

ON THE SPECTRAL DENSITY FUNCTION OF THE LAPLACIAN OF A GRAPH

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ABSTRACT. Let X be a finite graph. Let E be the number of its edges and d be its degree. Denote by $F_1(X)$ its first spectral density function which counts the number of eigenvalues $\leq \lambda^2$ of the associated Laplace operator. We prove the estimate $F_1(X)(\lambda) - F_1(X)(0) \leq 2 \cdot E \cdot d \cdot \lambda$ for $0 \leq \lambda < 1$. We explain how this gives evidence for conjectures about approximating Fuglede-Kadison determinants and L^2 -torsion.

1. INTRODUCTION

Our main result is

Theorem 1.1 (Estimate on the first spectral density function). *Let X be a finite graph. Let $|E(X)|$ be the number of its edges and $\deg(X)$ be its degree. Denote by $F_1(X)$ its first spectral density function.*

Then we obtain for $0 \leq \lambda < 1$

$$F_1(X)(\lambda) - F_1(X)(0) \leq 2 \cdot |E(X)| \cdot \deg(X) \cdot \lambda.$$

The notion of degree and spectral density function will be recalled in Section 2, where also the proof of Theorem 1.1 is presented.

Our main motivation is the following conjecture. For information about torsion and L^2 -torsion we refer to [6, Chapter 3]. We will explain in Section 4 why Theorem 1.1 gives some evidence for it.

Conjecture 1.2 (Approximation Conjecture for analytic L^2 -torsion).

Let G be a group together with a sequence of in G normal subgroups with finite index $[G : G_i]$

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$$

such that $\bigcap_{i \geq 0} G_i = \{1\}$. Let M be a closed Riemannian manifold and let $\overline{M} \rightarrow M$ be a G -covering. Equip \overline{M} and $M_i := \overline{M}/G_i$ with the Riemannian metrics for which the projections $\overline{M} \rightarrow M$ and $M_i \rightarrow M$ are local isometries. Denote by $\rho_{\text{an}}^{(2)}(\overline{M}; \mathcal{N}(G))$ the analytic L^2 -torsion with respect to the cocompact proper free isometric G -action on \overline{M} and denote by $\rho_{\text{an}}(M[i])$ the (analytic) Ray-Singer torsion of M_i .

Then

$$\rho_{\text{an}}^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \rightarrow \infty} \frac{\rho_{\text{an}}^{(2)}(M[i])}{[G : G_i]}.$$

Remark 1.3 (Conjecture by Bergeron-Venkatesh). Conjecture 1.2 is related to a conjecture by Bergeron-Venkatesh [1, Conjecture 1.3] which expresses L^2 -torsion in terms of the growth of the torsion of the singular homology of M_i in dimension n if M is a locally symmetric space of dimension $\dim(M) = 2n + 1$. Their conjecture

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makes also sense if one requires only that M is an aspherical closed $(2n + 1)$ -dimensional manifold.

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2. ESTIMATE ON THE FIRST SPECTRAL DENSITY FUNCTION OF A FINITE GRAPH

2.1. The spectral density function. Let X be a finite directed graph with set of vertices $V = V(X)$ and of edges $E = E(X)$, where each edge e has an initial vertex $v_0(e)$ and a terminal vertex $v_1(e)$. Given a vertex $v \in V$, we denote by d_v its *degree*, i.e., the sum of the number of edges which have precisely one endpoint equal to v and two times the number of edges whose two endpoints agree with v . The *degree* $\deg(X)$ of X is the maximum of the set $\{d_v \mid v \in V\}$. Define the *volume* of X to be

$$(2.1) \quad \text{vol}(X) = \sum_{v \in V} d_v.$$

The elementary so called *handshaking lemma* says

$$(2.2) \quad \text{vol}(X) = 2 \cdot |E(X)|.$$

We equip the graph with the *path metric*, i.e., the distance of two points is the infimum over the length of all piecewise linear paths joining these points, where the length of a piecewise linear path is defined in the obvious way such that every edge has length one. The *diameter* $\text{diam}(X)$ is the maximum of the distances of any two vertices. Obviously we have

$$(2.3) \quad \text{diam}(X) \leq |E(X)|.$$

We define $C_1(X) = l^2(E(X))$ and $C_0(X) = l^2(V(X))$ to be the vector space of sequences of real numbers indexed by the elements in $E(X)$ and $V(X)$, equipped with the standard Euclidean inner product. We obtain a 1-dimensional chain complex whose first differential is

$$c_1: C_1(X) \rightarrow C_0(X), \quad (a_e)_{e \in E(X)} \mapsto \left(\sum_{e: v_1(e)=v} a_e - \sum_{e: v_0(e)=v} a_e \right)_{v \in V(X)}.$$

