_i^f]^{\epsilon}} &= 0, \end{aligned}$$

since we can take $F^s(i) = i^{-\epsilon/2}$ and $F^f(i) = i^{\epsilon/2-1}$.

The condition $\lim_{i\to\infty} i \cdot F^f(i) = \infty$ is reasonable. Namely, in most cases one would expect $\lim_{i\to\infty} b_n(X[i];F) = \infty$ and if this is true, we get

$$\lim_{i\to\infty} [G:G_i] \cdot \frac{b_n(X[i];F)}{[G:G_i]} = \infty.$$

5.2. **Rank gradient.**

Theorem 5.2. *Let G be the product of $\mathbb{Z}$ with a finitely generated free group of rank ≥ 2 or the product of $\mathbb{Z}$ with the fundamental group of a closed surface of genus ≥ 2.*

Then there are two chains $(G_i^s)_{i\geq 0}$ and $(G_i^f)_{i\geq 0}$ such that G_i^s and G_i^f are normal subgroups of G of finite p-power index, the intersections $\bigcap_{i\geq 0} G_i^s$ and $\bigcap_{i\geq 0} G_i^f$ are trivial, and we have

$$\begin{aligned} \mathrm{RG}(G,(G_i^s)_{i\geq 0}) &= 0; \\ \frac{d(G_i^s)-1}{[G:G_i^s]} &\geq F^s([G:G_i^s]); \\ \mathrm{RG}(G,(G_i^f)_{i\geq 0}) &= 0; \\ \frac{d(G_i^f)-1}{[G:G_i^f]} &\leq F^f([G:G_i^f]). \end{aligned}$$

Proof. Essentially the same argument as in the proof of Theorem 5.1 also applies to the rank gradient. Let K be a finitely presented group with $b_1^{(2)}(K) > 0$. Choose any chain $(K_i)_{i\geq 0}$ of normal subgroups of K of finite index $[K:K_i]$ which is a power of p and with trivial intersection $\bigcap_{i\geq 0} K_i = \{1\}$. Then $\left(\frac{d(K_i)-1}{[K:K_i]}\right)_{i\geq 0}$ is a monotone decreasing sequence. Its limit $\mathrm{RG}(K,(K_i)_{i\geq 0}) := \lim_{i\to\infty} \frac{d(K_i)-1}{[K:K_i]}$ exists and is greater than $b_1^{(2)}(K)$. Hence we can choose constants $C_1 > 0$ and $C_2 > 0$ such that such that for each $i \geq 1$ we get

$$C_1 \leq \frac{d(K_i)-1}{[K:K_i]} \leq \frac{d(K_i)}{[K:K_i]} \leq C_2.$$

Put $G = K \times \mathbb{Z}$. Now construct the sequences $(j_i)_{i\geq 0}$ and $(n_i^s)_{i\geq 0}$ and define G_i^s and G_i^f as in the proof of Theorem 5.1. Then $d(K_{j_i}) \leq d(G_i^s) \leq d(K_{j_i}) + 1$ and $d(K_{j_i}) \leq d(G_i^f) \leq d(K_{j_i}) + 1$ holds. Now a calculation analogous to the one in the proof of Theorem 5.1 shows

$$\begin{aligned} \frac{d(G_i^s)-1}{[G:G_i^s]} &\geq F^s([G^s:G_i^s]); \\ \frac{d(G_i^f)-1}{[G:G_i^f]} &\leq F^f([G^f:G_i^f]). \end{aligned}$$

If K is a finitely generated free group of rank ≥ 2 or the fundamental group of a closed surface of genus ≥ 2, then K is finitely presented, residually p-finite and $b_1^{(2)}(K) > 0$. □

6. The Approximation Conjecture for Fuglede-Kadison determinants

Let $A \in M_{r,s}(\mathbb{Q}G)$ be a matrix. It induces by right multiplication a G-equivariant bounded operator $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$. We denote by $\det_{\mathcal{N}(G)}^{(2)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr)$ its Fuglede-Kadison determinant.

Denote by $A[i] \in M_{r,s}(\mathbb{Q}[G/G_i])$ the matrix obtained from A by applying elementwise the ring homomorphisms $\mathbb{Q}G \to \mathbb{Q}[G/G_i]$ induced by the projection

$G \to G/G_i$. It induces a $\mathbb{C}$-homomorphism of finite-dimensional complex Hilbert spaces $r_{A[i]}^{(2)} \colon \mathbb{C}[G/G_i]^r \to \mathbb{C}[G/G_i]^s$.

Consider a $\mathbb{C}$-homomorphism of finite-dimensional Hilbert spaces $f \colon V \to W$. It induces an endomorphism $f^*f \colon V \to V$. We have $\ker(f) = \ker(f^*f)$. Denote by $\ker(f)^\perp$ the orthogonal complement of $\ker(f)$. Then f^*f induces an automorphism $(f^*f)^\perp \colon \ker(f)^\perp \to \ker(f)^\perp$. Define

$$\det{}'(f) \quad := \quad \sqrt{\det\bigl((f^*f)^\perp\bigr)}. \tag{6.1}$$

If $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ are the non-zero eigenvalues (listed with multiplicity) of the positive operator $f^*f \colon V \to V$, then

$$\det{}'(f) = \prod_{j \ge 1} \sqrt{\lambda_j}.$$

If f is an isomorphism, then $\det'(f)$ reduces to $\sqrt{\det(f^*f)}$.

Conjecture 6.2 (Approximation Conjecture for Fuglede-Kadison determinants)**.** *Consider a matrix $A \in M_{r,s}(\mathbb{Q}G)$. Then we get*

$$\ln\bigl(\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)})\bigr) = \lim_{i \to \infty} \frac{\ln\bigl(\det{}'(r_{A[i]}^{(2)})\bigr)}{[G : G_i]}.$$

Remark 6.3. If $r = s$ and $A \in M_{r,r}(\mathbb{Q}G)$ is invertible, then the following equality is always true

$$\ln\bigl(\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)})\bigr) = \lim_{i \to \infty} \frac{\ln\bigl(\det(r_{A[i]}^{(2)})\bigr)}{[G : G_i]}.$$

However, for applications to L^2-torsion we have to consider the case, where r and s may be different and the maps $(r_A^{(2)})^* r_A^{(2)}$ and $(r_{A[i]}^{(2)})^* r_{A[i]}^{(2)}$ may not be injective.

Remark 6.4 ($\mathbb{Q}$ coefficients are necessary)**.** Conjecture 6.2 does not hold if one replaces $\mathbb{Q}$ by $\mathbb{C}$ by the following result appearing in [66, Example 13.69 on page 481]. There exists a sequence of integers $2 \le n_1 < n_2 < n_3 < \cdots$ and a real number s such that for $G = \mathbb{Z}$ and $G_i = n_i \cdot \mathbb{Z}$ and the $(1,1)$-matrix A given by the element $z - \exp(2\pi i s)$ in $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ we get for all $i \ge 1$

$$\begin{aligned} \ln\bigl(\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)})\bigr) \quad &= \quad 0; \\ \frac{\ln\bigl(\det(r_{A[i]}^{(2)})\bigr)}{[G : G_i]} \quad &\le \quad -1/2. \end{aligned}$$

Remark 6.5 (Status of Conjecture 6.2)**.** Conjecture 6.2 has been proved for $G = \mathbb{Z}$ by Schmidt [91], see also [66, Lemma 13.53 on page 478]. To the author's knowledge infinite virtually cyclic groups are the only infinite groups for which Conjecture 6.2 is known to be true. One does know in general the inequality, see Theorem 14.11,

$$\ln\bigl(\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)})\bigr) \quad \ge \quad \limsup_{i \to \infty} \frac{\ln\bigl(\det{}'(r_{A[i]}^{(2)})\bigr)}{[G : G_i]}. \tag{6.6}$$

For $G = \mathbb{Z}^n$ the last inequality for the limit superior is known to be an equality by Lê [55]. But nothing seems to be known beyond virtually finitely generated free abelian groups.

7. Torsion invariants

7.1. **L^2-torsion.** Let D_* be a finite based free $\mathbb{Z}$-chain complex, for instance the cellular chain complex $C_*(Y)$ of a finite CW-complex Y. The $\mathbb{C}$-chain complex $D_* \otimes_{\mathbb{Z}} \mathbb{C}$ inherits from the $\mathbb{Z}$-basis on D_* and the standard Hilbert space structure on $\mathbb{C}$ the structure of a Hilbert space and the resulting L^2-chain complex is denoted by $D_*^{(2)}$ with differentials $d_p^{(2)} := d_p \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{C}}$. Define its *$L^2$-torsion* by

$$(7.1) \qquad \rho^{(2)}(D_*^{(2)}; \mathcal{N}(\{1\})) := -\sum_{n \geq 1} (-1)^n \cdot \ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}(d_n^{(2)})\bigr) \quad \in \mathbb{R}.$$

Notice that $\det^{(2)}_{\mathcal{N}(\{1\})}(c_n^{(2)})$ is the same as $\det'(c_n^{(2)})$ which we have introduced in (6.1).

More generally, if C_* is a finite based free $\mathbb{Z}G$-chain complex, we obtain a finite Hilbert $\mathcal{N}(G)$-chain complex $C_*^{(2)} := C_* \otimes_{\mathbb{Z}G} L^2(G)$ and we define its *L^2-torsion*

$$(7.2) \qquad \rho^{(2)}(C_*^{(2)}; \mathcal{N}(G)) := -\sum_{n \geq 1} (-1)^n \cdot \ln\bigl(\det^{(2)}_{\mathcal{N}(G)}(c_n^{(2)})\bigr) \quad \in \mathbb{R}.$$

Conjecture 7.3 (Approximation Conjecture for Fuglede-Kadison determinants)**.** *Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Denote by $C[i]_*$ the finite free $\mathbb{Z}$-chain complex given by $C[i]_* = C_* \otimes_{\mathbb{Z}[G_i]} \mathbb{Z} = C_* \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G_i]$.*

Then we get

$$\rho^{(2)}(C_*^{(2)}; \mathcal{N}(G)) = \lim_{i \to \infty} \frac{\rho^{(2)}(C[i]_*^{(2)}; \mathcal{N}(\{1\}))}{[G : G_i]}.$$

Obviously Conjecture 7.3 is just the chain complex version of Conjecture 6.2 and these two conjectures are equivalent for a given group G.

7.2. **Analytic and topological L^2-torsion.** Let X be a closed Riemannian manifold. Let $\rho_{\mathrm{an}}(X[i])$ be the analytic torsion in the sense of Ray and Singer of the closed Riemannian manifold $X[i]$. Denote by $\rho_{\mathrm{an}}^{(2)}(\overline{X})$ the analytic L^2-torsion of the Riemannian manifold $\overline{X}$ with isometric free cocompact G-action.

Conjecture 7.4 (Approximation Conjecture for analytic torsion)**.** *Let X be a closed Riemannian manifold. Then*

$$\rho_{\mathrm{an}}^{(2)}(\overline{X}; \mathcal{N}(G)) = \lim_{i \to \infty} \frac{\ln(\rho_{\mathrm{an}}(X[i]))}{[G : G_i]}.$$

There are topological counterparts which we will denote by $\rho_{\mathrm{top}}(X[i])$ and $\rho_{\mathrm{top}}^{(2)}(\overline{X})$ which agree with their analytic versions by results Cheeger [21] and Müller [77] and Burghelea-Friedlander-Kappeler-McDonald [16]. So the conjecture above is equivalent to its topological counterpart.

Conjecture 7.5 (Approximation Conjecture for topological torsion)**.** *Let X be a closed Riemannian manifold. Then*

$$\rho_{\mathrm{top}}^{(2)}(\overline{X}; \mathcal{N}(G)) = \lim_{i \to \infty} \frac{\ln(\rho_{\mathrm{top}}(X[i]))}{[G : G_i]}.$$

Remark 7.6 (Dependency on the triangulation and the Riemannian metric)**.** Let X be a closed smooth manifold. Fix a smooth triangulation. Since this induces a structure of a free finite G-CW-complex on $\overline{X}$, we get a $\mathbb{Z}G$-basis for $C_*(\overline{X})$ and hence can consider $\rho^{(2)}(C_*^{(2)}(\overline{X}); \mathcal{N}(G))$. The cellular $\mathbb{Z}G$-basis for $C_*(\overline{X})$ is not unique, only up to permutation of the basis elements and multiplying base elements with trivial units, i.e., elements of the shape $\pm g$ for $g \in G$, but it turns out that $\rho^{(2)}(C_*^{(2)}(\overline{X}); \mathcal{N}(G))$ is independent of these choices after we have fixed a smooth triangulation of X. However, if we pass to a subdivision of the smooth triangulation of C, then $\rho^{(2)}(C_*^{(2)}(\overline{X});\mathcal{N}(G))$ changes in general.

Let X be a closed smooth Riemannian manifold. Then $\rho_{\mathrm{an}}^{(2)}(\overline{X};\mathcal{N}(G))$ and $\rho_{\mathrm{top}}^{(2)}(\overline{X};\mathcal{N}(G))$ are independent of the choice of smooth triangulation and hence depend only on the isometric diffeomorphism type of X. However, changing the Riemannian metric does in general change $\rho_{\mathrm{an}}^{(2)}(\overline{X};\mathcal{N}(G))$ and $\rho_{\mathrm{top}}^{(2)}(\overline{X};\mathcal{N}(G))$. If we have $b_n^{(2)}(\overline{X};\mathcal{N}(G)) = 0$ for all $n \geq 0$, then $\rho_{\mathrm{an}}^{(2)}(\overline{X};\mathcal{N}(G))$ and $\rho_{\mathrm{top}}^{(2)}(\overline{X};\mathcal{N}(G))$ are independent of the Riemannian metric and depend only on the diffeomorphism type of X, actually, they depend only on the simple homotopy type of X. There is a lot of evidence that in this situation only the homotopy type of X matters.

The next result is a special case of Theorem 14.10.

Theorem 7.7 (Relating the Approximation Conjecture for Fuglede-Kadison determinant and torsion invariants)**.** *Suppose that X is a closed Riemannian manifold such that $b_n^{(2)}(\overline{X},\mathcal{N}(G))$ vanishes for all $n \geq 0$. If G satisfies Conjecture 6.2 for all matrices $A \in M_{r,s}(\mathbb{Q}G)$ and all natural numbers r, s, then Conjecture 7.4 and Conjecture 7.5 hold for X.*

Remark 7.8 (On the L^2-acyclicity assumption)**.** Recall that in Theorem 7.7 we require that $b_n^{(2)}(M;\mathcal{N}(G)\big) = 0$ holds for $n \geq 0$. This assumption is satisfied in many interesting cases. It is possible that this assumption is not needed for Theorem 7.7 to be true, but our proof does not work without it.

7.3. **Integral torsion.**

Definition 7.9 (Integral torsion)**.** Define for a finite $\mathbb{Z}$-chain complex D_* its *integral torsion*

$$\rho^{\mathbb{Z}}(D_*) \quad := \quad \sum_{n \geq 0} (-1)^n \cdot \ln\left(\left|\mathrm{tors}(H_n(D_*))\right|\right) \quad \in \mathbb{R},$$

where $\left|\mathrm{tors}(H_n(D_*))\right|$ is the order of the torsion subgroup of the finitely generated abelian group $H_n(D_*)$.

Given a finite CW-complex X, define its *integral torsion* $\rho^{\mathbb{Z}}(X)$ by $\rho^{\mathbb{Z}}(C_*(X))$, where $C_*(X)$ is its cellular $\mathbb{Z}$-complex.

Remark 7.10 (Integral torsion and Milnor's torsion)**.** Let C_* be a finite free $\mathbb{Z}$-chain complex. Fix for each $n \geq 0$ a $\mathbb{Z}$-basis for C_n and for $H_n(C)/\mathrm{tors}(H_n(C))$. They induce $\mathbb{Q}$-bases for $\mathbb{Q}\otimes_{\mathbb{Z}} C_n$ and $H_n\big(\mathbb{Q}\otimes_{\mathbb{Z}} C_*\big) \cong \mathbb{Q}\otimes_{\mathbb{Z}}\big(H_n(C)/\mathrm{tors}(H_n(C))\big)$. Then the torsion in the sense of Milnor [76, page 365] is $\rho^{\mathbb{Z}}(C_*)$.

The following two conjectures are motivated by [12, Conjecture 1.3] and [66, Conjecture 11.3 on page 418 and Question 13.52 on page 478]. They are true in special cases by Theorem 10.4. The assumption that $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$ ensures that the definition of the topological L^2-torsion $\rho_{\mathrm{top}}^{(2)}(\overline{X};\mathcal{N}(G))$ makes sense for X also in the case of a connected finite CW-complex.

Conjecture 7.11 (Approximation Conjecture for integral torsion)**.** *Let X be a finite connected CW-complex. Suppose that $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$. Then*

$$\rho_{\mathrm{top}}^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(X[i])}{[G:G_i]}.$$

The chain complex version of Conjectures 7.11 is

Conjecture 7.12 (Approximating Fuglede-Kadison determinants and L^2-torsion by homology)**.**

(1) *Let $f\colon \mathbb{Z}G^r \to \mathbb{Z}G^r$ be a $\mathbb{Z}G$-homomorphism such that $f^{(2)}\colon L^2(G)^r \to L^2(G)^r$ is a weak isomorphism of Hilbert $\mathcal{N}(G)$-modules. Let $f[i] := f \otimes_{\mathbb{Z}G_i} \mathbb{Z} = f \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G_i]\colon \mathbb{Z}[G/G_i]^r \to \mathbb{Z}[G/G_i]^r$ be the induced $\mathbb{Z}$-homomorphism. Then*

$$\det{}^{(2)}_{\mathcal{N}(G)}(f^{(2)}) \quad = \quad \lim_{i\to\infty} \left|\operatorname{tors}\bigl(\operatorname{coker}(f[i])\bigr)\right|^{1/[G:G_i]};$$

(2) *Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Suppose that $C_*^{(2)}$ is L^2-acyclic, i.e., $b_p^{(2)}\bigl(C_*^{(2)}\bigr) = 0$ for all $p \geq 0$. Let $C[i]_* := C_* \otimes_{\mathbb{Z}G_i} \mathbb{Z} = C_* \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G_i]$ be the induced finite based free $\mathbb{Z}$-chain complex. Then*

$$\rho^{(2)}\bigl(C_*^{(2)}\bigr) \quad = \quad \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(C[i]_*)}{[G:G_i]};$$

In Conjecture 7.11 and Conjecture 7.12 it is necessary to demand that f is a weak isomorphism and that C_* and X are L^2-acyclic, otherwise there are counterexamples, see Remark 9.2.

Here are some results about the conjecture above which will be proved in Section 18.

Theorem 7.13.

(1) *Let $f\colon \mathbb{Z}G^r \to \mathbb{Z}G^s$ be a $\mathbb{Z}G$-homomorphism. Then*

$$\ln\bigl(\det{}^{(2)}_{\mathcal{N}(G)}(f^{(2)})\bigr) \quad \geq \quad \limsup_{i\to\infty} \frac{\left|\operatorname{tors}\bigl(\operatorname{coker}(f[i])\bigr)\right|}{[G:G_i]};$$

(2) *Suppose in the situation of assertion* (1) *of Conjecture 7.12 that $f \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{Q}}\colon \mathbb{Q}[G]^r \to \mathbb{Q}[G]^r$ is bijective. Then the conclusion appearing there is true.*

Suppose in the situation of assertion (2) *of Conjecture 7.12 that $H_p(C_*)\otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $p \geq 0$. Then the conclusion appearing there is true;*

(3) *If G is the infinite cyclic group $\mathbb{Z}$, then Conjecture 7.12 is true;*

Assertion (2) of Theorem 7.13 is generalized in Lemma 18.4. Assertion (3) of Theorem 7.13 has already been proved by Bergeron-Venkatesh [12, Theorem 7.3]. Applied to cyclic coverings of a knot complement this reduces to a theorem of Silver-Williams [94, Theorem 2.1]. An extension of the results in this paper is given by Raimbault [84].

8. On the relation of L^2-torsion and integral torsion

Let C_* be a finite based free $\mathbb{Z}$-chain complex, for instance the cellular chain complex $C_*(X)$ of a finite CW-complex. We have introduced in Subsection 7.1 the L^2-chain complex $C_*^{(2)} = C_* \otimes_{\mathbb{Z}} \mathbb{C}$ with differentials $c_n^{(2)} := c_n \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{C}}$ and its L^2-torsion $\rho^{(2)}(C_*^{(2)};\mathcal{N}(\{1\}))$.

Let $H_n^{(2)}(C_*^{(2)})$ be the L^2-homology of $C_*^{(2)}$ with respect to the von Neumann algebra $\mathcal{N}(\{1\}) = \mathbb{C}$. The underlying complex vector space is the homology $H_n(C_* \otimes_{\mathbb{I}} \mathbb{C})$ of $C_* \otimes_{\mathbb{Z}} \mathbb{C}$, but it comes now with the structure of a Hilbert space. For the reader's convenience we recall this Hilbert space structure. Let

$$\Delta_n^{(2)} = \bigl(c_n^{(2)}\bigr)^* \circ c_n^{(2)} + c_{n+1}^{(2)} \circ \bigl(c_{n+1}^{(2)}\bigr)^*\colon C_n^{(2)} \to C_n^{(2)}$$

be the associated Laplacian. Equip $\ker(\Delta_n^{(2)}) \subseteq C_n^{(2)}$ with the induced Hilbert space structure. Equip $H_n^{(2)}(C_*^{(2)})$ with the Hilbert space structure for which the obvious $\mathbb{C}$-isomorphism $\ker(\Delta_n^{(2)}) \to H_n^{(2)}(C_*^{(2)})$ becomes an isometric isomorphism. This is the same as the Hilbert subquotient structure with respect to the inclusion $\ker\bigl(c_n^{(2)}\bigr) \to C_n^{(2)}$ and the projection $\ker\bigl(c_n^{(2)}\bigr) \to H_n^{(2)}(C_*^{(2)})$.

