On the K- and L-theory of hyperbolic and virtually finitely generated abelian groups

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May 2012

Abstract

We investigate the algebraic K- and L-theory of the group ring RG, where G is a hyperbolic or virtually finitely generated abelian group and R is an associative ring with unit.

Key words: Algebraic K- and L-theory, hyperbolic groups, virtually finitely generated abelian groups.
Mathematics Subject Classification 2000: 19D99, 19G24, 19A31, 19B28.

Introduction

The goal of this paper is to compute the algebraic K- and L-groups of group rings RG, where G is a hyperbolic group or a virtually finitely generated abelian group and R is an associative ring with unit (and involution).

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The Farrell-Jones Conjecture for a group G and a ring R predicts that the assembly maps

$$\begin{aligned} H_n^G\big(\underline{\underline{E}}G;\mathbf{K}_R\big) &\xrightarrow{\cong} H_n^G\big(G/G;\mathbf{K}_R\big) = K_n(RG); \\ H_n^G\big(\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}\big) &\xrightarrow{\cong} H_n^G\big(G/G;\mathbf{L}_R^{\langle -\infty\rangle}\big) = L_n^{\langle -\infty\rangle}(RG), \end{aligned}$$

induced by the projection $\underline{\underline{E}}G \to G/G$, are bijective for every integer n. This conjecture, introduced by Farrell and Jones in their groundbreaking paper [21], has many consequences. Knowing that it is true for a given group implies several other well-known conjectures for that group, such as the ones due to Bass, Borel, Kadison and Novikov. The Farrell-Jones Conjecture also helps to calculate the K- and L-theory of group rings, since homology groups are equipped with tools such as spectral sequences that can simplify computations.

Recently it has been established that the Farrell-Jones Conjecture is true for word hyperbolic groups and virtually finitely generated abelian groups. Using this fact, we are able to compute the K- and L-theory of their group rings in several cases by analyzing the left-hand side of the assembly map. The key ingredients used to compute these groups are the *induction structure* that equivariant homology theories possess [26, Section 1] and the work of Lück-Weiermann [31], which investigates when a universal space for a given group G and a given family of subgroups $\mathcal{F}$ can be constructed from a universal space for G and a smaller family $\mathcal{F}' \subseteq \mathcal{F}$.

Even in basic situations determining the K- and L-groups is difficult. However, we are able to handle hyperbolic groups. The favorite situation is when the hyperbolic group G is torsion-free and $R = \mathbb{Z}$, in which case $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$, the reduced projective class group $\widetilde{K}_0(\mathbb{Z}G)$ and the Whitehead group $\mathrm{Wh}(G)$ vanish, and $K_n(\mathbb{Z}G)$ is computed by $H_n(BG;\mathbf{K}(\mathbb{Z}))$, i.e., the homology with coefficients in the K-theory spectrum $\mathbf{K}(\mathbb{Z})$ of the classifying space BG. Moreover, the L-theory $L_n^{\langle i\rangle}(\mathbb{Z}G)$ is independent of the decoration $\langle i\rangle$ and is given by $H_n(BG;\mathbf{L}(\mathbb{Z}))$, where $\mathbf{L}(\mathbb{Z})$ is the (periodic) L-theory spectrum of $\mathbf{L}(\mathbb{Z})$. Recall that $\pi_n(\mathbf{L}(\mathbb{Z}))$ is $\mathbb{Z}$ if $n \equiv 0 \mod 4$, $\mathbb{Z}/2$ if if $n \equiv 2 \mod 4$, and vanishes otherwise. We also give a complete answer for $G = \mathbb{Z}^d$. However, for a group G appearing in an exact sequence $1 \to \mathbb{Z}^d \to G \to Q \to 1$ for a finite group Q, our computations can only be carried out under the additional assumption that the conjugation action of Q on $\mathbb{Z}^d$ is free away from 0 or that Q is cyclic of prime order.

In Section 1 precise statements of our results are given, as well as several examples. Section 2 contains the necessary background for the proofs, which are presented in Section 3.

The paper was supported by the Sonderforschungsbereich SFB 878 – Groups, Geometry and Actions –, the Leibniz-Preis of the first author, and the Fulbright Scholars award of the second author. The authors thank the referee for several useful comments.

Contents

1 Statement of Results

Let $K_n(RG)$ denote the *algebraic K-groups* of the group ring RG in the sense of Quillen for $n \geq 0$ and in the sense of Bass for $n \leq -1$, and let $L_n^{\langle -\infty \rangle}(RG)$ be the *ultimate lower quadratic L-groups* of RG (see Ranicki [37]). When considering L-theory, we will tacitly assume that R is a ring with an involution. Let $NK_n(R)$ denote the *Bass-Nil-groups* of R, which are defined as the cokernel of the map $K_n(R) \to K_n(R[x])$. Recall that the Bass-Heller-Swan decomposition says

$$K_n(R\mathbb{Z}) \quad \cong \quad K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \tag{1.1}$$

If R is a regular ring, then $NK_n(R) = 0$ for every $n \in \mathbb{Z}$ (see for instance Rosenberg [38, Theorems 3.3.3 and 5.3.30]).

The *n-th Whitehead group* $\mathrm{Wh}_n(G;R)$ is defined as

$$\mathrm{Wh}_n(G;R) \quad := \quad H_n^G\big(EG \to \{\bullet\}; \mathbf{K}_R\big),$$

where $H_n^G\big(EG \to \{\bullet\}; \mathbf{K}_R\big)$ is the relative term in the equivariant homology theory $H_*^G\big(-; \mathbf{K}_R\big)$ associated to the equivariant map $EG \to \{\bullet\}$ (see 2.4).

Thus, $\mathrm{Wh}_n(G;R)$ fits into a long exact sequence

$$\cdots \to H_n\big(BG;\mathbf{K}(R)\big) \to K_n(RG) \to \mathrm{Wh}_n(G;R) \\ \to H_{n-1}\big(BG;\mathbf{K}(R)\big) \to K_{n-1}(RG) \to \cdots ,$$

where $\mathbf{K}(R)$ is the non-connective K-theory spectrum associated to R and $H_*\big(-;\mathbf{K}(R)\big)$ is the associated homology theory. Notice that $\mathrm{Wh}_1(G;\mathbb{Z})$ agrees with the classical Whitehead group $\mathrm{Wh}(G)$. Whitehead groups arise naturally when studying h-cobordisms, pseudoisotopy, and Waldhausen's A-theory. Their geometric significance can be reviewed, for example, in Dwyer-Weiss-Williams [20, Section 9] and Lück-Reich [29, Section 1.4.1], where additional references can also be found. When $G=\mathbb{Z}$, it follows from (1.1) and the fact that $H_n^{\mathbb{Z}}\big(E\mathbb{Z};\mathbf{K}_R\big) \cong K_n(R)\oplus K_{n-1}(R)$ that there is an identification

$$H_n^{\mathbb{Z}}\big(E\mathbb{Z}\to\{\bullet\};\mathbf{K}_R\big) \quad\cong\quad NK_n(R)\oplus NK_n(R). \tag{1.2}$$

Define the *periodic n-th structure group with decoration $\langle-\infty\rangle$* to be

$$\mathcal{S}_n^{\mathrm{per},\langle-\infty\rangle}(G;R) \quad:=\quad H_n^G\big(EG\to\{\bullet\};\mathbf{L}_R^{\langle-\infty\rangle}\big).$$

These groups fit into the periodic version of the long exact surgery sequence with decoration $\langle-\infty\rangle$,

$$\cdots \to H_n\big(BG;\mathbf{L}^{\langle-\infty\rangle}(R)\big) \to L_n^{\langle-\infty\rangle}(RG) \to \mathcal{S}_n^{\mathrm{per},\langle-\infty\rangle}(G;R) \\ \to H_{n-1}\big(BG;\mathbf{L}^{\langle-\infty\rangle}(R)\big) \to L_{n-1}^{\langle-\infty\rangle}(RG) \to \cdots ,$$

where $\mathbf{L}^{\langle-\infty\rangle}(R)$ is the spectrum whose homotopy groups are the ultimate lower quadratic L-groups. This periodic surgery sequence (with a different decoration) for $R=\mathbb{Z}$ appears in the classification of ANR-homology manifolds in Bryant-Ferry-Mio-Weinberger [9, Main Theorem]. It is related to the algebraic surgery exact sequence and thus to the classical surgery sequence (see Ranicki [36, Section 18]). For $n\in\mathbb{Z}$ define

$$\mathrm{UNil}_n^{\langle-\infty\rangle}(D_\infty;R) \quad:=\quad H_n^{D_\infty}\big(\underline{E}D_\infty\to\{\bullet\};\mathbf{L}_R^{\langle-\infty\rangle}\big). \tag{1.3}$$

These groups are related to Cappell's UNil-groups, as explained in Section 2.

Now we are able to state our main results.

1.1 Hyperbolic groups

Theorem 1.4 (Hyperbolic groups). *Let G be a hyperbolic group in the sense of Gromov [24], and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of G.*

(i) For each $n\in\mathbb{Z}$ there is an isomorphism

$$H_n^G\big(\underline{E}G;\mathbf{K}_R\big)\oplus\bigoplus_{V\in\mathcal{M}}H_n^V\big(\underline{E}V\to\{\bullet\};\mathbf{K}_R\big) \quad\xrightarrow{\cong}\quad K_n(RG);$$

(ii) For each $n \in \mathbb{Z}$ there is an isomorphism

$$H_n^G\big(\underline{E}G; \mathbf{L}_R^{\langle -\infty\rangle}\big) \oplus \bigoplus_{V \in \mathcal{M}} H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG),$$

provided that there exists $n_0 \leq -2$ such that $K_n(RV) = 0$ holds for all $n \leq n_0$ and all virtually cyclic subgroups $V \subseteq G$. (The latter condition is satisfied if $R = \mathbb{Z}$ or if R is regular with $\mathbb{Q} \subseteq R$.)

A good model for $\underline{E}G$ is given by the Rips complex of G (see Meintrup-Schick [32]). Tools for computing $H_n^G\big(\underline{E}G; \mathbf{K}_R\big)$ are the equivariant version of the Atiyah-Hirzebruch spectral sequence (see Davis-Lück [16, Theorem 4.7]), the p-chain spectral sequence (see Davis-Lück [17]) and equivariant Chern characters (see Lück [26]). More information about the groups $H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{K}_R\big)$ and $H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)$ is given in Section 2.

1.2 Torsion-free hyperbolic groups

The situation simplifies if G is assumed to be a torsion-free hyperbolic group.

Theorem 1.5 (Torsion-free hyperbolic groups)**.** *Let G be a torsion-free hyperbolic group, and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of G.*

(i) For each $n \in \mathbb{Z}$ there is an isomorphism

$$H_n\big(BG; \mathbf{K}(R)\big) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(RG);$$

(ii) For each $n \in \mathbb{Z}$ there is an isomorphism

$$H_n\big(BG; \mathbf{L}^{\langle -\infty\rangle}(R)\big) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG).$$

In particular, it follows that for a torsion-free hyperbolic group G and a regular ring R,

$$K_n(RG) = \{0\} \quad \text{for } n \leq 1$$

and the obvious map

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective. Moreover, $\widetilde{K}_0(\mathbb{Z}G)$, $\mathrm{Wh}(G)$ and $K_n(\mathbb{Z}G)$ for $n \leq -1$ all vanish.

Example 1.6 (Finitely generated free groups)**.** Let F_r be the finitely generated free group $*_{i=1}^r \mathbb{Z}$ of rank r. Since F_r acts freely on a tree it is hyperbolic. By Theorem 1.5,

$$K_n(RF_r) \cong K_n(R) \oplus K_{n-1}(R)^r \oplus \bigoplus_{V \in \mathcal{M}} \big(NK_n(R) \oplus NK_n(R)\big)$$

and

$$L_n^{\langle -\infty\rangle}(RF_r) \cong L_n^{\langle -\infty\rangle}(R) \oplus L_{n-1}^{\langle -\infty\rangle}(R)^r,$$

where $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of F_r.

1.3 Finitely generated free abelian groups

Before tackling the virtually finitely generated abelian case, we first consider finitely generated free abelian groups. The K-theory case of Theorem 1.7 below is also proved in [14].

Theorem 1.7 ($\mathbb{Z}^d$)**.** *Let $d \geq 1$ be an integer. Let $\mathcal{MICY}$ be the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$. Then there are isomorphisms*

$$\begin{aligned}
\mathrm{Wh}_n(\mathbb{Z}^d;R) &\cong \bigoplus_{C\in\mathcal{MICY}} \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(R)\oplus NK_{n-i}(R)\big)^{\binom{d-1}{i}};\\
K_n(R[\mathbb{Z}^d]) &\cong \left(\bigoplus_{i=0}^{d} K_{n-i}(R)^{\binom{d}{i}}\right) \oplus \mathrm{Wh}_n(\mathbb{Z}^d;R);\\
L_n^{\langle -\infty\rangle}(R[\mathbb{Z}^d]) &\cong \bigoplus_{i=0}^{d} L_{n-i}^{\langle -\infty\rangle}(R)^{\binom{d}{i}}.
\end{aligned}$$

Example 1.8 ($\mathbb{Z}^d \times G$)**.** Let G be a group. By Theorem 1.7,

$$\begin{aligned}
&K_n(R[G\times\mathbb{Z}^d]) \cong K_n(RG[\mathbb{Z}^d])\\
&\cong \bigoplus_{i=0}^{d} K_{n-i}(RG)^{\binom{d}{i}} \oplus \bigoplus_{C\in\mathcal{MICY}(\mathbb{Z}^d)} \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(RG)\oplus NK_{n-i}(RG)\big)^{\binom{d-1}{i}},
\end{aligned}$$

where $\mathcal{MICY}(\mathbb{Z}^d)$ is the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$. Since

$$H_n\big(B(G\times\mathbb{Z}^d);\mathbf{K}(R)\big) \cong \bigoplus_{i=0}^{d} H_n\big(BG;\mathbf{K}(R)\big)^{\binom{d}{i}},$$

this implies

$$\begin{aligned}
\mathrm{Wh}_n(G\times\mathbb{Z}^d;R) \;\cong\; & \bigoplus_{i=0}^{d} \mathrm{Wh}_{n-i}(G;R)^{\binom{d}{i}}\\
& \oplus \bigoplus_{C\in\mathcal{MICY}(\mathbb{Z}^d)} \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(RG)\oplus NK_{n-i}(RG)\big)^{\binom{d-1}{i}}.
\end{aligned}$$

Example 1.9 (Surface groups)**.** Let Γ_g be the fundamental group of the orientable closed surface of genus g, and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of G. If $g = 0$, then Γ_g is trivial. If $g = 1$, then Γ_g is $\mathbb{Z}^2$ and Theorem 1.7 implies

$$K_n(R\Gamma_1) \;\cong\; K_n(R)\oplus K_{n-1}(R)^2\oplus K_{n-2}(R)\oplus \bigoplus_{V\in\mathcal{M}} \big(NK_n(R)^2\oplus NK_{n-1}(R)^2\big)$$

and

$$L_n^{\langle -\infty\rangle}(R\Gamma_1) \;\cong\; L_n^{\langle -\infty\rangle}(R)\oplus L_{n-1}^{\langle -\infty\rangle}(R)^2\oplus L_{n-2}^{\langle -\infty\rangle}(R).$$

If $g \geq 2$, then Γ_g is hyperbolic and torsion-free, so by Theorem 1.5

$$K_n(R\Gamma_g) \;\cong\; H_n\big(B\Gamma_g;\mathbf{K}(R)\big) \oplus \bigoplus_{V\in\mathcal{M}} NK_n(R)^2,$$

and

$$L_n^{\langle -\infty\rangle}(R\Gamma_g) \;\cong\; H_n\big(B\Gamma_g;\mathbf{L}^{\langle -\infty\rangle}(R)\big).$$

Since Γ_g is stably a product of spheres,

$$H_n\big(B\Gamma_g;\mathbf{K}(R)\big) \;\cong\; K_n(R)\oplus K_{n-1}(R)^{2g}\oplus K_{n-2}(R),$$

and

$$H_n\big(B\Gamma_g;\mathbf{L}^{\langle -\infty\rangle}(R)\big) \;\cong\; L_n^{\langle -\infty\rangle}(R)\oplus L_{n-1}^{\langle -\infty\rangle}(R)^{2g}\oplus L_{n-2}^{\langle -\infty\rangle}(R).$$

If $R=\mathbb{Z}$, then for every $i\in\{1,0,-1,\ldots\}\amalg\{-\infty\}$, $L_n^{\langle i\rangle}(\mathbb{Z})$ is $\mathbb{Z}$ if $n\equiv 0 \mod 4$, $\mathbb{Z}/2$ if $n\equiv 2 \mod 4$, and is trivial otherwise. Therefore,

$$L_n^{\langle i\rangle}(\mathbb{Z}\Gamma_g)\cong\begin{cases}\mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } n\equiv 0,2 \mod 4;\\ \mathbb{Z}^g & \text{if } n\equiv 1 \mod 4;\\ (\mathbb{Z}/2)^g & \text{if } n\equiv 3 \mod 4.\end{cases}$$

More generally, cocompact planar groups (sometimes called cocompact non-Euclidean crystallographic groups), e.g., cocompact Fuchsian groups, are treated in Lück-Stamm [30] for $R=\mathbb{Z}$. These computations can be carried over to arbitrary R.