We denote the *adjoint* of c_1 by $c_1^*: C_0(X) \rightarrow C_1(X)$. The *zeroth Laplace operator of X*

$$\Delta_0: C_0(X) \rightarrow C_0(X)$$

is given by $c_1 \circ c_1^*$.

The *spectral density function* $F(f)$ of a map $f: V \rightarrow W$ of finite-dimensional Hilbert spaces is the function

$$F(f): [0, \infty) \rightarrow [0, \infty)$$

which sends $\lambda \in [0, \infty)$ to the supremum of the dimensions of all subvector spaces $V_0 \subseteq V$ for which $|f(v)| \leq \lambda \cdot |v|$ holds for all $v \in V_0$. The *first spectral density function* of X is

$$F_1(X) := F(c_1: C_1(X) \rightarrow C_0(X)).$$

We will be interested in $F_1(X)(\lambda) - F_1(X)(0)$.

The dimension of the kernel of c_1 is called the *first Betti number* $b_1(X)$, and the dimension of the kernel of c_1^* is called the *zeroth Betti number* $b_0(X)$, which is also the number of components of X .

2.2. Singular values. The *singular value decomposition* states that, given a $n \times m$ matrix A , there exist orthogonal matrices U and V and a sequence of nonnegative numbers σ_i called singular values so that

$$(2.4) \quad A = U^T \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots \\ \vdots & \ddots & & \vdots \end{pmatrix} V.$$

The number σ_i are uniquely determined up to their order. They are called *singular values*. Equivalently, if $f: V \rightarrow W$ is a linear map between finite dimensional Euclidean vector spaces, then there exist an orthonormal basis in V and an orthonormal basis in W so that the matrix representing f in that basis is the diagonal matrix appearing in (2.4).

The nonzero eigenvalues of $A^T A$ and AA^T coincide. They are the squares of the nonzero singular values. The matrices A and A^T have the same nonzero singular values. The number of non-zero singular values of f less or equal to λ is $F(f)(\lambda) - F(f)(0)$.

We collect some of the elementary statements discussed above.

Lemma 2.5. *We get for $\lambda \geq 0$:*

- (1) $F_1(X)(\lambda) - F_1(X)(0) = F(c_1^*)(\lambda) - F(c_1^*)(0)$;
- (2) $F(c_1^*)(\lambda) = F(\Delta_0)(\lambda^2)$ is the sum of all eigenvalues of Δ_0 counted with multiplicity which are less or equal to λ^2 ;
- (3) We have

$$b_0(X) := \dim(\ker(c_1^*)) = F(c_1^*)(0) = F(\Delta_0)(0) = \dim(\operatorname{coker}(c_1)),$$

and

$$b_1(X) := \dim(\ker(c_1)) = F_1(X)(0).$$

See [6, Lemma 2.4 on page 74, Lemma 2.11 (11) on page 77, Example 2.5 on page 75, Lemma 1.18 on page 24] for analogous statements to Lemma 2.5.

Observe that the Laplace operator and $F_1(X)$ do not depend on the orientation of the edges.

2.3. One-dimensional CW complexes. A finite one-dimensional CW complex is the same as a finite graph. One-cells correspond to edges, and zero-cells to vertices. A finite one-dimensional CW complex with an orientation for each cell is the same as a directed graph. The cellular chain complex with real coefficients agrees with the chain complex $C_*(X)$ of the graph introduced above. The Betti numbers above are the Betti numbers of the CW complex defined as the dimensions of its homology groups.