Notation 8.1. If M is a finitely generated abelian group, define

$$M_f \quad := \quad M/\operatorname{tors}(M).$$

We have the canonical $\mathbb{C}$-isomorphism

$$(8.2) \quad \alpha_n \colon \big(H_n(C_*)_f\big)^{(2)} := \big(H_n(C_*)/\operatorname{tors}(H_n(C_*)\big) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} H_n^{(2)}(C_*^{(2)}).$$

Choose a $\mathbb{Z}$-basis on $H_n(C_*)_f$. This and the standard Hilbert space structure on $\mathbb{C}$ induces a Hilbert space structure on $\big(H_n(C_*)_f\big)^{(2)}$. Now we can consider the logarithm of the Fuglede-Kadison determinant

$$(8.3) \qquad R_n(C_*) \quad = \quad \ln\Big(\det_{\mathcal{N}(\{1\})}\big(\alpha_n \colon \big(H_n(C_*)_f\big)^{(2)} \to H_n^{(2)}(C_*^{(2)})\big)\Big),$$

which is some times called the *nth regulator*. It is independent of the choice of the $\mathbb{Z}$-basis of $H_n(C_*)_f$, since the absolute value of the determinant of an invertible matrix over $\mathbb{Z}$ is always 1. If $\{b_1, b_2, \ldots, b_r\}$ is an integral basis of $H_n(C_*)_f$ and we equip $H_n(C_*)_f \otimes_{\mathbb{Z}} \mathbb{C}$ with an inner product $\langle -, - \rangle$ for which the map α_n of (8.2) becomes an isometry, then

$$R_n(C_*) = \frac{\ln\big(\det_{\mathbb{C}}(B)\big)}{2},$$

where B is the Gram-Schmidt matrix $\big(\langle b_i, b_j\rangle\big)_{i,j}$.

The next result is proved for instance in [68, Lemma 2.3].

Lemma 8.4. *Let C_* be a finite based free $\mathbb{Z}$-chain complex. Then*

$$\rho^{\mathbb{Z}}(C_*) - \rho^{(2)}\big(C_*^{(2)}; \mathcal{N}(\{1\})\big) \quad = \quad \sum_{n \geq 0} (-1)^n \cdot R_n(C_*).$$

Remark 8.5 (Comparing conjectures for L^2-torsion and integral torsion)**.** Consider the following three statements:

(1) Every finite based free $\mathbb{Z}G$-chain complex C_* with $b_n^{(2)}\big(C_*^{(2)}\big) = 0$ for all $n \geq 0$ satisfies

$$\rho^{(2)}\big(C_*^{(2)}\big) = \lim_{i \to \infty} \frac{\rho^{(2)}\big(C[i]_*^{(2)}\big)}{[G : G_i]};$$

(2) Every finite based free $\mathbb{Z}G$-chain complex C_* with $b_n^{(2)}\big(C_*^{(2)}\big) = 0$ for all $n \geq 0$ satisfies assertion (2) of Conjecture 7.12, i.e.,

$$\rho^{(2)}\big(C_*^{(2)}\big) \quad = \quad \lim_{i \to \infty} \frac{\rho^{\mathbb{Z}}(C[i]_*)}{[G : G_i]};$$

(3) Every finite based free $\mathbb{Z}G$-chain complex C_* with $b_n^{(2)}\big(C_*^{(2)}\big) = 0$ for all $n \geq 0$ satisfies

$$\lim_{i \to \infty} \sum_{n \geq 0} (-1)^n \cdot \frac{R_n(C[i]_*)}{[G : G_i]} \quad = \quad 0,$$

By Lemma 8.4 they are all true if two of them hold.

Lemma 8.6. *Let X be an oriented closed smooth manifold of dimension d. Fix a smooth triangulation. Let s_n be the number of n-simplices of the triangulation of X. Then we get*

$$\begin{aligned} R_d(C_*(X[i])) \quad &= \quad \frac{\ln([G : G_i] \cdot s_d)}{2}; \\ R_0(C_*(X[i])) \quad &= \quad -\frac{\ln([G : G_i] \cdot s_0)}{2}; \\ \lim_{i \to \infty} \frac{R_n(C_*(X[i]))}{[G : G_i]} \quad &= \quad 0 \quad \textit{for } n = 0, d. \end{aligned}$$

Proof. The fundamental class $[X[i]]$ is a generator of the infinite cyclic group $H_d(X[i];\mathbb{Z})$, and is represented by the cycle $\sigma[i]_d$ in $C_d(X[i])$ given by the sum over all d-dimensional simplices. The number of d-simplices in $X[i]$ is $[G:G_i]\cdot s_d$. If we consider $\sigma[i]_d$ as an element in $C_d^{(2)}(X[i])$, it belongs to the kernel of $\Delta[i]_d^{(2)}$ and has norm $\sqrt{[G:G_i]\cdot s_d}$. Hence $R_d(C_*(X[i]))$, which is the logarithm of the norm of $\sigma[i]_d$ considered as an element in $C_d^{(2)}(X[i])$, is $\frac{\ln([G:G_i]\cdot s_d)}{2}$.

Consider the element $\sigma[i]_0\in C_0(X[i])$ given by the sum of the 0-simplices of $X[i]$. The number of 0-simplices in $X[i]$ is $[G:G_i]\cdot s_0$. The element $\sigma[i]_0$ considered as element in $C_0^{(2)}(X[i])$ has norm $\sqrt{[G:G_i]\cdot s_0}$ and lies in the kernel of $\big(c[i]_1^{(2)}\big)^*$ and hence in the kernel of $\Delta[i]_0^{(2)}$ since it is orthogonal to any element of the shape e_1 - e_0 for 0-simplices e_0 and e_1 and hence to the image of $c_1^{(2)}\colon C_1^{(2)}(X[i])\to C_0^{(2)}(X[i])$. The augmentation map $C_0(X[i])\to\mathbb{Z}$ sending a 0-simplex to 1 induces an isomorphism $H_0(C_*(X[i]))\xrightarrow{\cong}\mathbb{Z}$. This shows that $\sigma[i]_0$ represents $[G:G_i]\cdot s_0$ times the generator of $H_0(X[i];\mathbb{Z})$. Hence $R_0(C_*(X[i]))$ which is the logarithm of the norm of $\frac{\sigma[i]_0}{[G:G_i]\cdot s_0}$ is $-\frac{\ln([G:G_i]\cdot s_0)}{2}$.

In particular we get $\lim_{i\to\infty}\frac{R_n(C_*(X[i]))}{[G:G_i]}=0$ for $n=0,d$. □

9. Elementary example about L^2-torsion and integral torsion

Consider integers a, b, k, l, and $g\geq 1$, such that $(a,b)=(1)$ and $(k,l)=(1)$.

Consider the following finite based free $\mathbb{Z}$-chain complex C_* which is concentrated in dimensions 0, 1, 2 and 3 and given there by

$$0\cdots\to 0\to\mathbb{Z}\xrightarrow{c_3=\begin{pmatrix}-l\\ k\end{pmatrix}}\mathbb{Z}^2\xrightarrow{c_2=\begin{pmatrix}gka & gla\\ gkb & glb\end{pmatrix}}\mathbb{Z}^2\xrightarrow{c_1=\begin{pmatrix}-b & a\end{pmatrix}}\mathbb{Z}\to 0\to\cdots.$$

Notice that any matrix homomorphism $c_2\colon\mathbb{Z}^2\to\mathbb{Z}^2$ whose kernel has rank one is of the shape above.

One easily checks that

$$\begin{aligned}
\ker(c_3) &= \{0\};\\
\operatorname{im}(c_3) &= \{n\cdot\begin{pmatrix}-l\\ k\end{pmatrix}\mid n\in\mathbb{Z}\};\\
\ker(c_2) &= \{n\cdot\begin{pmatrix}-l\\ k\end{pmatrix}\mid n\in\mathbb{Z}\};\\
\operatorname{im}(c_2) &= \{n\cdot\begin{pmatrix}ga\\ gb\end{pmatrix}\mid n\in\mathbb{Z}\};\\
\ker(c_1) &= \{n\cdot\begin{pmatrix}a\\ b\end{pmatrix}\mid n\in\mathbb{Z}\};\\
\operatorname{im}(c_1) &= \mathbb{Z}.
\end{aligned}$$

This implies

$$H_i(C_*)=\begin{cases}\mathbb{Z}/g & i=1;\\ \{0\} & i\neq 1.\end{cases}$$

We conclude from [66, Lemma 3.15 (4) on page 129]

$$\begin{aligned}
\ln\big(\det\nolimits^{(2)}_{\mathcal{N}(\{1\})}\big(c_3^{(2)}\big)\big) &= \frac{1}{2}\cdot\ln\big(\det\nolimits^{(2)}_{\mathcal{N}(\{1\})}\big((c_3^{(2)})^*\circ c_3^{(2)}\big)\big)\\
&= \frac{1}{2}\cdot\ln\big(\det\nolimits^{(2)}_{\mathcal{N}(\{1\})}\big((k^2+l^2)\colon\mathbb{C}\to\mathbb{C}\big)\big)=\frac{\ln(k^2+l^2)}{2}.
\end{aligned}$$

Analogously one shows

$$\ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(c_1^{(2)}\bigr)\bigr) = \frac{\ln(a^2+b^2)}{2}.$$

The kernel of $c_2^{(2)}$ is the subvector space of $\mathbb{C}^2$ generated by $\frac{1}{\sqrt{k^2+l^2}} \cdot \begin{pmatrix} -l \\ k \end{pmatrix}$ and the image of $c_2^{(2)}$ is the subvector space of $\mathbb{C}^2$ generated by $\frac{1}{\sqrt{a^2+b^2}} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$. Hence the orthogonal complement $\ker\bigl(c_2^{(2)}\bigr)^{\perp}$ of the kernel of $c_2^{(2)}$ is the subvector space of $\mathbb{C}^2$ generated by $\frac{1}{\sqrt{k^2+l^2}} \cdot \begin{pmatrix} k \\ l \end{pmatrix}$. Since $\frac{1}{\sqrt{a^2+b^2}} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ and $\frac{1}{\sqrt{k^2+l^2}} \cdot \begin{pmatrix} k \\ l \end{pmatrix}$ have norm 1 and

$$c_2^{(2)}\left(\frac{1}{\sqrt{k^2+l^2}} \cdot \begin{pmatrix} k \\ l \end{pmatrix}\right) = \left(g \cdot \sqrt{k^2+l^2} \cdot \sqrt{a^2+b^2}\right) \cdot \left(\frac{1}{\sqrt{a^2+b^2}} \cdot \begin{pmatrix} a \\ b \end{pmatrix}\right),$$

we conclude from [66, Lemma 3.15 (3) on page 129]

$$\ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(c_2^{(2)}\bigr)\bigr) = \ln(g) + \frac{\ln(a^2+b^2)+\ln(k^2+l^2)}{2}.$$

Notice that Lemma 8.4 predicts $\rho^{(2)}\bigl(C_*^{(2)}\bigr) = \rho^{\mathbb{Z}}(C_*)$ which is consistent with the direct computation

$$\begin{aligned}
\rho^{(2)}\bigl(C_*^{(2)}\bigr) &= -\ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(c_3^{(2)}\bigr)\bigr) + \ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(c_2^{(2)}\bigr)\bigr) - \ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(c_1^{(2)}\bigr)\bigr) \\
&= \ln(g) \\
&= \ln\bigl(\operatorname{tors}\bigl(H_1(C_*)\bigr)\bigr) \\
&= \rho^{\mathbb{Z}}(C_*).
\end{aligned}$$

We also compute the combinatorial Laplace operators of C_*. We get for their matrices

$$\begin{aligned}
\Delta_3 &= (k^2+l^2); \\
\Delta_2 &= \begin{pmatrix} g^2k^2a^2+g^2k^2b^2+l^2 & g^2kla^2+g^2klb^2-kl \\ g^2kla^2+g^2klb^2-kl & g^2l^2a^2+g^2l^2b^2+k^2 \end{pmatrix}; \\
\Delta_1 &= \begin{pmatrix} g^2k^2a^2+g^2l^2a^2+b^2 & g^2k^2ab+g^2l^2ab-ab \\ g^2k^2ab+g^2l^2ab-ab & g^2k^2b^2+g^2l^2b^2+a^2 \end{pmatrix}; \\
\Delta_0 &= (a^2+b^2).
\end{aligned}$$

This implies

$$\det_{\mathbb{Z}}(\Delta_i) = \det^{(2)}_{\mathcal{N}(\{1\})}\bigl(\Delta_i^{(2)}\bigr) = \begin{cases} k^2+l^2 & i=3; \\ (a^2+b^2)\cdot g^2 \cdot (k^2+l^2)^2 & i=2; \\ (a^2+b^2)^2 \cdot g^2 \cdot (k^2+l^2) & i=1; \\ (a^2+b^2) & i=0. \end{cases}$$

This is consistent with the formula

$$\rho^{(2)}\bigl(C_*^{(2)};\mathcal{N}(\{1\})\bigr) = -\frac{1}{2} \cdot \sum_{i\geq 0} (-1)^i \cdot i \cdot \ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\})}\bigl(\Delta_i^{(2)}\bigr)\bigr).$$

Remark 9.1 (No relationship between the differentials and homology in each degree)**.** We see that there is no relationship between $\ln\bigl(\det_{\mathcal{N}(\{1\})}\bigl(\Delta_i^{(2)}\bigr)\bigr)$ and $\ln\bigl(\operatorname{tors}\bigl(H_i(C_*)\bigr)\bigr)$ or between $\ln\bigl(\det^{(2)}_{\mathcal{N}(\{1\}}\bigl(c_i^{(2)}\bigr)\bigr)$ and $\ln\bigl(\operatorname{tors}\bigl(H_i(C_*)\bigr)\bigr)$ for each individual $i \in \mathbb{Z}$, there is only a relationship after taking the alternating sum over $i \geq 0$.

This shows that a potential proof of Conjecture 7.12 will require more input than one would expect for a potential proof of Conjecture 7.3, Conjecture 7.4, or Conjecture 7.5.

Remark 9.2 (L^2-acyclicity is necessary for the homological version)**.** This example can also be used to show that the condition of L^2-acyclicity appearing in Conjecture 7.12 is necessary. This is not a surprise since $\rho^{\mathbb{Z}}(C[n]_*)$ depends only on the $\mathbb{Z}[\mathbb{Z}]$-chain homotopy type of C_* which is not true for $\rho^{(2)}(C_*^{(2)})$ unless $C_*^{(2)}$ is L^2-acyclic.

Namely, consider the 1-dimensional $\mathbb{Z}$-chain complex chain complex D_* whose only non-trivial differential d_1 is the differential c_2 in the chain complex C_* above. Let $E_* := D_* \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$. Put $E_*[n] = E_* \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[\mathbb{Z}/n]$. Then $E_*[n] = D_* \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/n]$. We conclude from the computations above and [66, Theorem 3.14 (5) and (6) on page 128]

$$\begin{aligned}
\rho^{(2)}\bigl(E_*^{(2)};\mathcal{N}(\mathbb{Z})\bigr) &= \ln(g) + \frac{\ln(a^2+b^2)+\ln(k^2+l^2)}{2};\\
\frac{\rho^{(2)}\bigl(E[n]_*^{(2)};\mathcal{N}(\{1\})\bigr)}{n} &= \ln(g) + \frac{\ln(a^2+b^2)+\ln(k^2+l^2)}{2};\\
\frac{\rho^{\mathbb{Z}}(E[n]_*)}{n} &= \ln(g).
\end{aligned}$$

Hence we have

$$\rho^{(2)}\bigl(E_*^{(2)};\mathcal{N}(\mathbb{Z})\bigr) = \lim_{n\to\infty} \frac{\rho^{(2)}\bigl(E[n]_*^{(2)};\mathcal{N}(\{1\})\bigr)}{n}$$

but

$$\rho^{(2)}\bigl(E_*^{(2)};\mathcal{N}(\mathbb{Z})\bigr) \neq \lim_{n\to\infty} \frac{\rho^{\mathbb{Z}}(E[n]_*)}{n}.$$

Notice that the condition of L^2-acyclicity is not demanded in Conjecture 7.3, Conjecture 7.4, Conjecture 7.5, Conjecture 14.5, and Conjecture 14.8.

10. Aspherical manifolds

The following conjecture is in our view the most advanced and interesting one. It combines Conjecture 7.11 that one can approximate L^2-torsion by integral torsion in the L^2-acyclic case with the conjecture that for closed aspherical manifolds X the L^2-cohomology of $\widetilde{X}$ and asymptotically the homology of $X[i]$ are concentrated in the middle dimension.

Conjecture 10.1 (Homological growth and L^2-torsion for aspherical closed manifolds)**.** *Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then*

(1) For any natural number n with $2n \neq d$ we get

$$b_n^{(2)}(\widetilde{M}) = 0.$$

If $d = 2n$, we have

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \geq 0.$$

If $d = 2n$ and M carries a Riemannian metric of negative sectional curvature, then

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) > 0;$$

(2) *Let $(G_i)_{i\geq 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G : G_i]$ and trivial intersection $\bigcap_{i\geq 0} G_i = \{1\}$. Put $M[i] = G_i\backslash\widetilde{M}$.*

Then we get for any natural number n and any field F

$$b_n^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_n(M[i];F)}{[G : G_i]} = \lim_{i\to\infty} \frac{d\big(H_n(M[i];\mathbb{Z})\big)}{[G : G_i]};$$

and for $n = 1$

$$b_1^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_1(M[i];F)}{[G : G_i]} = \lim_{i\to\infty} \frac{d\big(G_i/[G_i,G_i]\big)}{[G : G_i]}$$
$$= RG(G,(G_i)_{i\geq 0}) = \begin{cases} 0 & d \neq 2; \\ -\chi(M) & d = 2; \end{cases}$$

(3) *We get for $m \geq 0$*

$$\lim_{i\to\infty} \frac{\chi_m^{\mathrm{trun}}(M[i])}{[G : G_i]} = \begin{cases} \chi(M) & \text{if } d \text{ is even and } 2m \geq d; \\ 0 & \text{otherwise}; \end{cases}$$

(4) *If $d = 2n+1$ is odd, we have*

$$(-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\big(\widetilde{M}\big) \geq 0;$$

If $d = 2n+1$ is odd and M carries a Riemannian metric with negative sectional curvature, we have

$$(-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\big(\widetilde{M}\big) > 0;$$

(5) *Let $(G_i)_{i\geq 0}$ be any chain of normal subgroups $G_i \subseteq G$ of finite index $[G : G_i]$ and trivial intersection $\bigcap_{i\geq 0} G_i = \{1\}$. Put $M[i] = G_i\backslash\widetilde{M}$.*

Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i\to\infty} \frac{\ln\Big(\big|\mathrm{tors}\big(H_n(M[i])\big)\big|\Big)}{[G : G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i\to\infty} \frac{\ln\Big(\big|\mathrm{tors}\big(H_n(M[i])\big)\big|\Big)}{[G : G_i]} = (-1)^n \cdot \rho_{\mathrm{an}}^{(2)}\big(\widetilde{M}\big) \geq 0.$$

Notice that in assertion (1) and (4) we are not demanding that $G = \pi_1(M)$ is residually finite. This assumption only enters in assertions (2) and (5) , where the chain $(G_i)_{i\geq 0}$ occurs.

Remark 10.2 (Rank growth versus torsion growth)**.** Let us summarize what Conjecture 10.1 means for an aspherical closed manifold M of dimension d. It predicts that the rank of the singular homology grows in dimension m sublinear if $2m \neq d$, and grows linearly if $d = 2m$ and M carries a Riemannian metric of negative sectional curvature. It also predicts that the cardinality of the torsion of the singular homology grows in dimension m subexponentially if $2m+1 \neq d$ and grows exponentially if $d = 2m+1$ and M carries a Riemannian metric of negative sectional curvature. Roughly speaking, the free part of the singular homology is asymptotically concentrated in dimension m if $d = 2m$ and the torsion part is asymptotically concentrated in dimension m if $d = 2m+1$. A vague explanation for this phenomenon could be that Poincaré duality links the rank in dimensions m and $d-m$, whereas the torsion is linked in dimensions m and $d-1-m$ and there must be some reason that except in the middle dimension the growth of the rank and the growth of the cardinality of the torsion block one another in dual dimensions.

Remark 10.3 (Finite Poincaré complexes)**.** One may replace in the formulation of Conjecture 10.1 the aspherical closed manifold M by an aspherical finite Poincaré complex. In the formulation of the part of assertion (1), where negative sectional curvature is required, one has to add an assumption on $\pi_1(X)$, for instance that $\pi_1(X)$ is a CAT(-1)-group.

Assertion (1) of Conjecture 10.1 in the case that M carries a Riemannian metric with non-positive sectional curvature is the Singer Conjecture The Singer Conjecture and also the related Hopf Conjecture are discussed in detail in [66, Section 11].

Assertion (2) is closely related to Conjecture 2.4, Conjecture 2.5 and Question 3.4.

The parity condition about the L^2-torsion appearing in assertion (4) of Conjecture 10.1 is already considered in [66, Conjecture 11.3 on page 418].

Assertion (5) appearing in Conjecture 10.1 in the case, that M is a locally symmetric space, is discussed in Bergeron-Venkatesh [12], where also twisting with a finite-dimensional integral representation is considered.

Some evidence for Conjecture 10.1 comes from the following result of Lück [68, Corollary 1.13].

Theorem 10.4. *Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$ and fields F*

$$\begin{aligned}
\lim_{i\to\infty} \frac{b_n(M[i];F)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{d\bigl(H_n(M[i];\mathbb{Z})\bigr)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\ln\bigl(\bigl|\operatorname{tors}\bigl(H_n(M[i])\bigr)\bigr|\bigr)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\rho_{\mathrm{an}}\bigl(M[i];\mathcal{N}(\{1\})\bigr)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}\bigl(M[i]\bigr)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M}) &= 0;\\
\rho_{\mathrm{an}}^{(2)}(\widetilde{M}) &= 0.
\end{aligned}$$

In particular Conjecture 2.4, Conjecture 2.5, Conjecture 7.4, Conjecture 7.5, Conjecture 7.11 and Conjecture 10.1 with the exception of assertion (3) *are known to be true for $G = \pi_1(M)$ and $X = M$.*

Estimates of the growth of the torsion in the homology in terms of the volume of the underlying manifold and examples of aspherical manifolds, where this growth is subexponential, are given in [87].

Sometimes one can express for certain classes of closed Riemannian manifolds M the L^2-torsion of the universal covering $\widetilde{M}$ by the volume

$$\rho_{\mathrm{an}}^{(2)}(\widetilde{M}) = C_{\dim(M)} \cdot \operatorname{vol}(M),$$

where $C_{\dim(M)} \in \mathbb{R}$ is a dimension constant depending only on the class but not on the specific M. This follows from the Proportionality Principle due to Gromov, see for instance [66, Theorem 1.183 on page 201]. Typical examples are locally symmetric spaces of non-compact type, for instance hyperbolic manifolds, see [66, Theorem 5.12 on page 228]. Since $\rho_{\mathrm{an}}^{(2)}(\widetilde{M})$ vanishes for even-dimensional closed

Riemannian manifolds, only the case of odd dimension is interesting. For locally symmetric spaces of non-compact type with odd dimension d one can show that $(-1)^{(d-1)/2} \cdot C_d \geq 0$ holds. Thus one obtains some computational evidence for Conjecture 10.1.

Here is a concrete and already very interesting special case.