1.4 Virtually finitely generated abelian groups

Consider the group extension

$$1\to A\to G\xrightarrow{q} Q\to 1, \tag{1.10}$$

where A is isomorphic to $\mathbb{Z}^d$ for some $d\geq 0$ and Q is a finite group. The conjugation action of G on the normal abelian subgroup A induces an action of Q on A via a group homomorphism which we denote by $\rho\colon Q\to \operatorname{aut}(A)$.

Let $\mathcal{MICY}(A)$ be the set of maximal infinite cyclic subgroups of A. Since any automorphism of A sends a maximal infinite cyclic subgroup to a maximal infinite cyclic subgroup, ρ induces a Q-action on $\mathcal{MICY}(A)$. Fix a subset

$$I \;\subseteq\; \mathcal{MICY}(A)$$

such that the intersection of every Q-orbit in $\mathcal{MICY}(A)$ with I consists of precisely one element.

For $C\in I$, let

$$Q_C \;\subseteq\; Q$$

be the isotropy group of $C \in \mathcal{MICY}(A)$ under the Q-action. Let

$$\begin{aligned} I_1 &= \bigl\{C \in I \mid Q_C = \{1\}\bigr\}, \\ I_2 &= \bigl\{C \in I \mid Q_C = \mathbb{Z}/2\bigr\}, \end{aligned}$$

and let J be a complete system of representatives of maximal non-trivial finite subgroups of G.

1.4.1 K-theory in the case of a free conjugation action

Theorem 1.11. *Consider the group extension $1 \to A \to G \xrightarrow{q} Q \to 1$, where A is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and Q is a finite group. Suppose that the Q-action on A is free away from $0 \in A$.*

(i) For each $n \in \mathbb{Z}$ there is an isomorphism induced by the various inclusions

$$\left(\bigoplus_{F \in J} \mathrm{Wh}_n(F;R)\right) \oplus \bigl(\mathbb{Z} \otimes_{\mathbb{Z}Q} \mathrm{Wh}_n(A;R)\bigr) \xrightarrow{\cong} \mathrm{Wh}_n(G;R).$$

(ii) For each integer n, $\mathbb{Z} \otimes_{\mathbb{Z}Q} \mathrm{Wh}_n(A;R)$ is isomorphic to

$$\left(\bigoplus_{C \in I_1} \bigoplus_{i=0}^{d-1} \bigl(NK_{n-i}(R) \oplus NK_{n-i}(R)\bigr)^{\binom{d-1}{i}}\right) \oplus \left(\bigoplus_{C \in I_2} \bigoplus_{i=0}^{d-1} NK_{n-i}(R)^{\binom{d-1}{i}}\right).$$

If $Q = \{1\}$, then Theorem 1.11 reduces to Theorem 1.7 since $J = \emptyset$, $I_2 = \emptyset$ and I_1 is the set of maximal infinite cyclic subgroups of $G = \mathbb{Z}^d$.

Theorem 1.12. *Under the assumptions of Theorem 1.11:*

(i) If R is regular, then for each $n \in \mathbb{Z}$

$$\bigoplus_{F \in J} \mathrm{Wh}_n(F;R) \cong \mathrm{Wh}_n(G;R).$$

In particular,

$$\bigoplus_{F \in J} K_n(RF) \cong K_n(RG) \quad \text{for } n \leq -1$$

and

$$\bigoplus_{F \in J} \mathrm{coker}\bigl(K_0(R) \to K_0(RF)\bigr) \cong \mathrm{coker}\bigl(K_0(R) \to K_0(RG)\bigr).$$

(ii) If R is a Dedekind ring of characteristic zero, then

$$K_n(RG) = \{0\} \quad \text{for } n \leq -2.$$

Applying Theorem 1.12 to the special case $R = \mathbb{Z}$, we recover the fact that if G satisfies the assumptions of Theorem 1.11, then:

$$\begin{aligned}
\bigoplus_{F \in J} \mathrm{Wh}(F) &\xrightarrow{\cong} \mathrm{Wh}(G); \\
\bigoplus_{F \in J} \widetilde{K}_0(\mathbb{Z}F) &\xrightarrow{\cong} \widetilde{K}_0(\mathbb{Z}G); \\
\bigoplus_{F \in J} K_{-1}(\mathbb{Z}F) &\xrightarrow{\cong} K_{-1}(\mathbb{Z}G); \\
K_n(\mathbb{Z}G) &\cong \{0\} \quad \text{for } n \le -2.
\end{aligned}$$

1.4.2 L-theory in the case of a free conjugation action

Theorem 1.13. *Consider the group extension $1 \to A \to G \xrightarrow{q} Q \to 1$, where A is isomorphic to $\mathbb{Z}^d$ for some $d \ge 0$ and Q is a finite group. Suppose that the Q-action on A is free away from $0 \in A$. Assume that there exists $n_0 \le -2$ such that $K_n(RV) = 0$ for all $n \le n_0$ and all virtually cyclic subgroups $V \subseteq G$. (By Theorem 1.12 (ii), this condition is satisfied if R is a Dedekind ring of characteristic zero).*

Then there is an isomorphism

$$\left(\bigoplus_{F \in J} \mathcal{S}_n^{\mathrm{per},\langle -\infty \rangle}(F;R)\right) \oplus \left(\bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \mathrm{UNil}_n^{\langle -\infty \rangle}(D_\infty;R)\right) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle -\infty \rangle}(G;R),$$

where J_C is a complete system of representatives of the conjugacy classes of maximal finite subgroups of $W_G C = N_G C/C$.

If Q has odd order, then

$$\bigoplus_{F \in J} \mathcal{S}_n^{\mathrm{per},\langle -\infty \rangle}(F;R) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle -\infty \rangle}(G;R).$$

Dealing with groups extensions of the type described above when the conjugation action of Q on A is not free is considerably harder than the free case. However, we are able to say something when Q is a cyclic group of prime order and R is regular.

1.4.3 K-theory in the case $Q = \mathbb{Z}/p$ for a prime p and regular R

Theorem 1.14. *Consider the group extension $1 \to A \to G \xrightarrow{q} \mathbb{Z}/p \to 1$, where A is isomorphic to $\mathbb{Z}^d$ for some $d \ge 0$ and p is a prime number. Let e be the natural number given by $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$, J be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of G, and $\mathcal{MICY}(A^{\mathbb{Z}/p})$ be the set of maximal infinite cyclic subgroups of $A^{\mathbb{Z}/p}$.*

If R is regular, then there is an isomorphism

$$\mathrm{Wh}_n(G;R) \cong \left(\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathrm{Wh}_{n-i}(H;R)^{\binom{e}{i}}\right)$$
$$\oplus\left(\bigoplus_{H\in J}\bigoplus_{C\in\mathcal{MICY}(A^{\mathbb{Z}/p})}\bigoplus_{i=0}^{e-1}\big(NK_{n-i}(R[\mathbb{Z}/p])\oplus NK_{n-i}(R[\mathbb{Z}/p])\big)^{\binom{e-1}{i}}\right).$$

Remark 1.15 (Cardinality of J)**.** Assume that the group G appearing in Theorem 1.14 is not torsion-free, or, equivalently, that J is non-empty. Consider A as a $\mathbb{Z}[\mathbb{Z}/p]$-module via the conjugation action $\rho\colon \mathbb{Z}/p \to \mathrm{aut}(A)$. Then there is a bijection

$$H^1(\mathbb{Z}/p;A) \quad \xrightarrow{\cong} \quad J,$$

defined as follows. Fix an element $t \in G$ of order p. Every $\overline{x} \in H^1(\mathbb{Z}/p;A)$ is represented by an element x in the kernel of $\sum_{i=0}^{p-1}\rho^i\colon A \to A$. Then xt has order p. Send $\overline{x}$ to the unique element of J that is conjugate to $\langle xt\rangle$ in G.

If p is prime and R is regular, then Example 1.8 in the case $G = \mathbb{Z}/p$ is consistent with Theorem 1.14 and Remark 1.15. Namely, $\mathbb{Z}/p$ acts trivially on $\mathbb{Z}^d$, and so $A = A^{\mathbb{Z}/p}$, $J = \{\{0\}\times\mathbb{Z}/p\}$, and $d = e$.

1.4.4 L-theory in the case $Q = \mathbb{Z}/p$ for an odd prime p.

Theorem 1.16. *Consider the group extension $1 \to A \to G \xrightarrow{q} \mathbb{Z}/p \to 1$, where A is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and p is an odd prime number. Let e be the natural number given by $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$ and J be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of G.*

Then there is an isomorphism

$$\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}_{n-i}^{\mathrm{per},\langle-\infty\rangle}(H;R)^{\binom{e}{i}} \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle-\infty\rangle}(G;R).$$

Two-dimensional crystallographic groups are treated in Pearson [33] and Lück-Stamm [30] for $R = \mathbb{Z}$. Some of the computations there can be carried over to arbitrary R. The Whitehead groups of three-dimensional crystallographic groups are computed in Alves-Ontaneda [1].

In the case $R = \mathbb{Z}$, we get a computation for all decorations.

Theorem 1.17. *Consider the group extension $1 \to A \to G \xrightarrow{q} \mathbb{Z}/p \to 1$, where A is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and p is an odd prime number. Let e be the natural number given by $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$, J be a complete system of representatives of the conjugacy classes of non-trivial finite subgroups of G, and ϵ be any of the decorations s, h, p or $\{\langle j\rangle \mid j = 1,0,-1,\ldots\} \amalg \{-\infty\}$. Define*

$$\mathcal{S}_n^{\mathrm{per},\epsilon}(G;\mathbb{Z}) \quad := \quad H_n^G\big(EG \to \{\bullet\};\mathbf{L}_{\mathbb{Z}}^{\epsilon}\big).$$

Then there is an isomorphism

$$\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}^{\mathrm{per},\epsilon}_{n-i}(H;\mathbb{Z})^{\binom{e}{i}}\xrightarrow{\cong}\mathcal{S}^{\mathrm{per},\epsilon}_{n}(G;\mathbb{Z}).$$

Remark 1.18. In general, the structure sets $\mathcal{S}^{\mathrm{per},\epsilon}_{n-i}(H;\mathbb{Z})$ depend on the decoration ϵ. We mention without proof that for an odd prime p

$$\mathcal{S}^{\mathrm{per},s}_{n}(\mathbb{Z}/p;\mathbb{Z})\cong\widetilde{L}^{s}_{n}(\mathbb{Z}[\mathbb{Z}/p])[1/p]\cong\begin{cases}\mathbb{Z}[1/p]^{(p-1)/2} & n \text{ even};\\ \{0\} & n \text{ odd}.\end{cases}$$

Remark 1.19. The computation of the structure set $\mathcal{S}^{\mathrm{per},\epsilon}_{n}(G;\mathbb{Z})$ when the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^d$ is free plays a role in a forthcoming paper by Davis and Lück, in which this case is further analyzed to compute the geometric structure sets of certain manifolds that occur as total spaces of a bundle over lens spaces with d-dimensional tori as fibers.

Remark 1.20 (Topological K-theory of reduced group C^*-algebras)**.** All of the above computations also apply to the topological K-theory of the reduced group C^*-algebra, since the Baum-Connes Conjecture is true for these groups. In the Baum-Connes setting the situation simplifies considerably because one works with $\underline{E}G$ instead of $\underline{\underline{E}}G$ and hence there are no Nil-phenomena. For instance, if G is a hyperbolic group, then there is an isomorphism

$$K^G_n(\underline{E}G)\xrightarrow{\cong}K_n(C^*_r(G))$$

from the equivariant topological K-theory of $\underline{E}G$ to the topological K-theory of the reduced group C^*-algebra $C^*_r(G)$. In the case of a torsion-free hyperbolic group, this reduces to an isomorphism

$$K_n(BG)\xrightarrow{\cong}K_n(C^*_r(G)).$$

The case $G=\mathbb{Z}^d\rtimes_\rho\mathbb{Z}/p$ for a free conjugation action has been carried out for both complex and real topological K-theory in detail in [18, Theorem 0.3 and Theorem 0.6].

2 Background

In this section we give some background about the Farrell-Jones Conjecture and related topics.

2.1 Classifying Spaces for Families

Let G be a group. A *family of subgroups* of G is a collection of subgroups that is closed under conjugation and taking subgroups. Examples of such families are

$$\begin{aligned}
\{1\} &= \{\text{trivial subgroup}\};\\
\mathcal{FIN} &= \{\text{finite subgroups}\};\\
\mathcal{VCY} &= \{\text{virtually cyclic subgroups}\};\\
\mathcal{ALL} &= \{\text{all subgroups}\}.
\end{aligned}$$

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *universal space* $E_{\mathcal{F}}(G)$ *for* $\mathcal{F}$ is a G-CW-complex X with isotropy groups in $\mathcal{F}$ such that for any G-CW-complex Y with isotropy groups in $\mathcal{F}$ there exists a G-map $Y \to X$ that is unique up to G-homotopy. In other words, X is a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups belong to $\mathcal{F}$. In particular, any two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent, and for two families $\mathcal{F}_0 \subseteq \mathcal{F}_1$, there is precisely one G-map $E_{\mathcal{F}_0}(G) \to E_{\mathcal{F}_1}(G)$ up to G-homotopy.

For every group G and every family of subgroups $\mathcal{F}$ there exists a model for $E_{\mathcal{F}}(G)$. A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if the H-fixed point set X^H is contractible for every H in $\mathcal{F}$ and is empty otherwise. For example, a model for $E_{\mathcal{ALL}}(G)$ is $G/G = \{\bullet\}$, and a model for $E_{\{1\}}(G)$ is the same as a model for EG, the total space of the universal G-principal bundle $EG \to BG$. The *universal G-CW-complex for proper G-actions*, $E_{\mathcal{FIN}}(G)$, will be denoted by $\underline{E}G$, and the universal space $E_{\mathcal{VCY}}(G)$ for $\mathcal{VCY}$ will be denoted by $\underline{\underline{E}}G$. For more information on classifying spaces the reader is referred to the survey article by Lück [28].

2.2 Review of the Farrell-Jones Conjecture

Let $\mathcal{H}^?_*$ be an equivariant homology theory in the sense of Lück [26, Section 1]. Then, for every group G and every G-CW-pair (X, A) there is a $\mathbb{Z}$-graded abelian group $\mathcal{H}^G_*(X, A)$, and subsequently a G-homology theory $\mathcal{H}^G_*$. For every group homomorphism $\alpha\colon H \to G$, every H-CW-pair (X, A) and every $n \in \mathbb{Z}$, there is a natural homomorphism $\mathrm{ind}_\alpha : \mathcal{H}^H_*(X, A) \to \mathcal{H}^G_*(G \times_\alpha (X, A))$, known as the induction homomorphism. If the kernel of α operates relative freely on (X, A), then ind_α is an isomorphism.