2.4. Formulation of the main theorem. We want to show

Theorem 2.6 (Estimate on the first spectral density function). *Let X be a finite connected graph.*

- (1) *Suppose that $\deg(X) \geq 2$. Then we get*

$$F_1(X)(\lambda) - F_1(X)(0) \leq \begin{cases} 0 & \text{if } \lambda < \frac{1}{\sqrt{2} \cdot |E(X)|}; \\ 2 \cdot |E(X)| \cdot \deg(X) \cdot \lambda & \text{if } \frac{1}{2 \cdot (|E(X)| - 1) \cdot \deg(X)} \leq \lambda < 1; \end{cases}$$

- (2) *If $\deg(X) \leq 1$, then X is a point or the unit interval and $F_1(X)(\lambda) - F_1(X)(0) = 0$ for $0 \leq \lambda \leq 1$;*

(3) We obtain for $0 \leq \lambda < 1$

$$F_1(X)(\lambda) - F_1(X)(0) \leq 2 \cdot |E(X)| \cdot \deg(X) \cdot \lambda.$$

For connected graphs $F_1(X)(0) = 1$ and the statement could be simplified. We will take that into account in the sequel, but we still prefer to state the main result in a formulation motivated by the corresponding vector valued problem. This context will be discussed in the next section.

2.5. Proof of the main Theorem. For the proof of Theorem 2.6 we need the following three results. The first one is taken from [2, Lemma 1.9].

Theorem 2.7 (Estimate on the first non-trivial eigenvalue). *Let λ_1 be the smallest non-zero eigenvalue of the zero-th Laplace operator Δ_0 of X . Then*

$$\lambda_1 \geq \frac{1}{\text{diam}(X) \cdot \text{vol}(X)}.$$

Lemma 2.8. *Let X be a finite graph and $Y \subseteq X$ be a subgraph obtained from X by deleting some edges. Then we get for $0 \leq \lambda$*

$$F(c_1(X)^*)(\lambda) \leq F(c_1(Y)^*)(\lambda).$$

Proof. Let $i: C_1(Y) \rightarrow C_1(X)$ be the inclusion. It induces an isometric embedding $i_*: C_1(Y) \rightarrow C_1(X)$. Since $c_1(Y) = c_1(X) \circ i_*$, the claim follows directly from the definitions. \square

Lemma 2.9. *Let T be a finite tree. Let P be a real number with $P \leq (|E(T)| - 1) \cdot \deg(T)$. Then we can remove an appropriate collection of edges such that the resulting forrest is a disjoint union $\coprod_{i=1}^k T_i$ of trees T_1, T_2, \dots, T_k with the property that $1 \leq k \leq \frac{|E(T)| \cdot \deg(T)}{P} + 1$ and $E(T_i) \leq P$ for $i = 1, 2, \dots, k$ holds.*

Proof. Next we describe the following construction on T . Choose a leaf v , i.e., a vertex v with $\deg(v) = 1$. Such a leaf always exists in a finite tree. Since $|E(T)| - 1 \geq \frac{P}{\deg(T)}$, there exists an edge e with the property (P) that after removing e the tree splits into to trees T' and T'' such that T' contains at least $\frac{P}{\deg(T)}$ edges and T'' contains v . Namely, take for instance for e the edge which has v as vertex. Now choose an edge e which has property (P) and which has among all edges with property (P) maximal distance from v .

We claim that T' contains at most P edges. Let v' be the vertex in T' which belongs to e . Let e_1, e_2, \dots, e_l be the edges in T' which have v' as vertex. Since in T the degree of v' is bounded by $\deg(T)$, we have $l \leq \deg(T) - 1$. If we remove e_i from T , we obtain a disjoint union of trees $T'_i \amalg T''_i$ such that T''_i contains v . The distance in T of e_i from v is larger than the distance in T of e from v . Hence the tree T'_i contains at most $\frac{P}{\deg(T)} - 1$ edges. Every edge of T' is an edge of T'_i for some $i \in \{1, 2, \dots, l\}$ or belongs to $\{e_1, e_2, \dots, e_l\}$. Hence

$$\begin{aligned} |E(T')| &\leq l + \sum_{i=1}^l |E(T'_i)| \leq l + \sum_{i=1}^l \left(\frac{P}{\deg(T)} - 1 \right) \\ &= l + l \cdot \left(\frac{P}{\deg(T)} - 1 \right) = l \cdot \frac{P}{\deg(T)} \leq \frac{P \cdot (\deg(T) - 1)}{\deg(T)} < P. \end{aligned}$$