Example 10.5 (Hyperbolic 3-manifolds)**.** Suppose that M is a closed hyperbolic 3-manifold. Then $\rho_{\mathrm{an}}(\widetilde{M})$ is known to be $-\frac{1}{6\pi} \cdot \mathrm{vol}(M)$, see [66, Theorem 4.3 on page 216], and hence Conjecture 10.1 predicts

$$\lim_{i \to \infty} \frac{\ln\big(\big|\mathrm{tors}\big(H_1(G_i)\big|\big)}{[G : G_i]} = \frac{1}{6\pi} \cdot \mathrm{vol}(M).$$

Since the volume is always positive, the equation above implies that $|\operatorname{tors}\big(H_1(G_i)\big|$ growth exponentially in $[G : G_i]$. This is in contrast to the question appearing in the survey paper by Aschenbrenner-Friedl-Wilton [5, Question 9.13] whether a hyperbolic 3-manifold of finite volume admits a finite covering $N \to M$ such that $\operatorname{tors}\big(H_1(N)\big)$ is non-trivial. However, a positive answer to this question and evidence for Conjecture 10.1 for closed hyperbolic 3-manifolds is given in Sun [96, Corollary 1,6], where it is shown that for any finitely generated abelian group A, and any closed hyperbolic 3-manifold M, there exists a finite cover N of M, such that A is a direct summand of $H_1(N;\mathbb{Z})$.

Bergeron-Segun-Venkatesh [11] consider the equality above for arithmetic hyperbolic 3-manifolds and relate it to a conjecture about classes in the second integral homology.

Some numerical evidence for the equality above is given in Sengun [93].

The inequality

$$\limsup_{i \to \infty} \frac{\ln\big(\big|\mathrm{tors}\big(H_1(G_i)\big|\big)}{[G : G_i]} \leq \frac{1}{6\pi} \cdot \mathrm{vol}(M).$$

is proved by Thang [54] for a compact connected orientable irreducible 3-manifold M with infinite fundamental group and empty or toroidal boundary.

Remark 10.6 (Possible Scenarios)**.** Consider the situation of Conjecture 10.1. We can find for each $i \geq 0$, $n \geq 0$ and prime number p integers $r[i,n] \geq 0$, $t[i,n,p] \geq 0$, and $n[i,n,p]_1, n[i,n,p]_2, \ldots, n[i,n,p]_{t[i,n,p]} \geq 1$ such that the set $\{p \text{ prime} \mid t[i,n,p] > 0\}$ is finite and

$$H_n(M[i];\mathbb{Z}) \cong \mathbb{Z}^{r[i,n]} \oplus \bigoplus_{p \text{ prime}} \bigoplus_{j=1}^{t[i,n,p]} \mathbb{Z}/p^{n[i,n,p]_j}.$$

Then

$$\begin{aligned} b_n(M[i];\mathbb{Q}) &= r[i,n]; \\ b_n(M[i];\mathbb{F}_p) &= r[i,n] + t[i,n,p]; \\ d\big(H_n(M[i];\mathbb{Z})\big) &= r[i,n] + \max\{t[i,n,p] \mid p \text{ prime}\}; \\ \ln\big(\big|\mathrm{tors}\big(H_n(M[i])\big)\big|\big) &= \sum_p \sum_{j=1}^{t[i,n,p]} n[i,n,p]_j \cdot \ln(p). \end{aligned}$$

Let us discuss first the case, where M possesses a Riemannian metric with negative sectional curvature and $\dim(M) = 2n+1$. Then Conjecture 10.1 predicts

$$\begin{aligned}
\lim_{i\to\infty} \frac{r[i,n]}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\max\{t[i,n,p] \mid p \text{ prime}\}}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\sum_p \sum_{j=1}^{t[i,n,p]} n[i,n,p]_j \cdot \ln(p)}{[G:G_i]} &> 0.
\end{aligned}$$

There are two scenarios which can explain these expected statements. Of course there are other scenarios as well, but the two below illustrate nicely what may happen.

- The number $\max\{t[i,n,p] \mid p \text{ prime}\}$ grows sublinearly in comparison with $[G:G_i]$. The number of primes p for which $t[i,n,p] \geq 1$ grows linearly with $[G:G_i]$. A concrete example is the case, where
$$H_1(M[i];\mathbb{Z})) \cong \mathbb{Z}^{r[i,1]} \oplus \bigoplus_{p\in\mathcal{P}[i]} \mathbb{Z}/p,$$
where $\mathcal{P}[i]$ is a set of primes satisfying $\lim_{i\to\infty} \frac{|\mathcal{P}[i]|}{[G:G_i]} > 0$, and $\lim_{i\to\infty} \frac{r[i,1]}{[G:G_i]} = 0$ holds;
- The number $\max\{t[i,n,p] \mid p \text{ prime}\}$ grows sublinearly in comparison with $[G:G_i]$. There is a prime p such that the number $\sum_{j=1}^{t[i,n,p]} n[i,n,p]_j$ grows linearly with $[G:G_i]$. A concrete example is the case, when
$$H_1(M[i];\mathbb{Z})) \cong \mathbb{Z}^{r[i,1]} \oplus \mathbb{Z}/p^{m[i,1]}$$
for a prime p such that $\lim_{i\to\infty} \frac{m[i,1]}{[G:G_i]} > 0$ and $\lim_{i\to\infty} \frac{r[i,1]}{[G:G_i]} = 0$ holds.

Next we discuss the case, where M possesses a Riemannian metric with negative sectional curvature and $\dim(M) = 2n$. Then Conjecture 10.1 predicts

$$\begin{aligned}
\lim_{i\to\infty} \frac{r[i,n]}{[G:G_i]} &> 0;\\
\lim_{i\to\infty} \frac{\sum_p \sum_{j=1}^{t[i,n,p]} n[i,n,p]_j \cdot \ln(p)}{[G:G_i]} &= 0.
\end{aligned}$$

11. Mapping tori

A very interesting case is the example of a mapping torus T_f of a selfmap $f\colon Z \to Z$ of a connected finite CW-complex Z (which is not necessarily a homotopy equivalence). The canonical projection $q\colon T_f \to S^1$ induces an epimorphism $\mathrm{pr}\colon G = \pi_1(T_f) \to \mathbb{Z}$. Let K be its kernel which can be identified with the colimit of the direct system of groups indexed by $\mathbb{Z}$.

$$\cdots \xrightarrow{\pi_1(f)} \pi_1(Z) \xrightarrow{\pi_1(f)} \pi_1(Z) \xrightarrow{\pi_1(f)} \cdots$$

In particular the inclusion $Z \to T_f$ induces a homomorphism $j\colon \pi_1(Z) \to K$ which corresponds to the structure map in the description of K as a colimit at $0 \in \mathbb{Z}$. We obtain a short exact sequence $1 \to K \xrightarrow{j} G \xrightarrow{\mathrm{pr}} \mathbb{Z} \to 1$. We use the Setup (0.1) with $X = T_f$ and $\overline{X} = \widetilde{T_f}$. Put $K_i = j^{-1}(G_i)$. Let $d_i \in \mathbb{Z}$ be the integer for which $pr(G_i) = d_i \cdot \mathbb{Z}$. We obtain an induced exact sequence $1 \to K_i \xrightarrow{j_i} G_i \xrightarrow{\mathrm{pr}_i} d_i \cdot \mathbb{Z} \to 1$. We have

$$[G:G_i] = [K:K_i] \cdot d_i.$$

If $\pi_1(f)$ is an isomorphism, then $j\colon \pi_1(Z) \to K$ is an isomorphism.

Let $p_i\colon S^1 \to S^1$ be the d_i-sheeted covering given by $z \mapsto z^{d_i}$. It is the covering associated to $d_i \cdot \mathbb{Z} \subseteq \mathbb{Z}$. Let $\overline{p_i}\colon T_f[i]' \to T_f$ be the d_i-sheeted covering given by the pullback

$$\begin{array}{ccc} T_f[i]' & \xrightarrow{q[i]'} & S^1 \\ {\scriptstyle \overline{p_i}}\big\downarrow & & \big\downarrow{\scriptstyle p_i} \\ T_f & \xrightarrow[q]{} & S^1 \end{array}$$

It is the covering associated to $\operatorname{pr}^{-1}(d_i \cdot \mathbb{Z}) \subseteq G = \pi_1(T_f)$. Let $q_i\colon T_f[i] \to T_f[i]'$ be the $[K : K_i]$-sheeted covering which is associated to $G_i \subseteq \operatorname{pr}^{-1}(d_i \cdot \mathbb{Z}) = \pi_1(T_f[i]')$. The composite $T_f[i] \xrightarrow{q_i} T_f'[i] \xrightarrow{\overline{p_i}} T_f$ is the $[G : G_i]$-sheeted covering associated to $G_i \subseteq G = \pi_1(T_f)$. Let $\overline{q_i}\colon Z[i] \to Z$ be the $[K : K_i]$-sheeted covering given by the pullback

$$\begin{array}{ccc} Z[i] & \longrightarrow & T_f[i] \\ \big\downarrow{\scriptstyle \overline{q_i}} & & \big\downarrow{\scriptstyle q_i} \\ Z & \xrightarrow{i} & T_f[i]' \end{array}$$

It is the covering given by $K/K_i \times_{\pi_1(Z)/j^{-1}(K_i)} \widetilde{Z}$.

Since $T_f[i]'$ is obtained from the d_i-fold mapping telescope of f by identifying the left and the right end by the identity, there is an obvious map $T_f[i]' \to T_{f^{d_i}}$ which turns out to be a homotopy equivalence. Hence we can choose a homotopy equivalence

$$u\colon T_{f^{d_i}} \xrightarrow{\simeq} T_f[i]'.$$

Define the homotopy equivalence $\overline{u}\colon \overline{T_{f^{d_i}}} \xrightarrow{\simeq} T_f[i]$ by the pullback

$$\begin{array}{ccc} \overline{T_{f^{d_i}}} & \xrightarrow{\overline{u}} & T_f[i] \\ \big\downarrow{\scriptstyle \overline{q_i}} & & \big\downarrow{\scriptstyle q_i} \\ T_{f^{d_i}} & \xrightarrow{u} & T_f[i]' \end{array}$$

There is a finite CW-structure on $T_{f^{d_i}}$ such that the number of n-cells $c_n(T_{f^{d_i}})$ is $c_n(Z)+c_{n-1}(Z)$, where $c_n(Z)$ is the number of n-cells in Z. Since $\overline{q_i}\colon \overline{T_{f^{d_i}}} \to T_{f^{d_i}}$ is a $[K : K_i]$-sheeted covering, there is a finite CW-structure on $\overline{T_{f^{d_i}}}$ such that the number of n-cells $c_n(T_{f^{d_i}})$ is $[K : K_i] \cdot (c_n(Z) + c_{n-1}(Z))$. This implies

$$\frac{c_n(T_{f^{d_i}})}{[G : G_i]} \quad = \quad \frac{c_n(Z) + c_{n-1}(Z)}{d_i}.$$

Hence we get for any field F

$$\begin{aligned} \frac{b_n(T_f[i]; F)}{[G : G_i]} &\quad\leq\quad \frac{c_n(Z) + c_{n-1}(Z)}{d_i}; \\ d(\pi_1(T_f[i])) &\quad\leq\quad \frac{c_1(Z) + c_0(Z)}{d_i}; \\ \frac{\operatorname{def}(\pi_1(T_f[i]))}{[G : G_i]} &\quad\leq\quad \frac{c_2(Z) + c_1(Z) + c_0(Z)}{d_i}; \\ \frac{\chi_n^{\mathrm{trun}}(T_f[i])}{[G : G_i]} &\quad\leq\quad \frac{\sum_{i=0}^n c_i(Z)}{d_i}. \end{aligned}$$

11.1. **The case $\bigcap_{i\geq 0} d_i \cdot \mathbb{Z} = \{1\}$.** Suppose that $\bigcap_{i\geq 0} d_i \cdot \mathbb{Z} = \{1\}$, or, equivalently, $\lim_{i\to\infty} d_i = \infty$ holds. Then we conclude for any field F

$$\begin{aligned}
\lim_{i\to\infty} \frac{b_n(T_f[i];F)}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{d(\pi_1(T_f[i]))}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\operatorname{def}(\pi_1(T_f[i]))}{[G:G_i]} &= 0;\\
\lim_{i\to\infty} \frac{\chi_n^{\mathrm{trun}}(T_f[i])}{[G:G_i]} &= 0.
\end{aligned}$$

Since

$$b_n^{(2)}(\widetilde{T_f}) = 0 \tag{11.1}$$

holds for $n \geq 0$ by [63, Theorem 2.1], this gives evidence for Conjecture 2.4 and Conjecture 2.5, and positive answers to Questions 3.3 and Question 4.3, provided that Z is aspherical.

11.2. **The case $\bigcap_{i\geq 0} d_i \cdot \mathbb{Z} \neq \{1\}$.** Next we consider the hard case $\bigcap_{i\geq 0} d_i \cdot \mathbb{Z} \neq \{1\}$. Then there exists an integer i_0 such that $d_i = d_{i_0}$ for all $i \geq i_0$. We can assume without loss of generality that $d_i = 1$ holds for all $i \geq 0$, otherwise replace T_f by $T_f[i_0]$, G by G_{i_0}, $\mathbb{Z}$ by $n_{i_0} \cdot \mathbb{Z}$, and $(G_i)_{i\geq 0}$ by $(G_i)_{i\geq i_0}$.

We conclude from Theorem 2.1, Theorem 2.2 and (11.1)

$$\lim_{i\to\infty} \frac{b_n(T_f[i];F)}{[G:G_i]} = 0. \tag{11.2}$$

provided that F has characteristic zero. We get the same conclusion (11.2) for any field F provided that G is torsionfree elementary amenable and residually finite by Theorem 2.2 and (11.1), since Theorem 2.2 implies for torsionfree elementary G that $\lim_{i\to\infty} \frac{b_n(T_f[i];F)}{[G:G_i]}$ exists and is independent of the chain $(G_i)_{i\geq 0}$ and because of Subsection 11.1 there exists an appropriate chain, for instance $(G_i \cap \mathrm{pr}^{-1}(2^i \cdot \mathbb{Z}))_{i\geq 0}$, with $\lim_{i\to\infty} \frac{b_n(T_f[i];F)}{[G:G_i]} = 0$. We do not know whether (11.2) holds for arbitrary fields and arbitrary residually finite groups G, as predicted by Conjecture 2.4.

We do not know whether the rank gradient $RG(G,(G_i)_{i\geq 0})$ is zero for any chain $(G_i)_{i\geq 0}$ as predicted by Conjecture 3.3 in view of (11.2), but at least for chains with $\bigcap_{i\geq 0} d_i \cdot \mathbb{Z} = \{1\}$ this follows from Subsection 11.1. This illustrates why it would be very interesting to know whether the rank gradient $RG(G,(G_i)_{i\geq 0})$ is independent of the chain $(G_i)_{i\geq 0}$. The same remark applies to the more general Question 4.3.

One can express $b_n(T_f[i];F)$ in terms of f as follows. Obviously K_i is a normal subgroup of G, the automorphisms $\pi_1(f)\colon G = \pi_1(Z) \to G = \pi_1(Z)$ sends K_i to K_i and we have $[G:G_i] = [K:K_i]$ for all $i \geq 0$. Put $f[0] = f\colon Z = Z[0] \to Z = Z[0]$. We can choose for each $i \geq 1$ selfhomotopy equivalences $f[i]\colon Z[i] \to Z[i]$ for which the following diagram with the obvious coverings as vertical maps

$$\begin{array}{ccc}
Z[i] & \xrightarrow{f[i]} & Z[i]\\
\downarrow & & \downarrow\\
Z[i-1] & \xrightarrow{f[i-1]} & Z[i-1]
\end{array}$$

commutes. Then $T_f[i]$ is $T_{f[i]}$. We have the Wang sequence of R-modules for any commutative ring R

$$(11.3)\qquad \cdots \to H_n(Z[i];R) \xrightarrow{\operatorname{id}-H_n(f[i];R)} H_n(Z[i];R) \to H_n(T_f[i]) \\ \to H_{n-1}(Z[i];R) \xrightarrow{\operatorname{id}-H_{n-1}(f[i];R)} H_{n-1}(Z[i];R) \to \cdots$$

This implies

$$(11.4)\qquad b_n(T_f[i];F) \\ = \dim_F\bigl(\operatorname{coker}\bigl(\operatorname{id}-H_n(f[i];F)\bigr)\bigr) + \dim_F\bigl(\ker\bigl(\operatorname{id}-H_{n-1}(f[i];F)\bigr)\bigr).$$

11.3. **Selfhomeomorphism of a surface.** Now assume that Z is a closed orientable surface of genus g and $f\colon Z \to Z$ be an orientation preserving selfhomeomorphism.

If $g = 0$, we get $T_f = S^1 \times S^2$ and in this case everything can be computed directly.

If $g = 1$, then $\pi_1(T_f)$ is poly-$\mathbb{Z}$ and T_f is aspherical, and hence we know already that Conjecture 2.4 and Conjecture 2.5 are true, the answers to Questions 3.3 and Question 4.3 are positive, and Conjecture 10.1 is true, namely apply Remark 3.6, Lemma 4.7, Example 4.9 and Theorem 10.4.

So the interesting (and open) case is $g \geq 2$. In this situation equality (11.4) becomes

$$b_n(T_f[i];F) \quad = \quad \begin{cases} \dim_F\bigl(\operatorname{coker}\bigl(\operatorname{id}-H_1(f[i];F)\bigr)\bigr)+1 & n=1; \\ \dim_F\bigl(\ker\bigl(\operatorname{id}-H_1(f[i];F)\bigr)\bigr)+1 & n=2; \\ 1 & n=0,3; \\ 0 & n\geq 4. \end{cases}$$

We know for all $n \geq 0$

$$(11.5)\qquad \lim_{i\to\infty} \frac{b_n(T_f[i];F)}{[G:G_i]} \quad = \quad 0,$$

provided that F has characteristic zero. Notice that (11.5) for a field of characteristic zero is equivalent to

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{Q}}\bigl(\operatorname{coker}(\operatorname{id}-H_1(f[i];\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q}\bigr)}{[G:G_i]} \quad = \quad 0.$$

Next we consider the case that F is a field of prime characteristic p. Then we do know (11.5) in the situation of Subsection 11.1 but not in the situation of Subsection 11.2. Recall that Conjecture 2.4 predicts (11.5) in view of (11.1) also in this case. In order to prove (11.2) also for a field F of prime characteristic p for all $n \geq 0$ in the situation of Subsection 11.1, it suffices to show

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\operatorname{tors}(H_1(T_f[i];\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{F}_p\bigr)}{[G:G_i]} \quad = \quad 0;$$

or equivalently

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\operatorname{tors}\bigl(\operatorname{coker}(\operatorname{id}-H_1(f[i];\mathbb{Z}))\bigr)\otimes_{\mathbb{Z}}\mathbb{F}_p\bigr)}{[G:G_i]} \quad = \quad 0.$$

So one needs to understand more about the maps $\operatorname{id}-H_1(f[i];\mathbb{Z})\colon H_1(Z[i];\mathbb{Z}) \to H_1(Z[i];\mathbb{Z})$ for $i \geq 0$.

The status of Conjecture 10.1 (5) is even more mysterious. Suppose that $f\colon Z \to Z$ is an orientation preserving irreducible selfhomeomorphism of a closed orientable surface Z of genus $g \geq 2$. If f is periodic, T_f is finitely covered by $S^1 \times Z$ and Conjecture 10.1 is known to be true. Therefore we consider from now on the case, where f is not periodic. Then f is pseudo-Anosov, see [20, Theorem 6.3] and T_f carries the structure of a hyperbolic 3-manifold by [75, Theorem 3.6 on page 47, Theorem 3.9 on page 50]. Hence Conjecture 10.1 predicts, see Example 10.5,

$$\lim_{i\to\infty} \frac{\ln\bigl(\bigl|\operatorname{tors}(H_1(T_f[i];\mathbb{Z}))\bigr|\bigr)}{[G:G_i]} > 0,$$
$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\operatorname{tors}(H_1(T_f[i];\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{F}_p\bigr)}{[G:G_i]} = 0,$$
$$\lim_{i\to\infty} \frac{d\bigl(\operatorname{tors}(H_1(T_f[i];\mathbb{Z}))\bigr)}{[G:G_i]} = 0,$$

or, because of the Wang sequence (11.3) equivalently,

$$\lim_{i\to\infty} \frac{\ln\bigl(\bigl|\operatorname{tors}\bigl(\operatorname{coker}(\operatorname{id}-H_1(f[i];\mathbb{Z}))\bigr)\bigr|\bigr)}{[G:G_i]} > 0,$$
$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p}\bigl(\operatorname{tors}\bigl(\operatorname{coker}(\operatorname{id}-H_1(f[i];\mathbb{Z}))\bigr)\otimes_{\mathbb{Z}}\mathbb{F}_p\bigr)}{[G:G_i]} = 0,$$
$$\lim_{i\to\infty} \frac{d\bigl(\operatorname{tors}\bigl(\operatorname{coker}(\operatorname{id}-H_1(f[i];\mathbb{Z}))\bigr)\bigr)}{[G:G_i]} = 0.$$

12. Dropping the finite index condition

From now on we want to drop the condition that the index of the subgroups G_i in G is finite and that the index set for the chain is given by the natural numbers. So we will consider for the remainder of this paper the following more general situation:

Setup 12.1 (Inverse system)**.** Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i\in I} G_i = \{1\}$.

If I is given by the natural numbers, this boils down to a nested sequence of normal subgroups

$$G = G_0 \supset G_1 \supseteq G_2 \supseteq \cdots$$

satisfying $\bigcap_{n\geq 1} G_n = \{1\}$. If we additionally assume that $[G:G_i]$ is finite, we are back in the previous special situation (0.2). Some of the following conjectures reduce to previous conjectures in this special case. The reason is that for a finite group H and a based free finite $\mathbb{Z}H$-chain complex D_* we have

$$b_p^{(2)}(D_*^{(2)};\mathcal{N}(H)) = \frac{b_p^{(2)}(D_*^{(2)};\mathcal{N}(\{1\}))}{|H|};$$
$$\rho^{(2)}(D_*^{(2)};\mathcal{N}(H)) = \frac{\rho^{(2)}(D_*^{(2)};\mathcal{N}(\{1\}))}{|H|}.$$

13. Review of the Determinant Conjecture and the Approximation Conjecture for L^2-Betti numbers

We begin with the Determinant Conjecture (see [66, Conjecture 13.2 on page 454]).