Our main examples are the equivariant homology theories $H^?_*(-; \mathbf{K}_R)$ and $H^?_*(-; \mathbf{L}^{\langle -\infty \rangle}_R)$ appearing in the K-theoretic and L-theoretic Farrell-Jones Conjectures, where R is an associative ring with unit (and involution) (see Lück-Reich [29, Section 6]). The basic property of these two equivariant homology theories is that

$$\begin{array}{lllll}
H^G_n\big(G/H; \mathbf{K}_R\big) & = & H^H_n\big(\{\bullet\}; \mathbf{K}_R\big) & = & K_n(RH);\\
H^G_n\big(G/H; \mathbf{L}^{\langle -\infty \rangle}_R\big) & = & H^H_n\big(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}_R\big) & = & L^{\langle -\infty \rangle}_n(RH),
\end{array}$$

for every subgroup $H \subseteq G$.

The *Farrell-Jones Conjecture for a group G and a ring R* states that the assembly maps

$$\begin{aligned} H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) &\xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_R) = K_n(RG); \\ H_n^G(\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}) &\xrightarrow{\cong} H_n^G(G/G;\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG), \end{aligned}$$

induced by the projection $\underline{\underline{E}}G \to G/G$, are bijective for every $n \in \mathbb{Z}$ [21]. The Farrell-Jones Conjecture has been studied extensively because of its geometric significance. It implies, in dimensions ≥ 5, both the Novikov Conjecture about the homotopy invariance of higher signatures and the Borel Conjecture about the rigidity of manifolds with fundamental group G. It also implies other well-known conjectures, such as the ones due to Bass and Kadison. For a survey and applications of the Farrell-Jones Conjecture, see, for example, Lück-Reich [29, Section 6] and Bartels-Lück-Reich [7].

Theorem 2.1 (Farrell-Jones Conjecture for hyperbolic groups and virtually $\mathbb{Z}^d$-groups)**.** *The Farrell-Jones Conjecture is true if G is a hyperbolic group or a virtually finitely generated abelian group, and R is any ring.*

The K-theoretic Farrell-Jones Conjecture with coefficients in any additive G-category for a hyperbolic group G was proved by Bartels-Reich-Lück [6]. The version of the Farrell-Jones Conjecture with coefficients in an additive category encompasses the version with rings as coefficients. The L-theoretic Farrell-Jones Conjecture with coefficients in any additive G-category for hyperbolic groups and CAT(0)-groups was established by Bartels and Lück [5]. In that paper it is also shown that the K-theoretic assembly map is 1-connected for CAT(0)-groups. Note that a virtually finitely generated abelian group is CAT(0). Quinn [34, Theorem 1.2.2]) proved that the K-theoretic assembly map for virtually finitely generated abelian groups is bijective for every integer n if R is a commutative ring. However, the proof carries over to the non-commutative setting.

Remark 2.2 (The Interplay of K- and L-Theory)**.** L-theory $L_*^{\langle i\rangle}(RG)$ can have various decorations for $i \in \{2, 1, 0, -1, -2, \dots\} \amalg \{-\infty\}$. One also finds $L_*^{\epsilon}(RG)$ for $\epsilon = p, h, s$ in the literature. The decoration $\langle 1\rangle$ coincides with the decoration h, $\langle 0\rangle$ with the decoration p, and $\langle 2\rangle$ is related to the decoration s. For $j \leq 1$ there are forgetful maps $L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R)$ that fit into the so-called *Rothenberg sequence* (see Ranicki [35, Proposition 1.10.1 on page 104], [37, 17.2])

$$\begin{aligned} \cdots \to L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R) \to \widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R)) \\ \to L_{n-1}^{\langle j+1\rangle}(R) \to L_{n-1}^{\langle j\rangle}(R) \to \cdots . \end{aligned} \tag{2.3}$$

$\widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R))$ denotes the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$-module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R. There is a similar sequence relating $L_n^s(RG)$ and $L_n^h(RG)$, where the third term is the $\mathbb{Z}/2$-Tate-cohomology of $\mathrm{Wh}_1^R(G)$.

For geometric applications the most important case is $R = \mathbb{Z}$ with the decoration s. If $\widetilde{K}_0(\mathbb{Z}G)$, $\mathrm{Wh}(G)$ and $K_n(\mathbb{Z}G)$ for $n \le -1$ all vanish, then the Rothenberg sequence for $n \in \mathbb{Z}$ implies the bijectivity of the natural maps

$$L_n^s(\mathbb{Z}G) \xrightarrow{\cong} L_n^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^p(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

In the formulation of the Farrell-Jones Conjecture (see Section 2), one must use the decoration $\langle -\infty \rangle$ since the conjecture is false otherwise (see Farrell-Jones-Lück [23]).

2.3 Equivariant homology and relative assembly maps

Consider an equivariant homology theory $\mathcal{H}_*^?$ in the sense of [26, Section 1]. Given a G-map $f\colon X \to Y$ of G-CW-complexes and $n \in \mathbb{Z}$, define

$$\mathcal{H}_n^G(f) \quad := \quad \mathcal{H}_n^G\bigl(\mathrm{cyl}(f_0), X\bigr) \tag{2.4}$$

for any cellular G-map $f_0\colon X \to Y$ that is G-homotopic to f. Here, $\mathrm{cyl}(f_0)$ is the G-CW-complex given by the mapping cylinder of f. It contains X as a G-CW-subcomplex. Such an f_0 exists by the Equivariant Cellular Approximation Theorem (see [39, Theorem II.2.1 on page 104]). The definition is independent of the choice of f_0, since two cellular G-homotopic G-maps $f_0, f_1\colon X \to Y$ are cellularly G-homotopic by the Equivariant Cellular Approximation Theorem (see [39, Theorem II.2.1 on page 104]) and a cellular G-homotopy between f_0 and f_1 yields a G-homotopy equivalence $u\colon\ \mathrm{cyl}(f_0) \to \mathrm{cyl}(f_1)$ that is the identity on X. This implies that $\mathcal{H}_n^G(f)$ depends only on the G-homotopy class of f. From the axioms of an equivariant homology theory, $\mathcal{H}_n^G(f)$ fits into a long exact sequence

$$\cdots \to \mathcal{H}_{n+1}^G(f) \to \mathcal{H}_n^G(X) \to \mathcal{H}_n^G(Y) \to \mathcal{H}_n^G(f) \to \mathcal{H}_{n-1}^G(X) \to \cdots . \tag{2.5}$$

The following fact is proved in Bartels [3].

Lemma 2.6. *For every group G, every ring R, and every $n \in \mathbb{Z}$:*

(i) the relative assembly map

$$H_n^G\bigl(\underline{E}G; \mathbf{K}_R\bigr) \to H_n^G\bigl(\underline{\underline{E}}G; \mathbf{K}_R\bigr)$$

is split-injective;

(ii) the relative assembly map

$$H_n^G\bigl(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}\bigr) \to H_n^G\bigl(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}\bigr)$$

is split-injective, provided there is an $n_0 \le -2$ such that $K_n(RV) = 0$ for every $n \le n_0$ and every virtually cyclic subgroup $V \subseteq G$.

Remark 2.7. Notice that the condition appearing in assertion (ii) of the above lemma is automatically satisfied if one of the following stronger conditions holds:

(i) $R = \mathbb{Z}$;

(ii) R is regular with $\mathbb{Q} \subseteq R$.

If $R = \mathbb{Z}$, this follows from [22, Theorem 2.1(a)]. If R is regular and $\mathbb{Q} \subseteq R$, then for every virtually cyclic subgroup $V \subseteq G$ the ring RV is regular and hence $K_n(RV) = 0$ for all $n \leq -1$.

Lemma 2.6 tells us that the source of the assembly map appearing in the Farrell-Jones Conjecture can be computed in two steps, the computation of $H_n^G(\underline{E}G; \mathbf{K}_R)$ and the computation of the remaining term $H_n^G(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R)$ defined in (2.4). Furthermore, Lemma 2.6 implies

$$H_n^G(EG \to \underline{\underline{E}}G; \mathbf{K}_R) \cong H_n^G(EG \to \underline{E}G; \mathbf{K}_R) \oplus H_n^G(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R); \quad (2.8)$$

and

$$H_n^G(EG \to \underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}) \cong H_n^G(EG \to \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus H_n^G(\underline{E}G \to \underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}), \quad (2.9)$$

provided that there exists an $n_0 \leq -2$ such that $K_n(RV) = 0$ for every $n \leq n_0$ and every virtually cyclic subgroups $V \subseteq G$. This is useful for calculating the Whitehead groups for a given group G and ring R which satisfy the Farrell-Jones Conjecture.

Remark 2.10. The induction structure of $H_*^?(-; \mathbf{K}_R)$ can be used to define a $\mathbb{Z}/2$-action on $H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R)$ that is compatible with the $\mathbb{Z}/2$-action on $NK_n(R) \oplus NK_n(R)$ given by flipping the two factors. The action is defined by the composition of isomorphisms

$$\tau : H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R) \xrightarrow{\mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}}} H_n^{\mathbb{Z}}(\mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R) \to H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R),$$

where the second map is induced by the unique (up to equivariant homotopy) equivariant map $l : \mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} E\mathbb{Z} \to E\mathbb{Z}$; l is an equivariant homotopy equivalence since $\mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} E\mathbb{Z}$ is a model for $E\mathbb{Z}$.

To see that this corresponds to the flip action on $NK_n(R) \oplus NK_n(R)$, consider the following diagram coming from the long exact sequence (2.5).

$$\begin{array}{ccccc}
H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R) \\
\downarrow \mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} & & \downarrow \mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} & & \downarrow \mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} \\
H_n^{\mathbb{Z}}(\mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} E\mathbb{Z}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(\mathrm{ind}_{-\mathrm{id}_{\mathbb{Z}}} E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R) \\
\downarrow l_* & & \downarrow = & & \downarrow l_* \\
H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) & \longrightarrow & H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\}; \mathbf{K}_R)
\end{array}$$

Recall that $H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) \to H_n^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_R) \cong K_n(R\mathbb{Z}) \cong K_n(R[t,t^{-1}])$ is a split injection, and so the Bass-Heller-Swan decomposition (1.1) establishes the identification $H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\};\mathbf{K}_R) \cong NK_n(R) \oplus NK_n(R)$ (1.2). Also recall that the two copies of $NK_n(R)$ appearing in the decomposition of $K_n(R[t,t^{-1}])$ come from the embeddings $R[t] \hookrightarrow R[t,t^{-1}]$ and $R[t^{-1}] \hookrightarrow R[t,t^{-1}]$. By the definition of the induction structure for $H_*^?(-;\mathbf{K}_R)$,

$$\mathrm{ind}_{-\,\mathrm{id}_{\mathbb{Z}}} : H_n^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_R) \to H_n^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_R)$$

corresponds to the homomorphism $K_n(R[t,t^{-1}]) \to K_n(R[t,t^{-1}])$ induced by interchanging t and t^{-1} (see, for example, [29, Section 6]), which swaps the two copies of $NK_n(R)$ in the decomposition of $K_n(R[t,t^{-1}])$. Therefore the above diagram implies that τ coincides with the flip action on $NK_n(R) \oplus NK_n(R)$. A proof of this fact can also be found in [19, Lemma 3.22].

Remark 2.11 (Relative assembly and Nil-terms)**.** Let V be an infinite virtually cyclic group. If V is of type I, then V can be written as a semi-direct product $F \rtimes \mathbb{Z}$, and $H_*^V(\underline{E}V \to \{\bullet\};\mathbf{K}_R)$ can be identified with the non-connective version of Waldhausen's Nil-term associated to this semi-direct product (see [4, Sections 9 and 10]). If V is of type II, then it can be written as an amalgamated product $V_1 *_{V_0} V_2$ of finite groups, where V_0 has index two in both V_1 and V_2. In this case, $H_*^V(\underline{E}V \to \{\bullet\};\mathbf{K}_R)$ can be identified with the non-connective version of Waldhausen's Nil-term associated to this amalgamated product [4]. The identifications come from the Five-Lemma and the fact that both groups fit into the same long exact sequence associated to the semi-direct product, or amalgamated product, respectively. This is analogous to the L-theory case which is explained below. If R is regular and $\mathbb{Q} \subseteq R$, e.g., R is a field of characteristic zero, then $H_n^V(\underline{E}V \to \{\bullet\};\mathbf{K}_R) = 0$ for every virtually cyclic group V, and hence, for any group G the map

$$H_n^G(\underline{E}G;\mathbf{K}_R) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G;\mathbf{K}_R)$$

is bijective (see [29, Proposition 2.6]).

Let $\mathrm{UNil}_n^h(R;R,R)$ denote the Cappell UNil-groups [10] associated to the amalgamated product $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$. It is a direct summand in $L_n^h(R[D_\infty])$ and there is a Mayer-Vietoris sequence

$$\cdots \to L_n^h(R) \to L_n^h(R[\mathbb{Z}/2]) \oplus L_n^h(R[\mathbb{Z}/2]) \to L_n^h(R[D_\infty])/\,\mathrm{UNil}_n(R;R,R) \\ \to L_n^h(R) \to L_n^h(R[\mathbb{Z}/2]) \oplus L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/2]) \to \cdots .$$

There is also a Mayer-Vietoris sequence that maps to the one above which comes from the model for $\underline{E}D_\infty$ given by the obvious D_∞-action on $\mathbb{R}$:

$$\cdots \to L_n^h(R) \to L_n^h(R[\mathbb{Z}/2]) \oplus L_n^h(R[\mathbb{Z}/2]) \to H_n^{D_\infty}(\underline{E}D_\infty;\mathbf{L}_R^h) \\ \to L_n^h(R) \to L_n^h(R[\mathbb{Z}/2]) \oplus L_n^h(R[\mathbb{Z}/2]) \to \cdots .$$

This implies

$$\mathrm{UNil}_n^h(R;R,R) \quad\cong\quad H_*^{D_\infty}\big(\underline{E}D_\infty \to \{\bullet\}; \mathbf{L}_R^h\big).$$

Lemma 2.12. *Assume that* $\widetilde{K}_i(R) \cong \widetilde{K}_i(R[\mathbb{Z}/2]) \cong \widetilde{K}_i(R[D_\infty]) = 0$ *for all* $i \leq 0$. *(This condition is satisfied, for example, if* $R = \mathbb{Z}$.*) Then:*

$$\mathrm{UNil}_n^h(R;R,R) \cong \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty;R).$$

Proof. Using the Rothenberg sequences (2.3), one obtains natural isomorphisms

$$\begin{array}{rcl} L_n^{\langle -\infty\rangle}(R) & \cong & L_n^h(R); \\ L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/2]) & \cong & L_n^h(R[\mathbb{Z}/2]); \\ L_n^{\langle -\infty\rangle}(R[D_\infty]) & \cong & L_n^h(R[D_\infty]). \end{array}$$

Thus, a comparison argument involving the Atiyah-Hirzebruch spectral sequences shows that the obvious map

$$H_n^{D_\infty}\big(\underline{E}D_\infty \to \{\bullet\}; \mathbf{L}_R^h\big) \xrightarrow{\cong} H_n^{D_\infty}\big(\underline{E}D_\infty \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)$$

is bijective for all $n \in \mathbb{Z}$. □

Cappell's UNil-terms [10] have been further investigated in [2],[12] and [13]. The Waldhausen Nil-terms have been analyzed in [15] and [25]. If 2 is inverted, the situation in L-theory simplifies. Namely, for every $n \in \mathbb{Z}$ and every virtually cyclic group V, $H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{L}_R^{\langle\infty\rangle}\big)[1/2] = 0$. Therefore the map

$$H_n^G\big(\underline{E}G; \mathbf{L}_R^{\langle -\infty\rangle}\big)[1/2] \xrightarrow{\cong} H_n^G\big(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty\rangle}\big)[1/2]$$

is an isomorphism for any group G, and the decorations do not play a role (see [29, Proposition 2.10]).