Now put $T_1 := T'$. Recall that T_1 has at least $\frac{P}{\deg(T)}$ and at most P edges. If $P > (|E(T'')| - 1) \cdot \deg(T)$, we put $T_2 = T''$ and $k = 2$. Obviously T_2 has at most P edges. If $P \leq (|E(T'')| - 1) \cdot \deg(T)$, we apply the procedure to T'' and the resulting tree is a disjoint union of a tree T_2 and a tree T''' such that T_2 has at least $\frac{P}{\deg(T)}$ and at most P edges. Either T''' becomes T_3 and the procedure stops,

or we apply the construction to T''' . This procedure has to stop after finitely many steps, since T contains only finitely many edges. Thus we have removed edges from T such that the resulting tree is a disjoint union $\coprod_{i=1}^k T_i$ of trees T_1, T_2, \dots, T_k with the property that $|E(T_i)| \leq P$ for $i = 1, 2, \dots, k$ and $\frac{P}{\deg(T)} \leq |E(T_i)|$ holds for $i = 1, 2, \dots, (k-1)$. The latter inequality implies $1 \leq k \leq \frac{|E(T)| \cdot \deg(T)}{P} + 1$ since the sum of the number of edges in T_i for $i = 1, 2, \dots, (k-1)$ is bounded by $|E(T)|$. \square

Now we are ready to prove Theorem 2.6.

Proof of Theorem 2.6.

(2) This follows from a direct elementary calculation.

(1) Let $T \subseteq X$ be a maximal tree. Then $\deg(T) \leq \deg(X)$, $\text{diam}(T) \leq \text{diam}(X)$, $|E(T)| \leq |E(X)|$, $|V(T)| = |V(X)|$, and $b_0(X) = b_0(T) = 1$. We have

$$\begin{aligned} \frac{1}{\sqrt{2} \cdot |E(X)|} &\leq \frac{1}{\sqrt{2} \cdot |E(T)|}; \\ \frac{1}{2 \cdot (|E(X)| - 1) \cdot \deg(X)} &\leq \frac{1}{\sqrt{2} \cdot |E(T)|}; \\ \frac{1}{2 \cdot (|E(T)| - 1) \cdot \deg(T)} &\leq \frac{1}{\sqrt{2} \cdot |E(T)|} \quad \text{if } |E(T)| \geq 2; \\ 2 \cdot |E(T)| \cdot \deg(T) \cdot \lambda &\leq 2 \cdot |E(X)| \cdot \deg(X) \cdot \lambda. \end{aligned}$$

We conclude from Lemma 2.5 and Lemma 2.8

$$\begin{aligned} F_1(X)(\lambda) - F_1(X)(0) &= F(c_1(X)^*)(\lambda) - F((c_1(X)^*)(0)) \\ &= F(c_1(X)^*)(\lambda) - b_0(X) \\ &\leq F(c_1(T)^*)(\lambda) - b_0(X) \\ &= F(c_1(T)^*)(\lambda) - b_0(T) \\ &= F(c_1(T)^*)(\lambda) - F(c_1(T)^*)(0) \\ &= F_1(T)(\lambda) - F_1(T)(0). \end{aligned}$$

The inequalities above and assertion (2) show that it suffices to prove the claim in the special case $X = T$ and $\deg(T) \geq 2$. Notice that $\deg(T) \geq 2$ implies $|E(T)| \geq 2$.