Conjecture 13.1 (Determinant Conjecture for a group G)**.** *For any matrix $A \in M_{r,s}(\mathbb{Z}G)$, the Fuglede-Kadison determinant of the morphism of Hilbert modules $r_A^{(2)}\colon L^2(G)^r \to L^2(G)^s$ given by right multiplication with A satisfies*

$$\det_{\mathcal{N}(G)}^{(2)}\bigl(r_A^{(2)}\bigr) \geq 1.$$

Remark 13.2 (Status of the Determinant Conjecture)**.** We will often have to assume that the Determinant Conjecture 13.1 is true. This is an acceptable condition since it is known for a large class of groups. Namely, the following is known (see [31, Theorem 5], [66, Section 13.2 on pages 459 ff], [89, Theorem 1.21]). Let $\mathcal{F}$ be the class of groups for which the Determinant Conjecture 13.1 is true. Then:

(1) Amenable quotient
Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{F}$ and the quotient G/H is amenable. Then $G \in \mathcal{F}$;
(2) Colimits
If $G = \operatorname{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I (with not necessarily injective structure maps) and each G_i belongs to $\mathcal{F}$, then G belongs to $\mathcal{F}$;
(3) Inverse limits
If $G = \lim_{i \in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{F}$, then G belongs to $\mathcal{F}$;
(4) Subgroups
If H is isomorphic to a subgroup of a group G with $G \in \mathcal{F}$, then $H \in \mathcal{F}$;
(5) Quotients with finite kernel
Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. If K is finite and G belongs to $\mathcal{F}$, then Q belongs to $\mathcal{F}$;
(6) Sofic groups belong to $\mathcal{F}$.

The class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and is closed under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups. One expects that there exists non-sofic groups but no example is known. More information about sofic groups can be found for instance in [32] and [82].

Notation 13.3 (Inverse systems and matrices)**.** Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$. Given a matrix $A \in M_{r,s}(RG)$, let $A[i] \in M_{r,s}(R[G/G_i])$ be the matrix obtained from A by applying elementwise the ring homomorphism $RG \to R[G/G_i]$ induced by the projection $G \to G/G_i$. Let $r_A \colon RG^r \to RG^s$ and $r_{A[i]} \colon R[G/G_i]^r \to R[G/G_i]^s$ be the RG- and $R[G/G_i]$-homomorphism given by right multiplication with A and $A[i]$. Let $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$ and $r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s$ be the morphisms of Hilbert $\mathcal{N}(G)$- and Hilbert $\mathcal{N}(G/G_i)$-modules given by right multiplication with A and $A[i]$.

Next we deal with the Approximation Conjecture for L^2-Betti numbers (see [89, Conjecture 1.10], [66, Conjecture 13.1 on page 453]).

Conjecture 13.4 (Approximation Conjecture for L^2-Betti numbers)**.** *A group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1 satisfies the* Approximation Conjecture for L^2-Betti numbers *if one of the following equivalent conditions hold:*

(1) Matrix version
Let $A \in M_{r,s}(\mathbb{Q}G)$ be a matrix. Then

$$\begin{aligned} &\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr)\bigr) \\ &\quad = \lim_{i \in I} \dim_{\mathcal{N}(G/G_i)} \bigl(\ker \bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr)\bigr); \end{aligned}$$

(2) CW-complex version

Let X be a G-CW-complex of finite type. Then $X[i] := G_i\backslash X$ is a G/G_i-CW-complex of finite type and

$$b_p^{(2)}(X;\mathcal{N}(G)) \quad = \quad \lim_{i\in I} b_p^{(2)}(X[i];\mathcal{N}(G/G_i)).$$

The two conditions appearing in Conjecture 13.4 are equivalent by [66, Lemma 13.4 on page 455].

We will frequently make the following assumption:

Assumption 13.5 (Determinant Conjecture)**.** For each $i \in I$ the quotient G/G_i satisfies the Determinant Conjecture 13.1.

Theorem 13.6 (The Determinant Conjecture implies the Approximation Conjecture for L^2-Betti numbers)**.** *If Assumption 13.5 holds, then the conclusion of the Approximation Conjecture 13.4 holds for $\{G_i \mid i \in I\}$.*

Proof. See [66, Theorem 13.3 (1) on page 454] and [89]. □

Suppose that each quotient G/G_i is finite. Then Assumption 13.5 is fulfilled by Remark 13.2, and we recover Theorem 2.1 from Theorem 13.6.

For more information about the Approximation Conjecture and the Determinant Conjecture we refer to [66, Chapter 13 on pages 453 ff] and [89].

14. The Approximation Conjecture for Fuglede-Kadison determinants and L^2-torsion

Next we turn to Fuglede-Kadison determinants and L^2-torsion.

14.1. The chain complex version.

Conjecture 14.1 (Approximation Conjecture for Fuglede-Kadison determinants)**.** *A group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1 satisfies the* Approximation Conjecture for Fuglede-Kadison determinants *if for any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant*

$$\begin{aligned}&\det_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr)\\ &\qquad = \lim_{i\in I} \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr),\end{aligned}$$

where the existence of the limit above is part of the claim.

Notation 14.2 (Inverse systems and chain complexes)**.** Let C_* be a finite based free $\mathbb{Q}G$-chain complex. In the sequel we denote by $C[i]_*$ the $\mathbb{Q}[G/G_i]$-chain complex $\mathbb{Q}[G/G_i] \otimes_{\mathbb{Q}G} C_*$, by $C_*^{(2)}$ the finite Hilbert $\mathcal{N}(G)$-chain complex $L^2(G) \otimes_{\mathbb{Q}G} C_*$, and by $C[i]_*^{(2)}$ the finite Hilbert $\mathcal{N}(G/G_i)$-chain complex $L^2(G/G_i) \otimes_{\mathbb{Q}[G/G_i]} C[i]_*$. The $\mathbb{Q}G$-basis for C_* induces a $\mathbb{Q}[G/G_i]$-basis for $C[i]_*$ and Hilbert space structures on $C_*^{(2)}$ and $C[i]_*^{(2)}$ using the standard Hilbert structure on $L^2(G)$ and $L^2(G/G_i)$. We emphasize that in the sequel after fixing a $\mathbb{Q}G$-basis for C_* the $\mathbb{Q}[G/G_i]$-basis for $C_*[i]$ and the Hilbert structures on $C_*^{(2)}$ and $C[i]_*^{(2)}$ has to be chosen in this particular way.

Denote by

$$\rho^{(2)}\bigl(C_*^{(2)}\bigr) \quad := \quad -\sum_{p\geq 0} (-1)^p \cdot \ln\bigl(\det{}_{\mathcal{N}(G)}^{(2)}\bigl(c_p^{(2)}\bigr)\bigr); \tag{14.3}$$

$$\rho^{(2)}\bigl(C[i]_*^{(2)}\bigr) \quad := \quad -\sum_{p\geq 0} (-1)^p \cdot \ln\bigl(\det{}_{\mathcal{N}(G/G_i)}^{(2)}\bigl(c[i]_p^{(2)}\bigr)\bigr), \tag{14.4}$$

their L^2-*torsion* over $\mathcal{N}(G)$ and $\mathcal{N}(G/G_i)$ respectively.

We have the following chain complex version which is obviously equivalent.

Conjecture 14.5 (Approximation Conjecture for L^2-torsion of chain complexes)**.** *A group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1 satisfies the* Approximation Conjecture for L^2-torsion of chain complexes *if for any finite based free $\mathbb{Q}G$-chain complex C_* we have*

$$\rho^{(2)}\big(C_*^{(2)}\big) = \lim_{i \in I} \rho^{(2)}\big(C[i]_*^{(2)}\big).$$

14.2. **L^2-torsion.** Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Denote by $M[i]$ the Riemannian manifold obtained from $\overline{M}$ by dividing out the G_i-action. The Riemannian metric on $M[i]$ is induced by the one on M. There is an obvious proper free cocompact isometric G/G_i-action on $M[i]$ induced by the given G-action on $\overline{M}$. Notice that $M = \overline{M}/G$ is a closed Riemannian manifold and we get a G-covering $\overline{M} \to M$ and a G/G_i-covering $M[i] \to M$ which are compatible with the Riemannian metrics. Denote by

$$\rho_{\mathrm{an}}^{(2)}\big(\overline{M};\mathcal{N}(G)\big) \in \mathbb{R}; \tag{14.6}$$

$$\rho_{\mathrm{an}}^{(2)}\big(M[i];\mathcal{N}(G/G_i)\big) \in \mathbb{R}, \tag{14.7}$$

their *analytic L^2-torsion* over $\mathcal{N}(G)$ and $\mathcal{N}(G/G_i)$ respectively.

Conjecture 14.8 (Approximation Conjecture for analytic L^2-torsion)**.** *Consider a group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1. Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Then*

$$\rho_{\mathrm{an}}^{(2)}\big(\overline{M};\mathcal{N}(G)\big) = \lim_{i \in I} \rho_{\mathrm{an}}^{(2)}\big(M[i];\mathcal{N}(G/G_i)\big).$$

Remark 14.9. The conjectures above imply a positive answer to [24, Question 21] and [66, Question 13.52 on page 478 and Question 13.73 on page 483]. They also would settle [50, Problem 4.4 and Problem 6.4] and [51, Conjecture 3.5]. One may wonder whether it is related to the Volume Conjecture due to Kashaev [49] and H. and J. Murakami [80, Conjecture 5.1 on page 102].

We will prove in Section 15 the following result which in the weakly acyclic case reduces Conjecture 14.8 to Conjecture 14.1.

Theorem 14.10. *Consider a group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1. Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action. Suppose that $b_p^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ for all $p \geq 0$. Assume that the Approximation Conjecture for L^2-torsion of chain complexes 14.5 (or, equivalently, Conjecture 14.1) holds for G.*

Then Conjecture 14.8 holds for M, i.e.,

$$\rho_{\mathrm{an}}^{(2)}\big(\overline{M};\mathcal{N}(G)\big) = \lim_{i \in I} \rho_{\mathrm{an}}^{(2)}\big(M[i];\mathcal{N}(G/G_i)\big).$$

It is conceivable that Theorem 14.10 remains to true if we drop the assumption that $b_p^{(2)}(M;\mathcal{N}(G))$ vanishes for all $p \geq 0$, but our present proof works only under this assumption (see Remark 15.7).

14.3. **An inequality.** We always have the following inequality.

Theorem 14.11 (Inequality)**.** *Consider a group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1. Suppose that Assumption 13.5 holds. Consider a matrix $A \in M_{r,s}(\mathbb{Q}G)$ with coefficients in $\mathbb{Q}G$.*

Then we get the inequality

$$\det^{(2)}_{\mathcal{N}(G)}\big(r^{(2)}_A\colon L^2(G)^r \to L^2(G)^s\big) \geq \limsup_{i\in I} \det^{(2)}_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{A[i]}\colon L^2(G/G_i)^r \to L^2(G/G_i)^s\big).$$

The proof of Theorem 14.11 will be given in Section 17.

14.4. Matrices invertible in $L^1(G)$.

Theorem 14.12 (Invertible matrices over $L^1(G)$)**.** *Consider a group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1. Consider an invertible matrix $A \in \mathrm{GL}_d(L^1(G))$ with coefficients in $L^1(G)$. The projection $G \to G/G_i$ induces a ring homomorphism $L^1(G) \to L^1(G/G_i)$. Thus we obtain for each $i \in I$ an invertible matrix $A[i] \in \mathrm{GL}_d(L^1(G/G_i))$.*

Then the Approximation Conjecture for Fuglede-Kadison determinants 14.1 holds for A, i.e.,

$$\det^{(2)}_{\mathcal{N}(G)}\big(r^{(2)}_A\colon L^2(G)^d \to L^2(G)^d\big) = \lim_{i\in I} \det^{(2)}_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{A[i]}\colon L^2(G/G_i)^d \to L^2(G/G_i)^d\big).$$

Theorem 14.12 has already been proved by Deninger [24, Theorem 17] in the case $d = 1$. Notice that Deninger [24, page 46] uses a different definition of Fuglede-Kadison determinant which agrees with ours for injective operators by [66, Lemma 3.15 (5) on page 129].

Corollary 14.13. *Consider a group G together with an inverse system $\{G_i \mid i \in I\}$ as in Setup 12.1.*

1. *Let C_* be a finite based free $L^1(G)$-chain complex which is acyclic. Then*
$$\rho^{(2)}\big(C_*^{(2)}\big) = \lim_{i\in I} \rho^{(2)}\big(C_*[i]^{(2)}\big);$$
2. *Let C_* and D_* be finite based free $L^1(G)$-chain complexes. Suppose that they are $L^1(G)$-chain homotopy equivalent. Then*
$$\rho^{(2)}\big(C_*^{(2)}\big) - \rho^{(2)}\big(D_*^{(2)}\big) = \lim_{i\in I} \rho^{(2)}\big(C_*[i]^{(2)}\big) - \rho^{(2)}\big(D_*[i]^{(2)}\big).$$

The proofs of Theorem 14.12 and Corollary 14.13 will be given in Section 17.

15. The L^2-de Rham isomorphism and the proof of Theorem 14.10

In this section we investigate the L^2-de Rham isomorphism in order to give the proof of Theorem 14.10.

Let M be a closed Riemannian manifold. Fix a smooth triangulation K of M. Consider a (discrete) group G and a G-covering $\mathrm{pr}\colon \overline{M} \to M$. The smooth triangulation K of M lifts to G-equivariant smooth triangulation $\overline{K}$ of $\overline{M}$. Denote by $\mathrm{pr}\colon \overline{K} \to K$ the associated G-covering. Equip $\overline{M}$ with the Riemannian metric for which $\mathrm{pr}\colon \overline{M} \to M$ becomes a local isometry.

In the sequel we will consider the de Rham isomorphism

$$\mathrm{Int}^p\colon \mathcal{H}^p_{(2)}(\overline{M}) \xrightarrow{\cong} H^p_{(2)}(\overline{K}). \tag{15.1}$$

from the space of harmonic L^2-integrable p-forms on $\overline{M}$ to the L^2-cohomology of the free simplicial G-complex $\overline{K}$. It is essentially given by integrating a p-form over a p-simplex and is an isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. For more details we refer to [26] or [66, Theorem 1.59 on page 52].

There is the de Rham cochain map (for large enough fixed k) (see [26] or [66, (1.77) on page 61]

$$A^*\colon H^{k-*}\Omega^p(\overline{M}) \to C^*_{(2)}(\overline{K}) \tag{15.2}$$

where $H^{k-*}\Omega^p(\overline{M})$ denotes the Sobolev space of p-forms on $\overline{M}$.

Lemma 15.3. *Assume that for every simplex σ of K we can find a neighborhood V_σ together with a diffeomorphism $\eta_\sigma \colon \mathbb{R}^m \to V_\sigma$. (This can be arranged by possibly passing to a d-fold barycentric subdivision of K.) Fix an integer p with $0 \le p \le \dim(M)$.*

Then there exist constants $L_1, L_2 > 0$ which depend on data coming from M and K, but do not depend on G and $\operatorname{pr}\colon \overline{M} \to M$ such that for every $\overline{\omega} \in H^{k-p}\Omega^p(\overline{M})$ we have

$$L_1 \cdot ||\overline{\omega}||_{H^{k-p}} \le ||A^p(\overline{\omega})||_{L^2} \le L_2 \cdot ||\overline{\omega}||_{H^{k-p}},$$

and we get for the operator norm of the operator Int^p *of* (15.1)

$$L_1 \le ||\operatorname{Int}^p|| \le L_2.$$

Proof. Dodziuk [26, Lemma 3.2] proves for a given G-covering $\operatorname{pr}\colon \overline{M} \to M$ and $p \ge 0$ that the map $A^p\colon H^{k-p}\Omega^p(\overline{M}) \xrightarrow{\cong} C^p_{(2)}(\overline{K})$ is bounded, i.e., there exists a constant L_2 such that for $\overline{\omega} \in H^{k-p}\Omega^p(\overline{M})$ we have

$$||A^p(\overline{\omega})||_{L^2} \le L_2 \cdot ||\overline{\omega}||_{H^{k-p}}.$$

Next we will analyze Dodziuk's proof and explain why the constant L_2 depends only on data coming from M and K and does not depend on G and $\operatorname{pr}\colon \overline{M} \to M$.

For every p-simplex σ in K we choose a relatively compact neighborhood U_σ of σ with $U_\sigma \subseteq V_\sigma$. Choose N to be an integer such that every point $x \in M$ belongs to at most N of the sets U_σ, where σ runs through the p-simplices of K, e.g., take N to be the number of p-simplices of K. We can apply [26, Lemma 3.1] to $\sigma \subseteq V_\sigma$ and obtain a constant $C_\sigma > 0$ such that for any p-form ω in $H^{k-p}\Omega^p(M)$ and any p-simplex σ of K

$$\sup_{x\in\sigma} |\omega(x)| \quad \le \quad C_\sigma \cdot \left(||\omega||^{U_\sigma}_{H^{k-p}} + ||\omega||^{U_\sigma}_{L^2}\right)$$

holds. The real number $|\omega(x)|$ is the norm of $\omega(x)$ as an element in $\Lambda^p T^*_x M$, and $||\omega||^{U_\sigma}_{H^{k-p}}$ and $||\omega||^{U_\sigma}_{L^2}$ are the Sobolev norm and L^2-norm of ω restricted to U_σ.

Let C be the maximum of the numbers C_σ, where σ runs through all p-simplices of K. Let E be the maximum over the volumes of the p-simplices of K. Obviously the numbers C, E and N depend only on data coming from M and K, but do not depend on G and $\operatorname{pr}\colon \overline{M} \to M$.

Since V_σ is contractible, the restriction of $\operatorname{pr}\colon \overline{M} \to M$ to $\operatorname{pr}^{-1}(V_\sigma)$ is trivial and hence $\operatorname{pr}^{-1}(V_\sigma)$ is G-diffeomorphic to $G \times V_\sigma$. Hence there are for every p-simplex $\overline{\sigma} \in \overline{M}$ open neighborhoods $U_{\overline{\sigma}}$ and $V_{\overline{\sigma}}$ which are uniquely determined by the property that they are mapped diffeomorphically under $\operatorname{pr}\colon \overline{M} \to M$ onto $U_{\operatorname{pr}(\overline{\sigma})}$ and $V_{\operatorname{pr}(\overline{\sigma})}$. Notice for the sequel that $\operatorname{pr}|_{V_{\overline{\sigma}}}\colon V_{\overline{\sigma}} \to V_{\operatorname{pr}(\overline{\sigma})}$ and $\operatorname{pr}|_{U_{\overline{\sigma}}}\colon U_{\overline{\sigma}} \to U_{\operatorname{pr}(\overline{\sigma})}$ are isometric diffeomorphisms. Hence we get for every p-form $\overline{\omega}$ in $H^{k-p}\Omega^p(\overline{M})$ and any p-simplex $\overline{\sigma}$ of $\overline{K}$

$$\sup_{\overline{x}\in\overline{\sigma}} |\omega(\overline{x})| \quad \le \quad C \cdot \left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{H^{k-p}} + ||\overline{\omega}||^{U_{\overline{\sigma}}}_{L^2}\right).$$

Here the real number $|\overline{\omega}(\overline{x})|$ is the norm of $\overline{\omega}(\overline{x})$ as an element in $\Lambda^p T^*_{\overline{x}}\overline{M}$, and $||\overline{\omega}||^{V_{\overline{\sigma}}}_{H^{k-p}}$ and $||\omega||^{V_{\overline{\sigma}}}_{L^2}$ are the Sobolev norm and L^2-norm of the restriction of $\overline{\omega}$ to $V_{\overline{\sigma}}$. One easily checks that every point $\overline{x} \in \overline{M}$ belongs to at most N of the sets $U_{\overline{\sigma}}$, where $\overline{\sigma}$ runs through the p-simplices of $\overline{K}$ and that the volume of every p-simplex of $\overline{K}$ is bounded by E. Put

$$L_2 := \sqrt{4 \cdot C^2 \cdot E^2 \cdot N}.$$

Then L_2 depends only on data coming from M and K, but does not depend on G and $\operatorname{pr}\colon \overline{M} \to M$.

Next we perform essentially the same calculation as in [26, Lemma 3.2] We estimate for a p-simplex $\overline{\sigma}$ of $\overline{K}$ and an element $\overline{\omega} \in H^{k-p}\Omega^p(\overline{M})$

$$\begin{aligned}
\left(\int_{\overline{\sigma}} \overline{\omega}\right)^2 & \leq \left(\sup_{\overline{x}\in\overline{\sigma}}\{|\overline{\omega}(\overline{x})|\} \cdot \operatorname{vol}(\overline{\sigma})\right)^2 \\
& \leq \left(C \cdot \left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{H^{k-p}} + ||\overline{\omega}||^{U_{\overline{\sigma}}}_{L^2}\right) \cdot \operatorname{vol}(\overline{\sigma})\right)^2 \\
& \leq C^2 \cdot 2 \cdot \left(\left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{H^{k-p}}\right)^2 + \left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{L^2}\right)^2\right) \cdot E^2.
\end{aligned}$$

This implies for $\overline{\omega} \in H^{k-p}\Omega^p(\overline{M})$, where $\overline{\sigma}$ runs through the p-simplices of $\overline{K}$.

$$\begin{aligned}
\sum_{\overline{\sigma}} \left(\int_{\overline{\sigma}} \overline{\omega}\right)^2 & \leq \sum_{\overline{\sigma}} C^2 \cdot 2 \cdot \left(\left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{H^{k-p}}\right)^2 + \left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{L^2}\right)^2\right) \cdot E^2 \\
& \leq 2 \cdot C^2 \cdot E^2 \cdot \left(\sum_{\overline{\sigma}} \left(\left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{H^{k-p}}\right)^2 + \left(||\overline{\omega}||^{U_{\overline{\sigma}}}_{L^2}\right)^2\right)\right) \\
& \leq 2 \cdot C^2 \cdot E^2 \cdot N \cdot \left(\left(||\overline{\omega}||_{H^{k-p}}\right)^2 + \left(||\overline{\omega}||_{L^2}\right)^2\right) \\
& \leq 2 \cdot C^2 \cdot E^2 \cdot N \cdot 2 \cdot ||\overline{\omega}||^2_{H^{k-p}} \\
& = L_2^2 \cdot ||\overline{\omega}||^2_{H^{k-p}}.
\end{aligned}$$

We conclude that the de Rham map

$$A^p \colon H^{k-p}\Omega^p(\overline{M}) \to C^p_{(2)}(\overline{K})$$

is a bounded operator whose norm is less or equal to L_2.