Remark 2.13 (Role of type I and II)**.** Let $\mathcal{VCY}_I$ be the family of subgroups that are either finite or infinite virtually cyclic of type I, i.e., groups admitting an epimorphism onto $\mathbb{Z}$ with finite kernel. Then the following maps are bijections

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCY}_I}(G); \mathbf{K}_R) & \xrightarrow{\cong} & H_n^G(\underline{\underline{E}}G; \mathbf{K}_R); \\ H_n^G\big(\underline{E}G; \mathbf{L}_R^{\langle -\infty\rangle}\big) & \xrightarrow{\cong} & H_n^G\big(E_{\mathcal{VCY}_I}(G); \mathbf{L}_R^{\langle -\infty\rangle}\big). \end{array}$$

For the K-theory case, see, for instance, [15, 19]. The L-theory case is proven in Lück [27, Lemma 4.2]). In particular, for a torsion-free group G, the map

$$H_n^G\big(EG; \mathbf{L}_R^{\langle -\infty\rangle}\big) \quad\xrightarrow{\cong}\quad H_n^G\big(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty\rangle}\big)$$

is a bijection.

3 Proofs of Results

We now prove the results stated in Section 1.

3.1 Hyperbolic groups

Proof of Theorem 1.4. By [31, Corollary 2.11, Theorem 3.1 and Example 3.5] there is a G-pushout

$$\begin{array}{ccc} \coprod_{V\in\mathcal{M}} G\times_V \underline{E}V & \xrightarrow{i} & \underline{E}G \\ \downarrow{\scriptstyle \coprod_{V\in\mathcal{M}} p} & & \downarrow \\ \coprod_{V\in\mathcal{M}} G/V & \longrightarrow & \underline{\underline{E}}G \end{array}$$

where i is an inclusion of G-CW-complexes, p is the obvious projection and $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of G. Now Theorem 1.4 follows from Theorem 2.1 and Lemma 2.6. ☐

Proof of Theorem 1.5. (i) This follows from (1.2) and Theorem 1.4 (i).

(ii) Since any virtually cyclic subgroup of G is trivial or infinite cyclic, the claim follows from Theorem 2.1 and Remark 2.13. ☐

3.2 K- and L-theory of $\mathbb{Z}^d$

As a warm-up for virtually free abelian groups, we compute the K and L-theory of $R[\mathbb{Z}^d]$.

Proof of Theorem 1.7. Using the induction structure of the equivariant homology theory $H_*^?(-;\mathbf{K}_R)$ and the fact that $B\mathbb{Z}^d$ is the d-dimensional torus, it follows that

$$\begin{aligned} H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d;\mathbf{K}_R\big) &\cong H_n^{\{1\}}\big(B\mathbb{Z}^d;\mathbf{K}_R\big) \\ &\cong \bigoplus_{i=0}^{d} H_{n-i}^{\{1\}}\big(\{\bullet\};\mathbf{K}_R\big)^{\binom{d}{i}} \\ &= \bigoplus_{i=0}^{d} K_{n-i}(R)^{\binom{d}{i}}. \end{aligned} \tag{3.1}$$

Similarly one shows that

$$H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d;\mathbf{L}_R^{\langle-\infty\rangle}\big) \cong \bigoplus_{i=0}^{d} L_{n-i}^{\langle-\infty\rangle}(R)^{\binom{d}{i}}. \tag{3.2}$$

Theorem 2.1 implies that

$$\mathrm{Wh}_n(\mathbb{Z}^d;R) := H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d\to\{\bullet\};\mathbf{K}_R\big) \cong H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d\to\underline{\underline{E}}\mathbb{Z}^d;\mathbf{K}_R\big).$$

From [31, Corollary 2.10] it follows that

$$\bigoplus_{C\in\mathcal{MICY}} H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d \to E\mathbb{Z}^d/C;\mathbf{K}_R\big) \cong H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d \to \underline{\underline{E}}\mathbb{Z}^d;\mathbf{K}_R\big).$$

Since $C \subseteq \mathbb{Z}^d$ is maximal infinite cyclic, $\mathbb{Z}^d \cong C \oplus \mathbb{Z}^{d-1}$. Therefore the induction structure and (1.2) imply

$$\begin{aligned}
H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d \to E(\mathbb{Z}^d/C);\mathbf{K}_R\big) &\cong H_n^{C\oplus\mathbb{Z}^{d-1}}\big(EC\times E\mathbb{Z}^{d-1} \to E\mathbb{Z}^{d-1};\mathbf{K}_R\big)\\
&\cong H_n^{C}\big(EC\times B\mathbb{Z}^{d-1} \to B\mathbb{Z}^{d-1};\mathbf{K}_R\big)\\
&\cong H_n^{C}\big((EC\to\{\bullet\})\times B\mathbb{Z}^{d-1};\mathbf{K}_R\big)\\
&\cong \bigoplus_{i=0}^{d-1} H_{n-i}^{C}\big(EC\to\{\bullet\};\mathbf{K}_R\big)^{\binom{d-1}{i}}\\
&\cong \bigoplus_{i=0}^{d-1}\big(NK_{n-i}(R)\oplus NK_{n-i}(R)\big)^{\binom{d-1}{i}}. \qquad (3.3)
\end{aligned}$$

Hence,

$$\mathrm{Wh}_n(\mathbb{Z}^d;R) \;\cong\; \bigoplus_{C\in\mathcal{MICY}}\bigoplus_{i=0}^{d-1}\big(NK_{n-i}(R)\oplus NK_{n-i}(R)\big)^{\binom{d-1}{i}}.$$

From Theorem 2.1 and Lemma 2.6 (i)

$$K_n(R[\mathbb{Z}^d]) \;\cong\; H_n^{\mathbb{Z}^d}(E\mathbb{Z}^d;\mathbf{K}_R)\oplus \mathrm{Wh}_n(\mathbb{Z}^d;R),$$

which, by (3.1), is isomorphic to

$$\left(\bigoplus_{i=0}^{d} K_{n-i}(R)^{\binom{d}{i}}\right)\oplus \mathrm{Wh}_n(\mathbb{Z}^d;R).$$

Finally,

$$L_n^{\langle-\infty\rangle}(R[\mathbb{Z}^d]) \;\cong\; H_n^{\mathbb{Z}^d}\big(E\mathbb{Z}^d;\mathbf{L}_R^{\langle-\infty\rangle}\big) \;\cong\; \bigoplus_{i=0}^{d} L_{n-i}^{\langle-\infty\rangle}(R)^{\binom{d}{i}}$$

by Theorem 2.1, Remark 2.13 and (3.2). □

3.3 Virtually finitely generated abelian groups

For the remainder of the paper we will use the notation introduced in Subsection 1.4 and the following notation. For $C \in I$ there is an obvious extension

$$1 \to C \to N_GC \xrightarrow{p_C} W_GC \to 1,$$

where p_C is the canonical projection. We also have the extension

$$1 \to A/C \to W_G C \xrightarrow{\overline{q_C}} Q_C \to 1, \tag{3.4}$$

which is induced by the given extension $1 \to A \to G \xrightarrow{q} Q \to 1$. Since $C \subseteq A$ is a maximal infinite cyclic subgroup, $A/C \cong \mathbb{Z}^{d-1}$.

Notice that any infinite cyclic subgroup C of A is contained in a unique maximal infinite cyclic subgroup $C_{\max}$ of A. In particular for two maximal infinite cyclic subgroups $C, D \subseteq A$, either $C \cap D = \{0\}$ or $C = D$. Let

$$N_G[C] := \{g \in G \mid |gCg^{-1} \cap C| = \infty\}.$$

For every $C \in I$

$$N_G[C] = N_G C = q^{-1}(Q_C).$$

Consider the following equivalence relation on the set of infinite virtually cyclic subgroups of G. We call V_1 and V_2 equivalent if and only if $(A \cap V_1)_{\max} = (A \cap V_2)_{\max}$. Then for every infinite virtually cyclic subgroup V of G there is precisely one $C \in I$ such that V is equivalent to gCg^{-1}, for some $g \in G$. We obtain from [31, Theorem 2.3] isomorphisms

$$\bigoplus_{C \in I} H_n^{N_G C}\big(\underline{E}N_G C \to p_C^* \underline{E} W_G C; \mathbf{K}_R\big) \quad \cong \quad H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big); \tag{3.5}$$

$$\bigoplus_{C \in I} H_n^{N_G C}\big(\underline{E}N_G C \to p_C^* \underline{E} W_G C; \mathbf{L}_R^{\langle -\infty \rangle}\big) \quad \cong \quad H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}\big), \tag{3.6}$$

where $p_C \colon N_G C = q^{-1}(Q_C) \to W_G C = q^{-1}(Q_C)/C$ is the canonical projection.

Lemma 3.7. *Let $f \colon G_1 \to G_2$ be a surjective group homomorphism. Consider a subgroup $H \subset G_2$. Let Y be a G_1-space and Z be an H-space. Denote by $f_H \colon f^{-1}(H) \to H$ the map induced by f.*

Then there is a natural G_1-homeomorphism

$$G_1 \times_{f^{-1}(H)} \big(\operatorname{res}_{G_1}^{f^{-1}(H)} Y \times f_H^* Z\big) \xrightarrow{\cong} Y \times f^*(G_2 \times_H Z),$$

where f_H^*, f^* and $\operatorname{res}_{G_1}^{f^{-1}(H)}$ denote restriction and the actions on products are the diagonal actions.

Proof. The map sends $\big(g, (y, z)\big)$ to $\big(gy, (f(g), z)\big)$. Its inverse sends $\big(y, (k, z)\big)$ to $\big(h, (h^{-1}y, z)\big)$ for any $h \in G_1$ with $f(h) = k$. □

3.3.1 K-theory in the case of a free conjugation action

Proof of Theorem 1.11. We prove assertions (i) and (ii) simultaneously by a direct computation.

By Theorem 2.1 $H_n^G\big(\underline{\underline{E}}G; \mathbf{K}_R\big) \cong H_n^G\big(\{\bullet\}; \mathbf{K}_R\big)$. Thus,

$$\begin{aligned}
\operatorname{Wh}_n(G; R) \quad &:= \quad H_n^G\big(EG \to \{\bullet\}; \mathbf{K}_R\big) \\
&\cong \quad H_n^G\big(EG \to \underline{\underline{E}}G; \mathbf{K}_R\big) \\
&\cong \quad H_n^G\big(EG \to \underline{E}G; \mathbf{K}_R\big) \oplus H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big),
\end{aligned}$$

using (2.8). From [30, Lemma 6.3] and [31, Corollary 2.11], there is a G-pushout

$$\begin{array}{ccc} \coprod_{F \in J} G \times_F EF & \xrightarrow{i} & EG \\ \downarrow \coprod_{F\in J} p & & \downarrow \\ \coprod_{F \in J} G/F & \longrightarrow & \underline{E}G \end{array} \tag{3.8}$$

where J is a complete system of representatives of maximal finite subgroups of G. This produces an isomorphism

$$\bigoplus_{F\in J} \mathrm{Wh}_n(F;R) := \bigoplus_{F\in J} H_n^F\big(EF \to \{\bullet\}; \mathbf{K}_R\big) \xrightarrow{\cong} H_n^G\big(EG \to \underline{E}G; \mathbf{K}_R\big).$$

By (3.5),

$$\bigoplus_{C\in I} H_n^{N_G C}\big(\underline{E}N_G C \to p_C^* \underline{E}W_G C; \mathbf{K}_R\big) \;\cong\; H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big).$$

Fix C in I. Since the conjugation action of Q on A is free away from 0, it induces an embedding of Q_C into $\mathrm{aut}(C)$. Hence Q_C is either trivial or isomorphic to $\mathbb{Z}/2$. Therefore, $I = I_1 \amalg I_2$, where $I_1 = \big\{C \in I \mid Q_C = \{1\}\big\}$ and $I_2 = \big\{C \in I \mid Q_C = \mathbb{Z}/2\big\}$.

From [31, Corollary 2.10] there is an isomorphism

$$\begin{aligned} \bigoplus_{C\in\mathcal{MICY}(A)} & H_n^{N_A C}\big(EN_A C \to p_C^* EW_A C; \mathbf{K}_R\big) \\ & \cong H_n^A\big(EA \to \underline{\underline{E}}A; \mathbf{K}_R\big) = \mathrm{Wh}_n(A;R), \end{aligned}$$

where $p_C \colon N_A C = A \to W_A C = A/C$ denotes the projection. The conjugation action of Q on A induces an action of Q on $H_n^A\big(EA \to \underline{\underline{E}}A; \mathbf{K}_R\big)$. By the definition of the index set I and the subgroup $Q_C \subseteq Q$, we obtain a $\mathbb{Z}Q$-isomorphism

$$\bigoplus_{C\in I} \mathbb{Z}Q \otimes_{\mathbb{Z}[Q_C]} H_n^A\big(EA \to p_C^* E(A/C); \mathbf{K}_R\big) \cong H_n^A\big(EA \to \underline{\underline{E}}A; \mathbf{K}_R\big),$$

and hence an isomorphism

$$\bigoplus_{C\in I} \mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_n^A\big(EA \to p_C^* E(A/C); \mathbf{K}_R\big) \cong \mathbb{Z} \otimes_{\mathbb{Z}Q} H_n^A\big(EA \to \underline{\underline{E}}A; \mathbf{K}_R\big).$$

If $Q_C = \{1\}$, then $N_G C = A \cong C \oplus \mathbb{Z}^{d-1}$ and $W_G C \cong \mathbb{Z}^{d-1}$. Thus, by (3.3),

$$\begin{aligned} H_n^{N_G C}\big(\underline{E}N_G C \to p_C^* \underline{E}W_G C; \mathbf{K}_R\big) &\cong H_n^A\big(EA \to p_C^* E(A/C); \mathbf{K}_R\big) \\ &\cong H_n^{C\oplus\mathbb{Z}^{d-1}}\big(EC \times E\mathbb{Z}^{d-1} \to E\mathbb{Z}^{d-1}; \mathbf{K}_R\big) \\ &\cong \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(R) \oplus NK_{n-i}(R)\big)^{\binom{d-1}{i}}. \end{aligned}$$

Therefore, the proof of the theorem will be finished once it is established that for $C \in I_2$,

$$H_n^{N_GC}\big(\underline{E}N_GC \to p_C^*\underline{E}W_GC; \mathbf{K}_R\big) \cong \mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_n^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big) \cong \bigoplus_{i=0}^{d-1} NK_{n-i}(R)^{\binom{d-1}{i}}.$$

Assume that $Q_C = \mathbb{Z}/2$. Since the projection $\underline{E}N_GC \times p_C^*\underline{E}W_GC \to \underline{E}N_GC$ is an N_GC-homotopy equivalence, $H_n^{N_GC}\big(\underline{E}N_GC \to p_C^*\underline{E}W_GC; \mathbf{K}_R\big)$ is isomorphic to $H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*\underline{E}W_GC \to p_C^*\underline{E}W_GC; \mathbf{K}_R\big)$, which, by Lemma 3.9(i) below, is isomorphic to

$$H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^*EQ_C) \to p_C^*(EW_GC \times \overline{q_C}^*EQ_C); \mathbf{K}_R\big).$$

Sending a Q_C-CW-complex Y to the $\mathbb{Z}$-graded abelian group

$$H_*^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^*Y) \to p_C^*(EW_GC \times \overline{q_C}^*Y); \mathbf{K}_R\big)$$

yields a Q_C-homology theory. Since EQ_C is a free Q_C-CW-complex, there is an equivariant Atiyah-Hirzebruch spectral sequence converging to

$$H_{i+j}^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^*EQ_C) \to p_C^*(EW_GC \times \overline{q_C}^*EQ_C); \mathbf{K}_R\big)$$

whose E^2-term is given by

$$E_{i,j}^2 = H_i^{Q_C}\Big(EQ_C; H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^*Q_C) \to p_C^*(EW_GC \times \overline{q_C}^*Q_C); \mathbf{K}_R\big)\Big).$$

Lemma 3.9 (ii) implies that:

$$\begin{aligned}
E_{i,j}^2 &\cong H_i^{Q_C}\left(EQ_C; H_j^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big)\right)\\
&\cong H_i^{Q_C}\left(EQ_C; \mathbb{Z}[Q_C] \otimes_{\mathbb{Z}} \left(\mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_j^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big)\right)\right)\\
&\cong H_i^{\{1\}}\big(\operatorname{res}_{Q_C}^{\{1\}} EQ_C; \mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_j^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big)\big)\\
&\cong \begin{cases} \mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_j^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big) & i = 0\\ \{0\} & i \neq 0.\end{cases}\\
&\cong \begin{cases} \bigoplus_{l=0}^{d-1} NK_{j-l}(R)^{\binom{d-1}{l}} & i = 0\\ \{0\} & i \neq 0.\end{cases}
\end{aligned}$$