Suppose that $0 \leq \lambda < \frac{1}{\sqrt{2} \cdot |E(T)|}$. Let $\lambda_1(T)$ be the smallest non-zero eigenvalue of Δ_0 on T . Since $\text{vol}(T) = 2 \cdot |E(T)|$ by (2.2) and $\text{diam}(T) \leq |E(T)|$ by (2.3), we conclude from Theorem 2.7.

$$0 \leq \lambda^2 < \frac{1}{2 \cdot |E(T)|^2} \leq \frac{1}{\text{diam}(T) \cdot \text{vol}(T)} \leq \lambda_1(T),$$

and hence

$$F(\Delta_0)(\lambda^2) - F(\Delta_0)(0) = 0.$$

We conclude from Lemma 2.5

$$F_1(T)(\lambda) - F_1(T)(0) = 0.$$

Now suppose $\frac{1}{2 \cdot (|E(T)| - 1) \cdot \deg(T)} \leq \lambda < 1$. Put

$$P := \frac{1}{2 \cdot \lambda}.$$

Then $P \leq (|E(T)| - 1) \cdot \deg(T)$. Hence we can apply Lemma 2.9. So we can remove edges from T such that the resulting tree is a disjoint union $\coprod_{i=1}^k T_i$ of trees T_1, T_2, \dots, T_k with the property that $1 \leq k \leq \frac{|E(T)| \cdot \deg(T)}{P} + 1$ and $|E(T_i)| \leq P$ for $i = 1, 2, \dots, k$ holds. Let $\lambda_1(T_i)$ be the smallest non-zero eigenvalue of $\Delta_0(T_i)$ on

T_i for $i = 1, 2, \dots, k$. Since $\text{vol}(T_i) = 2 \cdot |E(T_i)|$ by (2.2) and $\text{diam}(T_i) \leq |E(T_i)|$ by (2.3), we conclude from Theorem 2.7 for $i = 1, 2, \dots, k$

$$\lambda_1(T_i) \geq \frac{1}{\text{diam}(T_i) \cdot \text{vol}(T_i)} \geq \frac{1}{2 \cdot |E(T_i)|^2} \geq \frac{1}{2 \cdot P^2} > \frac{1}{4 \cdot P^2} = \lambda^2.$$

Hence $F(\Delta_0(T_i))(\lambda^2) - F(\Delta_0(T_i))(0) = 0$. Lemma 2.5 implies

$$F(c_1(T_i)^*)(\lambda) - F(c_1(T_i)^*)(0) = F(\Delta_0(T_i))(\lambda^2) - F(\Delta_0(T_i))(0) = 0.$$

This together with Lemma 2.5 and Lemma 2.8 implies

$$\begin{aligned} F_1(T)(\lambda) &= F_1(T)(\lambda) - b_1(T) \\ &= F_1(T)(\lambda) - F_1(T)(0) \\ &= F(c_1(T)^*)(\lambda) - F(c_1(T)^*)(0) \\ &\leq F(c_1(\coprod_{i=1}^k T_i)^*)(\lambda) - F(c_1(T)^*)(0) \\ &= \left(\sum_{i=1}^k F(c_1(T_i)^*)(\lambda) \right) - F(c_1(T)^*)(0) \\ &= \left(\sum_{i=1}^k (F(c_1(T_i)^*)(\lambda) - F(c_1(T_i)^*)(0)) \right) \\ &\quad + \left(\sum_{i=1}^k F(c_1(T_i)^*)(0) \right) - F(c_1(T)^*)(0) \\ &= \left(\sum_{i=1}^k F(c_1(T_i)^*)(0) \right) - F(c_1(T)^*)(0) \\ &= \left(\sum_{i=1}^k b_0(T_i) \right) - b_0(T) \\ &= k - 1 \\ &\leq \frac{|E(T)| \cdot \deg(T)}{P} \\ &= 2 \cdot |E(T)| \cdot \deg(T) \cdot \lambda. \end{aligned}$$

This finishes the proof of assertion (1).

(3) This follows from assertions (1) and (2) since $\frac{1}{2 \cdot (|E(X)| - 1) \cdot \deg(X)} < \frac{1}{\sqrt{2} \cdot |E(X)|}$ holds for $\deg(X) \geq 2$. \square

Remark 2.10 (General strategy). Roughly speaking, our method of proof deduces from the estimate for the lowest non-zero eigenvalue (see Lemma 2.7) an estimate about the spectral density function for small λ , i.e., an estimate for the distribution of the small eigenvalues, see Theorem 1.1. This strategy has already been carried out in the paper by Grigor'yan-Yau [3], where the distribution of the small eigenvalues of an elliptic operators on a manifold is studied, provided that one has an estimate on the first non-zero eigenvalue.