Dodziuk [26, Lemma 3.7] (see also [66, (1.78) on page 61]) constructs a bounded G-equivariant operator

$$W^p \colon C^p_{(2)}(\overline{K}) \quad \to \quad H^{k-p}\Omega^p(\overline{M}), \tag{15.4}$$

and gives an upper bound for its operator norm by a number

$$\left|\{p\text{-simplices of } K\}\right| \cdot \max\left\{||W^p\sigma||_{H^{k-p}} \mid \sigma\, p\text{-simplex of } K\right\}.$$

Define

$$L_1 := \frac{1}{\left|\{p\text{-simplices of } K\}\right| \cdot \max\left\{||W\sigma||_{H^{k-p}} \mid \sigma\, p\text{-simplex of } K\right\}}.$$

Obviously L_1 depends only on data coming from M and K, but not on G and $\operatorname{pr}\colon \overline{M} \to M$. The maps A^p of (15.2) and W^p of (15.4) induce bounded G-operators (see [26, Corollary on page 162 and Corollary on page 163])

$$\begin{aligned}
H^p_{(2)}(A^*) \colon H^p_{(2)}\left(H^{k-*}\Omega^*(\overline{M})\right) & \to H^p_{(2)}\left(C^*_{(2)}(\overline{K})\right); \\
H^p_{(2)}(W^*) \colon H^p_{(2)}\left(C^*_{(2)}(\overline{K})\right) & \to H^p_{(2)}\left(H^{k-*}\Omega^*(\overline{M})\right),
\end{aligned}$$

such that we obtain for their operator norms

$$\begin{aligned}
\left|\left|H^p_{(2)}(A^*)\right|\right| & \leq L_2; \\
\left|\left|H^p_{(2)}(W^*)\right|\right| & \leq \frac{1}{L_1}.
\end{aligned}$$

Since $W^p \circ A^p = \operatorname{id}$ and $H^p_{(2)}(A^p)$ is an isomorphism (see [26, (3.6), Lemma 3.8 and Lemma 3.10] and [66, (1.79) and (1.80) on page 61], the map $H^p_{(2)}(W^*)$ is the inverse of $H^p_{(2)}(A^*)$. This implies

$$L_1 \leq \left|\left|H^p_{(2)}(A^*)\right|\right| \leq L_2.$$

Since the canonical inclusion

$$i^p\colon \mathcal{H}^p_{(2)} \xrightarrow{\cong} H^p_{(2)}\bigl(H^{k-*}\Omega^*(\overline{M})\bigr)$$

is an isometric G-isomorphism (see [66, Lemma 1.75 on page 59]) and the map $\operatorname{Int}^p\colon \mathcal{H}^p_{(2)} \to H^p_{(2)}(\overline{K})$ is the composite $H^p_{(2)}(A^*) \circ i^p$, Lemma 15.3 follows. □

Now we are ready to give

Proof of Theorem 14.10. Let $\overline{M}$ be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G-action such that $b_p^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for all $p \geq 0$. Fix a smooth triangulation K of $M = G\backslash\overline{M}$. By possibly subdividing it we can arrange that Lemma 15.3 will apply to M and K. The triangulation K lifts to a G-equivariant smooth triangulation $\overline{K}$ of $\overline{M}$ and to a G/G_i-equivariant smooth triangulation $\overline{K}[i]$ of $M[i] := G_i\backslash\overline{M}$.

We assume that the Approximation Conjecture for L^2-torsion of chain complexes 14.5 is true. Hence we get

$$\rho^{(2)}\bigl(\overline{K};\mathcal{N}(G)\bigr) \quad = \quad \lim_{i\in I}\, \rho^{(2)}\bigl(\overline{K}[i];\mathcal{N}(G/G_i)\bigr). \tag{15.5}$$

In the sequel we will use the theorem of Burghelea, Friedlander, Kappeler and McDonald [16] that the topological and the analytic L^2-torsion agree. Since M and hence $\overline{K}$ is L^2-acyclic, we get from the definitions

$$\begin{aligned}
\rho^{(2)}_{\mathrm{an}}\bigl(\overline{M};\mathcal{N}(G)\bigr) &= \rho^{(2)}\bigl(\overline{K};\mathcal{N}(G)\bigr);\\
\rho^{(2)}_{\mathrm{an}}\bigl(M[i];\mathcal{N}(G/G_i)\bigr) &= \rho^{(2)}\bigl(\overline{K}[i];\mathcal{N}(G/G_i)\bigr) - \sum_{p\geq 0}(-1)^p\cdot \det\nolimits_{\mathcal{N}(G/G_i)}(\operatorname{Int}[i]^p),
\end{aligned}$$

where $\operatorname{Int}[i]^p\colon \mathcal{H}^p_{(2)}(M[i]) \xrightarrow{\cong} H^p_{(2)}(\overline{K}[i])$ is the L^2-de Rham isomorphism of (15.1). Hence it suffices to show for $p \geq 0$

$$\lim_{i\in I}\, \ln\bigl(\det\nolimits^{(2)}_{\mathcal{N}(G/G_i)}(\operatorname{Int}[i]^p)\bigr) \quad = \quad 0. \tag{15.6}$$

We obtain from Lemma 15.3 constants $L_1 > 0$ and $L_2 > 0$ which are independent of $i \in I$ such that for every $i \in I$

$$L_1 \leq ||\operatorname{Int}[i]^p|| \leq L_2$$

holds for the operator norm of $\operatorname{Int}[i]^p$. Since

$$b_p^{(2)}\bigl(M[i];\mathcal{N}(G/G_i)\bigr) = \dim_{\mathcal{N}(G/G_i)}\bigl(\mathcal{H}^p_{(2)}(M[i])\bigr) = \dim_{\mathcal{N}(G/G_i)}\bigl(H^p_{(2)}(K[i])\bigr),$$

we conclude

$$\begin{aligned}
\ln(L_1)\cdot b_p^{(2)}\bigl(M[i];\mathcal{N}(G/G_i)\bigr) &\leq \ln\bigl(\det\nolimits^{(2)}_{\mathcal{N}(G/G_i)}(\operatorname{Int}[i]^p)\bigr)\\
&\leq \ln(L_2)\cdot b_p^{(2)}\bigl(M[i];\mathcal{N}(G/G_i)\bigr).
\end{aligned}$$

Since the Approximation Conjecture 13.4 holds for $\overline{M}$ by Theorem 13.6 and we have $b_p^{(2)}\bigl(\overline{M};\mathcal{N}(G)\bigr) = 0$ for $p \geq 0$ by assumption, we have

$$\lim_{i\in I}\, b_p^{(2)}\bigl(M[i];\mathcal{N}(G/G_i)\bigr) \quad = \quad 0.$$

Now (15.6) follows. This finishes the proof of Theorem 14.10. □

Remark 15.7 (On the L^2-acyclicity assumption)**.** Recall that in Theorem 14.10 we require that $b_p^{(2)}\bigl(\overline{M};\mathcal{N}(G)\bigr) = 0$ holds for $p \geq 0$. This assumption is satisfied in many interesting cases. It is possible that this assumption is not needed for Theorem 14.10 to be true, but our proof does not work without it. We can drop this assumption, if we can generalize (15.6) to

$$\lim_{i\in I}\, \ln\bigl(\det\nolimits_{\mathcal{N}(G)}(\operatorname{Int}[i]^p)\bigr) \quad = \quad \ln\bigl(\det\nolimits_{\mathcal{N}(G)}(\operatorname{Int}^p)\bigr).$$

16. A strategy to prove the Approximation Conjecture for Fuglede-Kadison determinants 14.1

16.1. **The general setup.** Throughout this section we will consider the following data:

- G is a group, B is a matrix in $M_d(\mathcal{N}(G))$ and $\operatorname{tr}\colon M_d(\mathcal{N}(G)) \to \mathbb{C}$ is a faithful finite normal trace.
- I is a directed set. For each $i \in I$ we have a group Q_i, a matrix $B[i] \in M_d(\mathcal{N}(Q_i))$ and a faithful finite normal trace $\operatorname{tr}_i\colon M_d(\mathcal{N}(Q_i)) \to \mathbb{C}$ such that $\operatorname{tr}_i(I_d) = d$ holds for the unit matrix $I_d \in M_d(\mathcal{N}(Q_i))$.

Faithful finite normal trace tr means that tr is $\mathbb{C}$-linear, satisfies $\operatorname{tr}(B_1B_2) = \operatorname{tr}_i(B_2B_1)$, sends B^*B to a real number $\operatorname{tr}(B^*B) \geq 0$ such that $\operatorname{tr}(B^*B) = 0 \Leftrightarrow B = 0$, and for $f \in \mathcal{N}(G)$, which is the supremum with respect to the usual ordering $\leq$ of positive elements of some monotone increasing net $\{f_j \mid j \in J\}$ of positive elements in $\mathcal{N}(G)$, we get $\operatorname{tr}(f) = \sup\{\operatorname{tr}(f_j) \mid j \in J\}$. The trace tr or tr_i respectively may or may not be the von Neumann trace (see [66, Definition 1.2 on page 15]) $\operatorname{tr}_{\mathcal{N}(G)}$ or $\operatorname{tr}_{\mathcal{N}(Q_i)}$ respectively.

Let $F\colon [0,\infty) \to [0,\infty)$ be the spectral density function of $r_B^{(2)}\colon L^2(G)^d \to L^2(G)^d$ with respect to tr as defined in [66, Definition 2.1 on page 73]. Let $F[i]\colon [0,\infty) \to [0,\infty)$ be the spectral density function of $r_{B[i]}^{(2)}\colon L^2(Q_i)^d \to L^2(Q_i)^d$ with respect to the trace tr_i. If $r_B^{(2)}$ is positive, we get $F(\lambda) = \operatorname{tr}(E_\lambda)$ for $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ the family of spectral projections of $r_B^{(2)}$ (see [66, Lemma 2.3 on page 74 and Lemma 2.11 (11) on page 77]). The analogous statement holds for $F[i]$.

Recall that for a directed set I and a net $(x_i)_{i\in I}$ of real numbers one defines

$$\liminf_{i\in I} x_i \quad := \quad \sup\{\inf\{x_j \mid j \in I, j \geq i\} \mid i \in I\}; \tag{16.1}$$

$$\limsup_{i\in I} x_i \quad := \quad \inf\{\sup\{x_j \mid j \in I, j \geq i\} \mid i \in I\}. \tag{16.2}$$

16.2. **The general strategy.** To any monotone non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ we can assign a density function, i.e., a monotone non-decreasing right-continuous function,

$$f^+\colon [0,\infty) \to [0,\infty), \qquad \lambda \mapsto \lim_{\epsilon\to 0+} f(\lambda+\epsilon).$$

Put

$$\begin{aligned} \underline{F}(\lambda) &:= \liminf_{i\in I} F[i](\lambda); \\ \overline{F}(\lambda) &:= \limsup_{i\in I} F[i](\lambda). \end{aligned}$$

Let det and $\det_i$ be the Fuglede-Kadison determinant with respect to tr and tr_i (compare [66, Definition 3.11 on page 127]). If tr is the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)}$, then det is the Fuglede-Kadison determinant $\det_{\mathcal{N}(G)}$ as defined in [66, Definition 3.11 on page 127]. We want to prove

Theorem 16.3. *Consider the following conditions, where $K > 0$ and $\kappa > 0$ are some fixed real numbers:*

(i) The operator norms satisfy $||r_B^{(2)}|| \leq K$ and $||r_{B[i]}^{(2)}|| \leq K$ for all $i \in I$;

(ii) For every polynomial p with real coefficients we have

$$\operatorname{tr}(p(B)) = \lim_{i\in I} \operatorname{tr}_i(p(B[i]));$$

(iii) We have $\det_i\bigl(r_{B[i]}^{(2)}\colon L^2(Q_i)^d \to L^2(Q_i)^d\bigr) \geq \kappa$ for each $i \in I$,

(iv) Suppose $r_B^{(2)} \colon L^2(G)^d \to L^2(G)^d$ *and* $r_{B[i]}^{(2)} \colon L^2(Q_i)^d \to L^2(Q_i)^d$ *for* $i \in I$ *are positive;*

(v) The uniform integrability condition *is satisfied, i.e., there exists* $\epsilon > 0$ *such that*

$$\int_{0+}^{\epsilon} \sup\left\{ \frac{F[i](\lambda) - F[i](0)}{\lambda} \;\middle|\; i \in I\right\} \, d\lambda \quad < \quad \infty.$$

Then:

(1) If conditions (i), (ii) (iii), *and* (iv) *are satisfied, then*

$$\det\bigl(r_B^{(2)} \colon L^2(G)^d \to L^2(G)^d\bigr) \ge \limsup_{i \in I} \det_i\bigl(r_{B[i]}^{(2)} \colon L^2(Q_i)^d \to L^2(Q_i)^d\bigr);$$

(2) If conditions (i), (ii) (iii), (iv) *and* (v) *are satisfied, then*

$$\det\bigl(r_B^{(2)} \colon L^2(G)^d \to L^2(G)^d\bigr) = \lim_{i \in I} \det_i\bigl(r_{B[i]}^{(2)} \colon L^2(Q_i)^d \to L^2(Q_i)^d\bigr).$$

Proof. Completely analogously to the proof of [66, Theorem 13.19 on page 461] we prove

$$F(\lambda) \quad = \quad \underline{F}^+(\lambda) \;=\; \overline{F}^+(\lambda) \quad \text{for } \lambda \in \mathbb{R}; \tag{16.4}$$

$$F(0) \quad = \quad \lim_{i \in I} F[i](0); \tag{16.5}$$

$$\kappa \quad \le \quad \ln\bigl(\det{}^{(2)}\bigl(r_B^{(2)} \colon L^2(G)^d \to L^2(G)^d\bigr)\bigr). \tag{16.6}$$

The proof of [66, (13.22) and (13.23) on page 462] carries directly over and yields

$$\ln(\det(r_B^{(2)})) \quad = \quad \ln(K) \cdot (F(K) - F(0)) - \int_{0+}^{K} \frac{F(\lambda) - F(0)}{\lambda} \, d\lambda; \tag{16.7}$$

$$\ln(\det_i(r_{B[i]}^{(2)})) \quad = \quad \ln(K) \cdot (F[i](K) - F[i](0)) - \int_{0+}^{K} \frac{F[i](\lambda) - F[i](0)}{\lambda} \, d\lambda. \tag{16.8}$$

We conclude from (16.6) and (16.7)

$$0 \quad \le \quad \int_{0+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda < +\infty. \tag{16.9}$$

Since $\underline{F}$ and $\overline{F}$ are monotone increasing bounded functions, there are only countably many elements $\lambda \in [0, \infty)$ such that $\underline{F}(\lambda) \ne \underline{F}^+(\lambda)$ or $\overline{F}(\lambda) \ne \overline{F}^+(\lambda)$ hold. We conclude from (16.4) that there is a countable set $S \subseteq [0, \infty)$ such that for all $\lambda \in [0, \infty) \setminus S$ the limit $\lim_{i \to \infty} F[i](\lambda)$ exists and is equal to $F(\lambda)$. Since S has measure zero and we have (16.5), we get almost everywhere for $\lambda \in [0, \infty)$

$$\lim_{i \in I} \frac{F[i](\lambda) - F[i](0)}{\lambda} \quad = \quad \frac{F(\lambda) - F(0)}{\lambda} \tag{16.10}$$

Analogously to the proof of [66, (13.28) on page 463] one shows

$$\int_{0+}^{K} \frac{\lim_{i \in I}\bigl(F[i](\lambda) - F[i](0)\bigr)}{\lambda} d\lambda \quad \le \quad \liminf_{i \in I} \int_{0+}^{K} \frac{F[i](\lambda) - F[i](0)}{\lambda} d\lambda.$$

This implies

$$\int_{0+}^{K} \frac{F(\lambda) - F(0)}{\lambda} d\lambda \quad \le \quad \liminf_{i \in I} \int_{0+}^{K} \frac{F[i](\lambda) - F[i](0)}{\lambda} d\lambda. \tag{16.11}$$

Now assertion (1) follows from (16.7), (16.8), and (16.11).

Next we prove assertion (2). We can apply Lebesgue's Dominated Convergence Theorem to (16.10) because of the assumption (v) and obtain

$$\int_{0+}^{K} \frac{\lim_{i \in I}\bigl(F[i](\lambda) - F[i](0)\bigr)}{\lambda} d\lambda \quad = \quad \lim_{i \in I} \int_{0+}^{K} \frac{F[i](\lambda) - F[i](0)}{\lambda} d\lambda.$$

Now assertion (2) follows from (16.7), and (16.8). This finishes the proof of Theorem 16.3. □

16.3. **The uniform integrability condition is not automatically satisfied.** The main difficulty to apply Theorem 16.3 to the situations of interest is the verification of the uniform integrability condition (v) appearing in Theorem 16.3. We will illustrate by an example that one needs extra input to ensure this condition since there are examples where this condition is violated but all properties of the spectral density functions which are known so far are satisfied.

Define the following sequence of functions $f_n\colon [0,1] \to [0,1]$

$$f_n(\lambda) = \begin{cases} \lambda & 0 \le \lambda \le e^{-3n}; \\ \frac{(e^{-2n}-\lambda)\cdot e^{-3n} + (\lambda - e^{-3n})\cdot\left(\frac{1}{-\ln(e^{-2n})}+e^{-2n}\right)}{e^{-2n}-e^{-3n}} & e^{-3n} \le \lambda \le e^{-2n}; \\ \frac{1}{-\ln(\lambda)} + \lambda & e^{-2n} \le \lambda \le e^{-n}; \\ \frac{1}{-\ln(e^{-n})} + e^{-n} & e^{-n} \le \lambda \le \frac{1}{-\ln(e^{-n})} + e^{-n}; \\ \lambda & \frac{1}{-\ln(e^{-n})} + e^{-n} \le \lambda \le 1. \end{cases}$$

Lemma 16.12.

(1) *The function $f_n(\lambda)$ is monotone non-decreasing and continuous for $n \ge 1$;*
(2) *$f_n(0) = 0$ and $f_n(1) = 1$ for all $n \ge 1$;*
(3) *$\lim_{n\to\infty} f_n(\lambda) = \lambda$ for $\lambda \in [0,1]$;*
(4) *We have for all $n \ge 1$ and $\lambda \in [0,1)$*

$$\lambda \le f_n(\lambda) \le \frac{1}{-\ln(\lambda)} + \lambda \le \frac{2}{-\ln(\lambda)};$$

(5) *We have for $\lambda \in [0, e^{-1}]$*

$$\sup\{f_n(\lambda) \mid n \ge 0\} = \frac{1}{-\ln(\lambda)} + \lambda;$$

(6) *We have*

$$\int_{0+}^{1} \frac{\sup\{f_n(\lambda) \mid n \ge 0\}}{\lambda}\, d\lambda = \infty;$$

(7) *We get for all $n \ge 1$*

$$\int_{0+}^{1} \frac{f_n(\lambda)}{\lambda}\, d\lambda \ge \ln(2) + 1;$$

(8) *We have*

$$\int_{0+}^{1} \lim_{n\to\infty} \frac{f_n(\lambda)}{\lambda} d\lambda < \liminf_{n\to\infty} \int_{0+}^{1} \frac{f_n(\lambda)}{\lambda} d\lambda \le \limsup_{n\to\infty} \int_{0+}^{1} \frac{f_n(\lambda)}{\lambda} d\lambda;$$

(9) *We get for all $n \ge 1$*

$$\int_{0+}^{1} \frac{f_n(\lambda)}{\lambda}\, d\lambda \le 4.$$

Proof. (1) One easily checks that the definition of f_n makes sense, in particular at the values $\lambda = e^{-3n}, e^{-2n}, e^{-n}$.

The first derivative of $f_n(\lambda)$ exists with the exception of $\lambda = e^{-3n}, e^{-2n}, e^{-n}, \frac{1}{-\ln(e^{-n})} + e^{-n}$ and is given by

$$f_n'(\lambda) = \begin{cases} 1 & 0 \leq \lambda < e^{-3n}; \\ 1 + \frac{1}{2n \cdot (e^{-2n} - e^{-3n})} & e^{-3n} < \lambda < e^{-2n}; \\ \frac{1}{\lambda \cdot \ln(\lambda)^2} + 1 & e^{-2n} < \lambda < e^{-n}; \\ 0 & e^{-n} \leq \lambda \leq \frac{1}{-\ln(e^{-n})} + e^{-n}; \\ 1 & \frac{1}{-\ln(e^{-n})} + e^{-n} < \lambda \leq 1. \end{cases}$$

Hence for $n \geq 1$ the function f_n is smooth with non-negative derivative on the open intervals $(0, e^{-3n})$, (e^{-3n}, e^{-2n}), (e^{-2n}, e^{-n}), and $(e^{-n}, 1)$, and is continuous on the closed intervals $[0, e^{-3n}]$, $[e^{-3n}, e^{-2n}]$, $[e^{-2n}, e^{-n}]$, and $[e^{-n}, 1]$. Hence each f_n is continuous and monotone non-decreasing.

(2) This is obvious.

(3) This follows since $\lim_{n\to\infty} \frac{1}{-\ln(e^{-n})} + e^{-n} = 0$, $f_n(\lambda) = \lambda$ for $\frac{1}{-\ln(e^{-n})} + e^{-n} \leq \lambda \leq 1$ and $f_n(0) = 0$ for all $n \geq 1$.

(4) We conclude $\lambda \leq f_n(\lambda) \leq \frac{1}{-\ln(\lambda)} + \lambda$ for $\lambda \in [0, 1)$ by inspecting the definitions since $\lambda \leq \frac{1}{-\ln(\lambda)} + \lambda$ holds and $\frac{1}{-\ln(\lambda)} + \lambda$ is monotone non-decreasing. We have $\lambda \leq \frac{1}{-\ln(\lambda)}$ for $\lambda \in (0, 1)$.

(5) From assertion (4) we conclude $\sup\{f_n(\lambda) \mid n \geq 0\} \leq \frac{1}{-\ln(\lambda)} + \lambda$ for $\lambda \in [0, 1)$. Since for λ with $0 < \lambda \leq e^{-1}$ we can find $n \geq 1$ with $e^{-2n} \leq \lambda \leq e^{-n}$ and hence $f_n(\lambda) = \frac{1}{-\ln(\lambda)} + \lambda$ holds for that n, we conclude $\sup\{f_n(\lambda) \mid n \geq 0\} = \frac{1}{-\ln(\lambda)} + \lambda$ for $\lambda \in [0, e^{-1}]$.

(6) We compute using assertion (5) for every $\epsilon \in (0, e^{-1})$

$$\begin{aligned} \int_{0+}^{1} \frac{\sup\{f_n(\lambda) \mid n \geq 0\}}{\lambda} \, d\lambda & \geq \int_{\epsilon}^{e^{-1}} \frac{\sup\{f_n(\lambda) \mid n \geq 0\}}{\lambda} \, d\lambda \\ & = \int_{\epsilon}^{e^{-1}} 1 + \frac{1}{\lambda \cdot (-\ln(\lambda))} \, d\lambda \\ & = e^{-1} - \epsilon + [-\ln(-\ln(\lambda))]_{\epsilon}^{e^{-1}} \\ & = e^{-1} - \epsilon - \ln(-\ln(e^{-1}) + \ln(-\ln(\epsilon)) \\ & = e^{-1} - \epsilon + \ln(-\ln(\epsilon)) \\ & \geq \ln(-\ln(\epsilon)). \end{aligned}$$

Since $\lim_{\epsilon \to 0+} \ln(-\ln(\epsilon)) = \infty$, assertion (6) follows.