Hence, the Atiyah-Hirzebruch spectral sequence collapses at E^2. Therefore,

$$H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^*EQ_C) \to p_C^*(EW_GC \times \overline{q_C}^*EQ_C); \mathbf{K}_R\big) \cong \mathbb{Z} \otimes_{\mathbb{Z}[Q_C]} H_n^A\big(EA \to p_C^*E(A/C); \mathbf{K}_R\big) \cong \bigoplus_{l=0}^{d-1} NK_{n-l}(R)^{\binom{d-1}{l}}.$$

This completes the proof of Theorem 1.11 once we have proved Lemma 3.9. $\square$

Lemma 3.9. *Let G be a group satisfying the assumptions of Theorem 1.11, and let C be a maximal cyclic subgroup of G such that $Q_C = \mathbb{Z}/2$. Then:*

(i) For every $n \in \mathbb{Z}$, the projection $EW_GC \times \overline{q_C}^* EQ_C \to \underline{E}W_GC$ induces a bijection,

$$H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* EQ_C) \to p_C^*(EW_GC \times \overline{q_C}^* EQ_C); \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{N_GC}\big(\underline{E}N_GC \times p_C^* \underline{E}W_GC \to p_C^* \underline{E}W_GC; \mathbf{K}_R\big).$$

(ii) There are isomorphisms of $\mathbb{Z}[Q_C]$-modules

$$H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* Q_C) \to p_C^*(EW_GC \times \overline{q_C}^* Q_C); \mathbf{K}_R\big) \cong H_j^A\big(EA \to p_C^* E(A/C); \mathbf{K}_R\big) \cong \bigoplus_{l=0}^{d-1} \big(\mathbb{Z}[Q_C] \otimes_{\mathbb{Z}} NK_{j-l}(R)\big)^{\binom{d-1}{l}}.$$

Proof. (i) Note that the conjugation action of $Q_C = \mathbb{Z}/2$ on A/C is free away from $0 \in A/C$. To see this, let q be a nontrivial element in Q, and let $aC \in A/C$ such that aC is fixed under the conjugation action with q. Then there is a $c \in C$ such that $\rho(q)(a) = a + c$, where $\rho\colon Q \to \operatorname{aut}(A)$ is given by the conjugation action of Q in A. This implies $\rho(q)(2a) = 2a + 2c$. Thus,

$$\rho(q)(2a + c) = \rho(q)(2a) + \rho(q)(c) = \rho(q)(2a) - c = 2a + 2c - c = 2a + c.$$

By assumption, the Q-action on A is free away from $0 \in A$, so $2a + c = 0$ and thus $2a \in C$. Since C is maximal infinite cyclic, $aC = 0$ in A/C. Hence, the conjugation action of Q_C on A/C is free away from $0 \in A/C$.

Since $(A/C)^{Q_C} = \{0\}$, the extension (3.4) has a section, and W_GC is isomorphic to the semidirect product $\mathbb{Z}^{d-1} \rtimes \mathbb{Z}/2$ with respect to the involution $-\operatorname{id}\colon \mathbb{Z}^{d-1} \to \mathbb{Z}^{d-1}$. Let J_C be a complete system of representatives of the conjugacy classes of maximal finite subgroups of W_GC. Then every element in J_C is isomorphic to $\mathbb{Z}/2$ and J_C contains 2^{d-1} elements (see Remark 1.15). From [30, Lemma 6.3] and [31, Corollary 2.11], there is a W_GC-pushout

$$\begin{array}{ccc} \coprod_{H \in J_C} W_GC \times_H EH & \xrightarrow{i} & EW_GC \\ \big\downarrow {\scriptstyle \coprod_{H \in J_C} p} & & \big\downarrow \\ \coprod_{H \in J_C} W_GC/H & \longrightarrow & \underline{E}W_GC \end{array} \tag{3.10}$$

Let $H \in J_C$. Given any N_GC-space Y and any H-space Z, Lemma 3.7 produces an N_GC-homeomorphism

$$N_GC \times_{p_C^{-1}(H)} \big(\operatorname{res}_{N_GC}^{p_C^{-1}(H)}(Y) \times p_C^* Z\big) \xrightarrow{\cong} Y \times p_C^*(W_GC \times_H Z), \tag{3.11}$$

where p_C^* denotes restriction for both $p_C\colon N_GC \to W_GC$ and the homomorphism $p_C^{-1}(H) \to H$ induced by p_C. Since $\operatorname{res}_{N_GC}^{p_C^{-1}(H)}(\underline{E}N_GC)$ is a $p_C^{-1}(H)$-CW-model for $\underline{E}p_C^{-1}(H)$, we obtain identifications of N_GC-spaces

$$\begin{aligned}
\underline{E}N_GC \times p_C^*(W_GC \times_H EH) &\cong N_GC \times_{p_C^{-1}(H)} \big(\underline{E}p_C^{-1}(H) \times p_C^* EH\big);\\
\underline{E}N_GC \times p_C^*(W_GC/H) &\cong N_GC \times_{p_C^{-1}(H)} \big(\underline{E}p_C^{-1}(H)\big);\\
p_C^*(W_GC \times_H EH) &\cong N_GC \times_{p_C^{-1}(H)} p_C^* EH;\\
p_C^*(W_GC/H) &\cong N_GC \times_{p_C^{-1}(H)} \{\bullet\},
\end{aligned}$$

by substituting $Y = \underline{E}N_GC$ or $Y = \{\bullet\}$, and $Z = EH$ or $Z = \{\bullet\}$ into (3.11). Using the induction structure of the equivariant homology theory $H_*^?(-;\mathbf{K}_R)$ and the Five-Lemma, these identifications imply

$$\begin{aligned}
&H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(W_GC \times_H EH) \to p_C^*(W_GC \times_H EH); \mathbf{K}_R\big)\\
&\qquad \xrightarrow{\cong} H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \times p_C^* EH \to p_C^* EH; \mathbf{K}_R\big) \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
&H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \to \{\bullet\}; \mathbf{K}_R\big)\\
&\qquad \xleftarrow{\cong} H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(W_GC/H) \to p_C^*(W_GC/H); \mathbf{K}_R\big) \qquad (3.13)
\end{aligned}$$

are bijective for every $n \in \mathbb{Z}$.

Consider the following $p_C^{-1}(H)$-pushout, where the left vertical arrow is an inclusion of $p_C^{-1}(H)$-CW-complexes and the upper horizontal arrow is cellular.

$$\begin{array}{ccc}
Ep_C^{-1}(H) & \longrightarrow & p_C^* EH\\
\downarrow & & \downarrow\\
\underline{E}p_C^{-1}(H) & \longrightarrow & X
\end{array}$$

Then X is a $p_C^{-1}(H)$-CW-complex, and for every subgroup $K \subseteq p_C^{-1}(H)$, the above $p_C^{-1}(H)$-pushout induces a pushout of CW-complexes:

$$\begin{array}{ccc}
Ep_C^{-1}(H)^K & \longrightarrow & (p_C^* EH)^K\\
\downarrow & & \downarrow\\
\underline{E}p_C^{-1}(H)^K & \longrightarrow & X^K
\end{array}$$

If $K = \{1\}$, then the spaces $Ep_C^{-1}(H)^K$, $(p_C^* EH)^K$ and $\underline{E}p_C^{-1}(H)^K$ are contractible, and thus X^K is contractible. If K is a non-trivial finite subgroup, then the spaces $Ep_C^{-1}(H)^K$ and $(p_C^* EH)^K$ are empty, and the space $\underline{E}p_C^{-1}(H)^K$ is contractible. Hence X^K is contractible. If K is an infinite subgroup of C, then the spaces $Ep_C^{-1}(H)^K$ and $\underline{E}p_C^{-1}(H)^K$ are empty, and the space $(p_C^* EH)^K$ is contractible since $p_C(K) = \{1\}$. Therefore X^K is contractible. If K is infinite and not contained in C, then the spaces $Ep_C^{-1}(H)^K$, $(p_C^* EH)^K$ and $\underline{E}p_C^{-1}(H)^K$ are all empty, and so X^K is also empty. Since $p_C^{-1}(H) \cong D_\infty$, an infinite virtually cyclic subgroup K of $p_C^{-1}(H)$ is of type I if and only if it is an infinite subgroup of C. Hence, X is a model for $E_{\mathcal{VCY}_I}(p_C^{-1}(H))$. Notice that $\underline{E}p_C^{-1}(H) \times p_C^* EH$ is a model for $Ep_C^{-1}(H)$. Therefore, we have the following cellular $p_C^{-1}(H)$-pushout, where the left vertical arrow is an inclusion of $p_C^{-1}(H)$-CW-complexes and the upper horizontal arrow is cellular.

$$\begin{array}{ccc} \underline{E}p_C^{-1}(H) \times p_C^* EH & \longrightarrow & p_C^* EH \\ \downarrow & & \downarrow \\ \underline{E}p_C^{-1}(H) & \longrightarrow & E_{\mathcal{VCY}_I}(p_C^{-1}(H)) \end{array}$$

This induces an isomorphism

$$H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \times p_C^* EH \to p_C^* EH; \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \to E_{\mathcal{VCY}_I}(p_C^{-1}(H)); \mathbf{K}_R\big), \quad (3.14)$$

for every $n \in \mathbb{Z}$. Since $p_C^{-1}(H) \cong D_\infty$ is virtually cyclic, Remark 2.13 implies that the map

$$H_n^{p_C^{-1}(H)}\big(E_{\mathcal{VCY}_I}(p_C^{-1}(H)); \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{p_C^{-1}(H)}\big(\{\bullet\}; \mathbf{K}_R\big)$$

is bijective for every $n \in \mathbb{Z}$, and so by (3.14),

$$H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \times p_C^* EH \to p_C^* EH; \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \to \{\bullet\}; \mathbf{K}_R\big) \quad (3.15)$$

is bijective for every $n \in \mathbb{Z}$. Therefore, (3.12), (3.13) and (3.15) imply that

$$H_n^{N_G C}\big(\underline{E}N_G C \times p_C^*(W_G C \times_H EH) \to p_C^*(W_G C \times_H EH); \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{N_G C}\big(\underline{E}N_G C \times p_C^*(W_G C/H) \to p_C^*(W_G C/H); \mathbf{K}_R\big) \quad (3.16)$$

is an isomorphism for every $n \in \mathbb{Z}$.

We obtain a $W_G C$-homology theory by assigning to a $W_G C$-CW-complex Z the $\mathbb{Z}$-graded abelian group $H_n^{N_G C}\big(\underline{E}N_G C \times p_C^* Z \to p_C^* Z; \mathbf{K}_R\big)$. From the Mayer-Vietoris sequence associated to the $W_G C$-pushout (3.10) and the bijectivity of the map (3.16), there is an isomorphism

$$H_n^{N_G C}\big(\underline{E}N_G C \times p_C^* EW_G C \to p_C^* EW_G C; \mathbf{K}_R\big) \xrightarrow{\cong} H_n^{N_G C}\big(\underline{E}N_G C \times p_C^* \underline{E}W_G C \to p_C^* \underline{E}W_G C; \mathbf{K}_R\big). \quad (3.17)$$

Since the projection $EW_GC \times \overline{q_C}^* EQ_C \to EW_GC$ is a W_GC-homotopy equivalence, the desired isomorphism now follows from (3.17).

(ii) By Lemma 3.7

$$\begin{aligned}
&H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* Q_C) \to p_C^*(EW_GC \times \overline{q_C}^* Q_C); \mathbf{K}_R\big) \\
\cong\ & H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*(W_GC \times_{A/C} \operatorname{res}_{W_GC}^{A/C} EW_GC) \\
&\qquad \to p_C^*(W_GC \times_{A/C} \operatorname{res}_{W_GC}^{A/C} EW_GC); \mathbf{K}_R\big) \\
\cong\ & H_j^{N_GC}\Big(N_GC \times_A \big(\operatorname{res}_{N_GC}^{A} \underline{E}N_GC \times \overline{p}_C^* \operatorname{res}_{W_GC}^{A/C} EW_GC)\big) \\
&\qquad \to N_GC \times_A \big(\overline{p}_C^* \operatorname{res}_{W_GC}^{A/C} EW_GC\big); \mathbf{K}_R\Big).
\end{aligned} \tag{3.18}$$

The generator t of $Q_C = \mathbb{Z}/2$ acts on

$$H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* Q_C) \to p_C^*(EW_GC \times \overline{q_C}^* Q_C); \mathbf{K}_R\big)$$

by the square of N_GC-maps

$$\begin{array}{ccc}
\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* Q_C) & \longrightarrow & p_C^*(EW_GC \times \overline{q_C}^* Q_C) \\
\Big\downarrow{\scriptstyle \mathrm{id} \times p_C^*(\mathrm{id} \times r_t)} & & \Big\downarrow{\scriptstyle p_C^*(\mathrm{id} \times r_t)} \\
\underline{E}N_GC \times p_C^*(EW_GC \times \overline{q_C}^* Q_C) & \longrightarrow & p_C^*(EW_GC \times \overline{q_C}^* Q_C)
\end{array}$$

where $r_t \colon Q_C \to Q_C$ is right multiplication with t.

To simplify notation, let $X = \operatorname{res}_{N_GC}^{A} \underline{E}N_GC$ and $Y = \overline{p}_C^* \operatorname{res}_{W_GC}^{A/C} EW_GC$. Choose an element $\gamma \in N_GC$ that is mapped to t by $q_C \colon N_GC \to Q_C$, and equip

$$H_j^{N_GC}\big(N_GC \times_A (X \times Y) \to N_GC \times_A Y; \mathbf{K}_R\big)$$

with the Q_C-action coming from square of N_GC-maps

$$\begin{array}{ccc}
N_GC \times_A \big(X \times Y\big) & \longrightarrow & N_GC \times_A Y \\
\Big\downarrow{\scriptstyle r_\gamma \times l_{\gamma^{-1}}} & & \Big\downarrow{\scriptstyle r_\gamma \times l_{\gamma^{-1}}} \\
N_GC \times_A \big(X \times Y\big) & \longrightarrow & N_GC \times_A Y
\end{array}$$

where $l_{\gamma^{-1}}$ is given by $(x, y) \mapsto (\gamma^{-1} \cdot x, p_C(\gamma^{-1}) \cdot y)$ for $x \in X$ and $y \in Y$ and r_γ is right multiplication by γ. It is straightforward to check that the isomorphism (3.18) is compatible with this Q_C-action.

Recall that Q_C acts freely on A away from the identity. This implies that if $t \in Q_C$ is the generator and $a \in A$, then $t \cdot (a + ta) = a + ta$, and so $a + ta = 0$. Therefore the Q_C-action on A is given by $-\mathrm{id}_A \colon A \to A$. Since the action of t on A is defined by conjugation by γ^{-1}, the following diagram commutes.

$$\begin{array}{ccc}
N_GC & \xrightarrow{c(\gamma^{-1})} & N_GC \\
\Big\uparrow & & \Big\uparrow \\
A & \xrightarrow{-\mathrm{id}_A} & A
\end{array}$$

For any A-CW complex Z, this yields the commutative square

$$\begin{array}{ccc} H_j^{N_GC}\big(N_GC \times_A Z; \mathbf{K}_R\big) & \xrightarrow[\cong]{f_* \circ \operatorname{ind}_{c(\gamma^{-1})}} & H_j^{N_GC}\big(N_GC \times_A (A \times_{-\operatorname{id}_A} Z); \mathbf{K}_R\big) \\ \operatorname{ind}_A^{N_GC} \uparrow \cong & & \operatorname{ind}_A^{N_GC} \uparrow \cong \\ H_j^A(Z; \mathbf{K}_R) & \xrightarrow[\cong]{\operatorname{ind}_{-\operatorname{id}_A}} & H_j^A(A \times_{-\operatorname{id}_A} Z; \mathbf{K}_R), \end{array}$$

where $\operatorname{ind}_A^{N_GC}$ denotes induction with respect to inclusion, and f_* is induced by the map $f : N_GC \times_{c(\gamma^{-1})} (N_GC \times_A Z) \to N_GC \times_A (A \times_{-\operatorname{id}_A} Z)$, which sends $\big(k, (g, z\big)$ to $\big(kc(\gamma^{-1})(g), (e, z)\big)$. From the axioms of an induction structure, we also have that $\operatorname{ind}_{c(\gamma^{-1})}$ is induced by the map $f_2 : N_GC \times_A Z \to N_GC \times_{c(\gamma^{-1})} (N_GC \times_A Z)$, which sends (k, z) to $\big(e, (\gamma k, z)\big)$ (see [26, Section 1]).