Probably one could obtain Theorem 1.1 by modifying the arguments of [3] for graphs. Our proof, however, is independent, short, elementary (due to the more elementary situation), and it yields explicit constants which will be important for the applications in Section 3.

3. APPROXIMATING THE KADISON-FUGLEDE DETERMINANT FOR FINITE GRAPHS

Let G be a group together with a sequence of in G normal subgroups with finite index $[G : G_i]$

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$$

such that $\bigcap_{i \geq 0} G_i = \{1\}$. Let $p: \overline{X} \rightarrow X$ be a G -covering of the finite graph X such that \overline{X} is connected. Put $X_i := \overline{X}/G_i$. Then the projection $X_i \rightarrow X$ is a $[G : G_i]$ -sheeted regular covering of connected finite graphs.

Consider a linear map $f: V \rightarrow W$ of finite-dimensional Hilbert spaces. Let $\det^{(2)}(f)$ be its *Fuglede-Kadison determinant* with respect to the trivial group in the sense of [6, Definition 3.11 on page 127]. Notice that we do not require f to be injective or surjective. If $\lambda_1, \lambda_2, \dots, \lambda_r$ are the non-zero singular values of f , then

$$\det^{(2)}(f) = \prod_{i=1}^r \lambda_i.$$

With our convention the Fuglede-Kadison determinant of the zero map $0: V \rightarrow W$ is always 1. If f is injective, then $\det^{(2)}(f)$ is the square root of the classical determinant of the positive automorphism $f^*f: V \rightarrow V$. Notice that $\det^{(2)}(f)$ is a real number greater than zero and hence $\ln(\det^{(2)}(f))$ is a well-defined real number.

Consider the differential $c_1(\overline{X}): C_1(\overline{X}) \rightarrow C_0(\overline{X})$ in the cellular $\mathbb{Z}G$ -chain complex of the free cocompact G -CW-complex \overline{X} . Its $\mathbb{Z}G$ -chain modules are finitely generated free. Let $c_1^{(2)}(\overline{X}): C_1^{(2)}(\overline{X}) \rightarrow C_0^{(2)}(\overline{X})$ be the associated morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules, where $\mathcal{N}(G)$ is the group von Neumann algebra. Let $\det^{(2)}(c_1^{(2)}(\overline{X}); \mathcal{N}(G))$ be the associated Fuglede-Kadison determinant in the sense of [6, Definition 3.11 on page 127]. Then

Theorem 3.1 (Determinant approximation for graphs). *We have under the conditions above*

$$\ln \left(\det^{(2)}(c_1^{(2)}(\overline{X}); \mathcal{N}(G)) \right) = \lim_{i \rightarrow \infty} \frac{\ln(\det^{(2)}(c_1^{(2)}(X_i)))}{[G : G_i]}.$$

Proof. Put $C := 2 \cdot |E(X)| \cdot \deg(X)$. Since $X_i \rightarrow X$ is a regular $[G : G_i]$ -sheeted covering, we get

$$\begin{aligned} \deg(X_i) &= \deg(X); \\ \frac{|E(X_i)|}{[G : G_i]} &= |E(X)|. \end{aligned}$$

Theorem 1.1 implies for $\lambda \in [0, 1)$

$$(3.2) \quad \frac{F_1(X_i)(\lambda) - F_1(X_i)(0)}{[G : G_i]} \leq C \cdot \lambda.$$

Notice that C is independent of i . This implies

$$(3.3) \quad \int_0^1 \sup \left\{ \frac{F_1(X_i)(\lambda) - F_1(X_i)(0)}{[G : G_i] \cdot \lambda} \mid i = 1, 2, \dots \right\} d\lambda < \infty.$$