(7) We estimate for given $n \geq 1$ using the conclusion $f_n(\lambda) - \lambda \geq 0$ for $\lambda \in [0, 1]$

from assertion (4)

$$\begin{aligned}
\int_{0+}^{1} \frac{f_n(\lambda)}{\lambda}\, d\lambda &= \int_{0+}^{1} \frac{f_n(\lambda) - \lambda}{\lambda}\, d\lambda + 1 \\
&\geq \int_{e^{-2n}}^{e^{-n}} \frac{f_n(\lambda) - \lambda}{\lambda}\, d\lambda + 1 \\
&= \int_{e^{-2n}}^{e^{-n}} \frac{1}{\lambda \cdot (-\ln(\lambda))}\, d\lambda + 1 \\
&= \left[-\ln(-\ln(\lambda))\right]_{e^{-2n}}^{e^{-n}} + 1 \\
&= -\ln(-\ln(e^{-n}) + \ln(-\ln(e^{-2n})) + 1 \\
&= -\ln(n) + \ln(2n) + 1 \\
&= \ln(2) + 1.
\end{aligned}$$

(8) This follows from assertion (3), (7).

(9) We estimate

$$
\begin{aligned}
&\int_{0+}^{1} \frac{f_n(\lambda)}{\lambda}\,d\lambda\\
= \;& \int_{0+}^{e^{-3n}} \frac{f_n(\lambda)}{\lambda}\,d\lambda + \int_{e^{-3n}}^{e^{-2n}} \frac{f_n(\lambda)}{\lambda}\,d\lambda + \int_{e^{-2n}}^{e^{-n}} \frac{f_n(\lambda)}{\lambda}\,d\lambda\\
&\qquad + \int_{e^{-n}}^{\frac{1}{-\ln(e^{-n})}+e^{-n}} \frac{f_n(\lambda)}{\lambda}\,d\lambda + \int_{\frac{1}{-\ln(e^{-n})}+e^{-n}}^{1} \frac{f_n(\lambda)}{\lambda}\,d\lambda\\
= \;& \int_{0+}^{e^{-3n}} \frac{\lambda}{\lambda}\,d\lambda + \int_{e^{-3n}}^{e^{-2n}} \frac{(e^{-2n}-\lambda)\cdot e^{-3n} + (\lambda - e^{-3n})\cdot\left(\frac{1}{-\ln(e^{-2n})} + e^{-2n}\right)}{e^{-2n}-e^{-3n}}\cdot\frac{1}{\lambda}\,d\lambda\\
&+ \int_{e^{-2n}}^{e^{-n}} \frac{\frac{1}{-\ln(\lambda)}+\lambda}{\lambda}\,d\lambda + \int_{e^{-n}}^{\frac{1}{n}+e^{-n}} \frac{\frac{1}{-\ln(e^{-n})}+e^{-n}}{\lambda}\,d\lambda + \int_{\frac{1}{n}+e^{-n}}^{1} \frac{\lambda}{\lambda}\,d\lambda\\
= \;& \int_{0+}^{e^{-3n}} 1\,d\lambda + \int_{e^{-3n}}^{e^{-2n}} 1 + \frac{1}{2n\cdot(e^{-2n}-e^{-3n})} + \frac{e^{-3n}}{2n\cdot(e^{-2n}-e^{-3n})}\cdot\frac{1}{\lambda}\,d\lambda\\
&+ \int_{e^{-2n}}^{e^{-n}} 1 - \frac{1}{\ln(\lambda)\cdot\lambda}\,d\lambda + \int_{e^{-n}}^{\frac{1}{n}+e^{-n}} \frac{\frac{1}{n}+e^{-n}}{\lambda}\,d\lambda + \int_{\frac{1}{n}+e^{-n}}^{1} 1\,d\lambda\\
= \;& e^{-3n} + \int_{e^{-3n}}^{e^{-2n}} 1 + \frac{1}{2n\cdot(e^{-2n}-e^{-3n})}\,d\lambda + \int_{e^{-3n}}^{e^{-2n}} \frac{e^{-3n}}{2n\cdot(e^{-2n}-e^{-3n})}\cdot\frac{1}{\lambda}\,d\lambda\\
&+ \int_{e^{-2n}}^{e^{-n}} 1\,d\lambda - \int_{e^{-2n}}^{e^{-n}} \frac{1}{\ln(\lambda)\cdot\lambda}\,d\lambda + \int_{e^{-n}}^{\frac{1}{n}+e^{-n}} \frac{\frac{1}{n}+e^{-n}}{\lambda}\,d\lambda + 1 - \frac{1}{n} - e^{-n}\\
= \;& e^{-3n} + \left(e^{-2n}-e^{-3n}\right)\cdot\left(1 + \frac{1}{2n\cdot(e^{-2n}-e^{-3n})}\right) + \left[\frac{e^{-3n}\cdot\ln(\lambda)}{2n\cdot(e^{-2n}-e^{-3n})}\right]_{e^{-3n}}^{e^{-2n}}\\
&+e^{-n} - e^{-2n} - [\ln(-\ln(\lambda))]_{e^{-2n}}^{e^{-n}} + \left[\left(\frac{1}{n}+e^{-n}\right)\cdot\ln(\lambda)\right]_{e^{-n}}^{\frac{1}{n}+e^{-n}} + 1 - \frac{1}{n} - e^{-n}\\
= \;& 1 - \frac{3}{2n} + \frac{e^{-3n}\cdot(-2n+3n)}{2n\cdot(e^{-2n}-e^{-3n})}\\
&-\big(\ln(n)-\ln(2n)\big) + \left(\frac{1}{n}+e^{-n}\right)\cdot\left(\ln\left(\frac{1}{n}+e^{-n}\right) - \ln(e^{-n})\right)\\
= \;& 1 - \frac{3}{2n} + \frac{1}{2\cdot(e^n-1)} + \ln(2) + \left(\frac{1}{n}+e^{-n}\right)\cdot\ln\left(\frac{e^n}{n}+1\right)\\
\leq \;& 1 + \frac{1}{2\cdot(e-1)} + \ln(2) + \frac{2}{n}\cdot\ln(2e^n)\\
= \;& 1 + \frac{1}{2\cdot(e-1)} + \ln(2) + 2\cdot\ln(2)\\
\leq \;& 4.
\end{aligned}
$$

This finishes the proof of Lemma 16.12. □

Remark 16.13 (Exotic behavior at zero)**.** The sequence of functions $(f_n)_{n\geq 0}$ has an exotic behavior close to zero in a small range depending on n. There are no $C > 0$ and $\epsilon > 0$ such that $f_n'(\lambda) \leq C$ holds for all $n \geq 1$ and all $\lambda \in (0,\epsilon)$ for which the derivative exists.

This exotic behavior is responsible for the violation of the the uniform integrability condition, see Lemma 16.12 (6). It is very unlikely that such a sequence

$(f_n)_{n\geq 0}$ actually occurs as the sequence of spectral density functions of the manifolds $G_i\backslash M$ for some smooth manifold M with proper free cocompact G-action and G-invariant Riemannian metric. The example above shows that we need to have more information on such sequences of spectral density functions.

16.4. **Uniform estimate on spectral density functions.** The crudest way to ensure the uniform integrability condition (v) appearing in Theorem 16.3 is to assume a uniform gap in the spectrum, namely we have the obvious

Lemma 16.14. *Suppose that* the uniform gap in the spectrum at zero condition *is satisfied, i.e., there exists* $\epsilon > 0$ *such that for all* $i \in I$ *and* $\lambda \in [0, \epsilon]$ *we have* $F[i](\lambda) = F[i](0)$.

Then the uniform integrability condition (v) *appearing in Theorem 16.3 is satisfied.*

However, this is an unrealistic condition in our situation for closed aspherical manifolds because of the following remark.

Remark 16.15 (The Zero-in-the-Spectrum Conjecture)**.** Let M be an aspherical closed manifold. If one wants to use Lemma 16.14 in connection with Theorem 16.3 to prove Conjecture 6.2 or more generally Conjecture 14.8 for $\widetilde{M}$, one has to face the fact that the assumption that one has in each dimension a uniform gap in the spectrum at zero implies that $b_p^{(2)}(\widetilde{M})$ vanishes and the p-th Novikov-Shubin invariant satisfies $\alpha_p(\widetilde{M}) = \infty^*$ for all $p \geq 0$. In other words, M must be a counterexample to the Zero-in-the-Spectrum Conjecture which is discussed in detail in [66, Chapter 12 on pages 437 ff]. Such counterexample is not known to exist and it is evident that it is hard to find one. Therefore the the uniform gap in the spectrum at zero condition is not useful in this setting.

There are examples where Lemma 16.14 does apply when one allows to twist with representations in favorable case, see for instance [12, 74, 78, 79].

Here is a more promising version.

Theorem 16.16 (The uniform logarithmic estimate)**.** *Suppose that there exists constants* $C > 0$, $0 < \epsilon < 1$ *and* $\delta > 0$ *independent of* i *such that*

$$F[i](\lambda) - F[i](0) \leq \frac{C}{(-\ln(\lambda))^{1+\delta}},$$

Then the uniform integrability condition (v) *appearing in Theorem 16.3 is satisfied.*

Proof. This follows from the following calculation.

$$\begin{aligned}
\int_{+0}^{\epsilon} \frac{C}{\lambda \cdot (-\ln(\lambda))^{1+\delta}} d\lambda &\overset{\lambda=\exp(\mu)}{=} \int_{-\infty}^{\ln(\epsilon)} \frac{C}{\exp(\mu)\cdot(-\ln(\exp(\mu)))^{1+\delta}} \cdot \exp(\mu)\, d\mu \\
&= \int_{-\infty}^{\ln(\epsilon)} \frac{C}{(-\mu)^{1+\delta}} d\mu \\
&\overset{\mu=-\nu}{=} \int_{-\ln(\epsilon)}^{\infty} \frac{C}{\nu^{1+\delta}} d\nu \\
&= \lim_{x\to\infty} \int_{-\ln(\epsilon)}^{x} \frac{C}{\nu^{1+\delta}} d\nu \\
&= \lim_{x\to\infty} -\delta \cdot \left(x^{-\delta} - (-\ln(\epsilon))^{-\delta}\right) \\
&= \delta \cdot (-\ln(\epsilon))^{-\delta} \\
&< \infty,
\end{aligned}$$

□

The p-th spectral density function $F_p(X[i])$ of $X[i]$ is defined as the spectral density function $F\big(c_p^{(2)}(X[i])\big)$ of the p-th differential of $C_*^{(2)}(X[i])$. If we now consider the Setup 0.1, we get from [62, Theorem 0.3] for all $p \geq 0$ at least the inequality

$$\text{(16.17)} \qquad \frac{F_p(X[i])(\lambda) - F_p(X[i])(0)}{[G:G_i]} \quad \leq \quad \frac{C}{-\ln(\lambda)}.$$

But this is not enough, since one cannot take $\delta = 0$ in Theorem 16.16, namely, we have for every $C > 0$ and $0 < \epsilon < 1$

$$\begin{aligned} \int_{+0}^{\epsilon} \frac{C}{\lambda \cdot (-\ln(\lambda))} d\lambda &= \lim_{x \to 0+} \int_{x}^{\epsilon} \frac{C}{\lambda \cdot (-\ln(\lambda))} d\lambda \\ &= \lim_{x \to 0+} C \cdot (-\ln(-\ln(\epsilon)) + \ln(-\ln(x))) \\ &= \infty. \end{aligned}$$

Condition (16.17) is implied by the stronger condition that there exist for each $p \geq 0$ constants $C_p > 0$, $\epsilon_p > 0$ and $\alpha_p > 0$ independent of i such that we have for all $\lambda \in [0, \epsilon_p)$ and all $i = 1, 2, \ldots$

$$\text{(16.18)} \qquad \frac{F_p(X[i])(\lambda) - F_p(X[i])(0)}{[G:G_i]} \quad \leq \quad C \cdot \lambda^{\alpha_p}.$$

Condition (16.18) is known for $p = 0$, see [52, Theorem 1.1]. However, there is an unpublished manuscript by Grabowski and Virag [44], where they show that there exists an explicit element a in the integral group ring of the wreath product $\mathbb{Z}^3 \wr \mathbb{Z}$ such that the spectral density function of the associated $\mathcal{N}(\mathbb{Z}^3 \wr \mathbb{Z})$-map $r_a \colon \mathcal{N}(\mathbb{Z}^3 \wr \mathbb{Z}) \to \mathcal{N}(\mathbb{Z}^3 \wr \mathbb{Z})$ does not satisfy condition (16.18). This implies that there exists a closed Riemannian manifold M with fundamental group $G = \pi_1(X) = \mathbb{Z}^3 \wr \mathbb{Z}$ such that for some p condition (16.18) is not satisfied for $X = \widetilde{M}$ and the p-th Novikov-Shubin invariant of $\widetilde{M}$ is zero, disproving a conjecture of Lott and Lück, see [61, Conjecture 7.1 and 7.2]. It may still be possible that Condition 16.18 and the conjecture of Lott and Lück hold for an aspherical closed manifold M.

There is no counterexample known to the condition appearing in Theorem 16.16 but the constant δ has to be depend on the group G, see Grabowski [43].

17. Proof of Theorem 14.11, Theorem 14.12, and Corollary 14.13

In this section we derive the proofs of Theorem 14.11, 14.12 from Theorem 16.3. This needs some preparation.

Define for a matrix $B \in M_{r,s}(L^1(G))$ the real number

$$\text{(17.1)} \qquad K^G(B) \quad := \quad rs \cdot \max\{||b_{i,j}||_{L^1} \mid 1 \leq i \leq r, 1 \leq j \leq s\},$$

where for $a = \sum_{g \in G} \lambda_g \cdot g$ its L^1-norm $||a||_{L^1}$ is defined by $\sum_{g \in G} |\lambda_g|$. The proof of the following result is analogous to the proof of [66, Lemma 13.33 on page 466].

Lemma 17.2. *We get for $B \in M_{r,s}(L^1(G))$*

$$||r_B^{(2)} \colon L^2(G)^r \to L^2(G)^s|| \leq K^G(B).$$

Consider the setup 12.1.

Lemma 17.3. *Consider $B \in M_d(L^1(G))$. Let $B[i] \in M_d(L^1(G/G_i))$ be obtained from B by applying the map $L^1(G) \to L^1(G/G_i)$ induced by the projection $\psi_i \colon G \to G/G_i$. Then*

$$\operatorname{tr}_{\mathcal{N}(G)}(B) = \lim_{i \in I} \operatorname{tr}_{\mathcal{N}(G/G_i)}(B[i]).$$

Proof. Suppose that B looks like $\left(\sum_{g\in G}\lambda_g(r,s)\cdot g\right)_{r,s}$. Denote in the sequel by e the unit element in G or G/G_i. Then

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}(G)}(B) &= \sum_{r=1}^{d}\lambda_e(r,r);\\ \operatorname{tr}_{\mathcal{N}(G/G_i)}(B[i]) &= \sum_{r=1}^{d}\sum_{g\in G,\psi_i(g)=e}\lambda_g(r,r).\end{aligned}$$

Consider $\epsilon > 0$. We can choose a finite subset $S \subseteq G$ with $e \in S$ such that $\sum_{g\in G,g\notin S}|\lambda_g(r,r)| < \epsilon/d$ holds for all $r \in \{1,2,\ldots,d\}$. Since $\bigcap_{i\in I} G_i = \{1\}$, there exists an index i_S such that $\psi_i(g) = e \Rightarrow g = e$ holds for all $g \in S$ and $i \geq i_S$. This implies for $i \geq i_S$

$$\begin{aligned}\left|\operatorname{tr}_{\mathcal{N}(G)}(B) - \operatorname{tr}_{\mathcal{N}(G/G_i)}(B[i])\right| &= \left|\sum_{r=1}^{d}\lambda_e(r,r) - \sum_{r=1}^{d}\sum_{g\in G,\psi_i(g)=e}\lambda_g(r,r)\right|\\ &\leq \sum_{r=1}^{d}\left|\lambda_e(r,r) - \sum_{g\in G,\psi_i(g)=e}\lambda_g(r,r)\right|\\ &= \sum_{r=1}^{d}\left|\sum_{g\in G,g\notin S,\psi_i(g)=e}\lambda_g(r,r)\right|\\ &\leq \sum_{r=1}^{d}\sum_{g\in G,g\notin S,\psi_i(g)=e}\left|\lambda_g(r,r)\right|\\ &\leq \sum_{r=1}^{d}\sum_{g\in G,g\notin S}\left|\lambda_g(r,r)\right|\\ &\leq \sum_{r=1}^{d}\epsilon/d\\ &= \epsilon.\end{aligned}$$

□

Next we give the proof of Theorem 14.11.

Proof of Theorem 14.11. We have to show for $A \in M_{r,s}(\mathbb{Q}G)$.

(17.4) $$\begin{aligned}&\det_{\mathcal{N}(G)}\big(r_A^{(2)}\colon L^2(G)^r \to L^2(G)^s\big)\\ &\qquad\geq \limsup_{i\in I}\det_{\mathcal{N}(G/G_i)}\big(r_{A[i]}^{(2)}\colon L^2(G/G_i)^r \to L^2(G/G_i)^s\big).\end{aligned}$$

We first deal with the special case that $A \in M_{r,s}(\mathbb{Z}G)$.

We will apply Theorem 16.3 to the following special situation:

- $B = A^*A$;
- $Q_i = G/G_i$;
- $B[i] = A[i]^*A[i]$;
- tr is the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)}\colon \mathcal{N}(G) \to \mathbb{C}$;
- tr_i is the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G/G_i)}\colon \mathcal{N}(G/G_i) \to \mathbb{C}$.

We have to check that the conditions of Theorem 16.3 (1) are satisfied. We obtain Condition (i) appearing in Theorem 16.3 from Lemma 17.2 since the projection

$L^1(G) \to L^1(G/G_i)$ has operator norm at most 1 and hence we get for the number $K^G(B)$ defined in (17.1)

$$K^{Q_i}(B[i]) \le K^G(B).$$

Condition (ii) appearing in Theorem 16.3 follows from Lemma 17.3. Condition (iii) appearing in Theorem 16.3 follows from the Assumption 13.5 that each quotient G/G_i satisfies the Determinant Conjecture 13.1. Condition (iv) appearing in Theorem 16.3 is satisfied because of $B = A^*A$ and $B[i] = A[i]^*A[i]$. Hence we conclude from Theorem 16.3 (1)

$$\det_{\mathcal{N}(G)}\bigl(r_B^{(2)} \colon L^2(G)^r \to L^2(G)^r\bigr) \ge \limsup_{i \in I} \det_{\mathcal{N}(G)}\bigl(r_{B[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^r\bigr).$$

Since we get from [66, Lemma 3.15 (4) on page 129]

$$\begin{aligned}
\det_{\mathcal{N}(G)}\bigl(r_A^{(2)}\bigr) &= \sqrt{\det_{\mathcal{N}(G)}\bigl(r_B^{(2)}\bigr)};\\
\det_{\mathcal{N}(G(G_i)}\bigl(r_{A[i]}^{(2)}\bigr) &= \sqrt{\det_{\mathcal{N}(G(G_i}\bigl(r_{B[i]}^{(2)}\bigr)},
\end{aligned}$$

equation (17.4) follows from Theorem 16.3 for $A \in M_{r,s}(\mathbb{Z}G)$.

Next we reduce the general case $A \in M_{r,s}(\mathbb{Q}G)$ to the case above.

Consider any real number $m > 0$, any group H and any morphism $f \colon L^2(H)^r \to L^2(H)^s$. We conclude from [66, Lemma 1.18 on page 24 and Theorem 3.14 (1) on page 128 and Lemma 3.15 (3), (4) and (7) on page 129]

$$\begin{aligned}
&\det_{\mathcal{N}(H)}\bigl(f \circ m \operatorname{id}_{L^2(H)^r}\bigr)^2\\
&= \det_{\mathcal{N}(H)}\Bigl(\bigl(f \circ m \operatorname{id}_{L^2(H)^r}\bigr)^* \circ \bigl(f \circ m \operatorname{id}_{L^2(H)^r}\bigr)\Bigr)\\
&= \det_{\mathcal{N}(H)}\bigl(f^* \circ f \circ m^2 \operatorname{id}_{L^2(H)^r}\bigr)\\
&= \det_{\mathcal{N}(H)}\Bigl(\bigl(f^* \circ f \circ m^2 \operatorname{id}_{L^2(H)^r}\bigr)\big|_{\ker\left(f^*\circ f \circ m^2 \operatorname{id}_{L^2(H)^r}\right)^\perp}\Bigr)\\
&= \det_{\mathcal{N}(H)}\Bigl(\bigl(f^* \circ f\bigr)\big|_{\ker(f^*f)^\perp} \circ m^2 \operatorname{id}_{\ker(f^*f)^\perp}\Bigr)\\
&= \det_{\mathcal{N}(H)}\Bigl(\bigl(f^* \circ f\bigr)\big|_{\ker(f^*f)^\perp}\Bigr) \cdot \det_{\mathcal{N}(H)}\bigl(m^2 \operatorname{id}_{\ker(f^*f)^\perp}\bigr)\\
&= \det_{\mathcal{N}(H)}\bigl(f^* \circ f\bigr) \cdot m^{2\cdot \dim_{\mathcal{N}(H)} \ker(f^*f)^\perp}\\
&= \det_{\mathcal{N}(H)}(f)^2 \cdot m^{2r - 2\dim_{\mathcal{N}(H)}(\ker(f^*f))}\\
&= \Bigl(\det_{\mathcal{N}(H)}(f) \cdot m^{r - \dim_{\mathcal{N}(H)}(\ker(f))}\Bigr)^2.
\end{aligned}$$

Thus we have shown

$$\det_{\mathcal{N}(H)}\bigl(f \circ m \operatorname{id}_{L^2(H)^r}\bigr) = \det_{\mathcal{N}(H)}\bigl(f\bigr) \cdot m^{r - \dim_{\mathcal{N}(H)}(\ker(f))}. \tag{17.5}$$

Let $m \ge 1$ be an integer such that $mI_r \cdot A$ belongs to $M_{r,s}(\mathbb{Z}G)$, where mI_r is obtained from the identity matrix by multiplying all entries with m. If we apply (17.5) in the case $H = G$ and $H = G/G_i$ to $f = r_A^{(2)}$ and $f = r_{A[i]}^{(2)}$, we obtain

$$\begin{aligned}
\det_{\mathcal{N}(G)}\bigl(r_{mI_r\cdot A}^{(2)}\bigr) &= \det_{\mathcal{N}(G)}\bigl(r_A^{(2)}\bigr) \cdot m^{r - \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}))};\\
\det_{\mathcal{N}(G/G_i)}\bigl(r_{mI_r\cdot A[i]}^{(2)}\bigr) &= \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)}\bigr) \cdot m^{r - \dim_{\mathcal{N}(G/G_i)}(\ker(r_{A[i]}^{(2)}))}.
\end{aligned}$$

Since $\det_{\mathcal{N}(G)}\big(r^{(2)}_{mI_r\cdot A}\big) \geq 1$ follows from (17.4) and the assumption that we have $\det_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{mI_r\cdot A[i]}\big) \geq 1$ for $i \in I$, we get $\det_{\mathcal{N}(G)}\big(r^{(2)}_{A}\big) > 0$. We conclude

$$\text{(17.6)} \quad \frac{\det_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{A[i]}\big)}{\det_{\mathcal{N}(G)}\big(r^{(2)}_{A}\big)} = \frac{\det_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{mI_rA[i]}\big)}{\det_{\mathcal{N}(G)}\big(r^{(2)}_{mI_rA}\big)} \cdot m^{|\dim_{\mathcal{N}(G)}(\ker(r^{(2)}_A))-\dim_{\mathcal{N}(G/G_i)}(\ker(r^{(2)}_{A[i]}))|}.$$

We derive

$$\text{(17.7)} \qquad \lim_{i\in I}\Big(\dim_{\mathcal{N}(G)}\big(\ker\big(r^{(2)}_A\big)\big) - \dim_{\mathcal{N}(G/G_i)}\big(\ker\big(r^{(2)}_{A[i]}\big)\big)\Big) = 0$$

from Theorem 13.6. Since (17.4) holds for mI_rA, it holds also for A by (17.6) and (17.7). This finishes the proof of Theorem 14.11. □

Next we give the proof of Theorem 14.12.