When $Z = X \times Y$ or $Z = Y$, we can define a map $l : A \times_{-\operatorname{id}_A} Z \to Z$ such that $l(a, z) = a\gamma^{-1}z$. Since $\operatorname{ind}_A^{N_GC}$ is natural, this produces the commutative square

$$\begin{array}{ccc} H_j^{N_GC}\big(N_GC \times_A (A \times_{-\operatorname{id}_A} Z); \mathbf{K}_R\big) & \xrightarrow[\cong]{l'_*} & H_j^{N_GC}\big(N_GC \times_A Z; \mathbf{K}_R\big) \\ \operatorname{ind}_A^{N_GC} \uparrow \cong & & \operatorname{ind}_A^{N_GC} \uparrow \cong \\ H_j^A(A \times_{-\operatorname{id}_A} Z; \mathbf{K}_R) & \xrightarrow[\cong]{l_*} & H_j^A(Z; \mathbf{K}_R), \end{array}$$

where $l' : N_GC \times_A (A \times_{-\operatorname{id}_A} Z) \to N_GC \times_A Z$ maps $\big(k(a, z)\big)$ to $(k, a\gamma^{-1}z)$. Notice that the composition $l'_* \circ f_* \circ \operatorname{ind}_{c(\gamma^{-1})} = l'_* \circ f_* \circ (f_2)_*$ is precisely the map $r_\gamma \times l_{\gamma^{-1}}$ used to define the Q_C-action on $H_j^{N_GC}\big(N_GC \times_A (X \times Y) \to N_GC \times_A Y; \mathbf{K}_R\big)$. Thus, the equivalence

$$H_j^{N_GC}\big(N_GC \times_A (X \times Y) \to N_GC \times_A Y; \mathbf{K}_R\big) \quad \xleftarrow{\operatorname{ind}_A^{N_GC}} \quad H_j^A\big(X \times Y \to Y; \mathbf{K}_R\big)$$

is compatible with the Q_C-action, where the action on $H_j^A\big(X \times Y \to Y; \mathbf{K}_R\big)$ is induced by the composition $l_* \circ \operatorname{ind}_{-\operatorname{id}_A}$.

Notice that X is a model for EA and Y is a model for ED. Therefore, the map l defined above is unique up to A-homotopy. Since $C \subseteq A$ is maximal cyclic, there is a $D \subseteq A$ such that $C \oplus D \cong A$ and $D \cong \mathbb{Z}^{d-1}$. Thus, the identification

$$\begin{aligned} H_j^A\big(X \times Y \to Y; \mathbf{K}_R\big) &\cong H_j^{C\oplus D}\big((EC \times ED) \times ED \to ED; \mathbf{K}_R\big) \\ &\cong H_j^{C\oplus D}\big(EC \times ED \to ED; \mathbf{K}_R\big) \end{aligned}$$

agrees with the Q_C-action since the action on $H_j^{C\oplus D}\big(EC \times ED \to ED; \mathbf{K}_R\big)$ is induced by $\big(l_C, l_D)_* \circ \operatorname{ind}_{(-\operatorname{id}_C \oplus -\operatorname{id}_D)}$, where $l_C : \operatorname{ind}_{-\operatorname{id}_C} EC \to EC$ denotes the unique C-equivariant map (up to C-homotopy) and $l_D : \operatorname{ind}_{-\operatorname{id}_D} ED \to ED$

denotes the unique D-equivariant map (up to D-homotopy). Note that this establishes the first desired isomorphism of $\mathbb{Z}[Q_C]$-modules:

$$H_j^{N_G C}\big(\underline{E}N_G C \times p_C^*(EW_G C \times \overline{q_C}^* Q_C) \to p_C^*(EW_G C \times \overline{q_C}^* Q_C); \mathbf{K}_R\big) \\ \cong H_j^A\big(EA \to p_C^* E(A/C); \mathbf{K}_R\big).$$

Using the freeness of the action of D,

$$H_j^{C\oplus D}\big(EC \times ED \to ED; \mathbf{K}_R\big) \quad \cong \quad H_j^C\big(EC \times BD \to BD; \mathbf{K}_R\big).$$

As in (3.3), since BD is the $(d-1)$-dimensional torus,

$$\begin{aligned} H_j^C\big(BD \times EC \to BD; \mathbf{K}_R\big) \quad &\cong \quad \bigoplus_{l=0}^{d-1} H_{j-l}^C\big(EC \to \{\bullet\}; \mathbf{K}_R\big)^{\binom{d-1}{l}} \\ &\cong \quad \bigoplus_{l=0}^{d-1} \big(NK_{j-l}(R) \oplus NK_{j-l}(R)\big)^{\binom{d-1}{l}}. \end{aligned}$$

By Remark 2.10, the Q_C-action on $H_*^C\big(EC \to \{\bullet\}; \mathbf{K}_R\big)$, induced by $(l_C)_* \circ \operatorname{ind}_{-\operatorname{id}_C}$, corresponds to interchanging the two copies of $NK_*(R)$ in the decomposition

$$H_*^C\big(EC \to \{\bullet\}; \mathbf{K}_R\big) \xrightarrow{\cong} NK_*(R) \oplus NK_*(R).$$

Thus, we obtain an isomorphism of $\mathbb{Z}[Q_C]$-modules

$$\bigoplus_{l=0}^{d-1} \big(\mathbb{Z}[Q_C] \otimes_{\mathbb{Z}} NK_{j-l}(R)\big)^{\binom{d-1}{l}} \quad \xrightarrow{\cong} \quad H_j^C\big(BD \times EC \to BD; \mathbf{K}_R\big),$$

which yields the desired result. □

Proof of Theorem 1.12.
(i) Since R is a regular ring, $NK_j(R) = \{0\}$ for every $j \in \mathbb{Z}$. Therefore,

$$\bigoplus_{F \in J} \operatorname{Wh}_n(F; R) \cong \operatorname{Wh}_n(G; R)$$

for every $n \in \mathbb{Z}$ by Theorem 1.11. Furthermore, $K_n(R) = 0$ for every $n \leq -1$. Thus, for any group Γ, the Atiyah-Hirzebruch spectral sequence implies that

$$H_n(B\Gamma; \mathbf{K}(R)) = 0 \quad \text{for } n \leq -1,$$

and the edge homomorphism induces an isomorphism

$$H_0(B\Gamma; \mathbf{K}(R)) \xrightarrow{\cong} K_0(R).$$

Hence,

$$\operatorname{Wh}_n(\Gamma; R) \cong K_n(R\Gamma) \quad \text{for } n \leq -1$$

and

$$\operatorname{Wh}_0(\Gamma; R) \cong \operatorname{coker}\big(K_0(R) \to K_0(R\Gamma)\big).$$

(ii) By Carter [11],

$$K_n(RF) \cong \{0\} \quad \text{for } n \leq -2$$

for every $F \in J$. Now apply assertion (i) to complete the proof. □

3.3.2 L-theory in the case of a free conjugation action

Proof of Theorem 1.13. As in the K-theory proof, Theorem 2.1 and (2.9) imply:

$$\begin{aligned}\mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(G;R) &:= H_n^G\big(EG \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big)\\ &\cong H_n^G\big(EG \to \underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}\big)\\ &\cong H_n^G\big(EG \to \underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}\big) \oplus H_n^G\big(\underline{E}G \to \underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}\big).\end{aligned}$$

From the G-pushout (3.8),

$$\bigoplus_{F\in J} \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(F;R) := \bigoplus_{F\in J} H_n^F\big(EF \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big) \cong H_n^G\big(EG \to \underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}\big).$$

By (3.6),

$$\bigoplus_{C\in I} H_n^{N_GC}\big(\underline{E}N_GC \to p_C^*\underline{E}W_GC;\mathbf{L}_R^{\langle -\infty\rangle}\big) \;\cong\; H_n^G\big(\underline{E}G \to \underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}\big).$$

Recall from the proof of Theorem 1.11 that $I = I_1 \amalg I_2$.

If $C \in I_1$, then $N_GC = A \cong C \oplus \mathbb{Z}^{d-1}$ and $W_GC \cong \mathbb{Z}^{d-1}$. Therefore, as in (3.3), $H_n^{N_GC}\big(\underline{E}N_GC \to p_C^*\underline{E}W_GC;\mathbf{L}_R^{\langle -\infty\rangle}\big)$ is isomorphic to:

$$H_n^{C\oplus\mathbb{Z}^{d-1}}\big(EC\times E\mathbb{Z}^{d-1} \to E\mathbb{Z}^{d-1};\mathbf{L}_R^{\langle -\infty\rangle}\big) \;\cong\; \bigoplus_{i=0}^{d-1} H_{n-i}^C\big(EC \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big) \;=\; 0$$

by Remark 2.13.

Now assume $C \in I_2$. We obtain a W_GC-homology theory by assigning to a W_GC-CW-complex Z the $\mathbb{Z}$-graded abelian group

$$H_*^{N_GC}\big(\underline{E}N_GC \times p_C^*Z \to p_C^*Z;\mathbf{L}_R^{\langle -\infty\rangle}\big).$$

Consider any free W_GC-CW-complex Y. Then there is an equivariant Atiyah-Hirzebruch spectral sequence converging to

$$H_{i+j}^{N_GC}\big(\underline{E}N_GC \times p_C^*Y \to p_C^*Y;\mathbf{L}_R^{\langle -\infty\rangle}\big),$$

whose E^2-term is given by

$$E^2_{i,j} = H_i^{W_GC}\left(Y; H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*W_GC \to p_C^*W_GC;\mathbf{L}_R^{\langle -\infty\rangle}\big)\right).$$

Using Lemma 3.7,

$$\begin{aligned}&H_j^{N_GC}\big(\underline{E}N_GC \times p_C^*W_GC \to p_C^*W_GC;\mathbf{L}_R^{\langle -\infty\rangle}\big)\\ \cong\;& H_j^{N_GC}\big(N_GC \times_C \operatorname{res}^C_{N_GC}\underline{E}N_GC \to N_GC \times_C \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big)\\ \cong\;& H_j^C\big(\operatorname{res}^C_{N_GC}\underline{E}N_GC \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big)\\ \cong\;& H_j^C\big(EC \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}\big).\end{aligned}$$

But this is zero by Remark 2.13. Hence, $E^2_{i,j} = 0$ for all $i, j \in \mathbb{Z}$. This implies that for every free W_GC-CW-complex Y, $H^{N_GC}_*\big(\underline{E}N_GC \times p_C^*Y \to p_C^*Y; \mathbf{L}_R^{\langle -\infty\rangle}\big)$ vanishes. In particular,

$$H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*EW_GC \to p_C^*EW_GC; \mathbf{L}_R^{\langle -\infty\rangle}\big) = 0$$

and

$$H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*(W_GC \times_H EH) \to p_C^*(W_GC \times_H EH); \mathbf{L}_R^{\langle -\infty\rangle}\big) = 0.$$

Therefore the Mayer-Vietoris sequence associated to the W_GC-pushout (3.10) yields an isomorphism

$$\bigoplus_{H \in J_C} H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*W_GC/H \to p_C^*W_GC/H; \mathbf{L}_R^{\langle -\infty\rangle}\big) \xrightarrow{\cong} H_*^{N_GC}\big(\underline{E}N_GC \times p_C^*\underline{E}W_GC \to p_C^*\underline{E}W_GC; \mathbf{L}_R^{\langle -\infty\rangle}\big).$$

Since $p_C^{-1}(H) \cong D_\infty$, Lemma 3.7 implies:

$$\begin{aligned}
&H_n^{N_GC}\big(\underline{E}N_GC \times p_C^*W_GC/H \to p_C^*W_GC/H; \mathbf{L}_R^{\langle -\infty\rangle}\big)\\
&\quad\cong\quad H_n^{N_GC}\big(N_GC \times_{p_C^{-1}(H)} \operatorname{res}^{p_C^{-1}(H)}_{N_GC} \underline{E}N_GC \to N_GC \times_{p_C^{-1}(H)} \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)\\
&\quad\cong\quad H_n^{p_C^{-1}(H)}\big(\operatorname{res}^{p_C^{-1}(H)}_{N_GC} \underline{E}N_GC \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)\\
&\quad\cong\quad H_n^{p_C^{-1}(H)}\big(\underline{E}p_C^{-1}(H) \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)\\
&\quad\cong\quad H_n^{D_\infty}\big(\underline{E}D_\infty \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}\big)\\
&\quad=\quad \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty; R).
\end{aligned}$$

The projection $\underline{E}N_GC \times p_C^*\underline{E}W_GC \to \underline{E}N_GC$ is a N_GC-homotopy equivalence, thus

$$\bigoplus_{H \in J_C} \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty; R) \quad \xrightarrow{\cong} \quad H_n^{N_GC}\big(\underline{E}N_GC \to p_C^*\underline{E}W_GC; \mathbf{L}_R^{\langle -\infty\rangle}\big).$$

Therefore,

$$\bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty; R) \quad\cong\quad H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty}\big),$$

which implies that

$$\left(\bigoplus_{F \in J} \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(F; R)\right) \oplus \left(\bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty; R)\right) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(G; R).$$

Theorem 1.13 now follows from the fact that I_2 is empty if Q has odd order. $\square$

3.3.3 K-theory in the case $Q = \mathbb{Z}/p$ for a prime p and regular R

Lemma 3.19. *Let $f\colon G \to G'$ be a group homomorphism, $\mathcal{F}$ be a family of subgroups of G and R be a ring. Let X be a G'-CW-complex such that for every isotropy group $H' \subseteq G'$ of X and every $n \in \mathbb{Z}$,*

$$H_n^{f^{-1}(H')}\left(E_{\mathcal{F}\cap f^{-1}(H')}\big(f^{-1}(H') \to \{\bullet\}\big);\mathbf{K}_R\right) = 0,$$

where $\mathcal{F} \cap f^{-1}(H') := \{K \cap f^{-1}(H') \mid K \in \mathcal{F}\}$. Then, for every $n \in \mathbb{Z}$,

$$H_n^G\big(E_{\mathcal{F}}G \times f^*X \to f^*X;\mathbf{K}_R\big) = 0.$$

Proof. Sending X to $H_n^G\big(E_{\mathcal{F}}G \times f^*X \to f^*X;\mathbf{K}_R\big)$ defines a G'-homology theory. There is an equivariant version of the Atiyah-Hirzebruch spectral sequence converging to $H_{i+j}^G\big(E_{\mathcal{F}}G \times f^*X \to f^*X;\mathbf{K}_R\big)$ (see Davis-Lück [16, Theorem 4.7]). Its E^2-term is the Bredon homology of X

$$E^2_{i,j} = H_i^{\mathsf{Or}(G')}(X;V_j),$$

where the coefficients are given by the covariant functor

$$V_j\colon \mathsf{Or}(G') \to \mathbb{Z}\text{-}\mathsf{MODULES}$$

defined by

$$G'/H' \mapsto H_j^G\big(E_{\mathcal{F}}G \times f^*(G'/H') \to f^*(G'/H');\mathbf{K}_R\big).$$

It suffices to show that the E^2-term is trivial for all i, j. We will do this by showing that for every subgroup $H' \subseteq G'$ and every $j \in \mathbb{Z}$,

$$H_j^G\big(E_{\mathcal{F}}G \times f^*(G'/H') \to f^*(G'/H');\mathbf{K}_R\big) = 0.$$

Using the induction structure and Lemma 3.7

$$\begin{aligned}
&H_j^G\big(E_{\mathcal{F}}G \times f^*(G'/H') \to f^*(G'/H');\mathbf{K}_R\big)\\
\cong\;& H_j^G\left(G \times_{f^{-1}(H')}\left(\operatorname{res}_G^{f^{-1}(H')} E_{\mathcal{F}}G \to \{\bullet\}\right);\mathbf{K}_R\right)\\
\cong\;& H_j^G\left(G \times_{f^{-1}(H')}\left(E_{\mathcal{F}\cap f^{-1}(H')}\big(f^{-1}(H')\big) \to \{\bullet\}\right);\mathbf{K}_R\right)\\
\cong\;& H_j^{f^{-1}(H')}\left(E_{\mathcal{F}\cap f^{-1}(H')}\big(f^{-1}(H') \to \{\bullet\}\big);\mathbf{K}_R\right),
\end{aligned}$$

which is zero by assumption. □

The following general lemma will also be needed.