Now Theorem 3.1 follows from a general strategy which will be described in details in [7] and is a consequence of the material in [6, Subsection 13.2.1]. The basic idea

is, roughly speaking, that

$$\begin{aligned} \frac{\ln(\det^{(2)}(c_1^{(2)}(X_i)))}{[G : G_i]} &= - \int_0^K \frac{F_1(X_i)(\lambda) - F_1(X_i)(0)}{[G : G_i] \cdot \lambda} d\lambda \\ &\quad + \ln(K) \cdot (F_1(X_i)(K) - F_1(X_i)(0)); \\ \ln(\det^{(2)}(c_1^{(2)}(\overline{X}); \mathcal{N}(G))) &= - \int_0^K \frac{F_1(\overline{X})(\lambda) - F_1(\overline{X})(0)}{\lambda} d\lambda \\ &\quad + \ln(K) \cdot (F_1(\overline{X})(K) - F_1(\overline{X})(0)); \\ F_1(\overline{X})(K) - F_1(\overline{X})(0) &= \lim_{i \rightarrow \infty} (F_1(X_i)(K) - F_1(X_i)(0)), \end{aligned}$$

holds for the L^2 -spectral density function $F_1(\overline{X})(\lambda)$ of the cocompact free G -CW-complex \overline{X} and an large enough real number K , and one has almost everywhere

$$\frac{F_1(\overline{X})(\lambda) - F_1(\overline{X})(0)}{\lambda} = \lim_{i \rightarrow \infty} \frac{F_1(X_i)(\lambda) - F_1(X_i)(0)}{[G : G_i] \cdot \lambda},$$

so that an application of the Lebesgue's Dominated Convergence Theorem finishes the proof, provided that (3.3) holds. \square

Remark 3.4 (First Novikov-Shubin invariant). Another consequence of (3.2) is that the first Novikov-Shubin invariant of \overline{X} with respect to $\mathcal{N}(G)$ is greater or equal to 1. This has already been proved in general, actually, one does know the value in terms of G , see for instance [6, Theorem 2.55 (5) on page 98].

4. HIGHER DIMENSIONS

Conjecture 4.1 (Higher dimensions). *For every finite CW-complex X and $p \geq 1$, there exists positive constants C , ϵ and α such that the p -th spectral density function $F_p(X_i) = F(c_p(X_i))$ of X_i satisfies for all $\lambda \in [0, \epsilon]$ and all $i = 1, 2, \dots$*

$$\frac{F_p(X_i)(\lambda) - F_p(X_i)(0)}{[G : G_i]} \leq C \cdot \lambda^\alpha.$$

This conjecture seems to be hard but is very interesting. It would imply for instance the conjecture that all Novikov-Shubin invariant of a G -covering $\overline{M} \rightarrow M$ of a closed smooth Riemannian manifold M are positive, see [5, Conjecture 7.2] provided that G is residually finite.

Conjecture 4.1 is equivalent to the one described in the following Remark 4.2.

Remark 4.2 (Matrix formulation). Consider a matrix $A \in M_{m,n}(\mathbb{Z}G)$. Let $A[i] \in M_{m,n}(\mathbb{Z}[G/G_i])$ be its image under the canonical projection. Let $r_A : l^2(G)^m \rightarrow l^2(G)^n$ and $r_{A[i]} : l^2(G/G_i)^m \rightarrow l^2(G/G_i)^n$ be the induced operators given by right multiplication with A and A_i . Then there is the conjecture that there exists constants $C, \alpha, \epsilon > 0$ such that for all $i = 1, 2, \dots$ and $\lambda \in [0, \epsilon]$

$$\frac{F(r_{A[i]})(\lambda) - F(r_{A[i]})(0)}{[G : G_i]} \leq C \cdot \lambda^\alpha.$$

Remark 4.3 (Approximating Fuglede-Kadison determinants). The conjecture appearing in Remark 4.2 implies by the strategy mentioned in the proof of Theorem 3.1

$$\ln(\det^{(2)}(r_A); \mathcal{N}(G)) = \lim_{i \rightarrow \infty} \frac{\ln(\det^{(2)}(r_{A[i]}))}{[G : G_i]}.$$

The last equation has been proved for $G = \mathbb{Z}$ by Schmidt [8], see also [6, Theorem 15.53 on page 478] and hence holds for all virtually cyclic groups G . To the author's knowledge it seems to be unknown for all infinite groups which are not virtually cyclic. For more information we refer to the discussion in [4, Section 3.1].

Remark 4.4 (Approximating L^2 -torsion). If one can prove the equivalent claims in Conjecture 4.1 or Remark 4.2, then Conjecture 1.2 is true, provided that all L^2 -Betti number of \overline{M} vanish, see [7].

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