Proof of Theorem 14.12. We will apply Theorem 16.3 to the following special situation:

- $B = A^*A$;
- $Q_i = G/G_i$;
- $B[i] = A[i]^*A[i]$;
- tr is the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}$;
- tr_i is the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G/G_i)} \colon \mathcal{N}(G/G_i) \to \mathbb{C}$.

We have to check that the conditions of Theorem 16.3 (2) are satisfied. Condition (ii) appearing in Theorem 16.3 follows from Lemma 17.3. Condition (iv) appearing in Theorem 16.3 is satisfied because of $B = A^*A$ and $B[i] = A[i]^*A[i]$.

Let B^{-1} be the inverse of B in $\operatorname{GL}_d(L^1(G))$. Put $K := \max\{K^G(B), K^{G/G_i}(B[i])\}$. Since the projection $L^1(G) \to L^1(G/G_i)$ has operator norm at most 1, we get for the numbers $K^G(B)$ and $K^G(B^{-1})$ defined in (17.1) and all $i \in I$

$$\begin{aligned} K^{G/G_i}(B[i]) &\leq K; \\ K^{G/G_i}(B[i]^{-1}) &\leq K. \end{aligned}$$

hold. We conclude $||r^{(2)}_{B[i]^{-1}}|| \leq K$ and $||r^{(2)}_{B[i]^{-1}}|| \leq K$ for all $i \in I$ from Lemma 17.2. In particular condition (i) appearing in Theorem 16.3 is satisfied. Since $r^{(2)}_{B[i]^{-1}}$ is the inverse of $r^{(2)}_{B[i]}$, we conclude from [66, Lemma 213 (2) on page 78, Theorem 3.14 (1) on page 128 and Lemma 3.15 (6) on page 129]

$$\text{(17.8)} \qquad F[i](\lambda) = 0 \quad \text{for all } \lambda < K^{-1} \text{ and } i \in I;$$

$$\text{(17.9)} \qquad \det^{(2)}_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{B[i]}\big) \geq d \cdot \ln(K) \quad \text{for all } i \in I.$$

Hence condition (iii) is satisfied if we take $\kappa := d \cdot \ln(K)$. We conclude from (17.8) that also condition (v) is satisfied. We conclude from Theorem 16.3 (2)

$$\det^{(2)}_{\mathcal{N}(G)}\big(r^{(2)}_B \colon L^2(G)^d \to L^2(G)^d\big) = \lim_{i\in I} \det^2_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{B[i]} \colon L^2(G/G_i)^d \to L^2(G/G_i)^d\big).$$

Since we get from [66, Lemma 3.15 (4) on page 129]

$$\begin{aligned} \det^{(2)}_{\mathcal{N}(G)}\big(r^{(2)}_A\big) &= \sqrt{\det^{(2)}_{\mathcal{N}(G)}\big(r^{(2)}_B\big)}; \\ \det^{(2)}_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{A[i]}\big) &= \sqrt{\det^{(2)}_{\mathcal{N}(G/G_i)}\big(r^{(2)}_{B[i]}\big)}, \end{aligned}$$

Theorem 14.12 follows. □

Next we prove Corollary 14.13
Proof of Corollary 14.13. (1) Since C_* is acyclic over $L^1(G)$ and finitely generated free, we can choose an $L^1(G)$-chain contraction $\gamma\colon C_* \to C_{*+1}$. Then $(c+\gamma)_{\mathrm{odd}}\colon C_{\mathrm{odd}} \xrightarrow{\cong} C_{\mathrm{ev}}$ is an isomorphism of finitely generated based free $L^1(G)$-modules. It induces an isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules $(c+\gamma)^{(2)}_{\mathrm{odd}}\colon C^{(2)}_{\mathrm{odd}} \xrightarrow{\cong} C^{(2)}_{\mathrm{ev}}$. We conclude from [66, Lemma 3.41 on page 146]

$$\rho^{(2)}\big(C^{(2)}_*\big) := \ln\Big(\det{}^{(2)}_{\mathcal{N}(G)}\big((c+\gamma)^{(2)}_{\mathrm{odd}}\colon C^{(2)}_{\mathrm{odd}} \xrightarrow{\cong} C^{(2)}_{\mathrm{ev}}\big)\Big).$$

Analogously we prove for each $i \in I$

$$\rho^{(2)}\big(C[i]^{(2)}_*\big) := \ln\Big(\det{}^{(2)}_{\mathcal{N}(G/G_i)}\big((c[i]+\gamma[i])^{(2)}_{\mathrm{odd}}\colon C[i]^{(2)}_{\mathrm{odd}} \xrightarrow{\cong} C[i]^{(2)}_{\mathrm{ev}}\big)\Big).$$

Now assertion (1) follows from Theorem 14.12.

(2) We begin with the case of an isomorphism $f_*\colon C_* \xrightarrow{\cong} D_*$ of finitely generated based free $L^1(G)$-chain complexes. We conclude from [66, Lemma 3.41 on page 146] for all $i \in I$

$$\begin{aligned}
\rho^{(2)}\big(D^{(2)}_*\big) - \rho^{(2)}\big(C^{(2)}_*\big) &= \sum_{p\geq 0}(-1)^p \cdot \ln\big(\det\det{}^{(2)}_{\mathcal{N}(G)}(f^{(2)}_p)\big);\\
\rho^{(2)}\big(D[i]^{(2)}_*\big) - \rho^{(2)}\big(C[i]^{(2)}_*\big) &= \sum_{p\geq 0}(-1)^p \cdot \ln\big(\det\det{}^{(2)}_{\mathcal{N}(G/G_i)}(f[i]^{(2)}_p)\big).
\end{aligned}$$

Now the claim follows in this special case from Theorem 14.12.

Finally we consider an $L^1(G)$-chain homotopy equivalence $f_*\colon C_* \xrightarrow{\simeq} D_*$. Let $\mathrm{cyl}(f_*)$ be its mapping cylinder and $\mathrm{cone}(f_*)$ be its mapping cone. Let $\mathrm{cone}(C_*)$ be the mapping cone of C_*. We obtain based exact sequences of $L^1(G)$-chain complexes

$$0 \to C_* \to \mathrm{cyl}(f_*) \to \mathrm{cone}(f_*) \to 0$$

and

$$0 \to D_* \to \mathrm{cyl}(f_*) \to \mathrm{cone}(C_*) \to 0.$$

Since f_* is a $L^1(G)$-chain homotopy equivalence, $\mathrm{cone}(f_*)$ is contractible. Since $\mathrm{cone}(C_*)$ is contractible, we can find isomorphisms of $L^1(G)$-chain complexes (cf. [66, Lemma 3.42 on page 148])

$$\begin{aligned}
u_*\colon C_* \oplus \mathrm{cone}(f_*) &\xrightarrow{\cong} \mathrm{cyl}(f_*);\\
v_*\colon D_* \oplus \mathrm{cone}(C_*) &\xrightarrow{\cong} \mathrm{cyl}(f_*).
\end{aligned}$$

Since we have already treated the case of a chain isomorphism, we conclude

$$\begin{aligned}
&\rho^{(2)}\Big(\big(C_* \oplus \mathrm{cone}(f_*)\big)^{(2)}\Big) - \rho^{(2)}\big(\mathrm{cyl}(f_*)^{(2)}\big)\\
&\qquad = \lim_{i\in I}\rho^{(2)}\Big(\big(C[i]_* \oplus \mathrm{cone}(f[i]_*)\big)^{(2)}\Big) - \rho^{(2)}\big(\mathrm{cyl}(f[i]_*)^{(2)}\big);
\end{aligned}$$

and

$$\begin{aligned}
&\rho^{(2)}\Big(\big(D_* \oplus \mathrm{cone}(C_*)\big)^{(2)}\Big) - \rho^{(2)}\big(\mathrm{cyl}(f_*)^{(2)}\big)\\
&\qquad = \lim_{i\in I}\rho^{(2)}\Big(\big(D[i]_* \oplus \mathrm{cone}(C[i]_*)\big)^{(2)}\Big) - \rho^{(2)}\big(\mathrm{cyl}(f[i]_*)^{(2)}\big).
\end{aligned}$$

This implies

$$\begin{aligned}
&\rho^{(2)}\big(C^{(2)}_*\big) + \rho^{(2)}\big(\mathrm{cone}(f^{(2)}_*)\big) - \rho^{(2)}\big(D^{(2)}_*\big) - \rho^{(2)}\big(\mathrm{cone}(C_*)^{(2)}\big)\\
&= \lim_{i\in I}\Big(\rho^{(2)}\big(C[i]^{(2)}_*\big) + \rho^{(2)}\big(\mathrm{cone}(f[i]_*)^{(2)}\big) - \rho^{(2)}\big(D[i]^{(2)}_*\big) - \rho^{(2)}\big(\mathrm{cone}(C[i]_*)^{(2)}\big)\Big).
\end{aligned}$$

We conclude from assertion (1)

$$\begin{aligned}\rho^{(2)}\big(\operatorname{cone}(f_*^{(2)})\big) &= \lim_{i\in I}\rho^{(2)}\big(\operatorname{cone}(f[i]_*)^{(2)}\big);\\ \rho^{(2)}\big(\operatorname{cone}(C_*^{(2)})\big) &= \lim_{i\in I}\rho^{(2)}\big(\operatorname{cone}(C[i]_*)^{(2)}\big).\end{aligned}$$

This implies

$$\rho^{(2)}\big(C_*^{(2)}\big)-\rho^{(2)}\big(D_*^{(2)}\big) = \lim_{i\in I}\Big(\rho^{(2)}\big(C[i]_*^{(2)}\big)-\rho^{(2)}\big(D[i]_*^{(2)}\big)\Big).$$

This finishes the proof of Corollary 14.13. □

18. Proof of Theorem 7.13

Next we want to prove Theorem 7.13. First we deal with homotopy invariance and with the relationship between L^2-torsion and integral torsion.

Lemma 18.1. *Let G be a group for which the Determinant Conjecture 13.1 is true. Let $f_*\colon D_*\to E_*$ be a $\mathbb{Z}G$-chain homotopy equivalence equivalence of finite based free $\mathbb{Z}G$-chain complexes. Suppose that $D_*^{(2)}$ or $E_*^{(2)}$ is L^2-acyclic. Then both $D_*^{(2)}$ and $E_*^{(2)}$ are L^2-acyclic and*

$$\rho^{(2)}\big(D_*^{(2)}\big)=\rho^{(2)}\big(E_*^{(2)}\big).$$

Proof. This follows from [66, Theorem 3.93 (1) on page 161 and Lemma 13.6 on page 456]. □

Notation 18.2. Let A be a finitely generated free abelian group and let $B\subseteq A$ be a subgroup. Define the *closure* of B in A to be the subgroup

$$\overline{B} = \{x\in A \mid n\cdot x\in B \text{ for some non-zero integer } n\}.$$

Notice that $A/\overline{B}$ and $M_f := M$
$tors(M)$ are finitely generated free and we have $\overline{\ker(f)}=\ker(f)$ for a homomorphism $f\colon A_0\to A_1$ of finitely generated free abelian groups. The proof of the next result can be found in [68, Lemma 2.11].

Lemma 18.3. *Let $u\colon\mathbb{Z}^r\to\mathbb{Z}^s$ be a homomorphism of abelian groups. Let $j\colon\ker(u)\to\mathbb{Z}^r$ be the inclusion and $\operatorname{pr}\colon\mathbb{Z}^s\to\operatorname{coker}(u)_f$ be the canonical projection. Choose $\mathbb{Z}$-basis for $\ker(u)$ and $\operatorname{coker}(u)_f$.*

Then $\det_{\mathcal{N}(\{1\})}\big(j^{(2)}\big)$ and $\det_{\mathcal{N}(\{1\})}\big(\operatorname{pr}^{(2)}\big)$ are independent of the choice of the $\mathbb{Z}$-basis for $\ker(u)$ and $\operatorname{coker}(u)_f$, and we have

$$\det_{\mathcal{N}(\{1\})}(u^{(2)}) = \det_{\mathcal{N}(\{1\})}\big(j^{(2)}\big)\cdot\big|\operatorname{tors}(\operatorname{coker}(u))\big|\cdot\det_{\mathcal{N}(\{1\})}\big(\operatorname{pr}^{(2)}\big);$$

and

$$\begin{aligned}1 &\le \det_{\mathcal{N}(\{1\})}(j^{(2)}) &&\le \det_{\mathcal{N}(\{1\})}(u^{(2)});\\ 1 &\le \det_{\mathcal{N}(\{1\})}\big(\operatorname{pr}^{(2)}\big) &&\le \det_{\mathcal{N}(\{1\})}(u^{(2)});\\ 1 &\le \big|\operatorname{tors}(\operatorname{coker}(u))\big| &&\le \det_{\mathcal{N}(\{1\})}(u^{(2)}).\end{aligned}$$

The point of the next lemma is that the chain complexes live over $\mathbb{Z}G$ but the chain homotopy equivalence has only to exist over $\mathbb{Q}G$.

Lemma 18.4. *Let C_* and D_* be two finite free $\mathbb{Z}G$-chain complexes. Suppose that $C_*\otimes_{\mathbb{Z}}\mathbb{Q}$ and $D_*\otimes_{\mathbb{Z}}\mathbb{Q}$ are $\mathbb{Q}G$-chain homotopy equivalent and that $C_*^{(2)}$ is L^2-acyclic. Then $D_*^{(2)}$ is L^2-acyclic and*

$$\rho^{(2)}\big(D_*^{(2)}\big)-\rho^{(2)}\big(C_*^{(2)}\big) = \lim_{i\in I}\frac{\rho^{\mathbb{Z}}\big(D[i]_*\big)-\rho^{\mathbb{Z}}\big(C[i]_*\big)}{[G:G_i]}.$$

Proof. Let $g_*\colon C_* \otimes_{\mathbb{Z}} \mathbb{Q} \to D_* \otimes_{\mathbb{Z}} \mathbb{Q}$ be a $\mathbb{Q}G$-chain homotopy equivalence. Since C_* and D_* are finite based free $\mathbb{Z}G$-chain complexes, we can find a $\mathbb{Z}G$-chain map $f_*\colon C_* \to D_*$ and an integer l such that $f_* \otimes_{\mathbb{Z}} \mathbb{Q} = l \cdot g_*$. Obviously $f_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}G$-chain homotopy equivalence. In the sequel we abbreviate $C'_* := \operatorname{cyl}(f_*)$ and $C''_* := \operatorname{cone}(f_*)$. By the chain homotopy invariance of integral torsion and of L^2-torsion (see Lemma 18.1) it suffices to prove the claim for C_* and C'_* instead of C_* and D_*.

We have the obvious exact sequence of finite based free $\mathbb{Z}G$-chain complexes

$$0 \to C_* \xrightarrow{i_*} C'_* \xrightarrow{p_*} C''_* \to 0.$$

Since $f_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}G$-chain homotopy equivalence, we can choose a $\mathbb{Q}G$-chain contraction $\gamma_*\colon C''_* \otimes_{\mathbb{Z}} \mathbb{Q} \to C''_{*+1} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since each C''_* is a finite free $\mathbb{Z}G$-chain complex, we can find an integer m and $\mathbb{Z}G$-maps $\delta_p\colon C_p \to C''_{p+1}$ such that $m \cdot \gamma_p = \delta_p \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{Q}}$ holds for all $p \geq 0$. Hence $\delta_*\colon C''_* \to C''_{*+1}$ is a $\mathbb{Z}G$-chain homotopy from $m \cdot \operatorname{id}_{C_*}$ to the zero homomorphism. Moreover, $\delta[i]_*\colon C''[i]_* \to C''[i]_{*+1}$ is a $\mathbb{Z}[G/G_i]$-chain homotopy from $m \cdot \operatorname{id}_{C''[i]_*}$ to the zero homomorphism for all $i \in I$. Hence multiplication with m annihilates $H_p(C''[i])$ for all $n \geq 0$ and $p \in I$.

We have the long exact homology sequence

$$\cdots \to H_p(C[i]_*) \to H_p(C'[i]_*) \to H_p\big(C''[i]_*\big) \to H_{p-1}(C[i]_*) \to \cdots .$$

The group $H_p\big(C''[i]_*\big)$ is a finite abelian group for each $p \geq 0$. We obtain the following commutative diagram with exact rows

(18.5)

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{tors}\big(H_p(C[i]_*)\big) & \to & H_p(C[i]_*) & \to & H_p(C[i]_*)_f & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{tors}\big(H_p(C'[i]_*)\big) & \to & H_p(C'[i]_*) & \to & H_p(C'[i]_*)_f & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{tors}\big(H_p(C''[i]_*)\big) & \xrightarrow{\cong} & H_p(C''[i]_*) & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{tors}\big(H_{p-1}(C[i]_*)\big) & \to & H_{p-1}(C[i]_*) & \to & H_{p-1}(C[i]_*)_f & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We view it as a short exact sequence of $\mathbb{Z}$-chain complexes and hence can consider the associated long homology sequence. Notice that the chain complex given by the middle column is acyclic. Hence we obtain isomorphisms

(18.6)

$$\ker\left(\operatorname{tors}\big(H_p(C'[i]_*)\big) \to \operatorname{tors}\big(H_p(C''[i]_*)\big)\right)/\operatorname{im}\left(\operatorname{tors}\big(H_p(C[i]_*)\big) \to \operatorname{tors}\big(H_p(C'[i]_*)\big)\right)$$
$$\cong \ker\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right),$$

(18.7)
$$\ker\big(\operatorname{tors}\big(H_p(C''[i]_*)\big) \to \operatorname{tors}\big(H_{p-1}(C[i]_*)\big)\big)/\operatorname{im}\big(\operatorname{tors}\big(H_p(C'[i]_*)\big) \to \operatorname{tors}\big(H_p(C''[i]_*)\big)\big) \cong \operatorname{coker}\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right),$$

and

(18.8)
$$\ker\big(\operatorname{tors}\big(H_p(C[i]_*)\big) \to \operatorname{tors}\big(H_p(C'[i]_*)\big)\big)/\operatorname{im}\big(\operatorname{tors}\big(H_{p+1}(C[i]''_*)\big) \to \operatorname{tors}\big(H_p(C[i]_*)\big)\big) \cong 0.$$

Obviously $(H_p(C[i]_*)_f$ and hence $\ker\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right)$ are torsionfree. On the other hand $\ker\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right)$ is finite, since $\operatorname{tors}(H_p(C'[i]_*))$ is finite and $\ker\left(H_p(C[i]_*) \to H_p(C'[i]_*)\right)$ is a quotient of $H_{p+1}(C''[i]_*)$ and hence finite. Hence $\ker\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right)$ is trivial. We conclude from (18.6)

$$(18.9)\quad \ker\big(\operatorname{tors}\big(H_p(C'[i]_*)\big) \to \operatorname{tors}\big(H_p(C''[i]_*)\big)\big) = \operatorname{im}\big(\operatorname{tors}\big(H_p(C[i]_*)\big) \to \operatorname{tors}\big(H_p(C'[i]_*)\big)\big).$$

The cokernel of the map $H_p(C[i]_*) \to H_p(C'[i]_*)$ is a submodule of $H_{p-1}(C''[i]_*)$ and hence annihilated by multiplication with m. The cokernel of $H_p(C[i]_*)_f \to (H_p(C'[i]_*)_f$ is a quotient of the cokernel of $H_p(C[i]_*) \to H_p(C'[i]_*)$. Hence $\operatorname{coker}\big(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\big)$ is annihilated by multiplication with m. Therefore we obtain an epimorphism

$$H_p(C'[i]_*)_f/m \cdot H_p(C'[i]_*)_f \to \operatorname{coker}\big(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\big).$$

This implies

$$\left|\operatorname{coker}\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right)\right| \quad\leq\quad m^{\operatorname{rk}_{\mathbb{Z}}\left(H_p(C'[i]_*)\right)}.$$

Since $C_*^{(2)}$ is L^2-acyclic, and $C_* \otimes_{\mathbb{Z}} \mathbb{Q}$ and $C'_* \otimes_{\mathbb{Z}} \mathbb{Q}$ are $\mathbb{Q}G$-chain homotopy equivalent, $(C')_*^{(2)}$ is L^2-acyclic. We conclude from [62, Theorem 0.1] for all $p \geq 0$

$$\lim_{i \in I} \frac{\operatorname{rk}_{\mathbb{Z}}\big(H_p(C'_*[i])\big)}{[G : G_i]} = 0.$$

Since m is independent of p, we conclude

$$(18.10)\qquad \lim_{i \in I} \frac{\ln\left(\left|\operatorname{coker}\left(H_p(C[i]_*)_f \to H_p(C'[i]_*)_f\right)\right|\right)}{[G : G_i]} = 0.$$

Taking the logarithm of the order of a finite abelian group is additive under short exact sequences of finite abelian groups. Hence we get for any finite-dimensional chain complex E_* of finite abelian groups

$$\sum_{p \geq 0} (-1)^p \cdot |E_p| = \sum_{p \geq 0} (-1)^p \cdot |H_p(E_*)|.$$

If we apply this to the left column in the diagram (18.5), we conclude from (18.7), (18.8), and (18.9)

$$\begin{aligned}
&\Bigg|\sum_{p\geq 0}(-1)^p\cdot\ln\left(\left|\operatorname{tors}\left(H_p(C[i]_*)\right)\right|\right)-\sum_{p\geq 0}(-1)^p\cdot\ln\left(\left|\operatorname{tors}\left(H_p(C'[i]_*)\right)\right|\right)\\
&\qquad+\sum_{p\geq 0}(-1)^p\cdot\ln\left(\left|\operatorname{tors}\left(H_p(C''[i]_*)\right|\right)\Bigg|\\
&\qquad\qquad= \left|\sum_{p\geq 0}(-1)^p\cdot\ln\left(\left|\operatorname{coker}\left(H_p(C[i]_*)_f\to H_p(C'[i]_*)_f\right)\right|\right)\right|\\
&\qquad\qquad\leq \sum_{p\geq 0}\ln\left(\left|\operatorname{coker}\left(H_p(C[i]_*)_f\to H_p(C'[i]_*)_f\right)\right|\right).
\end{aligned}$$

This together with (18.10) implies

$$\lim_{i\in I}\left(\frac{\rho^{\mathbb{Z}}(C[i]_*)}{[G:G_i]}-\frac{\rho^{\mathbb{Z}}(C'[i]_*)}{[G:G_i]}+\frac{\rho^{\mathbb{Z}}(C''[i]_*)}{[G:G_i]}\right) = 0. \tag{18.11}$$

We conclude from [66, Lemma 3.68 on page 153]

$$\rho^{(2)}\left(C_*^{(2)}\right)-\rho^{(2)}\left((C')_*^{(2)}\right)+\rho^{(2)}\left((C'')_*^{(2)}\right) = 0. \tag{18.12}$$

Hence it suffices to show

$$\rho^{(2)}\left((C'')_*^{(2)}\right) = \lim_{i\in I}\frac{\rho^{\mathbb{Z}}(C''[i]_*)}{[G:G_i]}. \tag{18.13}$$

We conclude from Corollary 14.13 (1)

$$\rho^{(2)}\left((C'')_*^{(2)}\right) = \lim_{i\in I}\frac{\rho^{(2)}\left(C''[i]_*^{(2)}\right)}{[G:G_i]}.$$

Since $H_p\left((C''[i])_*\right)\otimes_{\mathbb{Z}}\mathbb{Q}$ vanishes for all $p\geq 0$ and $i\in I$, (18.13) follows from Lemma 8.4. This finishes the proof of Lemma 18.4. □

Now we are ready to prove Theorem 7.13.