Lemma 3.20. *Let G_1 and G_2 be two groups. Let X be a G_1-CW-complex, and let $\operatorname{pr}\colon G_1 \times G_2 \to G_1$ be projection. Then, for every $n \in \mathbb{Z}$, there are natural isomorphisms*

$$\begin{aligned}
H_n^{G_1\times G_2}\big(\operatorname{pr}^* X;\mathbf{K}_R\big) &\cong H_n^{G_1}\big(X;\mathbf{K}_{R[G_2]}\big);\\
H_n^{G_1\times G_2}\big(\operatorname{pr}^* X;\mathbf{L}_R^{\langle-\infty\rangle}\big) &\cong H_n^{G_1}\big(X;\mathbf{L}_{R[G_2]}^{\langle-\infty\rangle}\big).
\end{aligned}$$

Proof. We use the notation from [29, Section 6]). Let $\mathrm{pr}\colon \mathsf{Or}(G_1 \times G_2) \to \mathsf{Or}(G_2)$ be the functor given by induction with pr. It sends $(G_1 \times G_2)/H$ to $G_1/\operatorname{pr}(H)$. Because of the adjunction between induction and restriction (see [16, Lemma 1.9]) and [16, Lemma 4.6]), it suffices to construct a weak equivalence of covariant $\mathsf{Or}(G_1)$-spectra

$$\mathrm{pr}_*\bigl(\mathbf{K}_R \circ \mathcal{G}^{G_1 \times G_2}\bigr) \xrightarrow{\cong} \mathbf{K}_{RG_2} \circ \mathcal{G}^{G_1}.$$

By definition, $\mathrm{pr}_*\bigl(\mathbf{K}_R \circ \mathcal{G}^{G_1 \times G_2}\bigr)$ sends G_1/H to $\mathbf{K}_R\bigl(\mathcal{G}^{G_1}(G_1) \times \widehat{G_2}\bigr)$, where $\widehat{G_2}$ is the groupoid with one object and G_2 as its automorphism group. The desired weak equivalence of covariant $\mathsf{Or}(G_1)$-spectra is now obtained by comparing the definition of the category of free $R\bigl(\mathcal{G}^{G_1}(G_1) \times \widehat{G_2}\bigr)$-modules with the definition of the category of free $R[G_2]\mathcal{G}^{G_1}(G_1)$-modules. $\square$

Proof of Theorem 1.14. By Theorem 2.1 and (2.8),

$$\mathrm{Wh}_n(G;R) \;\cong\; H_n^G\bigl(EG \to \underline{E}G;\mathbf{K}_R\bigr) \oplus H_n^G\bigl(\underline{E}G \to \underline{\underline{E}}G;\mathbf{K}_R\bigr).$$

We begin by analyzing $H_n^G\bigl(EG \to \underline{E}G;\mathbf{K}_R\bigr)$.

Note that every non-trivial finite subgroup H of G is isomorphic to $\mathbb{Z}/p$, and is maximal. Furthermore, the normalizer N_GH is isomorphic to $A^{\mathbb{Z}/p} \times H$. This can be seen as follows. The homomorphism $q\colon G \to Q = \mathbb{Z}/p$ induces an injection $H \to \mathbb{Z}/p$, which must be an isomorphism for non-trivial H. Clearly $A^{\mathbb{Z}/p}$ and H belong to N_GH, and the subgroup generated by $A^{\mathbb{Z}/p}$ and H is isomorphic to $A^{\mathbb{Z}/p} \times H$. Thus, it remains to show that $N_GH \subseteq A^{\mathbb{Z}/p} \times H$. Let $t \in H$ be a generator of H. Then every element in G is of the form at^i for some $i \in \{0,1,2,\dots p-1\}$ and some $a \in A$. If $at^i \in N_GH$, then $at^it(at^i)^{-1} = ata^{-1} = t^j$ for some $j \in \{0,1,2,\dots p-1\}$. Since $q(t) = q(ata^{-1}) = q(t^j) = q(t)^j$, $j = 1$. Thus $ata^{-1} = t$ and $t^{-1}at = a$. This implies that $a \in A^{\mathbb{Z}/p}$, and hence, $at^i \in A^{\mathbb{Z}/p} \times H$.

From [31, Corollary 2.10], there is a G-pushout

$$\begin{array}{ccc}
\coprod_{H\in J} G \times_{A^{\mathbb{Z}/p}\times H} EA^{\mathbb{Z}/p} \times EH & \xrightarrow{\;i\;} & EG \\
\Big\downarrow{\scriptstyle \coprod_{H\in J} p} & & \Big\downarrow \\
\coprod_{H\in J} G \times_{A^{\mathbb{Z}/p}\times H} EA^{\mathbb{Z}/p} & \longrightarrow & \underline{E}G
\end{array}$$

which induces an isomorphism

$$\bigoplus_{H\in J} H_n^G\bigl(G \times_{A^{\mathbb{Z}/p}\times H} (EA^{\mathbb{Z}/p} \times EH) \to G \times_{A^{\mathbb{Z}/p}\times H} EA^{\mathbb{Z}/p};\mathbf{K}_R\bigr) \xrightarrow{\cong} H_n^G\bigl(EG \to \underline{E}G;\mathbf{K}_R\bigr),$$

where J is a complete system of representatives of the conjugacy classes of maximal finite subgroups of G. Since $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$, the induction structure implies

$$\begin{aligned}
& H_n^G\big(G \times_{A^{\mathbb{Z}/p} \times H} (EA^{\mathbb{Z}/p} \times EH) \to G \times_{A^{\mathbb{Z}/p} \times H} EA^{\mathbb{Z}/p}; \mathbf{K}_R\big) \\
\cong \quad & H^{A^{\mathbb{Z}/p} \times H}\big(EA^{\mathbb{Z}/p} \times EH \to EA^{\mathbb{Z}/p}; \mathbf{K}_R\big) \\
\cong \quad & H^H\big(BA^{\mathbb{Z}/p} \times (EH \to \{\bullet\}); \mathbf{K}_R\big) \\
\cong \quad & \bigoplus_{i=0}^{e} H_{n-i}^H\big(EH \to \{\bullet\}; \mathbf{K}_R\big)^{\binom{e}{i}} \\
\cong \quad & \bigoplus_{i=0}^{e} \mathrm{Wh}_{n-i}(H; R)^{\binom{e}{i}}.
\end{aligned}$$

Therefore,

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathrm{Wh}_{n-i}(H; R)^{\binom{e}{i}} \quad \xrightarrow{\cong} \quad H_n^G\big(EG \to \underline{E}G; \mathbf{K}_R\big). \tag{3.21}$$

Now we turn our attention to $H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big)$. Let $\overline{A} := A/A^{\mathbb{Z}/p}$ and $\overline{G} := G/A^{\mathbb{Z}/p}$. Then the exact sequence (1.10) induces the exact sequence

$$1 \to \overline{A} \to \overline{G} \xrightarrow{q} Q \to 1. \tag{3.22}$$

Let $\mathcal{F}$ be the family of subgroups K of G for which $\mathrm{pr}(K)$ is finite, where $\mathrm{pr} : G \to \overline{G}$ is the quotient homomorphism. Let $\mathcal{VCY}$ be the family of virtually cyclic subgroups of G, $\mathcal{VCY}_I$ be the family of virtually cyclic subgroups of type I, and $\mathcal{F}_1 = \mathcal{F} \cap \mathcal{VCY}_I$. The Farrell-Jones Conjecture in algebraic K-theory is true for any group appearing in $\mathcal{F}$, since every element in $\mathcal{F}$ is virtually finitely generated abelian (see Theorem 2.1). It is straightforward to check that $\mathrm{pr}^* \, \underline{E}\overline{G}$ is a model for $E_{\mathcal{F}}(G)$. By the Transitivity Principle (see [21, Theorem A.10] and [29, Theorem 65]) and Remark 2.13,

$$H_n^G\big(E_{\mathcal{F}_1}(G); \mathbf{K}_R\big) \quad \xrightarrow{\cong} \quad H_n^G\big(\mathrm{pr}^* \, \underline{E}\overline{G}; \mathbf{K}_R\big). \tag{3.23}$$

Since every virtually cyclic subgroup of type I in $\overline{G}$ is infinite cyclic, every element $K \in \mathcal{VCY}_I$ belongs to $\mathcal{F}_1$, or is infinite cyclic and $\{K' \mid K' \subseteq K \text{ and } K \in \mathcal{F}_1\}$ consists of just the trivial group. Since R is assumed to be regular, the map $H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to H_n^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_R)$ is bijective for every $n \in \mathbb{Z}$. Therefore, the Transitivity Principle implies that

$$H_n^G\big(E_{\mathcal{F}_1}(G); \mathbf{K}_R\big) \quad \cong \quad H_n^G\big(E_{\mathcal{VCY}_I}(G); \mathbf{K}_R\big). \tag{3.24}$$

By Remark 2.13,

$$H_n^G\big(E_{\mathcal{VCY}_I}(G); \mathbf{K}_R\big) \quad \xrightarrow{\cong} \quad H_n^G\big(\underline{\underline{E}}G; \mathbf{K}_R\big) \tag{3.25}$$

is also a bijection. Thus, (3.23), (3.24), (3.25) and the Five-Lemma imply that, for every $n \in \mathbb{Z}$,

$$H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big) \;\cong\; H_n^G\big(\underline{E}G \to \mathrm{pr}^* \, \underline{E}\overline{G}; \mathbf{K}_R\big). \tag{3.26}$$

Next we show that the conjugation action of $\mathbb{Z}/p$ on $\overline{A}$ is free away from $0 \in \overline{A}$. Let t be an element of G that is mapped by $q\colon G \to \mathbb{Z}/p$ to a generator of $\mathbb{Z}/p$. Let $a \in A$ be given such that $\operatorname{pr}(t)\operatorname{pr}(a)\operatorname{pr}(t)^{-1} = \operatorname{pr}(a)$. Thus, there is a $b \in A^{\mathbb{Z}/p}$ such that $tat^{-1} = ab$. If $\rho\colon A \to A$ denotes the conjugation action with t, then $\rho(a) - a = b$ in A, where the group operation in A is written additively. Since

$$p \cdot b = \sum_{i=0}^{p-1} \rho^i(b) = \sum_{i=0}^{p-1} \rho^i\bigl(\rho(a) - a\bigr) = \sum_{i=0}^{p-1} \rho^{i+1}(a) - \rho^i(a) = \rho^p(a) - a = 0,$$

it follows that $\rho(a) - a = b = 0$. This implies that $a \in A^{\mathbb{Z}/p}$, and thus $\overline{a} = 0$ in $\overline{A}$. Therefore, the conjugation action of $\mathbb{Z}/p$ on $\overline{A}$ is free away from $0 \in \overline{A}$.

Let $\overline{J}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of $\overline{G}$. From [30, Lemma 6.3] and [31, Corollary 2.11], there is a $\overline{G}$-pushout

$$\begin{array}{ccc} \coprod_{\overline{H}\in\overline{J}} \overline{G} \times_{\overline{H}} E\overline{H} & \xrightarrow{i} & E\overline{G} \\ \downarrow & & \downarrow \\ \coprod_{\overline{H}\in\overline{J}} \overline{G} \times_{\overline{H}} \{\bullet\} & \longrightarrow & \underline{E}\overline{G} \end{array} \tag{3.27}$$

Sending a $\overline{G}$-CW-complex $\overline{X}$ to the $\mathbb{Z}$-graded abelian group $H_*^G\bigl(\underline{E}G \times \operatorname{pr}^* \overline{X} \to \operatorname{pr}^* \overline{X}; \mathbf{K}_R\bigr)$ defines a $\overline{G}$-homology theory. Since $\operatorname{pr}^{-1}(\{1\}) = A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$ and R is regular, Theorem 1.7 and Lemma 3.19 imply that if $\overline{X}$ is a free $\overline{G}$-CW-complex, then $H_*^G\bigl(\underline{E}G \times \operatorname{pr}^* \overline{X} \to \operatorname{pr}^* \overline{X}; \mathbf{K}_R\bigr) = 0$. Therefore, because $\coprod_{\overline{H}\in\overline{J}} \overline{G} \times_{\overline{H}} E\overline{H}$ and $E\overline{G}$ are free $\overline{G}$-CW-complexes, the Mayer-Vietoris sequence associated to the $\overline{G}$-pushout (3.27) yields an isomorphism

$$\bigoplus_{\overline{H}\in\overline{J}} H_n^G\bigl(\underline{E}G \times \operatorname{pr}^* \overline{G}/\overline{H} \to \operatorname{pr}^* \overline{G}/\overline{H}; \mathbf{K}_R\bigr) \cong H_n^G\bigl(\underline{E}G \times \operatorname{pr}^* \underline{E}\overline{G} \to \operatorname{pr}^* \underline{E}\overline{G}; \mathbf{K}_R\bigr). \tag{3.28}$$

Notice that if $\operatorname{pr}^{-1}(\overline{H})$ is torsion-free, then it is isomorphic to $\mathbb{Z}^e$. If $\overline{H}$ is trivial, then this follows from $\operatorname{pr}^{-1}(\{1\}) = A^{\mathbb{Z}/p}$. If $\overline{H}$ is a non-trivial finite subgroup of $\overline{G}$, choose an element $t \in \operatorname{pr}^{-1}(\overline{H})$ such that $\operatorname{pr}(t)$ is a generator of $\overline{H}$. Then every element in $\operatorname{pr}^{-1}(\overline{H})$ is of the form at^u for $a \in A^{\mathbb{Z}/p}$ and $u \in \{0, 1, 2, \ldots, p-1\}$. For two such elements at^u and bt^v

$$at^u bt^v = abt^u t^v = bat^v t^u = bt^v at^u.$$

Hence $\operatorname{pr}^{-1}(\overline{H})$ is a torsion-free abelian group containing $\mathbb{Z}^e$ as subgroup of finite index, and so $\operatorname{pr}^{-1}(\overline{H}) \cong \mathbb{Z}^e$. Thus, if $\operatorname{pr}^{-1}(\overline{H})$ is torsion-free, then Theorem 1.7 and Lemma 3.19 imply that

$$H_n^G\bigl(\underline{E}G \times \operatorname{pr}^* \overline{G}/\overline{H} \to \operatorname{pr}^* \overline{G}/\overline{H}; \mathbf{K}_R\bigr) = 0.$$

Therefore, if $\overline{J}'$ is the subset of $\overline{J}$ consisting of those elements $\overline{H} \in J$ for which $\mathrm{pr}^{-1}(\overline{H})$ is not torsion-free, then (3.28) becomes

$$\bigoplus_{\overline{H}\in\overline{J}'} H_n^G\big(\underline{E}G \times \mathrm{pr}^*\, \overline{G}/\overline{H} \to \mathrm{pr}^*\, \overline{G}/\overline{H}; \mathbf{K}_R\big) \cong H_n^G\big(\underline{E}G \times \mathrm{pr}^*\, \underline{E}\overline{G} \to \mathrm{pr}^*\, \underline{E}\overline{G}; \mathbf{K}_R\big). \quad (3.29)$$