Proof of Theorem 7.13. (1) Notice that

$$\frac{\ln\left(\det_{\mathcal{N}(\{1\})}(f[i]^{(2)})\right)}{[G:G_i]} = \ln\left(\det_{\mathcal{N}(G/G_i)}(f[i]^{(2)})\right)$$

holds by [66, Theorem 3.14 (5) on page 128]. Theorem 14.11 implies

$$\ln\left(\det_{\mathcal{N}(G)}(f^{(2)})\right) \geq \limsup_{i\in I}\frac{\ln\left(\det_{\mathcal{N}(\{1\})}(f[i]^{(2)})\right)}{[G:G_i]}.$$

Now apply Lemma 18.3.

(2) Obviously it suffices to prove the claim for chain complexes. Notice that

$$\frac{\rho^{(2)}\left(C[i]_*^{(2)}\right)}{[G:G_i]} = \rho^{(2)}\left(C[i]_*^{(2)};\mathcal{N}(G/G_i)\right)$$

holds by [66, Theorem 3.35 (7) on page 143]. We conclude from Corollary 14.13 (1)

$$\rho^{(2)}\left(C_*^{(2)}\right) = \lim_{i\in I}\frac{\rho^{(2)}\left(C[i]_*^{(2)}\right)}{[G:G_i]}.$$

Since $H_p\left(C[i]_*\right)\otimes_{\mathbb{Z}}\mathbb{Q}$ vanishes for all $p\geq 0$ and $i\in I$, assertion (2) follows from Lemma 8.4

(3) Obviously it suffices to prove the chain complex version. Let C_* be a finite

based free $\mathbb{Z}[\mathbb{Z}]$-chain complex that is L^2-acyclic. If $\mathbb{Q}[\mathbb{Z}]_{(0)}$ is the quotient field of the integral domain $\mathbb{Q}[\mathbb{Z}]$, then $H_k(C_*) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Q}[\mathbb{Z}]_{(0)}$ is trivial for $k \geq 0$ because of [66, Lemma 1.34 (1) on page 35]. Since $\mathbb{Q}[\mathbb{Z}]$ is a principal ideal domain, we can find non-negative integers t_k and pairwise prime irreducible elements $p_{k,1}$, $p_{k,2}$, $\dots$, p_{k,t_k} in $\mathbb{Q}[\mathbb{Z}]$ and natural numbers $m_{k,1}$, $m_{k,2}$, $\dots m_{k,t_k}$ such that we have isomorphisms of $\mathbb{Q}[\mathbb{Z}]$-modules

$$H_k(C_*) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_k(C_*) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Q}[\mathbb{Z}] \cong \bigoplus_{j=1}^{t_k} \mathbb{Q}[\mathbb{Z}]/(p_{k,j}^{m_{k,j}}).$$

By multiplying the elements $p_{k,j}$ with some natural number and a power of the generator $t \in \mathbb{Z}$, we can arrange that the elements $p_{k,1}$, $p_{k,2}$, $\dots$, p_{k,t_k} belong to $\mathbb{Z}[t]$. Since $\mathbb{Q}[\mathbb{Z}]/(p_{k,j}^{m_{k,j}}) \cong \Big(\mathbb{Z}[\mathbb{Z}]/(p_{k,j}^{m_{k,j}})\Big) \otimes_{\mathbb{Z}} \mathbb{Q}$, there is a map of $\mathbb{Z}[\mathbb{Z}]$-modules

$$\xi_k \colon \bigoplus_{j=1}^{t_k} \mathbb{Z}[\mathbb{Z}]/(p_{k,j}^{m_{k,j}}) \to H_k(C_*)$$

which becomes an isomorphism of $\mathbb{Q}[\mathbb{Z}]$-modules after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$. By possibly enumerating the polynomials $p_{k,j}$ we can arrange, that for some integer s_k with $0 \leq s_k \leq t_k + 1$ a polynomial $p_{k,j}$ has some root of unity as a root if and only if $j \leq s_k$. Consider $j \in \{1, 2, \dots, s_k\}$. Let $d_{k,j} \geq 2$ be the natural number for which $p_{k,j}$ has a primitive $d_{k,j}$-th root of unity as zero. Recall the d-th cyclotomic polynomial Φ_d is a polynomial over $\mathbb{Z}[t]$ with $\Phi_{d_{k,j}}(0) = \pm 1$ and is irreducible over $\mathbb{Q}[t]$. Hence we can find a unit in $u \in \mathbb{Q}[\mathbb{Z}]$ such that $u \cdot \Phi_{d_{k,j}} = p_{k,j}$. Every unit in $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t, t^{-1}]$ is of the shape rt^l for some $r \in \mathbb{Q}, r \neq 0$ and $l \in \mathbb{Z}$. Since $p_{k,j}$ is a polynomial in $\mathbb{Z}[t]$, we can arrange $p_{k,j} = \Phi_{d_{k,j}}$. To summarize, we have achieved that $p_{k,j}$ is $\Phi_{d_{k,j}}$ for $j \in \{1, 2, \dots, s_k\}$ and that no root of unity is a root of $p_{k,j}$ for $j \in \{s_k + 1, s_k + 2, \dots, t_k\}$.

Let $F_*^{k,j}$ for $j \in \{1, 2, \dots, t_k\}$ be the $\mathbb{Z}[\mathbb{Z}]$-chain complex which is concentrated in dimensions $(k+1)$ and k and whose $(k+1)$-th differential is the $\mathbb{Z}[\mathbb{Z}]$-homomorphism $\mathbb{Z}[\mathbb{Z}] \xrightarrow{p_{k,j}} \mathbb{Z}[\mathbb{Z}]$ given by multiplication with $p_{k,j}$. There is an obvious identification of $\mathbb{Z}[\mathbb{Z}]$-modules

$$H_k\big(F_*^{k,j}\big) \cong \mathbb{Z}[\mathbb{Z}]/(p_{k,j})$$

and $H_i\big(F_*^{k,j}\big) = 0$ for $i \neq k$. Since $F_*^{k,j}$ has projective chain modules and is concentrated in dimensions $(k+1)$ and k and we have the exact sequence of $\mathbb{Z}[\mathbb{Z}]$-modules $C_{k+1} \xrightarrow{c_{k+1}} \ker(c_k) \to H_k(C_*)$, we can construct a $\mathbb{Z}[\mathbb{Z}|$-chain map

$$f_*^{k,j} \colon F_*^{k,j} \to C_*$$

such that $H_k\big(f_*^{k,j}\big)$ agrees with the restriction of ξ_k to the j-th summand. Define a $\mathbb{Z}[\mathbb{Z}]$-chain map

$$f_* := \bigoplus_{k \geq 0} \bigoplus_{j=1}^{t_k} f_*^{k,j} \colon \bigoplus_{k \geq 0} \bigoplus_{j=1}^{t_k} F_*^{k,j} \to C_*.$$

By construction $H_k(f_*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is bijective for all $k \geq 0$.

We conclude from Lemma 18.4 that we can assume without loss of generality

$$C_* = \bigoplus_{k \geq 0} \bigoplus_{j=1}^{t_k} F_*^{k,j}.$$

Obviously assertion (3) is satisfied for a direct sum $D_* \oplus E_*$ of two based free L^2-acyclic $\mathbb{Z}[\mathbb{Z}]$-chain complexes if both D_* and E_* satisfy assertion (3). Hence we only have to treat the case, where C_* is concentrated in dimension 0 and 1 and its first differential is given by $p \cdot \mathrm{id} \colon \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]$ for some non-trivial polynomial p

such that either p is of the shape ϕ_d^m for some natural numbers d and m or no root of unity is a root of p.

We begin with the case where p is of the shape ϕ_d^m for some natural numbers d and m. Then all roots of p have norm 1 and hence

$$\ln\bigl(\rho^{(2)}(C_*)\bigr) = \ln\bigl(\det_{\mathcal{N}(\mathbb{Z})}\bigl(p \cdot \mathrm{id}\colon L^2(\mathbb{Z}) \to L^2(\mathbb{Z})\bigr)\bigr) = 0$$

by [66, (3.23) on page 136]. Now the claim follows from assertion (1).

Finally we treat the case, where no root of unity is a root of p. Fix $i \in I$. Put $n = [\mathbb{Z} : \mathbb{Z}_i]$. Then $\mathbb{Z}/\mathbb{Z}_i = \mathbb{Z}/n$. For $l \in \mathbb{Z}/n$ let $\mathbb{C}_l$ be the unitary $\mathbb{Z}/n$-representation whose underlying Hilbert space is $\mathbb{C}$ and on which the generator in $\mathbb{Z}/n$ acts by multiplication with ζ_n^l, where we put $\zeta_n := \exp(2\pi i/n)$. We obtain a unitary $\mathbb{Z}/n$-isomorphism

$$\omega\colon \bigoplus_{l \in \mathbb{Z}/n} \mathbb{C}_l \xrightarrow{\cong} \mathbb{C}[\mathbb{Z}/n].$$

The following diagram of Hilbert $\mathcal{N}(\mathbb{Z}/n)$-modules commutes

$$\begin{array}{ccc} \bigoplus_{l\in\mathbb{Z}/n} \mathbb{C}_l & \xrightarrow[\cong]{\omega} & \mathbb{C}[\mathbb{Z}/n] \\ {\scriptstyle \bigoplus_{l\in\mathbb{Z}/n} p(\zeta_n^l)} \downarrow & & \downarrow {\scriptstyle p[i]} \\ \bigoplus_{l\in\mathbb{Z}/n} \mathbb{C}_l & \xrightarrow[\cong]{\omega} & \mathbb{C}[\mathbb{Z}/n] \end{array}$$

Hence $p[i]^{(2)}\colon \mathbb{Z}[\mathbb{Z}/n]^{(2)} \to \mathbb{Z}[\mathbb{Z}/n]^{(2)}$ is an isomorphism. Therefore $p[i]\colon \mathbb{Z}[\mathbb{Z}/n] \to \mathbb{Z}[\mathbb{Z}/n]$ is rationally an isomorphism. Now the claim follows from assertion (2). This finishes the proof of Theorem 7.13. □

19. Miscellaneous

We briefly mention some variations of the problems considered here or some other prominent open conjectures about L^2-invariants.

19.1. **Approximation for lattices.** In our setting we approximate the universal covering of a closed manifold or compact CW-complex by a tower of finite coverings corresponding to the normal chain $(G_i)_{i\geq 0}$ of normal subgroups of G with finite index and trivial intersection.

One can also look at a uniformly discrete sequence of lattices $(G_i)_{i\geq 0}$ in a connected center-free semisimple Lie group L without compact factors and study the quotients $M[i] = X/G_i$, where X is the associated symmetric space L/K for $K \subseteq L$ a maximal compact subgroup. There is a notion of BS-convergence for lattices which generalizes our setting. One can ask whether for such a convergence sequence of cocompact lattices the sequence $\frac{b_n(M[i];\mathbb{Q})}{\mathrm{vol}(M[i])}$ converges to the L^2-Betti number of X. This setup and various convergence questions are systematically examined in the papers by Abert-Bergeron-Biringer-Gelander-Nikolov-Raimbault-Samet [1, 2].

Another paper containing interesting information about these questions is Bergeron-Lipnowski [10].

19.2. **Twisting with representations.** We have already mentioned that one can twist the analytic torsion with special representations. This has in favorite situations the effect that one obtains a uniform gap for the spectrum of the Laplace operators and can prove the desired approximations results, see Remark 16.15. For more information we refer for instance to [12, 74, 78, 79].

In [70] twisted L^2-torsion for finite CW-complex X with $b_n^{(2)}(\widetilde{X}) = 0$ for all $n \geq 0$ will be introduced for finite-dimensional representations which are given by restricting finite-dimensional $\mathbb{Z}^d$-representations with any homomorphism $\pi_1(M) \to$

$\mathbb{Z}^d$. In particular one can twist the L^2-torsion for a given element $\phi \in H^1(X;\mathbb{Z})$ with the 1-dimensional representation whose underlying complex vector space is $\mathbb{C}$ and on which $g \in \pi_1(X)$ acts by multiplication with $t^{\phi(g)}$. This yields the L^2-torsion function $(0,\infty) \to \mathbb{R}$ whose value at 1 is the L^2-torsion itself. The proof that this function is well-defined is based on approximation techniques. This function seem to contain very interesting information, in particular for 3-manifolds, see for instance [27, 28, 29, 30]. In particular there is the conjecture that one can read off the Thurston norm of ϕ from the asymptotic behavior at 0 and ∞ if X is a connected compact orientable 3-manifold with infinite fundamental group and empty or toroidal boundary which is not $S^1 \times D^2$.

19.3. **Atiyah's Question.** Atiyah [6, page 72] asked the question, whether the L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ for a closed Riemannian manifold M are always rational numbers. Meanwhile it is known that the answer can be negative, see for instance [7, 42, 83]. However, the following problem, often referred to as the strong Atiyah Conjecture, remains open.

Question 19.1. *Let G be a group for which there exists natural number d such that the order of any finite subgroup divides d. Then:*

(1) *For any $A \in M_{m,n}(\mathbb{Z}G)$ we get for the von Neumann dimension of the kernel of the induced G-equivariant bounded operator $r_A^{(2)} \colon L^2(G)^m \to L^2(G)^n$*

$$d \cdot \dim_{\mathcal{N}(G)}\bigl(\ker(r_A^{(2)})\bigr) \in \mathbb{Z};$$

(2) *For every closed manifold M with $G \cong \pi_1(M)$ and $n \geq 0$ we have*

$$d \cdot b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

Notice that we can choose $d = 1$ if G is torsionfree. For a discussion, a survey on the literature and the status of this Question 19.1, we refer for instance to [66, Chapter 10].

The Approximation Conjecture 13.4, which is known by Remark 13.2 and Theorem 13.6 for a large class of groups, can be used to enlarge the class of groups for which the answer to part (1) of Question 19.1 is positive. Namely, if G is torsionfree and possesses a chain of normal subgroups $(G_i)_{i\geq 0}$ with trivial intersection $\bigcap_{i\geq 0} G_i = \{1\}$ such that the answer to part (1) of Question 19.1 is positive for each quotient G/G_i, then the answer to part (1) of Question 19.1 is positive for each quotient G/G_i. Here it becomes important that we could drop the condition that each G/G_i is finite. An example for G is a finitely generated free group whose descending central series gives such a chain $(G_i)_{i\geq 0}$ with torsionfree nilpotent quotients G/G_i.

Notice that Conjecture 10.1 implies a positive answer to part (2) of Question 19.1 if M is an aspherical closed manifold.

One can ask an analogous question in the mod p case as soon as one has a replacement for the L^2-Betti number in the mod p case. In some special cases this replacement exists and the answer is positive, see for instance Theorem 2.2 for torsionfree elementary amenable groups, and Theorem 2.3 for torsionfree G taking into account that is the nth mod p L^2-Betti numbers $b_n^{(2)}(\overline{X};F)$ occurring in [9, Definition 1.3] is an integer for torsionfree G.

19.4. **Simplicial volume.** The following conjecture is discussed in [66, Chapter 14].

Conjecture 19.2 (Simplicial volume and L^2-invariants)**.** *Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume*

$||M||$ *vanishes. Then*

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad \text{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

If the closed orientable manifold M has a selfmap $f\colon M \to M$ of degree different from -1, 0, 1, then one easily checks that its simplicial volume $||M||$ vanishes. If its minimal volume is zero, i.e., for every $\epsilon > 0$ one can find a Riemannian metric on M whose sectional curvature is pinched between -1 and 1 and for which the volume of M is less or equal to ϵ, then its simplicial volume $||M||$ vanishes. This follows from [45, page 37].

If one replaces in Conjecture 19.2 the simplicial volume by the minimal volume, whose vanishing implies the vanishing of the simplicial volume, then the claim for the L^2-Betti numbers in Conjecture 19.2 has been proved by Sauer [86, Second Corollary of Theorem A].

There are a versions of the simplicial volume such as the integral foliated simplicial volume and stable integral simplicial volume which are related to Conjecture 19.2 and may be helpful for a possible proof, and reflect a kind of approximation conjecture for the simplicial volume, see for instance [35, 60, 92].

More information about the simplicial volume and the literature can be found for instance in [45], [59], [66, Section 14.1].

19.5. **Entropy, Fuglede-Kadison determinants and amenable exhaustions.** In recent years the connection between entropy and Fuglede-Kadison determinant has been investigated in detail, see for instance [23, 25, 56, 57]. In particular the amenable exhaustion approximation result for Fuglede-Kadison determinants of Li-Thom [57, Theorem 0.7] for amenable groups G is very interesting, where the Fuglede-Kadison determinant of a matrix over $\mathbb{Z}G$ is approximated by finite-dimensional analogues of its "restrictions" to finite Fölner subsets of the group G.

19.6. **Lehmer's problem.** Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-trivial element. Its *Mahler measure* is defined by

$$M(p) \quad := \quad \exp\left(\int_{S^1} \ln(|p(z)|)d\mu\right). \tag{19.3}$$

By Jensen's inequality we have

$$\int_{S^1} \ln(|p(z)|)d\mu \quad = \quad \sum_{\substack{i=1,2,\ldots,r \\ |a_i|>1}} \ln(|a_i|), \tag{19.4}$$

if we write $p(z)$ as a product

$$p(z) = c \cdot z^k \cdot \prod_{i=1}^{r} (z - a_i)$$

for an integer $r \geq 0$, non-zero complex numbers c, a_1, $\ldots$, a_r and an integer k. This implies $M(p) \geq 1$.

Problem 19.5 (Lehmer's Problem)**.** *Does there exist a constant $\Lambda > 1$ such that for all non-trivial elements $p(z) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ with $M(p) \neq 1$ we have*

$$M(p) \geq \Lambda?$$

Remark 19.6 (Lehmer's polynomial)**.** There is even a candidate for which the minimal Mahler measure is attained, namely, *Lehmer's polynomial*

$$L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

It is conceivable that for any non-trivial element $p \in \mathbb{Z}[\mathbb{Z}]$ with $M(p) \neq 1$

$$M(p) \geq M(L) = 1.17628\ldots$$

holds.

For a survey on Lehmer's problem were refer for instance to [13, 14, 19, 95].

Consider an element $p = p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$. It defines a bounded $\mathbb{Z}$-operator $r_p^{(2)} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ by multiplication with p. Suppose that p is not zero. Then the Fuglede-Kadison determinant of $r_p^{(2)}$ agrees with the Mahler measure of p by [66, (3.23) on page 136].

Definition 19.7 (Lehmer's constant of a group)**.** Define *Lehmer's constant* $\Lambda(G)$ of a group G

$$\Lambda(G) \in [1, \infty)$$

to be the infimum of the set of Fuglede-Kadison determinants

$$\det{}^{(2)}_{\mathcal{N}(G)}\big(r_A^{(2)} : \mathcal{N}(G)^r \to \mathcal{N}(G)^s\big),$$

where A runs through all (r, s)-matrices $A \in M_{r,s}(\mathbb{Z}G)$ for all $r, s \in \mathbb{Z}$ with $r, s \geq 1$ for which $\det^{(2)}_{\mathcal{N}(G)}(r_A^{(2)}) > 1$ holds.

If we only allow square matrices A such that $r_A^{(2)} : \mathcal{N}(G)^r \to \mathcal{N}(G)^r$ is injective and $\det^{(2)}_{\mathcal{N}(G)}(r_A^{(2)}) > 1$, then we denote the corresponding infimum by

$$\Lambda^w(G) \in [1, \infty)$$

Obviously we have $\Lambda(G) \leq \Lambda^w(G)$. We suggest the following generalization of Lehmer's problem to arbitrary groups.

Problem 19.8 (Lehmer's problem for arbitrary groups)**.** *For which groups G is $\Lambda(G) > 1$ or $\Lambda^w(G) > 1$?*

For a discussion and results on this problems see [22, Question 4.7] and [69].

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