Let $\overline{H} \in \overline{J}'$. Then $\mathrm{pr}^{-1}(\overline{H}) \cong A^{\mathbb{Z}/p} \times \overline{H}$. Since $A^{\mathbb{Z}/p} \cong \mathbb{Z}^e$, the induction structure, Lemma 3.7, Lemma 3.20 and Theorem 1.7 imply

$$\begin{aligned}
& H_n^G\big(\underline{E}G \times \mathrm{pr}^*\, \overline{G}/\overline{H} \to \mathrm{pr}^*\, \overline{G}/\overline{H}; \mathbf{K}_R\big) \\
\cong\ & H_n^G\Big(G \times_{A^{\mathbb{Z}/p}\times\overline{H}} \big(\mathrm{res}_G^{A^{\mathbb{Z}/p}\times\overline{H}}\, \underline{E}G \to \{\bullet\}\big); \mathbf{K}_R\Big) \\
\cong\ & H_n^{A^{\mathbb{Z}/p}\times\overline{H}}\big(\mathrm{res}_G^{A^{\mathbb{Z}/p}\times\overline{H}}\, \underline{E}G \to \{\bullet\}; \mathbf{K}_R\big) \\
\cong\ & H_n^{A^{\mathbb{Z}/p}\times\overline{H}}\big(EA^{\mathbb{Z}/p} \to \{\bullet\}; \mathbf{K}_R\big) \\
\cong\ & H_n^{A^{\mathbb{Z}/p}}\big(EA^{\mathbb{Z}/p} \to \{\bullet\}; \mathbf{K}_{R[\overline{H}]}\big) \\
\cong\ & \mathrm{Wh}_n(A^{\mathbb{Z}/p}; R[\mathbb{Z}/p]) \\
\cong\ & \bigoplus_{C\in\mathcal{MICY}(A^{\mathbb{Z}/p})} \bigoplus_{i=0}^{e-1} \big(NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p])\big)^{\binom{e-1}{i}}.
\end{aligned}$$

Since the projection $\underline{E}G \times \mathrm{pr}^*\, \underline{E}\overline{G} \to \underline{E}G$ is a G-homotopy equivalence, (3.29) implies that

$$\bigoplus_{\overline{H}\in\overline{J}'} \bigoplus_{C\in\mathcal{MICY}(A^{\mathbb{Z}/p})} \bigoplus_{i=0}^{e-1} \big(NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p])\big)^{\binom{e-1}{i}} \cong H_n^G\big(\underline{E}G \to \mathrm{pr}^*\, \underline{E}\overline{G}; \mathbf{K}_R\big).$$

Together with (3.26), this produces the isomorphism

$$\bigoplus_{\overline{H}\in\overline{J}'} \bigoplus_{C\in\mathcal{MICY}(A^{\mathbb{Z}/p})} \left(\bigoplus_{i=0}^{e-1} \big(NK_{n-i}(R[\mathbb{Z}/p]) \oplus NK_{n-i}(R[\mathbb{Z}/p])\big)^{\binom{e}{i}}\right) \cong H_n^G\big(\underline{E}G \to \underline{\underline{E}}G; \mathbf{K}_R\big).$$

To complete the proof of the theorem, we must show that there is a bijection between the sets J and $\overline{J}'$. Send an element $H \in J$ to the element in $H' \in \overline{J}'$ that is uniquely determined by the property that $\mathrm{pr}(H)$ and H' are conjugated in $\overline{G}$. This map is well-defined and injective because two non-trivial finite subgroups H_1 and H_2 of G are conjugate if and only if $\mathrm{pr}(H_1)$ and $\mathrm{pr}(H_2)$ are conjugate in $\overline{G}$. The map is surjective, since for any element $\overline{H} \in \overline{J}'$ there exists a finite subgroup $H \subseteq G$ with $\overline{H} = \mathrm{pr}(H)$. This finishes the proof of Theorem 1.14. $\square$

3.3.4 L-theory in the case $Q = \mathbb{Z}/p$ for an odd prime p

Proof of Theorem 1.16. An argument completely analogous to the one that established (3.21) in the proof of Theorem 1.14 shows that

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathcal{S}_{n-i}^{\mathrm{per},\langle -\infty \rangle}(H;R)^{\binom{e}{i}} \xrightarrow{\cong} H_n^G\big(EG \to \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}\big). \tag{3.30}$$

Since p is odd, every infinite virtually cyclic subgroup of G is of type I. Thus the result follows from Theorem 2.1, Remark 2.13 and (3.30). □

Proof of Theorem 1.17. There are natural maps $\mathbf{L}_{\mathbb{Z}}^{\langle j \rangle} \to \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}$ of covariant functors from GROUPOIDS to SPECTRA for $j \in \{2, 1, 0, -1, \ldots\}$. Let $\mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}$ be the covariant functor from GROUPOIDS to SPECTRA that assigns to a groupoid $\mathcal{G}$ the homotopy cofiber of the map of spectra obtained by evaluating $\mathbf{L}_{\mathbb{Z}}^{\langle j \rangle} \to \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}$ at $\mathcal{G}$. We obtain equivariant homology theories $H_*^?\big(-; \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big)$, $H_*^?\big(-; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big)$ and $H_*^?\big(-; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big)$ (see Lück-Reich [29, Section 6]), such that for any group G and any G-CW-complex X, there is a long exact sequence

$$\begin{aligned} \cdots \to H_n^G\big(X; \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big) \to H_n^G\big(X; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \to H_n^G\big(X; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big) \\ \to H_{n-1}^G\big(X; \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big) \to H_{n-1}^G\big(X; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \to \cdots . \end{aligned} \tag{3.31}$$

If $X = \{\bullet\}$, then

$$\begin{aligned} H_n^G\big(\{\bullet\}; \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big) &\cong L_n^{\langle j+1 \rangle}(\mathbb{Z}G); \\ H_n^G\big(\{\bullet\}; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) &\cong L_n^{\langle j \rangle}(\mathbb{Z}G); \\ H_n^G\big(\{\bullet\}; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big) &\cong \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathbb{Z}G)), \end{aligned}$$

and the sequence (3.31) can be identified with the Rothenberg sequence (2.3). Since $L_n^{\langle j+1 \rangle}(\mathbb{Z}) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z})$ is a bijection for every $n \in \mathbb{Z}$ by (2.3), a spectral sequence argument shows that the map

$$H_n\big(BG; \mathbf{L}^{\langle j+1 \rangle}(\mathbb{Z})\big) \xrightarrow{\cong} H_n\big(BG; \mathbf{L}^{\langle j \rangle}(\mathbb{Z})\big)$$

is a bijection for every $n \in \mathbb{Z}$. Thus, $H_n^G\big(EG; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big) \cong H_n\big(BG; \mathbf{L}^{\langle j+1, j \rangle}(\mathbb{Z})\big)$ is zero for all $n \in \mathbb{Z}$. Hence, for every $n \in \mathbb{Z}$, there is an isomorphism

$$H_n^G\big(EG \to \{\bullet\}; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big) \xrightarrow{\cong} H_n^G\big(\{\bullet\}; \mathbf{L}_{\mathbb{Z}}^{\langle j+1, j \rangle}\big).$$

This implies that there is also a Rothenberg sequence for the structure sets

$$\begin{aligned} \cdots \to \mathcal{S}_n^{\mathrm{per},\langle j+1 \rangle}(G;\mathbb{Z}) \to \mathcal{S}_n^{\mathrm{per},\langle j \rangle}(G;\mathbb{Z}) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathbb{Z}G)) \\ \to \mathcal{S}_{n-1}^{\mathrm{per},\langle j+1 \rangle}(G;\mathbb{Z}) \to \mathcal{S}_{n-1}^{\mathrm{per},\langle j \rangle}(G;\mathbb{Z}) \to \cdots . \end{aligned} \tag{3.32}$$

Since $NK_n(\mathbb{Z}[\mathbb{Z}/p]) = 0$ for $n \leq 1$ (see Bass-Murthy [8]), Theorem 1.14 implies that, for every $n \leq 1$,

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathrm{Wh}_{n-i}(H;\mathbb{Z})^{\binom{e}{i}} \xrightarrow{\cong} \mathrm{Wh}_n(G;\mathbb{Z}). \tag{3.33}$$

For every $H \in J$ and $j \leq -2$, $K_j(\mathbb{Z}H) = 0$ by Carter [11]. Hence $K_j(\mathbb{Z}G)$ vanishes for $j \leq -2$ by (3.33). Theorem 1.16 and (3.32) imply that the map

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathcal{S}_n^{\mathrm{per},\langle j\rangle}(H;\mathbb{Z})^{\binom{e}{i}} \to \mathcal{S}_n^{\mathrm{per},\langle j\rangle}(G;\mathbb{Z})$$

is bijective for all $n \in \mathbb{Z}$ and $j \in \{-1,-2,\ldots\} \amalg \{-\infty\}$.

Next we use an inductive argument to show that this is also true for $j = 0, 1$. By taking the direct sum of the Rothenberg sequences (3.32) for the various elements $H \in J$ and mapping it to (3.32) for G, one obtains the following commutative diagram.

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}_{n-i}^{\mathrm{per},\langle j+1\rangle}(H;\mathbb{Z})^{\binom{e}{i}} & \longrightarrow & \mathcal{S}_{n-i}^{\mathrm{per},\langle j+1\rangle}(G;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}_{n-i}^{\mathrm{per},\langle j\rangle}(H;\mathbb{Z})^{\binom{e}{i}} & \longrightarrow & \mathcal{S}_{n}^{\mathrm{per},\langle j\rangle}(G;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\widehat{H}^{n-i}(\mathbb{Z}/2;\widetilde{K}_j(\mathbb{Z}H))^{\binom{e}{i}} & \xrightarrow{\cong} & \widehat{H}^{n}(\mathbb{Z}/2;\widetilde{K}_j(\mathbb{Z}G)) \\
\downarrow & & \downarrow \\
\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}_{n-1-i}^{\mathrm{per},\langle j+1\rangle}(H;\mathbb{Z})^{\binom{e}{i}} & \longrightarrow & \mathcal{S}_{n-1}^{\mathrm{per},\langle j+1\rangle}(G;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\bigoplus_{H\in J}\bigoplus_{i=0}^{e}\mathcal{S}_{n-1-i}^{\mathrm{per},\langle j\rangle}(H;\mathbb{Z})^{\binom{e}{i}} & \longrightarrow & \mathcal{S}_{n-1}^{\mathrm{per},\langle j\rangle}(G;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

The middle horizontal arrow is an isomorphism since it is induced by the isomorphism (3.33), which is compatible with the involutions. The Five-Lemma implies that the maps

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathcal{S}_{n-i}^{\mathrm{per},\langle j \rangle}(H;\mathbb{Z})^{\binom{e}{i}} \to \mathcal{S}_{n}^{\mathrm{per},\langle j \rangle}(G;\mathbb{Z})$$

are bijective for all $n \in \mathbb{Z}$ if and only if the maps

$$\bigoplus_{H \in J} \bigoplus_{i=0}^{e} \mathcal{S}_{n-i}^{\mathrm{per},\langle j+1 \rangle}(H;\mathbb{Z})^{\binom{e}{i}} \to \mathcal{S}_{n}^{\mathrm{per},\langle j+1 \rangle}(G;\mathbb{Z})$$

are bijective for all $n \in \mathbb{Z}$. This takes care of the decorations $j = 1$ and $j = 0$. A similar argument, where $\widetilde{K}_j(\mathbb{Z}H)$ and $\widetilde{K}_j(\mathbb{Z}G)$ are replaced by $\mathrm{Wh}(H)$ and $\mathrm{Wh}(G)$ in the Rothenberg sequence, shows that this is also true for the decoration s, since it is true for the decoration h which is the the same as $\langle 1 \rangle$. This completes the proof of the theorem. □

References

[1] A. Alves and P. Ontaneda. A formula for the Whitehead group of a three-dimensional crystallographic group. *Topology*, 45(1):1–25, 2006.

[2] M. Banagl and A. Ranicki. Generalized Arf invariants in algebraic L-theory. *Adv. Math.*, 199(2):542–668, 2006.

[3] A. Bartels. On the domain of the assembly map in algebraic K-theory. *Algebr. Geom. Topol.*, 3:1037–1050 (electronic), 2003.

[4] A. Bartels and W. Lück. Isomorphism conjecture for homotopy K-theory and groups acting on trees. *J. Pure Appl. Algebra*, 205(3):660–696, 2006.

[5] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)*, 175:631–689, 2012.

[6] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, 172(1):29–70, 2008.

[7] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones Conjecture and its applications. *Journal of Topology*, 1:57–86, 2008.

[8] H. Bass and M. P. Murthy. Grothendieck groups and Picard groups of abelian group rings. *Ann. of Math. (2)*, 86:16–73, 1967.

[9] J. Bryant, S. Ferry, W. Mio, and S. Weinberger. Topology of homology manifolds. *Ann. of Math. (2)*, 143(3):435–467, 1996.

[10] S. E. Cappell. Unitary nilpotent groups and Hermitian K-theory. I. *Bull. Amer. Math. Soc.*, 80:1117–1122, 1974.

[11] D. W. Carter. Lower K-theory of finite groups. *Comm. Algebra*, 8(20):1927–1937, 1980.

[12] F. Connolly and A. Ranicki. On the calculation of UNil. *Adv. Math.*, 195(1):205–258, 2005.

[13] F. X. Connolly and J. F. Davis. The surgery obstruction groups of the infinite dihedral group. *Geom. Topol.*, 8:1043–1078 (electronic), 2004.

[14] J. F. Davis. Some remarks on Nil-groups in algebraic K-theory. Preprint, arXiv:math.KT/0803.1641, 2008.

[15] J. F. Davis, Q. Khan, and A. Ranicki. Algebraic K-theory over the infinite dihedral group: an algebraic approach. *Algebr. Geom. Topol.*, 11(4):2391–2436, 2011.

[16] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[17] J. F. Davis and W. Lück. The p-chain spectral sequence. *K-Theory*, 30(1):71–104, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part I.

[18] J. F. Davis and W. Lück. The topological K-theory of certain crystallographic groups. Preprint, arXiv:1004.2660v1, to appear in Journal of Non-Commutative Geometry, 2010.

[19] J. F. Davis, F. Quinn, and H. Reich. Algebraic K-theory over the infinite dihedral group: a controlled topology approach. *J. Topol.*, 4(3):505–528, 2011.

[20] W. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta Math.*, 190(1):1–104, 2003.

[21] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K-theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993.

[22] F. T. Farrell and L. E. Jones. The lower algebraic K-theory of virtually infinite cyclic groups. *K-Theory*, 9(1):13–30, 1995.

[23] F. T. Farrell, L. E. Jones, and W. Lück. A caveat on the isomorphism conjecture in L-theory. *Forum Math.*, 14(3):413–418, 2002.

[24] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer-Verlag, New York, 1987.

[25] J. Grunewald. The behavior of Nil-groups under localization and the relative assembly map. *Topology*, 47(3):160–202, 2008.

[26] W. Lück. Chern characters for proper equivariant homology theories and applications to K- and L-theory. *J. Reine Angew. Math.*, 543:193–234, 2002.

[27] W. Lück. K- and L-theory of the semi-direct product of the discrete 3-dimensional Heisenberg group by $\mathbb{Z}/4$. *Geom. Topol.*, 9:1639–1676 (electronic), 2005.

[28] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[29] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[30] W. Lück and R. Stamm. Computations of K- and L-theory of cocompact planar groups. *K-Theory*, 21(3):249–292, 2000.

[31] W. Lück and M. Weiermann. On the classifying space of the family of virtually cyclic subgroups. *PAMQ*, 8(2):497–555, 2012.

[32] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.

[33] K. Pearson. Algebraic K-theory of two-dimensional crystallographic groups. *K-Theory*, 14(3):265–280, 1998.

[34] F. Quinn. Hyperelementary assembly for K-theory of virtually abelian groups. Preprint, arXiv:math.KT/0509294, 2005.

[35] A. A. Ranicki. *Exact sequences in the algebraic theory of surgery.* Princeton University Press, Princeton, N.J., 1981.

[36] A. A. Ranicki. *Algebraic L-theory and topological manifolds.* Cambridge University Press, Cambridge, 1992.

[37] A. A. Ranicki. *Lower K- and L-theory.* Cambridge University Press, Cambridge, 1992.

[38] J. Rosenberg. *Algebraic K-theory and its applications.* Springer-Verlag, New York, 1994.

[39] T. tom Dieck. *Transformation groups.* Walter de Gruyter & Co., Berlin, 1987.